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Direct and inverse scattering problems in dispersive media—Green's functions and invariant imbedding techniques

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Abstract

Transient electromagnetic wave propagation in a dispersive medium is reviewed. The medium is assumed to be 1) linear, 2) invariant to time translations, 3) causal, 4) continuous, and 5) isotropic. The constitutive relations are then uniquely represented by a Riemann-Stieltjes integral in the time variable. The kernel in this convolution is the susceptibility kernel. Two explicit examples of mathematical models of the susceptibility kernel are given. The medium treated in this paper is assumed to vary only with depth. In the direct problem the reflection and transmission data are computed. The inverse scattering problem is to find the susceptibility kernel from known reflexion data. It is, thus, a problem of finding a function depending on the time variable. In the spatially homogeneous case the inverse scattering problem is solved from reflexion data by solving a Volterra integral equation of the second kind. This inverse problem is therefore well-posed and easy to solve.

1 Dispersive media and Mathematical model

The electromagnetic field is modeled by the Maxwell equations, which in the absence of free charges are

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}, t) = -\partial_t \mathbf{B}(\mathbf{r}, t) \\ \nabla \times \mathbf{H}(\mathbf{r}, t) = \partial_t \mathbf{D}(\mathbf{r}, t). \end{cases} \quad (1.1)$$

These equations are, however, not complete. Six more equations, the constitutive relations, have to be added relating the electric field \mathbf{E} , the magnetic induction \mathbf{B} , the displacement field \mathbf{D} and the magnetic field \mathbf{H} to each other. These constitutive relations are completely independent of the Maxwell equations and are concerned with the equation of motion of the charges of the medium in an electromagnetic field [5]. The traditional way of describing these constitutive relations is as a relation at fixed frequency, viz.

$$\begin{cases} \mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} \\ \mathbf{H} = \mathbf{B} / \mu_0, \end{cases} \quad (1.2)$$

where the constants ϵ_0 and μ_0 are the permittivity and the permeability of vacuum, respectively, and ϵ_r is the relative permittivity of the medium.

In this paper, however, a time domain treatment is appropriate. Several different models for the constitutive relations, emphasizing different aspects of interaction, are found in the literature. The constitutive relations used in this paper are [1]

$$\begin{cases} \mathbf{D}(\mathbf{r}, t) = \epsilon_0 \{ \epsilon_1(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) + (G(\mathbf{r}, \cdot) * \mathbf{E}(\mathbf{r}, \cdot))(t) \} \\ \mathbf{H}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t) / \mu_0. \end{cases} \quad (1.3)$$

The time convolution integral in this equation is defined as

$$(G(\mathbf{r}, \cdot) * \mathbf{E}(\mathbf{r}, \cdot))(t) = \int_{-\infty}^t G(\mathbf{r}, t - t') \mathbf{E}(\mathbf{r}, t') dt'.$$

The function $\epsilon_1(\mathbf{r})$ models the instantaneous response of the medium and represents the relative permittivity of the medium at high frequencies, i.e. optical frequencies. The value of this function is usually taken as a constant equal to one, but to account for optical polarization effects a value different from one is sometime used. The kernel $G(\mathbf{r}, t)$ is the *susceptibility kernel* of the medium. The relation between the magnetic induction \mathbf{B} and the magnetic field \mathbf{H} is simply a multiplication factor, and no magnetic effects are therefore included in the treatment given here. The generalization to (linear) magnetic media can be made.

From the Maxwell equations, (1.1), and the constitutive relations, (1.3), it is easy to derive the wave equation for the electric field

$$\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2} \partial_t^2 \{ \epsilon_1 \mathbf{E} + G * \mathbf{E} \} = 0,$$

where $c = 1/\sqrt{\epsilon_0 \mu_0}$ is the velocity of light in vacuum.

In order to motivate the constitutive relations in (1.3), some mathematical definitions are appropriate.

Definition 1.1. *A function $f(t)$, defined on $(-\infty, \infty)$, is in H^n if $f(t) = 0$ on $(-\infty, 0)$, and if $f(t)$ is n times continuously differentiable on $[0, \infty)$.*

Definition 1.2. *The class \mathcal{C} is defined as all vector-valued functions with components that are continuous on $t \in (-\infty, \infty)$ and that are identically zero on $t \in (-\infty, 0)$.*

Denote the transformation that maps the vector field $\mathbf{E} \in \mathcal{C}$ to the vector field \mathbf{D} by L , i.e.

$$\mathbf{D}(\mathbf{r}, t) = L[\mathbf{E}(\mathbf{r}, \cdot)](t).$$

The transformation L between the displacement field \mathbf{D} and the electric field \mathbf{E} is assumed to satisfy the following conditions for each fixed \mathbf{r} .

