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Model Order Reduction Based on Semidefinite Programming

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Aivar Sootla

Abstract

The main topic of this PhD thesis is complexity reduction of linear time-invariant models. The complexity in such systems is measured by the number of differential equations forming the dynamical system. This number is called the order of the system. Order reduction is typically used as a tool to model complex systems, the simulation of which takes considerable time and/or has overwhelming memory requirements. Any model reflects an approximation of a real world system. Therefore, it is reasonable to sacrifice some model accuracy in order to obtain a simpler representation. Once a low-order model is obtained, the simulation becomes computationally cheaper, which saves time and resources. A low-order model still has to be "similar" to the full order one in some sense. There are many ways of measuring "similarity" and, typically, such a measure is chosen depending on the application.

Three different settings of model order reduction were investigated in the thesis. The first one is $\mathbb{H}_\infty$ model order reduction, i.e., the distance between two models is measured by the $\mathbb{H}_\infty$ norm. Although, the problem has been tackled by many researchers, all the optimal solutions are yet to be found. However, there are a large number of methods, which solve suboptimal problems and deliver accurate approximations. Recently, research community has devoted more attention to large-scale systems and computationally scalable extensions of existing model reduction techniques. The algorithm developed in the thesis is based on the frequency response samples matching. For a large class of systems the computation of the frequency response samples can be done very efficiently. Therefore, the developed algorithm is relatively computationally cheap. The proposed algorithm can be seen as a computationally scalable extension to the well-known Hankel model reduction, which is known to deliver very accurate solutions. One of the reasons for such an assessment is that the relaxation employed in the proposed algorithm is tightly related to the one used in Hankel model reduction. Numerical simulations also show that the accuracy of the method is comparable to the Hankel model reduction one.

The second part of the thesis is devoted to parameterized model order reduction. A parameterized model is essentially a family of models which depend on certain design parameters. The model reduction goal in this setting is to approximate the whole family of models for all values of parameters. The main motivation for such a model reduction setting is design of a model with an appropriate set of parameters. In order to make a good choice of parameters, the models need to be simulated for a large set of parameters. After inspecting the simulation results a model can be picked with suitable frequency or step responses. Parameterized

model reduction significantly simplifies this procedure. The proposed algorithm for parameterized model reduction is a straightforward extension of the one described above. The proposed algorithm is applicable to linear parameter-varying systems modeling as well.

Finally, the third topic is modeling interconnections of systems. In this thesis an interconnection is a collection of systems (or subsystems) connected in a typical block-diagram. In order to avoid confusion, throughout the thesis the entire model is called a supersystem, as opposed to subsystems, which a supersystem consists of. One of the specific cases of structured model reduction is controller reduction. In this problem there are two subsystems: the plant and the controller. Two directions of model reduction of interconnected systems are considered: model reduction in the nu-gap metric and structured model reduction. To some extent, using the nu-gap metric makes it possible to model subsystems without considering the supersystem at all. This property can be exploited for extremely large supersystems for which some forms of analysis (evaluating stability, computing step response, etc.) are intractable. However, a more systematic way of modeling is structured model reduction. There, the objective is to approximate certain subsystems in such a way that crucial characteristics of the given supersystem, such as stability, structure of interconnections, frequency response, are preserved. In structured model reduction all subsystems are taken into account, not only the approximated ones. In order to address structured model reduction, the supersystem is represented in a coprime factor form, where its structure also appears in coprime factors. Using this representation the problem is reduced to $\mathbb{H}_\infty$ model reduction, which is addressed by the presented framework.

All the presented methods are validated on academic or known benchmark problems. Since all the methods are based on semidefinite programming, adding new constraints is a matter of formulating a constraint as a semidefinite one. A number of extensions are presented, which illustrate the power of the approach. Properties of the methods are discussed throughout the thesis while some remaining problems conclude the manuscript.

Acknowledgment

First of all, I would like to thank my thesis advisor Anders Rantzer for giving me the wonderful opportunity of working at the Department of Automatic Control. Without his help on many levels, this work would simply be impossible. Not only did he suggest great research directions, but also provided me with two excellent office-mates Georgios Kotsalis and Kin Cheong Sou. Their insight into model reduction problems was priceless. Both Georgios and Kin supported me during their stay in the department and tried to steer my effort in the right direction. I am also thankful to Georgios and Kin for the work we have done together and for co-authoring papers. I would like also to thank Karl Johan Åström, Per Hagander, Andrey Ghulchak, Alexandru Aleman and Johan Åkesson, who made very valuable comments during different phases of my PhD project.

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Finally, I would like to thank the organizations that provided funding for my research: Estonian Ministry of Education and Research, Toyota Motor Corporation and Swedish Research Council through Linneaus Lund Center for Control of Complex Engineering Systems.

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Preface

Coming with a quite theoretical background to an engineering department was a big challenge. Therefore, I thought of doing more theoretical research and was somewhat hesitant to take on a more applied project. At the same time, I wanted to try something totally new. My PhD supervisor, Anders Rantzer, found the perfect trade-off project for us to work on: model order reduction. While the topic can be very theoretical, there are always applications to pick from. In my research project, I met with a vast variety of practical issues, which did not make sense at first, but later proved to be very important. Finding new questions to ask in model reduction was quite exciting, which will hopefully translate to the readers.

Outline

The outline of the thesis is as follows: the list of notations and frequently used acronyms is provided on pages 15 and 16. Chapter 1 is dedicated to introduction to this thesis. This chapter also provides the background essential for reading the thesis. In Chapter 2 the $\mathbb{H}_\infty$ model reduction is investigated and two algorithms are presented. These algorithms are the basis of the entire work, therefore the reader is recommended to familiarize with the contents of Chapter 2. Chapter 3 is devoted to parameterized model order reduction. Chapters 4 and 5 both deal with the approximation of interconnected models. In Chapter 4, ν-gap model reduction is presented, while in Chapter 5, the so called structured model reduction is investigated. The conclusion and future work directions are outlined in Chapter 6.

The contributions of the thesis are listed below with a technical description and publication references.

Chapter 2

This chapter is dedicated to model order reduction of linear time-invariant systems. Specifically, $\mathbb{H}_\infty$ model order reduction is investigated. One of the

early goals of this PhD project was to develop a scalable algorithm, which can efficiently compute a stable reduced order model of a reasonable quality in the $\mathbb{H}_\infty$ norm. Therefore, the main contribution of this chapter is a derivation of two scalable model reduction algorithms. Both algorithms provide a stable reduced order model. The algorithms perform a curve fitting procedure in the frequency domain using semidefinite programming methods. The input data to the algorithms are samples of the frequency response of a model, computation of which can be done efficiently even for large scale models. Both algorithms are obtained from a reformulation of the model reduction problem. One proposes a semidefinite relaxation, while the other is an iterative semidefinite approach. The relaxation approach is similar to the Hankel model reduction, which is a well-known and established method in the control literature. Due to this resemblance, the accuracy of approximation is also similar to the one of the Hankel model reduction.

An appealing quality of the proposed algorithms is ability to easily perform extensions, e.g., frequency-weighted, positive-real, bounded-real model reduction methods, which are also sketched in this chapter. Advantages of the approach are also illustrated on numerical examples.

Relevant publications are:

Aivar Sootla (2010): "Hankel-type Model Reduction Based on Frequency Response Matching" *In Proceedings of the Conference on Decision and Control.* Atlanta, GA, USA, pp. 5372–5377, Dec. 2010.

Aivar Sootla (2011): "Semidefinite Hankel-type Model Reduction Based on Frequency Response Matching" accepted for publication in *IEEE Transactions on Automatic Control.*

Chapter 3

In this chapter, a parameterized model order reduction framework is investigated. A parameterized model describes a linear time invariant system which also depends on a constant design parameter in addition to the frequency variable. This parameter defines a family of models. The model reduction goal in this setting is to approximate the whole family of models. The presented reduction framework is an extension of the methods developed in Chapter 2 to parameterized models. Therefore, it is also based on frequency response matching with a reasonable performance both in computational time and accuracy. Stability in this setting is guaranteed for every value of a parameter. Further investigation allowed the application of the framework to related problems, such as linear-parameter varying (LPV) system modeling. The theoretical result regarding the quality of approximation (a relaxation gap) is also obtained.

Relevant publications are:

with K. Sou (2010): "Frequency Domain Model Reduction Method for Parameter-Dependent Systems." *In Proceedings of the American Control Conference*, Baltimore, MD, USA, pp. 3082–3087, July 2010.

with K. Sou and A. Rantzer (2011): "Parameterized Model Order Reduction Based on the Semidefinite Programming." submitted to *Automatica*

Chapter 4

This chapter is concerned with a model reduction algorithm in the nu-gap metric. The metric was originally developed to evaluate robustness of a controller for a given plant. Actually, the nu-gap metric induces the weakest topology in the space of controllers, in which stability is a robust property. All in all, the nu-gap metric is perhaps the best metric to evaluate the distance between two systems in an arbitrary closed loop setup. In distributed control, if the approximation of subsystems is considered, such a metric can be vital for modeling purposes. The presented algorithm of the nu-gap model reduction is based on semidefinite programming methods and exploits the frequency domain representation of systems. Therefore, it may be easily extended to incorporate constraints on a frequency region of interest or the closed loop performance bound. The method is an application of the framework developed in Chapter 2 to the nu-gap model reduction problem.

Relevant publications are:

Aivar Sootla (2011): "Nu-gap Model Reduction in the Frequency Domain." *In Proceedings of the American Control Conference*. San Francisco, CA, USA, pp. 5025–5030, June 2011.

Aivar Sootla (2011): "Nu-gap Model Reduction in the Frequency Domain." submitted to *IEEE Transactions on Automatic Control*

Chapter 5

This chapter deals with modeling of structured systems. A structured system in this thesis refers to an interconnection of subsystems in a typical block-diagram. To deal with this problem, a structured model is rewritten in terms of coprime factors of subsystems, while introducing auxiliary inputs and outputs. These signals ensure that the obtained representation has a meaning of coprime factorization of the structured system. After the representation is obtained, the reduction problem is recast as an $\mathbb{H}_\infty$

model reduction one, which is addressed using the framework developed in Chapter 2.

Relevant publications are:

with A. Rantzer (2011): "Model Reduction of Spatially Distributed Systems Using Coprime Factors and Semidefinite Programming." *In Proceedings of the IFAC Wold Congress.* Milan, Italy, pp. 6663–6668, Aug. 2011.

with A. Rantzer (2012): "Convenient Representations of Structured Systems for Model Order Reduction." *submitted to American Control Conference 2012.* Montreal, Canada.

Other Publications:

with A. Rantzer and G. Kotsalis (2009): "Multivariable Optimization-Based Model Reduction " *IEEE Transactions on Automatic Control,* 54:10, pp. 2477–2480, Oct. 2009.

with A. Rantzer (2009): "Extensions to an Optimization-Based Multivariable Reduction Method" *In Proceedings of the European Control Conference,* Budapest, Hungary, pp. 1023–1028, Aug. 2009.

Aivar Sootla (2009): "Model Reduction Using Semidefinite Programming" Licentiate Thesis ISRN LUTFD2/TFRT–3247–SE, Department of Automatic Control, Lund University, Sweden, Nov. 2009.

