# An Information Theoretic Charachterization of Channel Shortening Receivers

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# An Information Theoretic Charachterization of Channel Shortening Receivers

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***Abstract*—Optimal data detection of a linear channel can always be implemented through the Viterbi algorithm (VA). However, in many cases of interest the memory of the channel prohibits application of the VA. A popular and conceptually simple method in this case, studied since the early 70s, is to first filter the received signal in order to shorten the memory of the channel, and then to apply a VA that operates with the shorter memory. We shall refer to this as a channel shortening (CS) receiver. Although studied for almost four decades, an information theoretic understanding of what such simple receiver solution is actually doing is not available.**

**In this paper we will show that an optimized CS receiver is implementing the chain rule of mutual information, but only up to the shortened memory that the receiver is operating with. Further, we will show that the tools for analyzing the ensuing achievable rates from an optimized CS receiver are precisely the same as the tools that are used for analyzing the achievable rates of an minimum mean square error (MMSE) receiver.**

***Index Terms*—Receiver design, channel shortening detection, reduced complexity detection, mismatched receivers, mismatched mutual information.**

## I. Introduction

In 1972, Forney [1] showed that the VA can be applied to intersymbol interference (ISI) channels in order to implement maximum likelihood (ML) detection. However, the complexity of the VA is exponential in the memory of the channel which prohibits the VA. As a remedy, Falconer and Magee proposed in 1973 the concept of channel shortening [2], also known as combined linear and Viterbi equalization. The concept is straightforward: (i) filter the received signal with a channel shortening filter so that the effective channel has much shorter duration than the original channel, (ii) Apply the VA to the shorter effective channel. Although Falconeer and Magee's original paper dealt solely with ISI, the concept extends straighforwardly to general linear channels in which case "filter with a channel shortening filter" should be interpreted as a matrix multiplication. After a QR factorization, the VA is then applied. Albeit simple, the achievable rates that can be supported by such receiver was first derived as late as 2012 in [3]. While [3] established the optimal parameters for the CS receiver, no insights into the nature of the optimized CS receiver was given. In this paper we analyze the optimal CS receiver from an information theoretic perspective. The two main findings are (i) An optimized CS receiver implements the chain rule of mutual information up to the reduced memory of the receiver and (ii) The tools for analyzing the achievable rates of CS are precisely the same as those used for analyzing the rates of MMSE receivers [7].

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## II. System model

We consider a system model according to

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}, \tag{1}$$

where $\boldsymbol{y}$ is the $n_{\mathrm{R}} \times 1$ received vector, $\boldsymbol{H}$ is an arbitrary channel matrix of dimension $n_{\mathrm{R}} \times n_{\mathrm{T}}$ that is perfectly known to the receiver, $\boldsymbol{x}$ is an $n_{\mathrm{T}} \times 1$ vector comprising the transmitted symbols from $\mathcal{X}$, and $\boldsymbol{n}$ is an $n_{\mathrm{R}} \times 1$ noise vector. We assume that $\boldsymbol{x}$ is distributed as a zero mean circularily symmetric complex Gaussian distributed vector with covariance $\boldsymbol{I}_{n_{\mathrm{T}}}$ and that $\boldsymbol{n}$ is distributed as a zero mean circularily symmetric complex Gaussian distributed vector with covariance $N_0\boldsymbol{I}_{n_{\mathrm{R}}}$. Note that we are not imposing any structure upon the matrix $\boldsymbol{H}$, so that (1) encompasses many communication systems, such as multiple-input multiple-output (MIMO), intersymbol interference (ISI), MIMO-ISI channels, intercarrier interference (ICI) etc. In our subsequent analysis the underlying structure of the channel matrix is irrelevant - the same results apply in all cases - but we point out that the Toeplitz structure of the matrix can be used to simplify the formulas for the ISI cases.