1. The transformation is linear, i.e. for every pair of real numbers α, β
 $L[\alpha \mathbf{E} + \beta \mathbf{E}'] = \alpha L[\mathbf{E}] + \beta L[\mathbf{E}']$.
2. The transformation is invariant to time translations, i.e. for every fixed time $\tau > 0$ the relation $\mathbf{E}'(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t - \tau)$ for all $t \in (-\infty, \infty)$ implies
 $\mathbf{D}'(\mathbf{r}, t) = L[\mathbf{E}'(\mathbf{r}, \cdot)](t) = \mathbf{D}(\mathbf{r}, t - \tau)$ for all $t \in (-\infty, \infty)$.
3. The transformation satisfies causality, i.e. for every fixed τ such that $\mathbf{E} = 0$ on $t \in (-\infty, \tau]$ implies $\mathbf{D} = 0$ on $t \in (-\infty, \tau]$.
4. The transformation is continuous, i.e. for every fixed τ and every $\epsilon > 0$ there exists a $\delta(\epsilon, \tau) > 0$ such that $|\mathbf{E}(\mathbf{r}, t)| < \delta(\epsilon, \tau)$ for all $t \in (-\infty, \tau]$ implies
 $|\mathbf{D}(\mathbf{r}, \tau)| < \epsilon$.
5. The transformation L is isotropic, i.e. the transformation L has the same coordinate representation in all rotated Cartesian coordinate systems.

These assumptions hold at each space point \mathbf{r} . The regularity requirements on the fields as functions of the spatial variables \mathbf{r} are not critical for the applications made in this paper. For convenience, assume all fields continuously differentiable in the spatial variables. Notice that the transformation L is local in the spatial variables \mathbf{r} and non-local in time t .

The transformation L maps \mathcal{C} into \mathcal{C} and, furthermore, L can uniquely be represented by the Riemann-Stieltjes convolution integral (for the proof of these statements in the scalar and vector cases, see, e.g., [9] and [8], respectively)

$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^t \mathbf{E}(\mathbf{r}, t - t') d\mathcal{G}(\mathbf{r}, t'),$$

where the function $\mathcal{G}(\mathbf{r}, t)$, defined on $t \in (-\infty, \infty)$, has the following properties as a function of time t

1. $\mathcal{G}(\mathbf{r}, t) = 0$ on $t \in (-\infty, 0)$.
2. $\mathcal{G}(\mathbf{r}, t)$ is of bounded variation on every closed subinterval of the real axis, as a function of time t .
3. $\mathcal{G}(\mathbf{r}, t)$ is continuous on the right on $t \in (-\infty, \infty)$, i.e. $\mathcal{G}(\mathbf{r}, t) = \mathcal{G}(\mathbf{r}, t+)$.

The motivation of the constitutive relations in (1.3) should now be clear. The constitutive relations in (1.3) can be seen as a further restriction of $\mathcal{G}(\mathbf{r}, t)$ to the class H^1 and $\epsilon_0 G(\mathbf{r}, t) = \partial_t \mathcal{G}(\mathbf{r}, t)$ and $\epsilon_0 \epsilon_1(\mathbf{r}) = \mathcal{G}(\mathbf{r}, 0^+)$ ¹.

The results obtained above are valid in the time domain. The conventional way of relating the displacement field $\mathbf{D}(\mathbf{r}, t)$ and the electric field $\mathbf{E}(\mathbf{r}, t)$ is as a relation in the Fourier domain, i.e. at fixed frequency ω . Denote the Fourier transform of the time variable t by a carat ($\hat{}$), i.e.

$$\begin{aligned} \hat{\mathbf{D}}(\mathbf{r}, \omega) &= \int_{-\infty}^{\infty} \mathbf{D}(\mathbf{r}, t) e^{i\omega t} dt \\ \mathbf{D}(\mathbf{r}, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\mathbf{D}}(\mathbf{r}, \omega) e^{-i\omega t} d\omega, \end{aligned}$$

and similarly for the electric field $\mathbf{E}(\mathbf{r}, t)$. The constitutive relation (1.3) is then transformed into

$$\begin{aligned} \hat{\mathbf{D}}(\mathbf{r}, \omega) &= \epsilon_0 \int_{-\infty}^{\infty} \{ \epsilon_1(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) + (G(\mathbf{r}, \cdot) * \mathbf{E}(\mathbf{r}, \cdot))(t) \} e^{i\omega t} dt \\ &= \epsilon_0 \epsilon_r(\mathbf{r}, \omega) \hat{\mathbf{E}}(\mathbf{r}, \omega), \end{aligned} \tag{1.4}$$

where

$$\epsilon_r(\mathbf{r}, \omega) = \epsilon_1(\mathbf{r}) + \int_0^{\infty} G(\mathbf{r}, t) e^{i\omega t} dt,$$

with inverse

$$G(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\epsilon_r(\mathbf{r}, \omega) - \epsilon_1(\mathbf{r})) e^{-i\omega t} d\omega.$$

The equation (1.4) is equivalent to the constitutive relation (1.2) above.