Nomenclature

Notation	Description
Vector spaces	
J	complex identity
X^T	transpose of a matrix X
X^*	Hermitian transpose of a complex-valued matrix X
$\overline{\sigma}(X)$	maximum singular value of a matrix X
$\mathbb{R}^n$	real vector space
$\mathbb{C}$	space of complex numbers
$\mathbb{D}$	unit disc $\{z \mid |z| < 1\}$ for $z \in \mathbb{C}$
$\partial\mathbb{D}$	unit circle $\{z \mid |z| = 1\}$ for $z \in \mathbb{C}$
$\mathbb{L}_2(\mathbb{A})$	space of square integrable functions on a set $\mathbb{A}$
$\mathbb{L}_\infty(\mathbb{A})$	space of essentially bounded, measurable functions on $\mathbb{A}$
$\mathbb{H}_2$	subspace of $\mathbb{L}_2(\partial\mathbb{D})$ analytic outside the unit disc $\mathbb{D}$ functions
$\mathbb{H}_\infty$	subspace of $\mathbb{L}_\infty(\partial\mathbb{D})$ analytic outside the unit disc $\mathbb{D}$ (for the discrete time) functions
	or subspace of $\mathbb{L}_\infty(\mathrm{Re}\,(s) \geq 0)$ analytic in the right half plane $\mathrm{Re}\,(s) > 0$ (for the continuous time) functions
$\mathbb{H}_\infty^{m_1 \times m_2}$	space of stable m_1 by m_2 matrix valued transfer functions
$\mathcal{R}^{m_1 \times m_2}$	subspace of rational transfer matrices of the space $\mathbb{H}_\infty^{m_1 \times m_2}$

Nomenclature

Notation	Description
Transfer functions	
$\eta(G)$	number of poles of G outside the unit circle
$G^{\sim}$	$G^{\sim}(z) = G^T(1/z)$ for discrete time functions
	or $G^{\sim}(s) = G^T(-s)$ for continuous time functions
$[G, K]$	$[G, K] = \begin{pmatrix} G \\ I \end{pmatrix} (I - KG)^{-1} \begin{pmatrix} -K & I \end{pmatrix}$
$\mathcal{F}_l(N, G)$	Lower fractional transformation between systems N and G
Norms	
$\|\cdot\|_\infty$	$\mathbb{L}_\infty$ norm of a function (see, Section 1.1 on page 19)
$\|\cdot\|_{\mathbb{H}_\infty}$	$\mathbb{H}_\infty$ norm of a function (see, Section 1.1 on page 19)
$\|\cdot\|_H$	Hankel norm of a function (see, page 23)
$\|\cdot\|_F$	Frobenius norm $\|X\|_F^2 = \sum\limits_{i,j=1}^{n} |x_{ij}|^2$, where x_{ij} are the elements of the matrix X
$\|\cdot\|_2$	Euclidean norm of a vector
Acronyms	
LTI	linear time-invariant (system)
LPV	linear parameter-varying (system)
LMI	linear matrix inequality
KYP	Kalman-Yakubovitch-Popov lemma (see, Lemma 1.3 on page 34)
SISO	single-input-single-output (transfer function)
MIMO	multiple-inputs-multiple-outputs (transfer function)
LFT	lower fractional transformation
NCF	normalized coprime factorization (see, Section 1.3)
MOR	model order reduction
PMOR	parameterized model order reduction
QCO	quasi-convex optimization (approach) to model reduction Algorithm 4 on page 37
SHMR	Semidefinite Hankel-type model reduction (method) Algorithm 5 on page 44

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Introduction and Background

Typically, the physical models account for various settings and modes, even those which can be hardly seen in experiments. Therefore, the physical models posses some degree of redundancy. The complexity reduction of a model can facilitate analysis and simulation, while preserving accuracy of a model. This thesis deals with linear-time invariant (LTI) models. These have an explicit complexity measure - the number of differential equations in the model or the order of the model. The accuracy can be measured by the $\mathbb{H}_\infty$ norm, which is one of the typical measures used in model order reduction. Therefore, the model reduction in this setting is called $\mathbb{H}_\infty$ model order reduction, which is one of the main topics of this thesis. All the discussed problems can be reduced to or generalized from the $\mathbb{H}_\infty$ reduction problem.

Most of the existing $\mathbb{H}_\infty$ model order reduction methods fall into two categories: singular value decomposition (SVD) based and Krylov based methods. The SVD based methods include balanced truncation and Hankel model reduction. Balanced truncation ([Moore, 1981]) proposes a simple, yet, very powerful algorithm with a stability guarantee for the reduced model and the approximation error bounds. Hankel model reduction ([Glover, 1984]) is more complicated than balanced truncation, but it has tighter error bounds. Both methods rely on solutions to Lyapunov equations to calculate the approximation, which makes them numerically heavy and thus non-applicable to large-scale models. The Krylov-based methods ([Antoulas *et al.*, 2001; Freund, 2003; Antoulas, 2009]) rely on moment matching techniques. These methods match the derivatives of the transfer functions at pre-defined frequencies, without their explicit computation. They provide much cheaper solutions, however, without explicit error bounds. Both SVD and Krylov methods compute the approximation from state-space representations of full order models. As an alternative,

one can use the frequency domain data, i.e., the frequency response samples, to obtain an approximation. Computing the frequency response for particular applications (e.g., modeling of electro-magnetic structures) can be even cheaper, than inverting the state-space matrix A, as shown in [Kamon *et al.*, 1997; Zhu *et al.*, 2003; Moselhy *et al.*, 2007]. One of the main tools for such an approximation is the interpolation techniques in Hardy spaces ([Anderson and Antoulas, 1986; Fulcheri and Olivi, 1998; Karlsson and Lindquist, 2008; Lefteriu and Antoulas, 2010]).

A different approach based on frequency response matching was introduced in [Sou *et al.*, 2008] and was called the Quasi-Convex Optimization (QCO) approach. It is not an interpolation technique and the frequency sampled data does not necessarily match, however, the objective is to minimize the distance between the frequency response samples of the full and reduced order models. Therefore, there is a bigger degree of flexibility in comparison to interpolation techniques. The method has many advantages in comparison to existing approaches namely:

1. The stability is preserved for a reasonably low computational cost.

2. The method exploits a Hankel type relaxation, for which the relaxation gap (ratio between upper and lower bounds on the solution) is estimated in [Megretski, 2006].

3. The method is based on semidefinite programing, which makes the extensions straightforward as long as new constraints can be expressed in a convex manner. Such extensions, as passive, bounded-real model reduction, are achieved *without* adding extra computational cost.

This chapter is organized as follows. In Section 1.1 system theory background is covered, which is relevant to the model reduction problems. Main topics such as energy functions, Gramians, Hankel operators are required to motivate the SVD methods. In Section 1.2 the SVD based methods, such as balanced truncation and Hankel model reduction, are described. A description of balanced truncation is given for a better understanding of generic mechanisms behind the model reduction algorithms. On the other hand, Hankel model reduction is tightly related to the framework presented in the thesis. The results described in Sections 1.3 and 1.4 will be used throughout the thesis. In Section 1.3 solutions to coprime and spectral factorization problems are presented, while the most common convex optimization techniques are sketched in Section 1.4. The algorithms in Chapter 2 are modifications of the quasi-convex optimization approach. Therefore, this approach is outlined in Section 1.5.

1.1 Systems Theory Background

Only the most relevant system theory concepts are sketched in this section. Most of the concepts are presented in the continuous time setting to simplify the presentation. Nevertheless, the same concepts are valid for the discrete time setting. If a statement or a definition is different in any way in the discrete time, it will be explicitly stated. For further reading see [Zhou *et al.*, 1996] and [Khalil, 2002].

A model in engineering is commonly represented by a system of differential equations. In this thesis, a simple but an important class of models is considered, which can be expressed by a system of linear differential equations with constant coefficients. In control engineering, besides the variables of the equations x (which are called the state-space variables) two additional ones are considered input (control) signals u and output (measurement) signals y. A linear time-invariant (LTI) dynamical system admits a following mathematical model

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^{m_1}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m_1 \times n}$, $D \in \mathbb{R}^{m_1 \times m}$. The matrices A, B, C and D are called *a state-space representation* of a model. In control engineering, signals u are typically designed based on the measurements y. Therefore, from the control theory perspective this dynamical system represents a mapping G_t from the space of control signals u into the space of output signals y and $y(t) = G_t u(t)$. The Laplace transformation of G_t with a complex variable s can be computed as:

$$G(s) = C(sI - A)^{-1}B + D$$

To shorten the notation, we write $G = (A, B, C, D)$, if the matrix D is equal to zero it is omitted in this notation. Sometimes the following notation will be also used:

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

Typically, the function G is evaluated on the imaginary axis, i.e., $s \in j\mathbb{R}$ or simply by replacing s with $j\omega$ where $\omega \in \mathbb{R}$. The variable ω is called a frequency and the function $G(s)$ - *the frequency domain representation* of a model. For every frequency domain representation G there exist infinitely many state-space representations. These representations can be obtained by a state-space transformation $\overline{x} = Tx$. The McMillan degree (or the order) of the function G is the minimal dimension of the state-space vector

x of all state-space realizations (A, B, C, D) of the transfer function G. The McMillan degree is denoted as $\deg(G)$ and a realization (A, B, C, D) with the dimension of the state-space vector x equal to $\deg(G)$ is called *minimal*.

In order reduction, the model is usually assumed to be *asymptotically stable*. For linear time-invariant systems, asymptotic stability is its ability to converge to the origin with a zero input from any point x_0 at zero time. An equivalent definition of asymptotic stability involves evaluating the eigenvalues of the A matrix. A system is called asymptotically stable if the eigenvalues of A lie in the left half of the complex plane, i.e., $\operatorname{Re}(\lambda_i(A)) < 0$ for all i. In the frequency domain, a corresponding stability criterion is that "the poles of G lie in the left half-plane". A system is called *anti-stable* if all the poles of G (or the eigenvalues of A) lie in the right half of the complex plane, i.e., all the poles are unstable.

For LTI systems, an important property is positive realness. A square transfer matrix G will be called *strictly positive real* if it is stable and $G + G^\sim$ is positive definite on the imaginary axis. The notation $G^\sim$ is defined as $G^\sim(s) = G^T(-s)$. Note if G is asymptotically stable, then $G^\sim$ is anti-stable. A positive real function has an interpretation of a passive system, that is, a system which does not generate energy (see [Khalil, 2002]).

In order to compare dynamical systems a metric or a norm should be introduced. One such norm comes from the frequency domain interpretation, which is called *the $\mathbb{H}_\infty$ norm*:

$$\|G\|_{\mathbb{H}_\infty} = \sup_{\omega \in [0, +\infty], G \text{ is stable}} \overline{\sigma}(G(\jmath\omega))$$

where $\overline{\sigma}$ denotes a maximum singular value of a matrix. Note that ω does not take negative values, since for linear systems $\overline{\sigma}(G(\jmath\omega)) = \overline{\sigma}(G(-\jmath\omega))$ and $\arg(G(\jmath\omega)) = -\arg(G(-\jmath\omega))$. Therefore, the norm is computed only for non-negative frequencies ω. The $\mathbb{H}_\infty$ norm also has a time-domain interpretation, it is an induced $\mathbb{L}_2$ norm

$$\|G\|_{\mathbb{H}_\infty} = \sup_{u(\cdot)} \left\{ \frac{\|y(t)\|_{\mathbb{L}_2(-\infty,+\infty)}}{\|u(t)\|_{\mathbb{L}_2(-\infty,+\infty)}} \;\middle|\; y(t) = G_t u(t) \right\}$$

The value $\|u(t)\|_{\mathbb{L}_2(-\infty,+\infty)}$ measures the amount of energy received by the system G, and the value $\|y(t)\|_{\mathbb{L}_2(-\infty,+\infty)}$ measures the amount of energy produced by G, given the received energy $\|u(t)\|_{\mathbb{L}_2(-\infty,+\infty)}$. This norm provides an estimate of how much energy can be produced by a system for an arbitrary input signal u. Note if G is unstable then the maximum produced energy is infinite and, therefore, the $\mathbb{H}_\infty$ norm is equal to infinity.