At the receiver side, a data detector is de-modulating the channel according to a certain detection rule that assigns a metric to each vector $\boldsymbol{x}$ for a given $\boldsymbol{y}$, $\mu(\boldsymbol{x}|\boldsymbol{y})$. In a coded system, not all vectors $\boldsymbol{x}$ can be transmitted, and, ideally, the detector should consider only the set of valid code words $\mathcal{C}$ and choose the one that maximizes the metric, i.e., it should solve

$$\hat{\boldsymbol{x}} = \arg\max_{\boldsymbol{x}\in\mathcal{C}} \mu(\boldsymbol{x}|\boldsymbol{y}). \tag{2}$$

In practice, such a procedure is difficult to carry out, and it is therefore common to apply iterative receivers where the receiver is iterating between channel detection and the decoding of the error control code. Nevertheless, the achievable rates that can be sustained depends ultimately on the detection rule $\mu(\boldsymbol{x}|\boldsymbol{y})$ that the date detector is equipped with. In this paper, we assume that some Search Algorithm (SA) operating with $\mu(\boldsymbol{x}|\boldsymbol{y})$ is adopted that can achieve close to optimal performance in the sense of (2). The pre-dominant approach would be to use the BCJR as the main component of the SA and then perform iterative detection.

An optimal SA for the channel (1) is operating on the basis of the conditional probability density function (pdf)

$$p(\boldsymbol{y}|\boldsymbol{x}) \propto \exp\left(-\frac{\|\boldsymbol{y}-\boldsymbol{H}\boldsymbol{x}\|^2}{N_0}\right). \tag{3}$$

The SA can now be implemented based on $p(\boldsymbol{y}|\boldsymbol{x})$ over a trellis with $|\mathcal{X}|^L$ states, where $L$ is the memory of the channel $\boldsymbol{H}$. Memory is a central theme of the paper and we therefore define it formally.

*Definition 1:* Let $\boldsymbol{G} = \boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$. If $G_{k,\ell} = 0$ for $|k-\ell| > L$ we say that the memory of the channel $\boldsymbol{H}$ is $L$.

For MIMO and ICI, the memory is typically "full" in the sense that $L = n_{\mathrm{T}} - 1$. In those cases, the optimal detector is operating over a tree with $|\mathcal{X}|^{n_{\mathrm{T}}}$ leaf nodes rather than over a trellis. Nevetheless, after linear filtering, we shall compress the memory of the channel, so that trellis processing can be applied. By applying a suitable permutation of the columns of $\boldsymbol{H}$ one can obtain a smaller memory. In this paper we do, however, not treat such permutation. We strongly point out that the aforementioned permutation is not the same as performing a minimum phase conversion of an ISI channel. Minimum phase conversions are covered by Definition 1.

### A. Classical CS

Classical CS [2] proceeds via the following steps,

1) Filter the signal $\boldsymbol{y}$ with a matrix $\boldsymbol{W}$, to obtain $\boldsymbol{r} = \boldsymbol{W}\boldsymbol{y}$.
2) Impose the structure $\boldsymbol{r} = \boldsymbol{F}\boldsymbol{x}+\boldsymbol{w}$, where $\boldsymbol{F}$ is a memory $K < L$ matrix and $\boldsymbol{w}$ a noise vector.
3) Apply the SA to the signal $\boldsymbol{r}$ as if $\boldsymbol{F}$ is the true channel and $\boldsymbol{w}$ a white noise.

In terms of a conditional pdf, classical CS can be expressed as if the SA is operating with the mismatched pdf

$$q(\boldsymbol{y}|\boldsymbol{x}) \propto \exp\left(-\frac{\|\boldsymbol{W}\boldsymbol{y}-\boldsymbol{F}\boldsymbol{x}\|^2}{N_0}\right). \tag{4}$$

Note that it is no loss of generality to assume the same noise density, as the two matrices $\boldsymbol{W}$ and $\boldsymbol{F}$ can be scaled at will.

### B. A new framework for CS

To the best of the authors' knowledge, all previous papers dealing with CS detection has been based on (4) and the goal has been to optimize the receiver parameters $\boldsymbol{W}$ and $\boldsymbol{F}$. However, the system model (4) is not the only system model for CS. As will be discussed in Property 1, the model (4) is not the most suitable model for CS. The conditional pdf $q(\boldsymbol{y}|\boldsymbol{x})$ can be written as

$$q(\boldsymbol{y}|\boldsymbol{x}) \propto \exp\left(\frac{2\mathcal{R}\left\{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{F}^{\mathrm{H}}\boldsymbol{W}\boldsymbol{y}\right\} - \boldsymbol{x}^{\mathrm{H}}\boldsymbol{F}^{\mathrm{H}}\boldsymbol{F}\boldsymbol{x}}{N_0}\right). \tag{5}$$

Based on (5), the SA can still be implemented, with no increase in computational complexity, using the formulation by Ungerboeck in [4] and its BCJR-version [5].