¹In the sections below, $\mathcal{G}(\mathbf{r}, t)$ is further restricted to the class H^2 .

2 Explicit examples

Two explicit examples of $G(\mathbf{r}, t)$ are presented in this section. More complex models can be constructed by generalizing the arguments in these examples. Note, however, that no specific assumptions of the form of the susceptibility kernel $G(\mathbf{r}, t)$ are made in the treatment of the direct and inverse scattering problems below (other than being a smooth function of time t).

2.1 Resonance model

The basic assumption in this model, which is an appropriate model for many solids, is that each charge-carrying particle, e.g. electron, is affected by three forces.

1. An electric force.
2. A restoring harmonic force.
3. A friction force proportional to the velocity of the charge.

The polarization vector $\mathbf{P}(\mathbf{r}, t)$ is defined by

$$\mathbf{D}(\mathbf{r}, t) - \epsilon_0 \mathbf{E}(\mathbf{r}, t) = \mathbf{P}(\mathbf{r}, t) = qN\mathbf{r},$$

where N is the number of charges per unit volume and \mathbf{r} is the average displacement of the charge. The equation of motion of the electron, then implies that the polarization vector $\mathbf{P}(\mathbf{r}, t)$ satisfies²

$$\partial_t^2 \mathbf{P}(\mathbf{r}, t) + \nu(\mathbf{r}) \partial_t \mathbf{P}(\mathbf{r}, t) + \omega_0^2(\mathbf{r}) \mathbf{P}(\mathbf{r}, t) - \omega_p^2(\mathbf{r}) \epsilon_0 \mathbf{E}(\mathbf{r}, t) = 0, \quad (2.1)$$

where $\omega_0(\mathbf{r})$ is the harmonic frequency of the restoring force, $\nu(\mathbf{r})$ is the collision frequency, and $\omega_p(\mathbf{r}) = \sqrt{Nq^2/\epsilon_0 m}$ is the plasma frequency of the medium. All three quantities can, in principle, be functions of \mathbf{r} . The charge and the mass of the particle are q and m , respectively.

Insert the constitutive relations (1.3) in the equation of motion (2.1), and the result is

$$\begin{aligned} & [\epsilon_1(\mathbf{r}) - 1] \partial_t^2 \mathbf{E}(\mathbf{r}, t) + [\nu(\mathbf{r})(\epsilon_1(\mathbf{r}) - 1) + G(\mathbf{r}, 0)] \partial_t \mathbf{E}(\mathbf{r}, t) \\ & + [\omega_0^2(\mathbf{r})(\epsilon_1(\mathbf{r}) - 1) + \partial_t G(\mathbf{r}, 0) + \nu(\mathbf{r})G(\mathbf{r}, 0) - \omega_p^2(\mathbf{r})] \mathbf{E}(\mathbf{r}, t) \\ & + \int_{-\infty}^t f(\mathbf{r}, t - t') \mathbf{E}(\mathbf{r}, t) dt' = 0, \end{aligned}$$

where

$$f(\mathbf{r}, t) = \partial_t^2 G(\mathbf{r}, t) + \nu(\mathbf{r}) \partial_t G(\mathbf{r}, t) + \omega_0^2 G(\mathbf{r}, t).$$

²More precisely, after a spatial averaging, see [6].

Since the electric field is arbitrary, the permittivity function $\epsilon_1(\mathbf{r})$ and the susceptibility kernel $G(\mathbf{r}, t)$ satisfy

$$\begin{cases} \epsilon_1(\mathbf{r}) = 1 \\ G(\mathbf{r}, 0) = 0 \\ \partial_t G(\mathbf{r}, 0) = \omega_p^2(\mathbf{r}) \\ \partial_t^2 G(\mathbf{r}, t) + \nu(\mathbf{r}) \partial_t G(\mathbf{r}, t) + \omega_0^2(\mathbf{r}) G(\mathbf{r}, t) = 0. \end{cases}$$

The unique solution to these equations is ($t > 0$)

$$G(\mathbf{r}, t) = \omega_p^2(\mathbf{r}) e^{-\nu(\mathbf{r})t/2} \frac{\sin \nu_0(\mathbf{r})t}{\nu_0(\mathbf{r})},$$

where $\nu_0^2(\mathbf{r}) = \omega_0^2(\mathbf{r}) - \nu^2(\mathbf{r})/4$. This special form of the susceptibility kernel $G(\mathbf{r}, t)$ is called the resonance or Lorentz model.