In order to compare unstable systems as well, *the $\mathbb{L}_\infty$ norm* is introduced.

$$\|G\|_\infty = \sup_{\omega \in [0,+\infty]} \overline{\sigma}(G(\jmath\omega))$$

To define dynamics in the discrete time instead of differential equations, difference equations are used. A dynamical system admits a similar representation:

$$x(t + \Delta t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{m_2}$, $y \in \mathbb{R}^{m_1}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m_2}$, $C \in \mathbb{R}^{m_1 \times n}$, $D \in \mathbb{R}^{m_1 \times m_2}$. A discrete-time dynamical system can be obtained from the continuous-time one by approximating the derivative as

$$\frac{dx}{dt} = \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

and computing the corresponding matrices A, B, C and D.

The transfer function G now is obtained using the Z-transformation $G(z) = C(zI - A)^{-1}B + D$. In the frequency domain discretization is performed as

$$s = \mu \frac{z - 1}{z + 1}$$

where μ depends on the sampling Δt. Such a mapping is one-to-one and it acts from the left half-plane onto the unit disc $\mathbb{D} = \{z \big| |z| < 1\}$. The frequencies are now computed on the unit circle, i.e., for $z = e^{\jmath\omega}$. The frequency ω is confined to an interval $[0, \pi]$, since a similar argument about negative frequencies can be applied as in the continuous time case.

A very common object in the discrete time analysis is *a finite impulse response (FIR) filter*, which is defined as:

$$F(z) = F_0 + F_1 z^{-1} + \cdots + F_k z^{-k}$$

The stability criterion is changed to $|\lambda_i(A)| < 1$ for all i. The condition in the frequency domain is changed as well, now all the poles of a asymptotically stable G lie inside the unit disc $\mathbb{D}$. The definition of the positive real functions and the $\mathbb{H}_\infty$ norm are introduced by replacing $G(\jmath\omega)$ with $G(e^{\jmath\omega})$. That is

$$\|G\|_{\mathbb{H}_\infty} = \sup_{\omega \in [0,\pi], G \text{ is stable}} \overline{\sigma}(G(e^{\jmath\omega}))$$

and the function is strictly positive real if it is asymptotically stable and $G(e^{\jmath\omega}) + G^T(e^{-\jmath\omega})$ is positive definite for all the frequencies ω in $[0, \pi]$.

Energy Functions, Gramians and Hankel operator

A central concept in balanced truncation and Hankel model reduction is a Gramian. It is tightly related to the observability and controllability energy functions defined below.

$$L_o = \int_0^\infty y(t)^T y(t) dt \qquad x(0) = x_0, \quad u(\cdot) = 0$$

$$L_c = \min_{u(\cdot)} \int_{-\infty}^0 u(t)^T u(t) dt \qquad x(-\infty) = 0, \quad x(0) = x_0$$

L_o describes the energy induced only by an initial state x_0 to the output signal y and L_c describes the minimal energy required to reach a state x_0 at the zero time. The energy functions can be computed as:

$$L_o = \langle x_0, Q x_0 \rangle_{\mathbb{R}^n} \qquad L_c = \langle x_0, P^{-1} x_0 \rangle_{\mathbb{R}^n},$$

where

$$Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt \qquad P = \int_0^\infty e^{At} B B^T e^{A^T t} dt$$

The matrices P and Q also satisfy the equations:

$$A^T Q + QA + C^T C = 0 \qquad AP + PA^T + BB^T = 0$$

The matrices P and Q are called the controllability and the observability Gramians respectively. The described equations are called the Lyapunov equations. If all the eigenvalues of A have negative real parts then P, Q are positive semidefinite.

A related object to the energy functions is the Hankel operator Γ_G, which is defined as follows.

$$\Gamma_G : \mathbb{L}(-\infty, 0) \to \mathbb{L}(0, \infty)$$

$$\Gamma_G u(t) = \int_{-\infty}^0 C e^{A(t-\tau)} B u(\tau) d\tau, \quad \text{for} \quad t \geq 0$$

and $\Gamma_G u$ has an interpretation of an output:

$$y(t) = \Gamma_G u(t) \quad t \geq 0$$

The operator maps past inputs into future outputs. Consider the operator $\Gamma_G^* \Gamma_G$. It can be verified that its non-zero eigenvalues are equal to the singular values of PQ. These are called the *Hankel singular values*. The Hankel norm of a transfer function is defined as the norm of its Hankel operator and computed as

$$\|G\|_H = \|\Gamma_G\| = \sqrt{\max \lambda(PQ)}$$

The relationship between the Hankel and the $\mathbb{L}_\infty$ norms is described by the well-known Nehari's theorem:

THEOREM 1.1—NEHARI
Suppose $G \in \mathbb{H}_\infty$, then

$$\inf_{\Delta_- \in \mathbb{H}_\infty} \|G - \Delta_-^{\sim}\|_\infty = \|\Gamma_G\| = \|G\|_H$$

and the infimum is achieved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

The famous Adamyam-Arov-Krein (AAK) theorem may be considered as a generalization of Nehari's theorem. Only a simplified formulation is presented here.

THEOREM 1.2—ADAMYAN-AROV-KREIN
Let $G(s)$ be an asymptotically stable, matrix-valued function bounded on the imaginary axis. Let $\sigma_1 \geq \cdots \geq \sigma_m \geq 0$ be m largest singular values of Γ_G. Then σ_m is the minimum of $\|G - \widehat{G}\|_H$ over the set of all stable systems $\widehat{G}$ of order less than m. $\qquad\qquad\qquad$ □

As a consequence of the AAK theorem one can rewrite the transfer function G as:

$$G(s) = D + \sigma_1 E_1(s) + \cdots + \sigma_n E_n(s)$$

where $E_i(s)$ are all-pass dilations, i.e., for all i the functions $E_i(j\omega)E_i^{\sim}(j\omega)$ are equal to the identity matrix. This representation gives a bound on the $\mathbb{H}_\infty$ norm through the Hankel singular values:

$$\|G\|_{\mathbb{H}_\infty} \leq 2(\sigma_1 + \cdots + \sigma_n)$$

A tighter bound is given by the next lemma:

LEMMA 1.1
Suppose $G \in \mathbb{H}_\infty$, and $\sigma_1 \geq \cdots \geq \sigma_n$ are the Hankel singular values of G, then there exists a constant real-valued matrix D_0 such that:

$$\|G - D_0\|_{\mathbb{H}_\infty} \leq \sigma_1 + \cdots + \sigma_n \leq n\sigma_1 = n\|G\|_H$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In the discrete time setting, a natural replacement for an integral is a sum, providing the definition of the energy functions.

$$L_o = \sum_{t=0}^{\infty} y(t)^T y(t) \qquad x(0) = x_0, \quad u(\cdot) = 0$$

$$L_c = \min_{u(\cdot)} \sum_{t=-\infty}^{0} u(t)^T u(t) \qquad x(-\infty) = 0, \quad x(0) = x_0$$

The functions can be computed in a similar manner, i.e.,

$$L_o = \langle x_0, Q x_0 \rangle_{\mathbb{R}^n} \qquad\qquad L_c = \langle x_0, P^{-1} x_0 \rangle_{\mathbb{R}^n}$$

where positive definite P and Q satisfy the Lyapunov equations:

$$A^T Q A - Q + C^T C = 0 \qquad A P A^T - P + B B^T = 0$$

The unique solutions to these equations exist if the eigenvalues of A lie inside the unit disc $\mathbb{D}$.

   The Nehari and AAK theorems, as well as Lemma 1.1, are also valid for the discrete-time case.

1.2 Basic Model Order Reduction Techniques

Basic model reduction techniques, that is, Hankel model reduction and balanced truncation, are presented only in continuous time. However, the methods were also developed for discrete-time systems. Since they use similar ideas, they are skipped. The main goal of this section is providing a general insight into model reduction mechanisms and not reviewing the existing methods. An interested reader may find a detailed review of the model reduction methods in [Antoulas, 2005] or [Obinata and Anderson, 2001].

   The order reduction problem is set to find a low-order approximation $\widehat{G} = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of the full order one G. The matrix $\hat{A} \in \mathbb{R}^{k \times k}$ and the dimensions of the rest of the matrices are changed, correspondingly. It means that the order for the reduced order model is less or equal to k, i.e., $\deg(\widehat{G}) \leq k$. Suppose $\deg(G) = n$ and n is much larger than k. Formally, one may write the problem as a minimization one:

$$\min_{\deg(\widehat{G}) \leq k} \| G - \widehat{G} \|_{\mathbb{H}_\infty}$$

Balanced Truncation [Moore, 1981]

The intuition behind $\mathbb{H}_\infty$ model reduction is quite simple: reduce the states, which induce a small amount energy into the output, and at the same time the states, which require a large amount of energy to control. Another interpretation of this intuition is reduction of near pole-zero cancellations. In order to explore the energy intuition, recall the energy interpretation of the $\mathbb{H}_\infty$ norm:

$$\| G \|_{\mathbb{H}_\infty} = \sup_{u(\cdot)} \left\{ \frac{\| y(t) \|_{\mathbb{L}_2(-\infty, +\infty)}}{\| u(t) \|_{\mathbb{L}_2(-\infty, +\infty)}} \;\middle|\; y(t) = G_t u(t) \right\}$$

To some extent the amount of received and induced energy can be measured by the energy functions L_c and L_o. These functions can be computed using the Gramians:

$$L_c = \langle x, P^{-1}x \rangle \qquad\qquad L_o = \langle x, Qx \rangle$$

Let P and Q be equal diagonal matrices, where $P_{ii} = Q_{ii} = \sigma_i$, then

$$L_c = \sum_{i=1}^{n} \sigma_i^{-1} x_i^2 \qquad\qquad L_o = \sum_{i=1}^{n} \sigma_i x_i^2$$

where x_i is the i-th entry of x. If σ_i^{-1} is large, then a state x_i requires a large amount of energy to control. Similarly, small σ_i correspond to a small amount of energy induced into the output. In summary, if σ_i is small, then by sending a large amount of energy to the state x_i, the amount of energy induced into the output will be very small. This way, the Gramians may be used to determine which states to truncate, if the matrices P and Q are diagonal. Simultaneous diagonalization may be performed as:

$$\tilde{Q} = T^{-T}QT^{-1} = \Sigma \qquad \tilde{P} = TPT^T = \Sigma \qquad \text{where}$$
$$T = \Sigma^{1/2}U^*P^{-1/2} \qquad \text{and} \qquad P^{1/2}QP^{1/2} = U\Sigma^2U^*$$

The state-space representation (TAT^{-1}, TB, CT^{-1}) is called a balanced realization of G which gave the name to the method. Since a similarity transformation does not effect G, the input-output relationship is still the same. The Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_n$ appear on the diagonal of balanced Gramians:

$$\tilde{P} = \tilde{Q} = \Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix}$$