We now modify $q(\boldsymbol{y}|\boldsymbol{x})$ in order to obtain an alternative framework for CS. We propose to abandon (4) in favor of

$$\tilde{q}(\boldsymbol{y}|\boldsymbol{x}) \propto \exp\left(2\mathcal{R}\left\{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{H}_{\mathrm{r}}\boldsymbol{y}\right\} - \boldsymbol{x}^{\mathrm{H}}\boldsymbol{G}_{\mathrm{r}}\boldsymbol{x}\right), \tag{6}$$

where $\boldsymbol{H}_{\mathrm{r}}$ is an arbitrary $n_{\mathrm{T}} \times n_{\mathrm{R}}$ matrix and $\boldsymbol{G}_{\mathrm{r}}$ is an Hermitian matrix of memory $K$. Again, based on [4] the SA can be implemented also for $\tilde{q}(\boldsymbol{y}|\boldsymbol{x})$. Altogether, from a conceptual and a computational complexity point of view, it is irrelevant whether the SA is implemented over $q(\boldsymbol{y}|\boldsymbol{x})$ or $\tilde{q}(\boldsymbol{y}|\boldsymbol{x})$, but as we discuss next, the latter pdf offers important advantages over the former.

*Property 1:* The pdf $\tilde{q}(\boldsymbol{y}|\boldsymbol{x})$ is a more general framework for CS than $q(\boldsymbol{y}|\boldsymbol{x})$ as the matrix $\boldsymbol{G}_{\mathrm{r}}$ need not be positive semi-definite as the matrix $\boldsymbol{F}^{\mathrm{H}}\boldsymbol{F}$ must be. For a given memory $K$, the complexity is identical in both cases.

Ostensibly, it may appear as if Property 1 lacks operational interest as one is tempted to assume that an optimized system would use an indefinite $\boldsymbol{G}_{\mathrm{r}}$ only in rare special cases, but this is not the case. Whenever the channel matrix $\boldsymbol{H}$ contains one or more small, but still strictly positive, eigenvalues, the optimal matrix $\boldsymbol{G}_{\mathrm{r}}$ to use is often indefinite.

If we set $K = n_{\mathrm{T}} - 1$ we reach the true conditional pdf, which means that the optimal SA is included as a special case of CS. Further, with $K = 0$ we can reach the linear MMSE equalizer. Hence, CS has these two well known detectors as limiting cases and, as we will show later, CS shares many properties with the MMSE equalizer.

### C. Special cases of CS

A popular special case of CS is to use a block diagonal form for $\boldsymbol{G}_{\mathrm{r}}$ [6]. We assume that $\boldsymbol{G}_{\mathrm{r}}$ contains $M$ blocks of dimensions $K_m \times K_m,\ 1 \le m \le M$, along the main diagonal, with

$$\sum_{m=1}^{M} K_m = n_{\mathrm{T}}.$$

Ideally, all blocks should have dimension $K_m = K$, but this may not be possible for certain parameter combinations. The rationale of this simplification is not that the detection is done over a single trellis of memory $K$, but rather that the detection is broken up into $M$ trees, of depths $K_m$. An important property of such scheme is

*Property 2:* With a block diagonal constraint on $\boldsymbol{G}_{\mathrm{r}}$, the optimal $\boldsymbol{G}_{\mathrm{r}}$ is always positive semi-definite.

We point out that "optimal" is with respect to generalized mutual information, which will be made more precise in Section III. This implies that for a block diagonal $\boldsymbol{G}_{\mathrm{r}}$ there is no gain in using the new framework from Section II-B as it can always be cast in the form of $q(\boldsymbol{y}|\boldsymbol{x})$ from Section II-A.

## III. Analysis of the Achievable Rates of CS

With an SA that operates with $\tilde{q}(\boldsymbol{y}|\boldsymbol{x})$ instead of the true $p(\boldsymbol{y}|\boldsymbol{x})$, the information rate of the channel

$$I_{\mathrm{R}} = I(\boldsymbol{y};\boldsymbol{x})$$

cannot be reached. (Note that limits and normalization must be included for ISI channels.) Instead, the relevant performance measure is the generalized mutual information (GMI). The GMI establishes a lower bound to the achievable rate that can

be supported with the mismatched conditional pdf. The GMI, in nats/channel input, equals

$$I_{\rm GMI} = -\mathbb{E}\log\left(\tilde{q}(\boldsymbol{y})\right) + \mathbb{E}\log\left(\tilde{q}(\boldsymbol{y}|\boldsymbol{x})\right),$$

where the expectations are with respect to the true conditional pdf $p(\boldsymbol{y}|\boldsymbol{x})$.