In a case where friction can be ignored $\nu = 0$, $G(\mathbf{r}, t)$ is

$$G(\mathbf{r}, t) = \frac{\omega_p^2(\mathbf{r}) \sin \omega_0(\mathbf{r})t}{\omega_0(\mathbf{r})},$$

and in the limiting case of no restoring force $\omega_0 = 0$, $G(\mathbf{r}, t)$ is

$$G(\mathbf{r}, t) = \frac{\omega_p^2(\mathbf{r})}{\nu(\mathbf{r})} \{1 - e^{-\nu(\mathbf{r})t}\}.$$

2.2 Relaxation model

The second model is appropriate when, e.g., the medium consists of molecules which have a permanent polarization that in an unperturbed state are randomly oriented. This model is based upon the assumption that the macroscopic polarization $\mathbf{P}(\mathbf{r}, t)$ of the medium changes with two competing processes. One part tries to align the molecular polarization with the electric field $\mathbf{E}(\mathbf{r}, t)$ and another process that tries to randomly orient it. The rate of alignment is modeled by the frequency $\alpha(\mathbf{r})$ and the process that tries to randomly orient the molecular polarization is modeled by the relaxation time $\tau(\mathbf{r})$. The basic equation, which describes the rate of change of the polarization $\mathbf{P}(\mathbf{r}, t)$, is then

$$\frac{d}{dt} \mathbf{P}(\mathbf{r}, t) = \epsilon_0 \alpha(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) - \frac{1}{\tau(\mathbf{r})} \mathbf{P}(\mathbf{r}, t).$$

Insert the polarization vector $\mathbf{P}(\mathbf{r}, t) = \mathbf{D}(\mathbf{r}, t) - \epsilon_0 \mathbf{E}(\mathbf{r}, t)$ and the constitutive relations (1.3) in this basic equation. The following identity is obtained

$$\begin{aligned} & [\epsilon_1(\mathbf{r}) - 1] \partial_t \mathbf{E}(\mathbf{r}, t) + \left[G(\mathbf{r}, 0) - \alpha(\mathbf{r}) + \frac{1}{\tau(\mathbf{r})} (\epsilon_1(\mathbf{r}) - 1) \right] \mathbf{E}(\mathbf{r}, t) \\ & + \int_{-\infty}^t f(\mathbf{r}, t - t') \mathbf{E}(\mathbf{r}, t') dt' = 0, \end{aligned}$$

where

$$f(\mathbf{r}, t) = \partial_t G(\mathbf{r}, t) + \frac{1}{\tau(\mathbf{r})} G(\mathbf{r}, t).$$

Since the electric field $\mathbf{E}(\mathbf{r}, t)$ is arbitrary, the permittivity function $\epsilon_1(\mathbf{r})$ and the susceptibility kernel $G(\mathbf{r}, t)$ satisfy

$$\begin{cases} \epsilon_1(\mathbf{r}) = 1 \\ G(\mathbf{r}, 0) = \alpha(\mathbf{r}) \\ \tau(\mathbf{r}) \partial_t G(\mathbf{r}, t) + G(\mathbf{r}, t) = 0, \end{cases}$$

with unique solution ($t > 0$)

$$G(\mathbf{r}, t) = \alpha(\mathbf{r}) e^{-t/\tau(\mathbf{r})}.$$

This special form of the susceptibility kernel $G(\mathbf{r}, t)$ is called the relaxation or Debye model.

3 Scattering representation and Wave splitting

In the remaining part of this paper the medium is assumed to vary only with depth z . The inhomogeneous region occupies the region $0 \leq z \leq L$. The medium is modelled by a permittivity kernel $G(z, t)$ and the permittivity $\epsilon_1 = 1$ for simplicity. A homogeneous dispersion-free medium is situated on either side of the slab, see Figure 1.

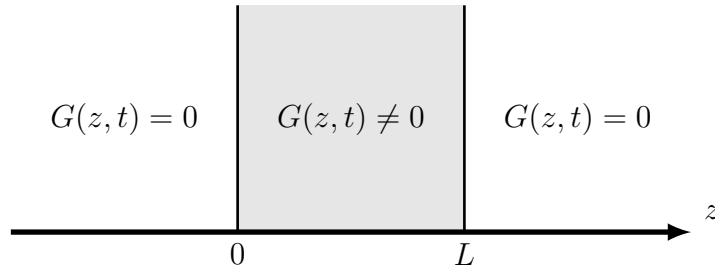


Figure 1

Assume that the electric field has only a transverse component $E(z, t)$ which satisfies the non-local wave equation

$$\partial_z^2 E(z, t) - \frac{1}{c^2} \left\{ \partial_t^2 E(z, t) + (G(z, \cdot) * (\partial_t^2 E(z, \cdot))) (t) \right\} = 0,$$

or as a system of first order equations

$$\partial_z \begin{pmatrix} E \\ \partial_z E \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{c^2} \{ \partial_t^2 + G * \partial_t^2 \} & 0 \end{pmatrix} \begin{pmatrix} E \\ \partial_z E \end{pmatrix} = \mathcal{D} \begin{pmatrix} E \\ \partial_z E \end{pmatrix}. \quad (3.1)$$