Algorithm 1 concludes the derivation. Properties of the reduced model are formulated in, for example, [Antoulas, 2005]:

THEOREM 1.3
Assume $\widehat{G}$ is obtained by balanced truncation of an asymptotically stable G. Then $\widehat{G}$ is asymptotically stable and

$$\|G - \widehat{G}\|_{\mathbb{H}_\infty} \leq 2 \sum_{i=k+1}^{n} \sigma_i$$

where $\{\sigma_i\}_{i=k+1}^{n}$ are the truncated Hankel singular values of G. $\qquad\square$

Algorithm 1 Balanced Truncation

- Let $G = (A, B, C)$ be an asymptotically stable system with $A \in \mathbb{R}^{n \times n}$, and B and C have corresponding sizes

- Solve Lyapunov equations for P and Q:

$$A^T Q + QA + C^T C = 0 \qquad AP + PA^T + BB^T = 0$$

- Calculate an invertible matrix $T \in \mathbb{R}^{n \times n}$ which is a state-space transformation, such that

$$TPT^T = T^{-T} Q T^{-1} = \Sigma = \text{diag}\{\sigma_1, \ldots, \sigma_n\}$$

- Let $W = T^{-T} \begin{pmatrix} I_k & 0_{k \times n-k} \end{pmatrix}$ $V = T \begin{pmatrix} I_k & 0_{k \times n-k} \end{pmatrix}$ and obtain the reduced model $(\hat{A}, \hat{B}, \hat{C}) = (W^T AV, W^T B, CV)$, where $\hat{A} \in \mathbb{R}^{k \times k}$ and $\hat{B}$ and $\hat{C}$ have corresponding sizes

---

Hankel Model Reduction [Glover, 1984]

Even though balanced truncation provided an excellent intuition for reducing the states, Hankel model reduction provides an approximation with tighter error bounds. The model reduction problem here is formulated in the Hankel norm:

$$\min_{\deg(\widehat{G}) \leq k, \widehat{G} \in \mathbb{H}_\infty} \|G - \widehat{G}\|_H$$

Using Nehari theorem it can be shown that this optimization problem is equivalent to:

$$\min_{\deg(\widehat{G}) \leq k, \widehat{G}, \Delta_- \in \mathbb{H}_\infty} \|G - \widehat{G} - \Delta_-^\sim\|_\infty \tag{1.1}$$

Note $\Delta_-^\sim$ is an anti-stable transfer matrix (all the poles are unstable). As the reader may recall, any transfer function can be written as:

$$\widehat{G}(s) = D + \sigma_1 E_1(s) + \cdots + \sigma_n E_n(s)$$

using all-pass dilations $E_i(s)$, for which $E_i E_i^\sim = I$ on the imaginary axis, and Hankel singular values σ_i. The idea of the Hankel model reduction boils down to calculating $E_i(s)$ and D_0 such that:

$$\widehat{G}(s) = D + D_0 + \sigma_1 E_1(s) + \cdots + \sigma_k E_k(s)$$

D_0 is required for a tighter error bound, obtained using Lemma 1.1. A detailed description of the algorithm is omitted, due to its technicality and insignificant relevancy to this thesis.

Algorithm 2 Hankel model reduction coupled with optimization

- Solve the optimal Hankel model reduction problem and obtain $\widehat{G} = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$

- Fix $\hat{A}, \hat{B}$ and solve the following optimization problem:

$$\min_{C,D} \|G - C(sI - \hat{A})^{-1}\hat{B} - D\|_{\mathbb{H}_\infty}$$

The properties of the reduced model obtained by Hankel norm minimization are formalized in a statement:

THEOREM 1.4
Suppose G is an asymptotically stable transfer function, $\widehat{G}$ is obtained by Hankel norm minimization. Then $\widehat{G}$ is asymptotically stable and

$$\sigma_{k+1} = \|G - \widehat{G}\|_H \le \|G - \widehat{G}\|_{\mathbb{H}_\infty} \le \sum_{i=k+1}^{n} \sigma_i$$

where $\{\sigma_i\}_{i=k+1}^n$ are the truncated Hankel singular values of G. □

Assume that the obtained $\widehat{G}$ with a state-space representation $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is an optimal solution in the Hankel norm. However, in the $\mathbb{H}_\infty$ norm a better one can be found using the already obtained data. Consider Algorithm 2, the constraint on matrices C, D is convex and the minimization can be solved using semidefinite or second order cone programming. A similar technique is used to obtain the reduced model in this thesis. Note that one can also fix the matrix $\hat{C}$ instead of $\hat{B}$ and optimize then over B and D.

Numerical Complexity

The balanced truncation algorithm typically requires $O(n^3)$ floating point operations (flops) to compute the reduced order models. Here, n is the order of the full order model. There are methods that can significantly lower the cost of Lyapunov equation solution under certain assumptions ([Reis and Stykel, 2010]). In this case, the cost of the balanced truncation is lowered to $O(n^2)$.

The Hankel model reduction involves solving Lyapunov equations as well, therefore, the cost is also $O(n^3)$. Although it is important to remark, that the cost is higher than in balanced truncation.

1.3 Transfer Function Factorizations

There are many ways of factorizing a transfer function. Only two are used in this thesis: coprime and spectral factorizations. In this section, all the definitions are introduced for the discrete time case. The principles behind coprime and spectral factorization are described in [Zhou *et al.*, 1996]. However, the presented algorithms were found more convenient to use for the problems arising in the thesis.

Coprime Factorization (e.g., [Bongers and Heuberger, 1990])

Consider a system G with a minimal state-space representation

$$x(t + \Delta t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

In order to investigate unstable G for controller design, it is factorized into stable transfer matrices M and N. If such matrices can be found, then the transfer matrix G is equal to $M^{-1}N$. The transfer matrices M and N should not have common zeros, which can be cancelled out while computing G. This concept is formalized as left coprimeness. A pair of transfer matrices M and N is called left coprime over $\mathbb{H}_\infty$ if there exist rational transfer matrices X_l and Y_l in $\mathbb{H}_\infty$ such that

$$[\,M \quad N\,] \begin{bmatrix} X_l \\ Y_l \end{bmatrix} = I$$

In this case, a state-space representation of coprime factorization can be computed as:

$$[\,M \quad N\,] = \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline C & I & D \end{array} \right]$$

where L is used to stabilize A. Such a factorization exists if there exists L, which can stabilize A. Note also if M and N is a left coprime factorization of G, then so is RM and RN for an invertible, real matrix R.

The norm of the coprime factors may significantly vary depending on the matrix L. Therefore, a normalized coprime factorization is introduced, i.e., such left coprime factors M and N that

$$M^\sim M + N^\sim N = I$$

With such a constraint, a state-space representation of normalized coprime factors is computed as:

$$[\,M \quad N\,] = \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline RC & R & RD \end{array} \right]$$

where R is the upper triangular Cholesky factor of $(I + DD^T + CPC^T)^{-1}$, $L = -(APC^T + BD^T)RR^T$ and the positive semidefinite P is the solution of the Riccati equation:

$$APA^T - P - (APC^T + BD^T)RR^T(APC^T + BD^T)^T + BB^T = 0$$

Similarly, a right coprime factorization $G = \widetilde{N}\widetilde{M}^{-1}$ can be introduced, by repeating the derivations for G^T. Then $G = N^T M^{-T}$, $\widetilde{M} = M^T$ and $\widetilde{N} = N^T$.

Symmetric Spectral Factorization (e.g., [Kučera, 1991])

Consider a square transfer matrix:

$$Q(z) = Q_k z^{-k} + Q_{k-1} z^{-k+1} + \cdots + Q_1 z^{-1} + Q_0$$

for a fixed k. Assume $G = Q + Q^\sim$ is positive definite on the unit circle. It implies that the spectrum (the zeros) of the transfer matrix G is symmetrically divided with respect to the unit circle. This means that for every zero p inside the unit circle $\partial\mathbb{D}$, there exist a zero $1/p$ outside the unit circle. The goal of the spectral factorization is to find such a transfer matrix

$$M(z) = M_k z^{-k} + \cdots + M_1 z^{-1} + M_0$$

that all the zeros of M lie inside the unit circle and $G = Q + Q^\sim = MM^\sim$. Let Q have the state-space representation:

$$Q = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Then the spectral factor M is computed as:

$$M = \left[\begin{array}{c|c} A & B \\ \hline F & R \end{array}\right]$$

where $R = (B^T PB + D + D^T)^{1/2}$, $F = R^{-1}(B^T PA + C)$ and P is the positive semidefinite solution to the Riccati equation

$$A^T PA - X - A^T PB(B^T PB + D + D^T)^{-1}(A^T PB)^T = 0$$

A factorization $G = N^\sim N$ may be also computed, where N has zeros inside the unit circle. In this case, a function G^T can be factorized using the algorithm above and the stable spectral factor is $N = M^T$.

Non-Symmetric Spectral Factorization ([Fairman *et al.*, 1992])

If a square matrix-valued polynomial is not positive semidefinite on the
unit circle, but there is no zeros on the unit circle and the number of zeros
inside is equal to the number of zeros outside the unit circle, this matrix
can be still factorized. In the non-symmetric case, a Riccati equation is
solved as well. Consider a transfer matrix:

$$G(z) = \frac{\sum\limits_{i=-k}^{i=k} Q_i z^{i+k}}{d(z)}$$

where $d(z)$ is a scalar valued polynomial of degree $2k$ with no zeros on
the unit circle, Q_i are square real matrices and Q_k is invertible. Since
the pseudo-polynomial Q has the same number of stable and unstable
zeros, $d(z)$ should also have the same property. In the thesis, the choice
$d(z) = z^k(z-2)^k$ provided satisfactory results.

   The goal is to find M_s and M_u such that $G = M_s M_u$, where poles
and zeros of M_s lie inside the unit circle, while poles and zeros of M_u lie
outside the unit circle. First, the transfer function is decoupled into stable
and anti-stable parts:

$$G(z) = Q_s(z) + Q_u(s) = \left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array}\right] + \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right]$$

where Q_s contains only stable modes (all the eigenvalues of A_s are smaller
than 1) and Q_u only unstable ones (all the eigenvalues of A_u are bigger
than 1). Then the following Riccati equation is solved:

$$(A_s - B_s D^{-1} C_s)P - P(A_u - B_u D^{-1} C_u) + P(B_u D^{-1} C_s)P - B_s D^{-1} C_u = 0$$

where $D = D_s + D_u$. Note that the solution P will not be symmetric.
Finally, introduce J_s and J_u such that $J_s J_u = D_s + D_u$ and spectral factors
are computed as:

$$M_s = \left[\begin{array}{c|c} A_s & (B_s - PB_u)J_u^{-1} \\ \hline C_s & J_s \end{array}\right] \qquad M_u = \left[\begin{array}{c|c} A_u & B_u \\ \hline J_s^{-1}(C_s P + C_u) & J_u \end{array}\right]$$

The factorization $G = N_u N_s$, where N_s has only stable poles and zeros,
and N_u has only unstable ones, can be performed similarly to the previous
cases. That is, factorize G^T into $M_s M_u$, and define $N_s = M_s^T$, $N_u = M_u^T$.

1.4 Convex Optimization

Only the main concepts are sketched here. For further reading, see [Boyd and Vandenberghe, 2004].