Maximization of $I_{\rm GMI}$ over the two matrices $\boldsymbol{H}_{\rm r}$ and $\boldsymbol{G}_{\rm r}$ was carried out in [3], and we state the optimal solutions next.

*Property 3:* The optimal matrix $\boldsymbol{H}_{\rm r}$ is

$$\boldsymbol{H}_{\rm r} = (\boldsymbol{G}_{\rm r} + \boldsymbol{I}_{n_{\rm T}})\boldsymbol{H}^{\rm H}\left[\boldsymbol{H}\boldsymbol{H}^{\rm H} + N_0\boldsymbol{I}_{n_{\rm R}}\right]^{-1}.$$

Let $\boldsymbol{G}_{\rm r} = \boldsymbol{U}\boldsymbol{U}^{\rm H} - \boldsymbol{I}_{n_{\rm T}}$ and define

$$\boldsymbol{B} \triangleq -\boldsymbol{H}^{\dagger}\left[\boldsymbol{H}\boldsymbol{H}^{\dagger} + N_0\boldsymbol{I}_{n_{\rm R}}\right]^{-1}\boldsymbol{H} + \boldsymbol{I}_{n_{\rm T}}. \tag{7}$$

To find the optimal $\boldsymbol{G}_{\rm r}$, one proceeds as follows. Let $\tilde{\boldsymbol{B}}_k^K$ denote the submatrix

$$\tilde{\boldsymbol{B}}_k^K = \begin{bmatrix} B_{k+1\,k+1} & \cdots & B_{k+1\,\kappa} \\ \vdots & \ddots & \vdots \\ B_{\kappa\,k+1} & \cdots & B_{\kappa\,\kappa} \end{bmatrix},$$

of $\boldsymbol{B}$, where $\kappa = \min(n_{\rm T}, k+K)$. Let $\boldsymbol{b}_k^K$ be the row vector $\boldsymbol{b}_k^K = [B_{k\,k+1}, \ldots B_{1\,\min(n_{\rm T},k+K)}]$. For $k = N$, $\tilde{\boldsymbol{B}}_k^{\nu} = 0$ and $\boldsymbol{b}_k^{\nu} = 0$. Let $\boldsymbol{u}_k^K$ denote the row vector $\boldsymbol{u}_k^K = [u_{k\,k+1}, \ldots u_{k\,\min(n_{\rm T},k+K)}]$, where $\{u_{mn}\}$ are the elements of $\boldsymbol{U}$. Then,

$$\max_{\boldsymbol{G}_{\rm r}} I_{\rm GMI} = \sum_{n=1}^{n_{\rm T}} \log\left(\frac{1}{c_n}\right), \tag{8}$$

where the constants $c_n$ are given by

$$c_n = B_{nn} - \boldsymbol{b}_n^K(\tilde{\boldsymbol{B}}_n^K)^{-1}(\boldsymbol{b}_n^K)^{\dagger}.$$

The optimal $\boldsymbol{G}^{\rm r} = \boldsymbol{U}\boldsymbol{U}^{\dagger} - \boldsymbol{I}$ is constructed from

$$u_{nn} = \frac{1}{\sqrt{c_n}}$$

and

$$\boldsymbol{u}_n^K = -u_{nn}\boldsymbol{b}_n^K(\tilde{\boldsymbol{B}}_n^K)^{-1}.$$

The recipe to find the optimal $\boldsymbol{G}_{\rm r}$ in property 3 is fairly involved, and our first result is a simpler characterization of it.

*Lemma 1:* Let $\mathrm{diag}_k(\boldsymbol{X})$ be a matrix of equal dimensions as $\boldsymbol{X}$ that equals $\boldsymbol{X}$ along the center $2K+1$ diagonals and zero elsewhere, i.e., if $\boldsymbol{Z} = \mathrm{diag}_k(\boldsymbol{X})$ then

$$Z_{k\ell} = \begin{cases} X_{k\ell}, & |k-\ell| \le K \\ 0, & |k-\ell| > K. \end{cases}$$

The optimal $\boldsymbol{G}_{\rm r}$ satisfies

$$\mathrm{diag}_K\left([\boldsymbol{G}_{\rm r} + \boldsymbol{I}_{n_{\rm T}}]^{-1}\right) = \mathrm{diag}_K\left([\boldsymbol{G} + N_0\boldsymbol{I}_{n_{\rm T}}]^{-1}\right).$$

Lemma 1 is sufficient to determine the optimal $\boldsymbol{G}_{\rm r}$ which can be verified by counting the number of variables in $\boldsymbol{G}_{\rm r}$ and the number of constraints specified by Lemma 1.