To the left of the slab, $z < 0$, the electric field is a sum of two parts, one right-going incident wave, $E^i(t)$, and one left-going reflected wave, $E^r(t)$. Similarly, to the right of the slab, $z > L$, the electric field consists of one right-going transmitted wave, $E^t(t)$. Thus, there are sources to the left of the slab, and no sources to the right of the slab. The total field $E(z, t)$ outside the slab is therefore

$$E(z, t) = \begin{cases} E^i(t - z/c) + E^r(t + z/c), & z < 0 \\ E^t(t - z/c), & z > L. \end{cases}$$

The incident and the scattered fields are related by scattering operators. These relations are integral operators represented by [1]

$$E^r(t) = \int_{-\infty}^t R(t - t') E^i(t') dt' \quad (3.2)$$

$$E^t(t) = \tau(L) \left\{ E^i(t) + \int_{-\infty}^t T(t - t') E^i(t') dt' \right\}, \quad (3.3)$$

where the kernels $R(t)$ and $T(t)$ are the reflection and the transmission kernels of the slab, respectively, for an incident wave from the left. These kernels are independent of how the slab is excited, i.e. uniquely determined by the susceptibility kernel $G(z, t)$. The constant $\tau(L) = \exp \left\{ - \int_0^L G(z, 0) dz / 2c \right\}$ gives the attenuation of the wave through the medium. Notice that if $E^i(t) = \delta(t)$ (where δ is the Dirac delta function), then it follows that $E^r(t) = R(t)$ and $E^t(t) = \tau(L) \{ \delta(t) + T(t) \}$. Hence, the scattering kernels $R(t)$ and $T(t)$ are the impulse responses of the medium.

The inverse scattering problem for this dispersive slab is to find the susceptibility kernel $G(z, t)$ from reflection data $R(t)$. This problem was first solved in Ref. [1]. The analogous case to find $G(z, t)$ from transmission data $T(t)$ is treated in [7]. The direct problem is the opposite situation, where the susceptibility kernel is known and reflection (and transmission) data are unknown.

One of the keystones in the theory of this paper is the wave splitting transformation, see, e.g., [2] and [3]. This is a transformation of dependent variables, from the pair $\{E, \partial_z E\}$ to another pair $\{E^+, E^-\}$ defined by

$$E^\pm(z, t) = \frac{1}{2} \left\{ E(z, t) \mp c \int_{-\infty}^t \partial_z E(z, t') dt' \right\}.$$

This wave splitting transformation can be written in a matrix shorthand notation as

$$\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -c\partial_t^{-1} \\ 1 & c\partial_t^{-1} \end{pmatrix} \begin{pmatrix} E \\ \partial_z E \end{pmatrix} = \mathcal{T} \begin{pmatrix} E \\ \partial_z E \end{pmatrix}. \quad (3.4)$$

The operator \mathcal{T} has a formal inverse

$$\mathcal{T}^{-1} = \begin{pmatrix} 1 & 1 \\ -c^{-1}\partial_t & c^{-1}\partial_t \end{pmatrix},$$

that will be used below.

In a homogeneous and non-dispersive region this wave splitting transformation has the effect of projecting out the left-going and the right-going parts of the field. More explicitly, in a non-dispersive region the general solution to (3.1) is

$$E(z, t) = f(t - z/c) + g(t + z/c),$$

where f and g are arbitrary functions. It is then easy to calculate the fields $E^+(z, t)$ and $E^-(z, t)$ defined in (3.4). They are

$$\begin{cases} E^+(z, t) = f(t - z/c) \\ E^-(z, t) = g(t + z/c). \end{cases}$$

In a dispersive region, however, the transformation defined by (3.4) is still well-defined. In this case, the fields $E^+(z, t)$ and $E^-(z, t)$ are *defined* as the left-going and the right-going parts of the field, respectively. The sum of $E^+(z, t)$ and $E^-(z, t)$ is always the total field, i.e.

$$E(z, t) = E^+(z, t) + E^-(z, t). \quad (3.5)$$

The new pair of fields $\{E^+, E^-\}$ satisfies a system of first order non-local equations

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \mathcal{T} \mathcal{D} \mathcal{T}^{-1} \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} E^+ \\ E^- \end{pmatrix}, \quad (3.6)$$

where

$$\begin{cases} \alpha = \frac{1}{c} \partial_t + \frac{1}{2c} G * \partial_t \\ \beta = \frac{1}{2c} G * \partial_t. \end{cases}$$

This equation is easily derived by a combination of (3.1) and (3.4).