One of the central concepts in optimization theory is convexity. A set A is called convex if

$$\forall x, y \in A, \quad \forall \theta \in [0,1]: \quad \theta x + (1 - \theta)y \in A$$

Convex sets are extremely convenient, since it is easy to perform a line search over them. If a point belongs to the border of a convex set, then there is no need to extend the line farther. The points beyond the border will not belong to the set, otherwise the set is not convex. The line search is the easiest form of optimization. Consider a more general optimization program - minimization:

$$
\begin{aligned}
&\text{minimize} && f_0(x) \\
&\text{subject to} && f_i(x) \leq 0 && i = 1, \ldots, N_1 \\
& && f_i(x) = 0 && i = N_1 + 1, \ldots, N_2 + N_1
\end{aligned}
$$

The minimization problem is called convex if for all $i = 0, \ldots, N_1 + N_2$ the functions f_i are convex, which means that they satisfy the inequality:

$$
\begin{aligned}
& f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y) \quad \forall i = 0, \ldots, N_1 + N_2 \\
& \forall \alpha, \beta \in \mathbb{R}^+ : \alpha + \beta = 1
\end{aligned}
$$

and x, y lie in the domain of f_i. If the functions f_i are convex then the decision variables x are confined to an intersection of convex sets. Minimization over convex functions with convex constraints is guaranteed to have a unique global minimum.

An important concept in optimization theory is a relaxation. A relaxation is removing some constraints from the problem, which creates an easier problem to solve. Typically, the removed constraints are non-convex. Consider a simple example:

$$
\begin{aligned}
&\gamma_n = \text{minimize}_{x,y} && c^T x \\
&\text{subject to} && b^T x + a^T y \leq 0 \\
& && x_i = 0 \text{ or } x_i = 1
\end{aligned}
$$

The binary constraint on x (the entries x_i are equal to 0 or 1) is not convex. It can be relaxed by replacing it with the constraint *"x lies in the closed interval* $[0,1]$*"*. Such a replacement can be seen as removing

the constraint x_i *does not belong to the open interval* $(0,1)$. The convex minimization problem is now as follows.

$$\gamma_r = \text{minimize}_{x,y} \qquad c^T x$$
$$\text{subject to} \qquad b^T x + a^T y \le 0$$
$$x_i \in [0,1]$$

Note that $\gamma_r \le \gamma_n$, since the relaxed problem has fewer constraints. It is typically required to estimate an upper bound on γ_n based on γ_r, i.e., estimate κ such that $\gamma_n \le \kappa \gamma_r$. The constant κ is usually called a relaxation gap. If κ is infinite, then this relaxation is not meaningful, since the programs are essentially different and the solutions are not close at all. In order to have solutions close as well, κ should be as close to 1 as possible.

Quasi-Convex and Solvability Programming

Some functions are not convex, yet, it is possible to include them into a convex program. These functions are called quasi-convex. Their main property is convexity of sub-level sets. A sub-level set of f_0 for a scalar γ is defined as

$$A_\gamma = \{x \,|\, f_0(x) \le \gamma\}$$

Let the function f_0 be quasi-convex and the other functions f_i be convex. A convex program may be obtained by introducing an extra constraint $f_0(x) \le \gamma$ and minimizing γ instead. Specifically, if f_0 is a rational function $f_0 = g_0/h_0$. If h_0 is positive for all x, the following program is obtained:

$$
\begin{aligned}
\text{minimize} \quad & \gamma \\
\text{subject to} \quad & g_0(x) \le \gamma h_0(x) \\
& f_i(x) \le 0 \qquad\qquad i = 1, \dots, N_1 \\
& f_i(x) = 0 \qquad\qquad i = 1 + N_1, \dots, N_2 + N_1
\end{aligned}
$$

One of the ways of solving the quasi-convex program is bisection, which is outlined in Algorithm 3. The program (1.2) can be replaced with a solvability one, i.e., set $s = 0$ and the objective is to find x satisfying the same constraints. However, in the optimization problems occurring in this thesis, solving a minimization problem is more numerically robust.

Semidefinite Programming and Relaxations

A semidefinite program is formulated as follows:

$$
\begin{aligned}
\text{minimize} \quad & b^T x \\
\text{subject to} \quad & x_1 A_1 + \cdots + x_n A_n + C \le 0 \\
& Ax = b
\end{aligned}
$$

Algorithm 3 Bisection algorithm applied to a quasi-convex program

---

Specify an upper bound γ_u, a lower bound γ_l and a tolerance level ε
repeat
   Set $\gamma = (\gamma_u - \gamma_l)/2$ and solve the following convex problem:

$$\begin{aligned}
\text{minimize} \quad & s & \text{(1.2)}\\
\text{subject to} \quad & g_0(x) \leq \gamma h_0(x) + s \\
& f_i(x) \leq 0 & i = 1, \ldots, N_1 \\
& f_i(x) = 0 & i = 1 + N_1, \ldots, N_2 + N_1
\end{aligned}$$

  if $s > 0$ **then**
    Set $\gamma_l = \gamma$
  else
    Set $\gamma_u = \gamma$
  end if
until $\gamma_u - \gamma_l \leq \varepsilon$

---

where $A_1, \ldots, A_n, C, b$ - constant matrices of suitable sizes. Any matrix inequality with matrix variable X constitutes a semidefinite program as well. A semidefinite program has a unique minimum and the solution can be obtained in polynomial time. Moreover, there are commercial and open-source solvers to compute the solution such as SPDT3 ([Tütüncü *et al.*, 1999]) and SᴇDᴜMᴵ ([Sturm, 1999]).

A useful result used in semidefinite programming is the following lemma. It is commonly used to obtain a semidefinite program from what appears to be a non-convex constraint.

Lᴇᴍᴍᴀ 1.2—Sᴄʜᴜʀ's Cᴏᴍᴘʟᴇᴍᴇɴᴛ
Given a complex-valued matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with positive definite D, the matrix $A - BD^{-1}C$ is positive (semi)definite if and only if and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is positive (semi)definite. $\qquad\square$

Positivity and Sum-of-Squares Constraints

Some constraints used in this thesis require a special treatment. These are concerned with the positive polynomials over specific domains. From the systems theory perspective, it is a positivity of transfer functions for all the frequencies ω in the interval $[0, \pi]$. The major result in this area is the Kalman-Yakubovitch-Popov (KYP) lemma (e.g. [Khalil, 2002; Willems,

1971]). This lemma allows to express the positivity as an algebraic condition instead of the frequency-dependent one. The following formulation is going to be used in the thesis:

LEMMA 1.3

Given A, B, M with $\det(e^{J\omega} I - A) \neq 0$ for $\omega \in [0, \pi]$ and (A, B) controllable, the following two statements are equivalent:

(i)
$$\begin{bmatrix} (e^{J\omega} I - A)^{-1} B \\ I \end{bmatrix}^{\sim} M \begin{bmatrix} (e^{J\omega} I - A)^{-1} B \\ I \end{bmatrix} \leq 0 \quad \forall \omega \in [0, \pi]$$

(ii) There exist a symmetric matrix P such that

$$M + \begin{bmatrix} A^T P A - P & A^T P B \\ B^T P A & B^T P B \end{bmatrix} \leq 0$$

The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable. $\qquad \square$

Unfortunately, a corresponding result for parameter-dependent transfer functions $G(e^{J\omega}, \theta)$ is not available in such a general formulation. Therefore, another approach is needed. Consider a trigonometric polynomial, which depends on two variables, i.e.:

$$a(\omega, \theta) = \sum_{i=-n_0}^{n_0} \sum_{k=-n_1}^{n_1} a_{ik} e^{-iJ\omega} e^{-kJ\theta}$$

Both variables ω and θ are in $[0, \pi]$. Instead of expressing the constraint $a \geq 0$, it is replaced with a is a sum-of-squares. It is common technique in such problems and is called a sum-of-squares relaxation.

THEOREM 1.5

This theorem is based on the results from [Dumitrescu, 2007].
The condition a is a sum-of-squares can be replaced by the following algebraic conditions

$$Q \text{ is a positive semidefinite matrix}$$
$$a_{ik} = \text{trace}\left((\Theta_i^{n_0} \otimes \Theta_k^{n_1})Q\right) \quad \forall (i, k) \in \mathfrak{I}$$

where Θ_i^n is an n by n elementary Toeplitz matrix, with ones on the i-th diagonal and $\Theta_i^n = (\Theta_{-i}^n)^T$. The index set $\mathfrak{I}$ describes an asymmetric half-plane. It means that if $i > 0$, then k is arbitrary, and if $i = 0$, then $k \geq 0$. Moreover, a can be expressed as:

$$a = \sum_{i=-n_0}^{n_0} \sum_{k=-n_1}^{n_1} \text{trace}\left((\Theta_i^{n_0} \otimes \Theta_k^{n_1})Q\right)e^{-ij\omega}e^{-kj\omega} \qquad \square$$

The result is extended to a larger number of parameters and matrix-valued pseudo-polynomials in [Dumitrescu, 2007]. Note that for non-parameterized problems, this approach is similar to the KYP lemma above.

1.5   Quasi-Convex Optimization Approach to Model Reduction

In [Sou *et al.*, 2005], it was suggested by Sou, Megretski and Daniel to solve directly the frequency domain problem

$$\min_{\deg(\widehat{G}) \le k} \|G - \widehat{G}\|_{\mathbb{H}_\infty}$$

where k is lower than $\deg(G)$ of the full order model G. The method was proposed for discrete-time SISO models, i.e., $G(z)$ is a scalar-valued, asymptotically stable, transfer function. An optimal solution of this non-smooth optimization is not practically computable. However, authors proposed a great suboptimal solution. Introduce a notation:

$$\widehat{G} = \frac{p}{q} \quad \text{where} \quad p(z) = \sum_{i=0}^{k} p_i z^{-i} \quad q(z) = \sum_{i=0}^{k} q_i z^{-i}$$

and all the zeros of q are inside the unit circle $\partial\mathbb{D}$, i.e., it is a stability constraint on $\widehat{G}$. Matching is performed on the unit circle $z = e^{j\omega}$, with p_i, q_i being the decision variables. Consider now only the norm constraint, later it will be shown that using the proposed technique it is possible to reconstruct a stable reduced order model. The infinity norm is reformulated as a minimization with an infinite number of constraints (for every frequency ω in $[0, \pi]$):

$$\min_{\gamma > 0, p_i, q_i} \gamma \qquad \text{subject to}$$

$$\left| G(e^{j\omega}) - \frac{p(e^{j\omega})}{q(e^{j\omega})} \right| < \gamma \quad \forall \omega \in [0, \pi]$$

Multiplication of both sides of the inequality with $|q(e^{j\omega})|^2 = q(e^{j\omega})q^\sim(e^{j\omega})$ yields:

$$\min_{\gamma > 0, p_i, q_i} \gamma \qquad \text{subject to}$$

$$\left| G(e^{j\omega})|q(e^{j\omega})|^2 - p(e^{j\omega})q^\sim(e^{j\omega}) \right| < \gamma |q(e^{j\omega})|^2 \quad \forall \omega \in [0, \pi]$$

These inequalities can be rewritten in a matrix form as follows:

$$\min_{\gamma>0,\,p_i,\,q_i} \gamma \quad \text{subject to for all } \omega \text{ in } [0,\pi]:$$

$$\begin{pmatrix} \gamma|q(e^{j\omega})|^2 & G(e^{j\omega})|q(e^{j\omega})|^2 - p(e^{j\omega})q^\sim(e^{j\omega}) \\ * & \gamma|q(e^{j\omega})|^2 \end{pmatrix} > 0$$

where the asterisk stands for $(G(e^{j\omega})|q(e^{j\omega})|^2 - p(e^{j\omega})q^\sim(e^{j\omega}))^\sim$. This program is still not convex in variables p_i and q_i. However, here a relaxation can be successfully applied. Let $a = |q|^2$ and $b = pq^\sim$, which yields a program:

$$\min_{\gamma>0,\,p_i,\,q_i,\,a,\,b} \gamma \quad \text{subject to:} \tag{1.3}$$

$$\begin{pmatrix} \gamma a(e^{j\omega}) & G(e^{j\omega})a(e^{j\omega}) - b(e^{j\omega}) \\ (G(e^{j\omega})a(e^{j\omega}) - b(e^{j\omega}))^\sim & \gamma a(e^{j\omega}) \end{pmatrix} > 0 \quad \forall \omega \in [0,\pi]$$

$$b(z) = p(z)q^\sim(z) \qquad \forall z \in \mathbb{C}$$

$$a(e^{j\omega}) > 0 \qquad \forall \omega \in [0,\pi]$$

The positivity condition $a > 0$ guarantees that a stable reduced order model can be reconstructed. If a is positive on the unit circle, then a can be always factorized into $qq^\sim$, where q has only stable zeros and poles. It means that the stable part of b/a can be taken as reduced order model. Note that the condition $a > 0$ is implied by the matrix inequality and, therefore, can be removed.