Our next Lemma, first derived in [8], simplifies the expression for the GMI.

*Lemma 2:* For the optimal $\boldsymbol{H}_{\rm r}$ and $\boldsymbol{G}_{\rm r}$, we have

$$I_{\rm GMI} = \log\det\left(\boldsymbol{I}_{n_{\rm T}} + \boldsymbol{G}_{\rm r}\right).$$

We next give our main theorem.

*Theorem 1:* Let $\boldsymbol{H}_{[k,n]}$ be the same matrix as $\boldsymbol{H}$ but with columns $[k, k+1, k+2, \ldots, n]$ removed. Let $\boldsymbol{G}_{[k,n]} = \boldsymbol{H}_{[k,n]}^{\rm H}\boldsymbol{H}_{[k,n]}$ and use the convention $\boldsymbol{G}_{[k,k-m]} = \boldsymbol{G}$ for $m > 0$. For the optimal $\boldsymbol{H}_{\rm r}$ and $\boldsymbol{G}_{\rm r}$, we have

$$\begin{aligned} I_{\rm GMI} &= \log\det\left(\boldsymbol{I}_{n_{\rm T}} + \frac{\boldsymbol{G}}{N_0}\right) \\ &\quad - \sum_{k=1}^{n_{\rm T}-K}\log\det\left(\boldsymbol{I}_{n_{\rm T}} + \frac{\boldsymbol{G}_{[k,k+K]}}{N_0}\right) \\ &\quad + \sum_{k=2}^{n_{\rm T}-K}\log\det\left(\boldsymbol{I}_{n_{\rm T}} + \frac{\boldsymbol{G}_{[k,k+K-1]}}{N_0}\right). \end{aligned} \tag{9}$$

In [7] the MMSE equalizer was analyzed and the following formula for the achievable rate was established,

$$\begin{aligned} I_{\rm MMSE} &= n_{\rm T}\log\det\left(\boldsymbol{I}_{n_{\rm T}} + \frac{\boldsymbol{G}}{N_0}\right) \\ &\quad - \sum_{k=1}^{n_{\rm T}}\log\det\left(\boldsymbol{I}_{n_{\rm T}} + \frac{\boldsymbol{G}_{[k,k]}}{N_0}\right). \end{aligned} \tag{10}$$

By inspection, it can be seen that by setting $K = 0$, Theorem 1 collapses into $I_{\rm MMSE}$ in (10). The structure of the formula for $I_{\rm GMI}$ of CS detection is closely related to that of $I_{\rm MMSE}$. In fact, if there is no transmit correlation, so that the columns of $\boldsymbol{H}$ are IID, then an analysis of $I_{\rm GMI}$ of CS detection for $n_{\rm R} \times n_{\rm T}$ MIMO is equivalent to an analysis of the achievable rates of MMSE for $n_{\rm R} \times (n_{\rm T} - K)$ MIMO. There is an abundance of literature dealing with analysis of the MMSE receiver, and essentially all of those results can be carried over to CS detection through Theorem 1. We will examplify this with an interesting example in Section IV.

We close this section with a re-formulation of Theorem 1 that sheds further light of the nature of CS detection. Recall that by using the chain rule of mutual information, the information rate of the channel can be expressed as

$$I_{\rm R} = \sum_{k=1}^{n_{\rm T}} I(\boldsymbol{y}; x_k|x_{k-1}, \ldots, x_1).$$

We have

*Theorem 2:* The rate $I_{\rm GMI}$ in Theorem 1 can be expressed as

$$I_{\rm GMI} = \sum_{k=1}^{n_{\rm T}} I(\boldsymbol{y}; x_k|x_{k-1}, \ldots, x_{k-K}).$$

Theorem 2 is most intuative: a properly optimized CS detector implements the chain rule of mutual information, but only up to the reduced memory of the receiver. With the block diagonal structure of $\boldsymbol{G}_{\rm r}$, we have the following corollary

*Corollary 1:* With a block diagonal structure of $\boldsymbol{G}_{\rm r}$ with $M$ blocks, each one of dimension $K_m \times K_m$, we have

$$I_{\rm GMI} = \sum_{m=1}^{M}\sum_{k=1}^{K_m} I\left(\boldsymbol{y}; x_{T_m+k}|x_{T_m+k-1}, ..., x_{T_m}\right),$$

where $T_m = \sum_{\ell=1}^{m-1} M_\ell$.
The meaning of Corollary 1 is that the chain rule of mutual information is implemented, but conditioning does not carry over across the blocks.