4 Imbedding equation

The results presented in this section is a review of the results in Ref. [1].

Consider a subsection $[z, L]$ of the region $[0, L]$, see Figure 2. Mathematically, the original problem, $[0, L]$, is imbedded in a family of problems with the left edge of the slab, z , as an imbedding parameter.

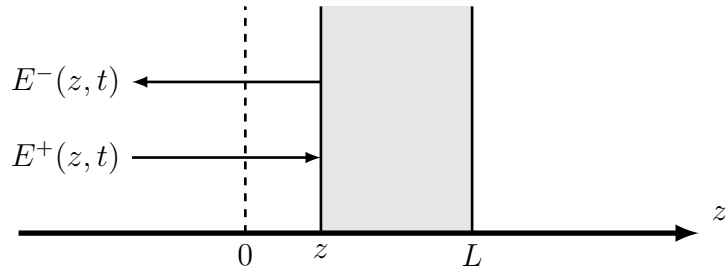


Figure 2

The fields $E^+(z, t)$ and $E^-(z, t)$, defined at the position z , are related in a similar fashion as the incident field and the scattered fields in (3.2) and (3.3) of the original physical problem, i.e. [1]³

$$E^-(z, t) = \int_{-\infty}^t R(z, t - t') E^+(z, t') dt' \quad (4.1)$$

$$E^+(L, t + \frac{L - z}{c}) = \frac{\tau(L)}{\tau(z)} \left\{ E^+(z, t) + \int_{-\infty}^t T(z, t - t') E^+(z, t') dt' \right\}, \quad (4.2)$$

where $\tau(z) = \exp \left\{ - \int_0^z G(z', 0) dz' / 2c \right\}$. The kernels $R(z, t)$ and $T(z, t)$ can be interpreted as the reflection and the transmission kernels, respectively, for the subsection $[z, L]$, where the medium to the left of z is non-dispersive, i.e. $G(z', t) = 0$ for $z' < z$. The field $E^+(z, t)$ serves as an incident field, while $E^-(z, t)$ is a reflected field, and $E^+(L, t)$ a transmitted field for this subsection problem. For the special value $z = 0$, the physical scattering kernels, $R(t)$ and $T(t)$, in (3.2) and (3.3), are identical to $R(0, t)$ and $T(0, t)$, respectively. Hence, the scattering kernels, $R(t)$ and $T(t)$, for the physical region $[0, L]$ are imbedded in a family of subsection problems $[z, L]$ with scattering kernels $R(z, t)$ and $T(z, t)$.

The differential equation for the fields $E^\pm(z, t)$, (3.6), and the relation between the fields $E^\pm(z, t)$, given by (4.1) and (4.2), imply that the reflection and transmission kernels $R(z, t)$ and $T(z, t)$ satisfy non-linear differential equations. Lengthy, but straightforward, calculations show [1] ($0 < z < L$, $t > 0$)

$$2c\partial_z R(z, t) - 4\partial_t R(z, t) = \partial_t G(z, t) + \partial_t \int_0^t G(z, t - t') \left\{ 2R(z, t') + \int_0^{t'} R(z, t' - t'') R(z, t'') dt'' \right\} dt' \quad (4.3)$$

$$R(z, 0) = -\frac{1}{4}G(z, 0)$$

$$R(L, t) = 0$$

$$[R(z, t)]_{\frac{2(L-z)}{c}-0}^{\frac{2(L-z)}{c}+0} = \frac{1}{4}G(z, 0) \exp \left\{ - \int_z^L G(z', 0) dz' / c \right\}$$

$$2c\partial_z T(z, t) = \partial_t G(z, t) - G(z, 0)T(z, t) + \partial_t \int_0^t G(z, t - t') \left\{ T(z, t') + R(z, t') + \int_0^{t'} T(z, t' - t'') R(z, t'') dt'' \right\} dt'. \quad (4.4)$$

The kernels $R(z, t)$ and $T(z, t)$ are continuous everywhere, except possibly $R(z, t)$, which can have a jump discontinuity along the curve $t = \frac{2(L-z)}{c}$. The jump discontinuity is denoted by a square bracket.