Removing the constraint $b = pq^\sim$ constitutes a relaxation. Therefore, b will not depend on q and p. The coefficients of the pseudo-polynomials $a(z) = \sum_{i=-k}^{k} a_i z^{-i}$ and $b(z) = \sum_{i=-k}^{k} b_i z^{-i}$ are the new decision variables. The positivity of a on the unit circle also implies that $a(z) = a(1/z)$, which is equivalent to $a_{-i} = a_i$ for all i. The resulting program is quasi-convex and can be solved using standard methods. The denominator is obtained by spectral factorization of a. To obtain an asymptotically stable reduced order model (1.4) is solved. This program does not significantly affect numerical complexity and provides an optimal numerator given a denominator q. The method is summarized in Algorithm 4. The error bounds for the algorithm were shown in [Megretski, 2006].

THEOREM 1.6—[MEGRETSKI, 2006]
If γ_*, p_*, q_* are obtained from (1.3, 1.4) then:

$$\sigma_{k+1}(G) \leq \gamma_* \leq \min_{p,q} \left\| G - \frac{p}{q} \right\|_{\mathbb{H}_\infty} \leq \left\| G - \frac{p_*}{q_*} \right\|_{\mathbb{H}_\infty} \leq (k+1)\gamma_*$$

Algorithm 4 Quasi-Convex Optimization (QCO) Approach to Model Reduction

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- Introduce new variables $a = \sum\limits_{i=-k}^{k} a_i z^{-i} \quad b = \sum\limits_{i=-k}^{k} b_i z^{-i}$

- Solve the quasi-convex program

$$\min_{\gamma > 0, a_i, b_i} \gamma \quad \text{subject to for all } \omega \text{ in } [0, \pi]:$$

$$\begin{pmatrix} \gamma a(e^{j\omega}) & G(e^{j\omega})a(e^{j\omega}) - b(e^{j\omega}) \\ (G(e^{j\omega})a(e^{j\omega}) - b(e^{j\omega}))^{\sim} & \gamma a(e^{j\omega}) \end{pmatrix} > 0$$

- Given a solution a_*, solve the spectral factorization problem $a_* = q_* q_*^{\sim}$

- Then given q_*, the numerator p_* is computed as:

$$\min_p \left\| G - \frac{p}{q_*} \right\|_{\infty} \tag{1.4}$$

- The reduced order model is computed as $\widehat{G} = p_* / q_*$

Here $\min\limits_{p,q} \left\| G - \frac{p}{q} \right\|_{\mathbb{H}_{\infty}}$ is the optimal error of reduction, and $\sigma_{k+1}(G)$ is the $k+1$-th largest Hankel singular value of G. □

To obtain a numerically efficient program, the norm constraints are enforced only on a finite frequency grid. To ensure stability the constraint $a > 0$ should be enforced for all the frequencies ω in $[0, \pi]$. This can be done efficiently using the Kalman-Yakubovitch-Popov lemma (see, Lemma 1.3).

The approach was also extended to the parameterized model reduction ([Sou *et al.*, 2005]) and frequency-weighted model reduction([Sandberg and Murray, 2007]).

The approach presented in Chapter 2 is a relaxation of the QCO algorithm. Therefore, a more detailed discussion on properties and interpretations of the method is presented not here, but in the upcoming chapters.

2

Semidefinite Hankel-type Model Order Reduction

This chapter deals with a problem of scalable model reduction in the $\mathbb{H}_\infty$ norm. Two algorithms are proposed to address the problem. The algorithms perform matching of frequency response samples of a transfer function. They result from a reformulation of the single-input-single-output model reduction problem, which will be called *a Hankel-type formulation*. This formulation is obtained by introducing an auxiliary variable into the objective function. In this formulation, two variables parameterize the denominator of the reduced order model to some extent. One of these variables has only unstable modes. This is similar to Hankel model reduction, in which a stable and an anti-stable functions are the decision variables (see, page 26 for the definition). Having this auxiliary variable allows two possibilities to obtain a convex problem: a relaxation and an iterative approach.

The employed relaxation is related to both quasi-convex optimization approach (QCO, in [Sou *et al.*, 2008]) and Hankel model reduction ([Glover, 1984]). The QCO is discussed in detail in this thesis (Algorithm 4 on page 37) and Hankel model reduction is sketched in Section 1.2. In these algorithms, optimization is performed over stable and anti-stable transfer functions. The stable one can be taken as a reduced order model. It can be also shown that Hankel model reduction and the presented approach are relaxations themselves of the QCO algorithm.

The proposed iterative algorithm solves a semidefinite program on every iteration. It also converges given any feasible starting point. However, given two different starting points, the converged solutions can be different. A starting point can be computed from an asymptotically stable low-order approximation of the full order model. Hence, the method is able to potentially improve the quality of any model order reduction method. Given such initial point the stability constraint is expressed as

a positive real constraint, i.e., a non-convex constraint is replaced by a semidefinite one. Similar idea to express stability is employed in [Rantzer and Megretski, 1994] and [Henrion *et al.*, 2003].

It can not be claimed that the proposed relaxation always provides a better model match than QCO Algorithm 4. Nevertheless, the proposed framework has a number of advantages:

- The quality of approximations obtained with QCO or the presented relaxation is comparable to the one of Hankel model reduction, which is known to deliver very accurate solutions. However, the computational cost of approximation may be significantly lower in comparison to the one of Hankel model reduction.

- The proposed approach is a relaxation of the QCO algorithm, which results in a few advantages in comparison to the QCO algorithm. Firstly, a better numerical robustness, which is illustrated in detail in Example 2.1. Secondly, results of the parameterized model reduction extension show a considerable improvement in the quality of approximation. The parameterized model order reduction is investigated in Chapter 3.

- The presented iterative approach is a powerful tool when systems with a structure are considered, i.e., decentralized structure, plant-controller systems, etc. The method is extended to such problems in Chapters 4 and 5.

- For particular models, it is possible that the actual approximation error of the proposed method is larger than the one of the QCO algorithm. Nonetheless, the presented iterative approach is able to significantly reduce the loss of approximation quality, if it occurs due to the looser relaxation.

A direct generalization of the Hankel-type formulation to the multi-input-multi-output (MIMO) model reduction appears to be impossible. Therefore, the MIMO extensions of the proposed approaches are, in fact, slightly different algorithms. These are obtained using optimization techniques employed in [Sootla and Sou, 2010] and [Tobenkin *et al.*, 2010].

The chapter is organized as follows: the Hankel-type formulation of model order reduction problem is presented in Section 2.1. Section 2.2 describes the proposed relaxation for the single-input-single-output case. Different subsections are devoted to a system theoretic interpretation, a discussion on relationship to the QCO algorithm, implementation details and different properties of the method. The iterative approach is briefly discussed in Section 2.3. Section 2.4 describes a MIMO, frequency-weighted, positive real extensions to model order reduction. Numerical examples are found in Section 2.5.

2.1 Hankel-type Formulation of Model Reduction Problem

The main focus of this chapter is reduction of discrete time models. Nevertheless, the algorithms can be extended to the continuous time case, as discussed in Section 2.5. Throughout the chapter, the assumptions are standard, i.e., the full order model G is an asymptotically stable, causal, rational transfer function. In this section, it is also assumed that G is a scalar-valued transfer function. The reduction problem for such models is formulated as:

$$\min_{p,q} \|G - p/q\|_{\mathbb{H}_\infty}$$

where p and q are FIR filters (i.e., $p(z) = \sum_{i=0}^{k} p_i z^{-i}$, $q(z) = \sum_{i=0}^{k} q_i z^{-i}$) q has a stable inverse and p/q is a sought-for approximation. It is also assumed that the order of the full order model G is much larger than k.

Minimizing the $\mathbb{H}_\infty$ norm is usually rewritten as a minimization of an approximation level γ subject to the norm constraints enforced only on the unit circle $\partial\mathbb{D} = \left\{ z \,\middle|\, |z| = 1 \right\}$ and the stability constraint. Therefore, z often will be substituted by $e^{j\omega}$. The resulting program reads as:

$$\gamma_{\mathrm{mor}} = \min_{p,q} \gamma \quad \text{subject to} \tag{2.1}$$

$$|G(e^{j\omega})q(e^{j\omega}) - p(e^{j\omega})| < \gamma|q(e^{j\omega})| \quad \forall\omega \in [0,\pi]$$

$q(z)$ has a stable inverse

The goal of this section is to show that the following formulation is equivalent to (2.1):

$$\gamma_{\mathrm{htf}} = \min_{\gamma>0,\,p,q,\varphi} \gamma \quad \text{subject to} \tag{2.2}$$

$$|G(e^{j\omega})q(e^{j\omega})\varphi^{\sim}(e^{j\omega}) - p(e^{j\omega})\varphi^{\sim}(e^{j\omega})| < \gamma\,\mathrm{Re}\left(q(e^{j\omega})\varphi^{\sim}(e^{j\omega})\right) \quad \forall\omega \in [0,\pi]$$

$\varphi(z)$ has a stable inverse

where $\varphi^{\sim}(z)$ is equal to $\varphi^T(1/z)$. The program (2.2) is called a *Hankel-type formulation* of model order reduction. The name of the formulation is explained in detail in Section 2.2. The main benefit of this formulation is the absence of the absolute value function on the right hand side, which gives two possibilities to obtain convex programs: a relaxation and an iterative approach, where φ is iterated over.