## IV. Applications

Let us now consider the ergodic achievable rates of $n_{\rm R} \times n_{\rm T}$ MIMO channels comprising IID complex Gaussian random variates, each one with zero mean and unit variance, with CS detection. Since we are interested in ergodic rates and the channel elements are IID, the formula for $\mathbb{E}[I_{\rm GMI}]$ simplifies. This is so since $\boldsymbol{G}_{[k,k+n]}$ is statistically equivalent to $\boldsymbol{G}_{[k+p,k+n+p]}$ for any $p$. Let us introduce the notation

$$\bar{I}[n_{\rm T}, n_{\rm R}, N_0] \triangleq \mathbb{E}\left[\log\det\left(\boldsymbol{I}_{n_{\rm T}} + \frac{\boldsymbol{H}^{\rm H}\boldsymbol{H}}{N_0}\right)\right]$$

where $\boldsymbol{H}$ is an IID complex Gaussian random matrix of dimension $n_{\rm R} \times n_{\rm T}$. Then,

$$\begin{aligned}\mathbb{E}[I_{\rm GMI}] &= \bar{I}[n_{\rm T}, n_{\rm R}, N_0] - (n_{\rm T}-K)\bar{I}[n_{\rm T}-K-1, n_{\rm R}, N_0] \\ &\quad + (n_{\rm T}-K-1)\bar{I}[n_{\rm T}-K, n_{\rm R}, N_0]. \qquad (11)\end{aligned}$$

Let us now consider the asymptotic slope of the achievable ergodic rate

$$S_\infty = \lim_{N_0\to 0} -\frac{\mathbb{E}[I_{\rm GMI}]}{\log(N_0)}.$$

A well known result is [7]

$$\lim_{N_0\to 0} -\frac{\bar{I}[n_{\rm T}, n_{\rm R}, N_0]}{\log(N_0)} = \min(n_{\rm R}, n_{\rm T}).$$

For the MMSE equalizer, i.e., a CS detector with $K = 0$, this gives after a few manipulations

$$S_\infty = \begin{cases} n_{\rm T}, & n_{\rm R} \geq n_{\rm T} \\ 0, & n_{\rm R} < n_{\rm T}. \end{cases}$$

Thus, for MMSE equalization, the asymptotic slope of the ergodic rate is zero if the number of receive antennas is less than the number of transmit antennas. As we shall see next, CS can compensate for the lack of receive antennas.

*Lemma 3:* For an optimized CS detector with memory $K$ we have

$$S_\infty = \begin{cases} n_{\rm T}, & n_{\rm R} \geq n_{\rm T} \\ n_{\rm R}, & n_{\rm R} < n_{\rm T}, n_{\rm R}+K \geq n_{\rm T} \\ 0, & \text{otherwise}. \end{cases}$$

Altogether, in the case of fewer receive antennas than transmit antennas, MMSE equalization is not effective at high SNR. CS detection can compensate for the lack of receive antennas by setting its memory equal to the difference between the antenna numbers. The trade-off between complexity and performance is clear.

Let us now return to the special case of a block diagonal structure of $\boldsymbol{G}_{\rm r}$. In this case we can show

*Lemma 4:* With a block diagonal structure of $\boldsymbol{G}_{\rm r}$ the asymptotic slope of the ergodic rate becomes

$$S_\infty = \begin{cases} n_{\rm T}, & n_{\rm R} \geq n_{\rm T} \\ \sum_{m:K_m > n_{\rm T}-n_{\rm R}} K_m - (n_{\rm T}-n_{\rm R}), & n_{\rm T} > n_{\rm R}. \end{cases}$$

It can be verified that whenever $\max_m K_m \leq K$, the slope in Lemma 3 is always superior to the slope in Lemma 4. We give an example next,

*Example 1:* Let $n_{\rm T} = 6$ and $n_{\rm R} = 4$. In Figure 1 we plot the ergodic rates $\mathbb{E}[I_{\rm GMI}]$ against SNR for different values of the memory of the CS detector. As we can see, whenever the memory $K$ equals the difference $K = n_{\rm T} - n_{\rm R} = 2$, the slope $S_\infty$ is optimal. The SNR penalty can be derived based on the results in [7].