These equations can be used to solve the direct scattering problem and to solve the inverse scattering problem for a homogeneous slab of finite length [1]. Numerical

³The definition of the transmission kernel $T(z, t)$ differs slightly compared to [1].

examples and algorithms for the direct and inverse scattering problems are presented in Ref. [1].

The scattering problem simplifies if the slab is homogeneous and of semi-infinite extent ($L \rightarrow \infty$). In the homogeneous, semi-infinite case the reflection kernel is independent of z and $R(z, t) = R(0, t) = R(t)$. The equation (4.3) can then be integrated in time and the result is (the susceptibility kernel is independent of z and denoted $G(t)$)

$$4R(t) + G(t) + (G * (2R + R * R))(t) = 0, \quad t > 0. \quad (4.5)$$

The solution of the inverse problem is particularly simple in this homogeneous case. Assuming $R(t)$ known, equation (4.5) is a Volterra integral equation of the second kind for the unknown susceptibility kernel $G(t)$. The inverse scattering problem is, therefore, well-posed and very easy to solve numerically. In Ref. [1] it is also shown that the direct scattering problem is well-posed in the homogeneous, semi-infinite case.

Some exactly soluble cases in the homogeneous, semi-infinite case can be obtained by Laplace transform techniques [1].

$$G(t) = \alpha e^{\beta t}, \quad (\alpha \text{ and } \beta \text{ are constants}), \quad t > 0,$$

implies

$$R(t) = -e^{(\beta - \alpha/2)t} \frac{I_1(\alpha t/2)}{t},$$

where I_1 is the modified Bessel function of the first kind. This is the homogeneous relaxation model found in Section 2.2. Another exactly soluble case is

$$G(t) = \alpha t e^{\beta t}, \quad (\alpha \text{ and } \beta \text{ are constants}), \quad t > 0,$$

which implies

$$R(t) = -2e^{\beta t} \frac{J_2(\alpha^{1/2} t)}{t},$$

where J_2 is the Bessel function of the first kind.

5 Green's function

The representation used to derive the imbedding equations in Section 4, given by (4.1) and (4.2), uses the scattering kernels for the subsection problem $[z, L]$. It is also possible to relate the fields $E^+(z, t)$ and $E^-(z, t)$ to the external excitation $E^+(0, t)$. These relations are integral operators represented by⁴

$$E^+(z, t) = \tau(z) \left\{ E^+(0, t - z/c) + \int_{-\infty}^{t-z/c} G_1(z, t - t') E^+(0, t') dt' \right\} \quad (5.1)$$

$$E^-(z, t) = \tau^{-1}(z) \int_{-\infty}^{t-z/c} G_2(z, t - t') E^+(0, t') dt', \quad (5.2)$$

⁴These representations can be derived by a variation of Duhamel's integral [4] p. 512.

where $\tau(z) = \exp \left\{ - \int_0^z G(z', 0) dz' / 2c \right\}$. The two kernels, $G_1(z, t)$ and $G_2(z, t)$, are the Green's functions of the time dependent problem. The sum of (5.1) and (5.2) gives the total internal field, $E(z, t)$, in the slab, see (3.5), and Figure 3.

The representations in (5.1) and (5.2) can be used to very efficiently calculate the internal field. The corresponding dissipative case is analyzed in Ref. [10] and numerical examples are also given in that reference.

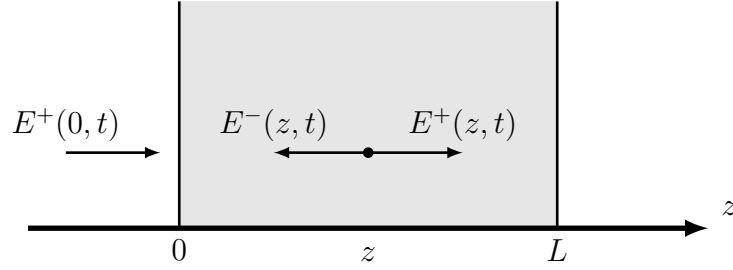


Figure 3

The Green's functions at $z = 0$ and $z = L$ are related to the reflection and transmission kernels in (4.1) and (4.2). The boundary values of G_1 and G_2 are

$$\begin{aligned} G_1(0, t) &= 0 \\ G_2(0, t) &= R(0, t) = R(t) \\ G_1(L, t + L/c) &= T(0, t) = T(t) \\ G_2(L, t) &= 0. \end{aligned}$$

These boundary values are easily found by comparing (4.1) and (4.2) with (5.1) and (5.2).