The Hankel-type formulation is derived in a few simple steps. First, consider a new variable φ, which is also an FIR filter of order k and

it is non-zero on the unit circle. Introduce φ in (2.1) by replacing the constraints

$$|G(e^{j\omega})q(e^{j\omega}) - p(e^{j\omega})| < \gamma|q(e^{j\omega})| \quad \forall\omega \in [0,\pi]$$

with the equivalent ones:

$$|G(e^{j\omega})q(e^{j\omega})\varphi^\sim(e^{j\omega}) - p(e^{j\omega})\varphi^\sim(e^{j\omega})| < \gamma|q(e^{j\omega})\varphi^\sim(e^{j\omega})| \quad \forall\omega \in [0,\pi]$$

The resulting minimum will not change since φ can be always canceled out. Now replace $|q\varphi^\sim|$ with $\mathrm{Re}\,(q\varphi^\sim)$.

$$\min_{\gamma>0,\,p,\,q,\,\varphi} \gamma \quad \text{subject to} \qquad\qquad\qquad\qquad (2.3)$$
$$|G(e^{j\omega})q(e^{j\omega})\varphi^\sim(e^{j\omega}) - p(e^{j\omega})\varphi^\sim(e^{j\omega})| < \gamma\mathrm{Re}\,(q(e^{j\omega})\varphi^\sim(e^{j\omega})) \quad \forall\omega \in [0,\pi]$$
$q(z)$ has a stable inverse and $\varphi \neq 0$ on $[0,\pi]$

Note that the left-hand side of the inequality is non-negative and, therefore, $\mathrm{Re}\,(q\varphi^\sim) \geq 0$. Since φ and q are not equal to zero, as an artifact a positivity constraint $\mathrm{Re}\,(q\varphi^\sim) > 0$ for all the frequencies ω in $[0,\pi]$ is obtained. For further use, a simple lemma is required.

LEMMA 2.1
Suppose q and φ are FIR filters of the same order. Assume also that $\mathrm{Re}\,(q\varphi^\sim)$ is positive for all the frequencies ω in $[0,\pi]$. Then q has a stable inverse if and only if φ has a stable inverse. $\qquad\square$

Proof. It is straightforward to show that the inequality $\mathrm{Re}\,(q\varphi^\sim) > 0$ implies that $\mathrm{Re}\,(q\varphi^{-1}) > 0$ and $\mathrm{Re}\,(\varphi q^{-1}) > 0$. If q^{-1} is stable, then the transfer function φq^{-1} is positive real and, therefore, it has a stable inverse. Hence, φ has a stable inverse. Similarly, the converse is shown. $\quad\square$

LEMMA 2.2
The optimal values γ_{mor} and γ_{htf} are equal. $\qquad\qquad\qquad\qquad\square$

Proof. Due to Lemma 2.1 the program (2.3) is equivalent to (2.2). Since $\mathrm{Re}\,(q\varphi^\sim) \leq |q\varphi^\sim|$ for all q, φ and the frequencies ω in $[0,\pi]$, we have:

$$\gamma_{\mathrm{htf}} \geq \min_{\gamma>0,\,p,\,q,\,\varphi} \gamma \quad \text{subject to}$$
$$|G(e^{j\omega})q(e^{j\omega})\varphi^\sim(e^{j\omega}) - p(e^{j\omega})\varphi^\sim(e^{j\omega})| < \gamma|q(e^{j\omega})\varphi^\sim(e^{j\omega})| \quad \forall\omega \in [0,\pi]$$
$q(z)$ has a stable inverse and $\varphi \neq 0$ on $[0,\pi]$

Now, we can divide both sides of the norm constraint with φ, and the minimization program reduces to (2.1). Therefore we have $\gamma_{\mathrm{mor}} \leq \gamma_{\mathrm{htf}}$.

To prove the converse, assume $p_* q_*^{-1}$ is an optimal solution to the model reduction problem (2.1) with the optimal approximation level $\gamma_* = \gamma_{\mathrm{mor}}$. If we choose $\varphi_* = q_*$, it is easy to verify that p_*, q_*, γ_*, φ_* satisfy the constraints of (2.2). Thus $\gamma_{\mathrm{mor}} \geq \gamma_{\mathrm{htf}}$. $\square$

Lemma 2.2 also provides an optimal choice of the auxiliary variable φ, which is simply equal to q. On the other hand, the variable $|q|$ is replaced with

$$\mathrm{Re}\left(q \cdot \frac{\varphi^{\sim}}{|\varphi|} \right)$$

This fact implies that the complex vector q is rotated in a way such that $\mathrm{Re}\,(q\varphi^{\sim})$ becomes positive. The value $\varphi^{\sim}/|\varphi|$ defines the angle of such a rotation. In the optimality, this angle is equal to $-\arg(q)$, which leaves only a positive part in expression $q\varphi^{\sim}$.

2.2 Semidefinite Hankel-type Model Reduction

First, a more systematic approach to convexification of the Hankel-type formulation (2.2) is employed, that is a relaxation. Introduce new variables $a :\triangleq q\varphi^{\sim}$ and $b :\triangleq p\varphi^{\sim}$. Since p, q, and φ are FIR filter of order k, the new variables are parameterized as $a = \sum_{i=-k}^{k} a_i e^{-ij\omega}$ and $b = \sum_{i=-k}^{k} b_i e^{-ij\omega}$. It is a standard technique in semidefinite programming and it yields a quasi-convex program:

$$\gamma_{\mathrm{shmr}}^{c} = \min_{\gamma>0,\,a,\,b} \quad \gamma \qquad \text{subject to} \qquad (2.4)$$
$$|G(e^{j\omega})a(e^{j\omega}) - b(e^{j\omega})| < \gamma\,\mathrm{Re}\,(a(e^{j\omega})) \quad \forall \omega \in [0,\pi]$$

The non-convex conditions *q and φ have only stable zeros*, which corresponds to *a (or $q\varphi^{\sim}$) has exactly k stable zeros*, is very hard to parameterize in a convex manner in a and b. However, this constraint is actually implied by positivity of the function $\mathrm{Re}\,(a)$.

LEMMA 2.3

Consider a function $a = \sum_{i=-k}^{k} a_i z^{-i}$, the unit disc $\mathbb{D}$ and the unit circle $\partial\mathbb{D}$. If $\mathrm{Re}\,(a(\partial\mathbb{D})) > 0$ then the pseudo-polynomial a has at most k zeros in $\mathbb{D}$ and no zeros on $\partial\mathbb{D}$. $\square$

Proof. The function $a(z)$ does not have zeros or poles on the unit circle (since $\mathrm{Re}\,(a(\partial\mathbb{D})) > 0$). It is also analytic in $\mathbb{D}$ except for a set of isolated points. Thus by Cauchy's argument principle $N_z - N_p = N_o$ where N_z is the number of zeros in $\mathbb{D}$, N_p is the number of poles in $\mathbb{D}$ and N_o is a

winding number of $a(\partial\mathbb{D})$ (number of times $a(\partial\mathbb{D})$ encircles the origin). Since $\text{Re}\,(a(\partial\mathbb{D})) > 0$ for all the frequencies ω in $[0,\pi]$, the curve $a(\partial\mathbb{D})$ lies only in the right half plane and thus $N_o = 0$ and $N_z = N_p$. Since N_p is at most k, so is N_z. $\square$

REMARK 2.1
If a is obtained using semidefinite programming, then the number of zeros in $\mathbb{D}$ is equal to k almost surely. The pseudo-polynomials with $a_k = 0$ (which would correspond to the case with $N_p \le k - 1$ and $N_z \le k - 1$) constitute a measure zero subspace of the pseudo-polynomials with $a_k \ne 0$. Therefore solutions of a semidefinite optimization procedure will have $a_k \ne 0$ almost surely. $\square$

After solving (2.4), the denominator q is obtained by solving the equation:

$$a = q\varphi^{\sim} \tag{2.5}$$

where φ, q have only stable zeroes and are the solutions to the non-symmetric spectral factorization problem (see, Section 1.3). Thus, the denominator q_* is computed. The numerator is obtained from:

$$p_* = \underset{p}{\text{argmin}}\, \|G - p/q_*\|_{\mathbb{H}_\infty} \tag{2.6}$$

and the reduced order model is simply p_*/q_*. Finally, the semidefinite Hankel-type model reduction reads as solving (2.4,2.5,2.6) consecutively.

Tractable Algorithm and its Computational Complexity

The programs (2.4) and (2.6) have an infinite number of constraints, one for each frequency ω in $[0,\pi]$. Therefore, these are not tractable problems. However, given that G is a rational transfer function, the frequency response can not change too fast. It means that it may be sufficient to impose some of the constraints of a finite number of frequencies $\{\omega_i\}_{i=1}^N \in [0,\pi]$. The norm constraints (2.8) clearly can be relaxed this way. This is outlined in SHMR Algorithm 5. The frequency griding is also a relaxation. If $\{\omega_i\}_{i=1}^N$ is an N element subset of some countable set $\{\omega_i\}_{i=1}^\infty$, then by construction $\gamma_{\text{shmr}}^N \le \gamma_{\text{shmr}}^\infty$ for any positive integer N. Moreover, if $\{\omega_i\}_{i=1}^\infty$ is dense in $[0,\pi]$ then $\gamma_{\text{shmr}}^\infty = \gamma_{\text{shmr}}^c$ (γ_{shmr}^c is the solution to (2.4)). In essence, with a large enough N the theoretical value γ_{shmr}^c can be approximated by γ_{shmr}^N.

To avoid over-fit, the number of points in the grid N should be at least $O(k^2)$, where k is the order of the reduced model. This griding approach may create unstable approximations, therefore the positivity constraint (2.9) is enforced for all the frequencies ω in $[0,\pi]$ using the

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Algorithm 5 SHMR. Single-Input-Single-Output Case

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1. Compute $G(e^{J\omega})$ for all ω in a given set $\Omega = \{\omega_i\}_{i=1}^{N} \in [0, \pi]$. Define

$$a = \sum_{i=-k}^{k} a_i e^{-iJ\omega} \qquad \text{and} \qquad b = \sum_{i=-k}^{k} b_i e^{-iJ\omega}$$

2. Solve the following quasi-convex program

$$\gamma_{\text{shmr}}^{N} = \min_{\gamma > 0, a, b} \gamma \qquad \text{subject to} \qquad (2.7)$$

$$|G(e^{J\omega})a(e^{J\omega}) - b(e^{J\omega})| < \gamma \operatorname{Re}\left(a(e^{J\omega})\right) \quad \forall \omega \in \Omega \qquad (2.8)$$

$$\operatorname{Re}(a) > 0 \quad \forall \ \omega \in [0, \pi] \qquad (2.9)$$

3. Given a solution a_*, perform the spectral factorization $a_* = q_* \tilde{\varphi_*}$, where φ_*, q_* have only stable zeros and poles

4. Solve the following semidefinite program

$$p_* = \operatorname*{argmin}_{p} \max_{\omega \in \Omega} |G(e^{J\omega}) - p(e^{J\omega})/q_*(e^{J\omega})| \qquad (2.10)$$

5. Construct the reduced order model as p_*/q_*

Kalman-Yakubovitch-Popov lemma (Lemma 1.3). Note that this lemma will add an LMI constraint with the size proportional to k and it will not depend on the order of G. The number of decision variables in this LMI constraint is proportional to k^2.