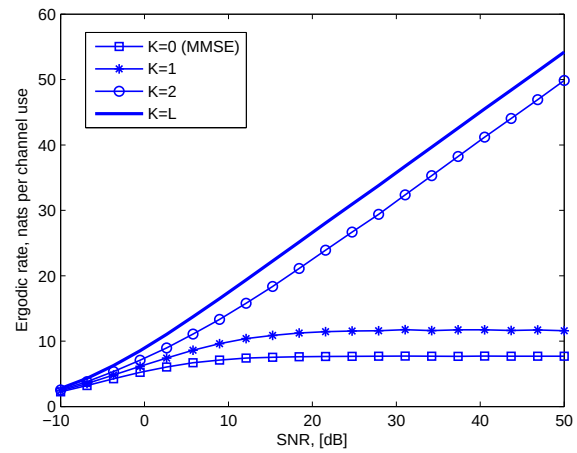


Fig. 1. Ergodic rates of $4 \times 6$ MIMO with IID complex Gaussian channel elementsw with CS detection.

In our next example, we consider the block diagonal special case in Section II-C.

*Example 2:* For the same parameter setup as in Example 1, i.e., $n_{\rm T} = 6$ and $n_{\rm R} = 4$ let us consider a block diagonal structure of $\boldsymbol{G}_{\rm r}$. The matrix $\boldsymbol{G}_{\rm r}$ has dimensions $n_{\rm T} \times n_{\rm T} = 6 \times 6$. What options do we have to select the block sizes? Clearly we can choose three blocks, i.e., $M = 3$, and each block would then be $2\times 2$, i.e., $K_1 = K_2 = K_3 = 2$. However, we then always have $K_k < n_{\rm T} - n_{\rm R} = 2$ so from Lemma 4 we get that $S_\infty = 0$. The conclusion of this is that although the block-diagonal detector with $M = 2$ is more complex than an MMSE equalizer, it does not improve much upon MMSE equalization at high SNR since $S_\infty = 0$.

Another choice would be to pick $M = 2$ and use $K_1 = K_2 = 3$. In view of Lemma 4, we now have $K_1 = K_2 > n_{\rm T} - n_{\rm R}$, and therefore we have that

$$S_\infty = K_1 - (n_{\rm T} - n_{\rm R}) + K_2 - (n_{\rm T} - n_{\rm R}) = 2.$$

This is still inferior to the slope of a CS detector with $K = 2$. An illustration is provided in Figure 2. Note that in this case, the detection complexity of the block diagonal structure is lower than that of a CS with $K = 2$. In the former case, we have two search *trees* of depth 3, while in the latter case we have one *trellis* of memory 2 with depth 6. However, performance is grossly reduced at high SNR.

In our next example we change the parameter settings.

*Example 3:* Let $n_{\rm T} = 6$ and $n_{\rm R} = 5$. In this case we already know from Lemma 3 that CS detection with $K = 1$ is sufficient to reach $S_\infty = \min(n_{\rm R}, n_{\rm T}) = n_{\rm R} = 5$. Detection can be made on the basis of a trellis with memory $K = 1$. The number of states in the trellis becomes $|\mathcal{X}|$.

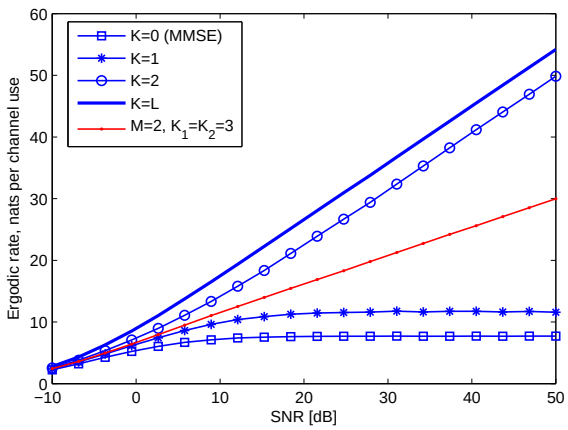


Fig. 2. Same case as in Example 1, but where the ergodic rate corresponding to a block diagonal $\boldsymbol{G}_{\mathrm{r}}$ with $K_1 = K_2 = 3$ has been added.