Following the same line of analysis as in Section 4, the differential equation, (3.6), and this new relations between the fields $E^+(z, t)$ and $E^-(z, t)$, given by (5.1) and (5.2), imply that the Green's functions $G_1(z, t)$ and $G_2(z, t)$ satisfy a system of first order linear differential equations. Lengthy, but straightforward, calculations show ($0 < z < L$, $t > z/c$)

$$\begin{aligned} 2c\partial_z G_1(z, t) + 2\partial_t G_1(z, t) &= -\partial_t G(z, t - z/c) - \int_{z/c}^t \partial_t G(z, t - t') G_1(z, t') dt' \\ &\quad - \tau^{-2}(z) \left\{ G(z, 0) G_2(z, t) + \int_{z/c}^t \partial_t G(z, t - t') G_2(z, t') dt' \right\} \\ 2c\partial_z G_2(z, t) - 2\partial_t G_2(z, t) &= \int_{z/c}^t \partial_t G(z, t - t') G_2(z, t') dt' \\ &\quad + \tau^2(z) \left\{ \partial_t G(z, t - z/c) + G(z, 0) G_1(z, t) + \int_{z/c}^t \partial_t G(z, t - t') G_1(z, t') dt' \right\} \end{aligned}$$

$$\begin{aligned}
G_1(z, z/c) &= -\frac{1}{2c} \int_0^z \left\{ \partial_t G(z', 0) - \frac{1}{4} G^2(z', 0) dz' \right\} \\
G_2(z, z/c) &= -\frac{1}{4} G(z, 0) \tau^2(z) \\
[G_2(z, t)]_{\frac{2L-z}{c}-0}^{\frac{2L-z}{c}+0} &= \frac{1}{4} G(L, 0) \tau^2(L).
\end{aligned}$$

The Green's functions $G_1(z, t)$ and $G_2(z, t)$ are continuous everywhere, except possibly $G_2(z, t)$, which can have a jump discontinuity along the curve $t = \frac{2L-z}{c}$. The jump discontinuity is denoted by a square bracket.

It is convenient to change the time variable t to $\tau = t - z/c$, i.e a time coordinate measured from the wave-front, and to introduce

$$\begin{cases} g_1(z, \tau) = G_1(z, \tau + z/c) \\ g_2(z, \tau) = \tau^{-2}(z) G_2(z, \tau + z/c). \end{cases}$$

The functions $g_1(z, \tau)$ and $g_2(z, \tau)$ satisfy ($0 < z < L$, $\tau > 0$)

$$\begin{aligned}
2c\partial_z g_1(z, \tau) &= G(z, 0) g_1(z, \tau) \\
-\partial_\tau \{G(z, \tau) + (G(z, \cdot) * g_1(z, \cdot))(\tau) + (G(z, \cdot) * g_2(z, \cdot))(\tau)\} & \quad (5.3)
\end{aligned}$$

$$\begin{aligned}
2c\partial_z g_2(z, \tau) - 4\partial_\tau g_2(z, \tau) &= G(z, 0) g_2(z, \tau) \\
+\partial_\tau \{G(z, \tau) + (G(z, \cdot) * g_1(z, \cdot))(\tau) + (G(z, \cdot) * g_2(z, \cdot))(\tau)\} & \quad (5.4)
\end{aligned}$$

$$g_1(z, 0) = -\frac{1}{2c} \int_0^z \left\{ \partial_\tau G(z', 0) - \frac{1}{4} G^2(z', 0) dz' \right\}$$

$$g_2(z, 0) = -\frac{1}{4} G(z, 0)$$

$$g_1(0, \tau) = 0$$

$$g_2(0, \tau) = R(\tau),$$

where the convolution $*$ is defined as

$$(f * g)(\tau) = \int_0^\tau f(\tau - \tau') g(\tau') d\tau'.$$

These Green's functions equations can be used to obtain a numerical algorithm for the electric field inside the dispersive medium. This is in contrast to the imbedding equations, (4.3) and (4.4), for which no simple solution of the internal fields seems accessible. For a solution of the inverse scattering problem in the homogeneous case, the Green's function equations, (5.3) and (5.4), are also of importance.

As a final remark, it is interesting to notice that for a homogeneous slab the reflection kernel $R(t)$ and the susceptibility kernel $G(t)$ satisfy (4.5). This identity implies that the Green's functions $g_1(z, \tau)$ and $g_2(z, \tau)$, evaluated at $z = 0$ satisfy

$$\partial_z g_2(0, \tau) = (g_2(0, \cdot) * \partial_z g_1(0, \tau))(\tau) = (R(\cdot) * \partial_z g_1(0, \tau))(\tau).$$

This relation implies that the two functions $\partial_z g_1(0, \tau)$ and $\partial_z g_2(0, \tau)$ are related by the reflection kernel $R(t)$, i.e. if $\partial_z g_1(0, \tau)$ is an incident field, then $\partial_z g_2(0, \tau)$ is the corresponding reflected field.

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