For the block diagonal structure, we still have the two choices $M = 3$, $K_1 = K_2 = K_3 = 2$ and $M = 2$, $K_1 = K_2 = 3$. For $M = 2$, we get from Lemma 4 that $S_\infty = 3$, while for $M = 3$ we get $S_\infty = 4$. Hence, both cases are worse than CS with $K = 1$. Complexity wise, the $M = 2$ case is less complex than the CS $K = 1$ case. However, for $M = 3$ we have two trees with depth 3. The number of leaf nodes becomes $|\mathcal{X}|^3$ and this is one order worse than the number of states in the trellis multiplied with its branching number $|\mathcal{X}|$. An illustration of the discussed slopes is provided in Figure 3.

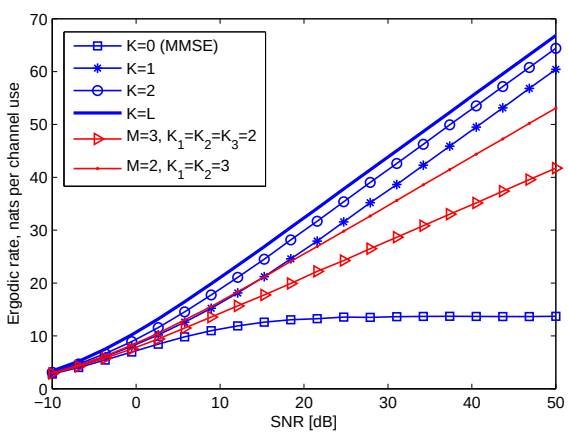


Fig. 3. Illustration of the ergodic rates discussed in Example 3.

We conclude by giving an illustration of what the optimized matrices $\boldsymbol{G}_{\mathrm{r}}$ may actually look like.

*Example 4:* Assume a channel matrix equal to

$$\boldsymbol{H} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

and that $N_0 = 1$. An optimized CS receiver with $K = 1$ has

$$\boldsymbol{G}_{\mathrm{r}} = \begin{bmatrix} 1.33 & -1 & 0 & 0 \\ -1 & 1.93 & 1.25 & 0 \\ 0 & 1.25 & 2.04 & 0.83 \\ 0 & 0 & 0.83 & 0.67 \end{bmatrix}.$$

The trellis structure is arising since there is no cross-coupling between the symbols $(x_1, x_3)$, $(x_1, x_4)$, and $(x_2, x_4)$. Now consider the block diagonal structure with $M = 2$. After optimization, for example via Lemma 1, we get

$$\boldsymbol{G}_{\mathrm{r}} = \begin{bmatrix} 1.33 & -1 & 0 & 0 \\ -1 & 1.33 & 0 & 0 \\ 0 & 0 & 1.417 & 0.83 \\ 0 & 0 & 0.83 & 0.67 \end{bmatrix}.$$

It is interesting to observe that the first and the last rows are not altered compared with the CS $K = 1$ case. This is so since the memory is still 1 at these two rows even with the block diagonal structure. At the two middle rows, the cross-coupling between symbols $(x_2, x_3)$ has been broken and this enforces a somewhat "weaker" matrix $\boldsymbol{G}_{\mathrm{r}}$ at these two rows. Further, due to the separated blocks, the trellis collapses into two trees. Finally, note that the matrix $\boldsymbol{G}_{\mathrm{r}}$ for the CS case is indefinite which means that the framework used in [2] will not be able to produce this particular receiver setting. The block diagonal $\boldsymbol{G}_{\mathrm{r}}$ is always positive semi-definite due to Property 2.

## V. Conclusion

In this paper we have investigated rate optimized channel shortening receivers. We have shown that an optimized receiver can reach the chain rule of mutual information, up to the reduced memory assumed by the receiver. Further, we have shown that the formula for the achievable rate of a receiver with memory $K$ is essentially the same as for an MMSE equlizer of a MIMO system with $K$ transmit antennas less. This results enables significant analytical treatment. As an example of this, we derived the asymptotic capacity slopes at high SNR, and we demonstrated that receiver memory can compensate for a lack of receive antennas.

We also discussed that the classical model for channel shortening is bounded away from the optimal solution due to an inappropriate system model. A better model should be based upon Ungerboeck's formulation of trellis detection for ISI channels.

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