There are two main contributors to complexity of the algorithm. The first one is the computation of frequency response samples and the second is the solution of the quasi-convex program. Note that spectral factorization is performed over the variables describing the reduced order model, therefore, the cost is not substantial (it is equal to $O(k^3)$). Computing the frequency samples costs in general $O(l^3)$ and can be lowered to $O(l \log(l))$ in certain relevant cases (e.g., [Kamon *et al.*, 1997; Zhu *et al.*, 2003; Moselhy *et al.*, 2007]), where l is the order of the full order system G. The cost of one iteration when optimization problems are solved with SeDuMi does not exceed $O(N_1^2 N_2^{2.5} + N_1^{3.5})$ floating point operations (e.g., [Peaucelle *et al.*, 2002]). Here N_1 is the number of decision variables and N_2 is the number of rows in the LMIs. In our case $N_1 = O(k^2)$ and $N_2 = O(N + k)$, where k is the order of the approximation and N is the

number of computed frequency samples. The overall cost is computed as

$$O(Nl \log(l)) + O(k^4 N^{2.5} + N^{3.5})$$

Based on numerical simulations, the computationally heaviest part for large-scale systems ($l > 10000$) is the computation of frequency response samples of G.

Error Bounds and System Theory Interpretation of the Relaxation

SHMR Algorithm 5 is interesting due to its connection to Hankel model reduction, which will be discussed in detail here. Rewriting the constraints in (2.4) with a norm constraint yields:

$$\min_{\mathrm{Re}\,(a)>0,\; b} \left\| \left(G - \frac{b}{a} \right) \frac{a}{\mathrm{Re}\,(a)} \right\|_\infty$$

where a and b are pseudo-polynomials in z with degrees spanning from $-k$ to k. Note that the weight $a/\mathrm{Re}\,(a)$ has the infinity norm bigger or equal to 1 since $|a| \geq |\mathrm{Re}\,(a)|$ for all the frequencies ω and thus:

$$\min_{\mathrm{Re}\,(a)>0,\; b} \left\| \left(G - \frac{b}{a} \right) \frac{a}{\mathrm{Re}\,(a)} \right\|_\infty \geq \min_{\mathrm{Re}\,(a)>0,\; b} \left\| G - \frac{b}{a} \right\|_\infty$$

Any pseudo-polynomial a with a positive real part can be decomposed into q and φ as $a = q\varphi^\sim$ using the spectral factorization. The FIR filters q and φ have stable inverse, therefore q and $\varphi^\sim$ are coprime. It implies that the pseudo-polynomial b can be also decomposed using the Diophantine equation for some p and r:

$$b(z) = p(z)\varphi^\sim(z) + z^{k-1}q(z)r(z)$$

where p, q, φ are FIR filters of order k, and r is an FIR filter of order $k - 1$. Such a factorization has a unique solution in variables p and r. Hence, a following decomposition is available

$$\frac{b}{a} = \frac{p}{q} + \frac{z^{k-1}r}{\varphi^\sim}$$

Going back to the optimization problems, the following inequality can be obtained by removing the positive real constraint.

$$\min_{\mathrm{Re}\,(a)>0,\; b} \left\| G - \frac{b}{a} \right\|_\infty \geq \min_{q^{-1},\; \varphi^{-1} \in \mathbb{H}_\infty} \left\| G - \frac{p}{q} - \frac{z^{k-1}r}{\varphi^\sim} \right\|_\infty$$

Since the optimization problem on the right hand side of the inequality is the definition of Hankel model order reduction we have:

$$\min_{\mathrm{Re}\,(a)>0,\ b}\left\|\left(G-\frac{b}{a}\right)\frac{a}{\mathrm{Re}\,(a)}\right\|_{\infty}\geq\min_{\frac{p}{q}\in\mathbb{H}_{\infty}}\left\|G-\frac{p}{q}\right\|_{H}$$

Thus, SHMR is Hankel model reduction with an extra weight in the objective function $(a/\mathrm{Re}\,(a))$ and an extra constraint on the denominator ($\mathrm{Re}\,(a)$ is positive for all the frequencies ω in $[0,\pi]$). However, the distinctive part of the Hankel norm optimization, i.e., optimization over an extra anti-stable transfer function, is preserved in SHMR. Now we are ready to formulate the main theorem of the section.

THEOREM 2.1
Assume (2.4), (2.5), (2.6) were consecutively solved providing as a solution $\gamma_{\mathrm{shmr}}^{c}$, a_*, b_*, p_*, q_*. Let γ_{mor} also be the optimal approximation level from (2.1). Then the following error bounds hold:

$$\sigma_{k+1}(G)\leq\gamma_{\mathrm{shmr}}^{c}\leq\gamma_{\mathrm{mor}}\leq\|G-p_*/q_*\|_{\infty} \qquad (2.11)$$

$$\|G-p_*/q_*\|_{\infty}\leq(k+1)\gamma_{\mathrm{shmr}}^{c} \qquad (2.12)$$

where $\sigma_{k+1}(G)$ is $k+1$-st Hankel singular value of G. □

Proof. The idea of the proof is taken from [Megretski, 2006].

First, consider the lower bounds (2.11). The inequality $\gamma_{\mathrm{mor}}\leq\|G-p_*/q_*\|_{\infty}$ follows from the fact that γ_{mor} is a solution of (2.1) and therefore γ_{mor} is a minimum value for all possible p and q.

The inequality $\gamma_{\mathrm{shmr}}^{c}\leq\gamma_{\mathrm{mor}}$ is satisfied by construction. Recall that (2.1) is general and (2.2) is Hankel-type formulations of model reduction. The Hankel-type formulation (2.2) has the optimal approximation level γ_{htf}. Furthermore, $\gamma_{\mathrm{mor}}=\gamma_{\mathrm{htf}}$ by Lemma 2.2. Since the proposed method (2.4) is a relaxation of the Hankel-type formulation (2.2), then $\gamma_{\mathrm{shmr}}^{c}\leq\gamma_{\mathrm{htf}}$ and the inequality follows.

Finally, the inequality $\sigma_{k+1}(G)\leq\gamma_{\mathrm{shmr}}^{c}$ follows from the fact that $\sigma_{k+1}(G)$ is a lower bound on Hankel model reduction and as discussed above is a lower bound on SHMR.

Now, prove the upper bound (2.12). Note that

$$\|G-b_*/a_*\|_{\infty}\leq\gamma_{\mathrm{shmr}}^{c}$$

a_* and b_* can be uniquely decomposed as mentioned above:

$$\frac{b_*}{a_*}=\frac{p}{q_*}+\frac{z^{k-1}r}{\varphi^{\sim}}$$

where p and r are FIR filters of order k and $k-1$ respectively. The inequality

$$\left\| G - p/q_* - z^{k-1}r/\varphi^{\sim} \right\|_\infty \leq \gamma^c_{\text{shmr}}$$

implies that

$$\left\| z^{k-1}r/\varphi^{\sim} \right\|_H \leq \gamma^c_{\text{shmr}}$$

by the famous Adamian-Arov-Krein theorem (see, for example [Zhou *et al.*, 1996]). A consequence to the theorem states that there exist such D that

$$\left\| z^{k-1}r/\varphi^{\sim} + D \right\|_\infty \leq k\gamma^c_{\text{shmr}}$$

Combining this bound with the triangle inequality yields:

$$\left\| G - \frac{p + Dq_*}{q_*} \right\|_\infty \leq (k+1)\gamma^c_{\text{shmr}}$$

Since the numerator p_* is obtained by means of optimization (2.6), the upper bound follows. $\qquad\square$

REMARK 2.2
The obtained upper bounds are conservative, since no information about the full order model G is used. However, an upper bound is required as a guarantee, that the employed relaxation will not create unreasonable approximations. The quality of a relaxation is estimated on numerical examples. $\qquad\square$

The difference between QCO Algorithm 4 and SHMR is a symmetry constraint $a(z) = a(1/z)$. It is straightforward to show that if $a(z) = a(1/z)$, then the imaginary part of $a(z)$ on the unit circle is equal to zero. Therefore the real part of a on the unit circle is equal a. Therefore, if this symmetry constraint is added to the SHMR algorithm, the QCO algorithm will be obtained. Therefore, the following chain of inequalities is available:

$$\gamma_{\text{mor}} \geq \gamma^c_{\text{qco}} = \min_{a>0,b} \left\| G - \frac{b}{a} \right\|_\infty \geq \gamma^c_{\text{shmr}} = \min_{\text{Re}(a)>0,b} \left\| \left(G - \frac{b}{a} \right) \frac{a}{\text{Re}(a)} \right\|_\infty \geq$$

$$\geq \sigma_{k+1}(G) = \min_{p/q \in \mathbb{H}_\infty} \left\| G - \frac{p}{q} \right\|_H$$

It is also possible to compare the upper bounds, given a specific way to compute the reduced order model. For example, if the reduced order models are obtained as in (2.6) then:

$$(k+1)\gamma^c_{\text{qco}} \geq (k+1)\gamma^c_{\text{shmr}} \geq \gamma_{\text{mor}} \geq \gamma^c_{\text{qco}} \geq \gamma^c_{\text{shmr}}$$

2.3 Iterative Approach to Hankel-type Formulation

The iterative algorithm is obtained from (2.2) simply by choosing an FIR filter ψ with a stable inverse in advance and solving (2.2) with a fixed $\varphi = \psi$:

$$\min_{\gamma > 0, p, q} \gamma \quad \text{subject to} \tag{2.13}$$

$$|(Gq - p)\psi| < \gamma \operatorname{Re}(q^{\sim}\psi) \quad \forall \omega \in [0, \pi]$$

The program slightly differs from (2.2). However, $|(Gq - p)\psi| = |(Gq - p)\psi^{\sim}|$ and $\operatorname{Re}(q^{\sim}\psi) = \operatorname{Re}(q\psi^{\sim})$. This form is taken to simplify the notation. Note that the stability constraint on q is satisfied by construction. It is possible to iterate this program by setting $\psi = q$ on the next step. The proposed approach is concluded in Algorithm 6. Its implementation is similar to the one of SHMR Algorithm 5. The same remarks about the number of samples and computational complexity of the algorithm are valid for Algorithm 6 as well. Using this algorithm is convenient since there is always an improvement in terms of γ^j on every step.

LEMMA 2.4
If the constraints are enforced for all the frequencies ω in the interval $[0, \pi]$, then $\{\gamma^j\}_{j=1}^{\infty}$ is a bounded, non-increasing sequence and there exist such a γ^{∞} that $\lim_{j \to \infty} \gamma^j = \gamma^{\infty}$. □

Algorithm 6 Iterative Approach to Model Reduction. Single-Input-Single-Output Case

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Compute $G(e^{j\omega})$ for all ω in $\Omega = \{\omega_i\}_{i=1}^N$ and and FIR filter ψ^0 with a stable inverse
Let $j = 1$, $\psi^j = \psi^0$, $q = 1 + \sum_{i=1}^k q_i e^{-ij\omega}$ and $p = \sum_{i=0}^k p_i e^{-ij\omega}$
repeat
  Given ψ^j, solve

$$\min_{\gamma > 0, p, q} \gamma \quad \text{subject to} \quad \operatorname{Re}(q^{\sim}(e^{j\omega})\psi^j(e^{j\omega})) > 0 \quad \forall \omega \in [0, \pi]$$

$$\left|(G(e^{j\omega})q(e^{j\omega}) - p(e^{j\omega}))\,\psi^j(e^{j\omega})\right| < \gamma \operatorname{Re}(q^{\sim}(e^{j\omega})\psi^j(e^{j\omega})) \quad \forall \omega \in \Omega$$

  Denote by q^j and p^j the solution of the program and by ψ_c^j the coefficients of ψ^j stacked in a vector. Set $\psi^{j+1} = q^j$ and $j = j + 1$.
until $\|\psi_c^j - \psi_c^{j-1}\|_2 \leq \varepsilon$ for all i
Construct the reduced order model as p^j/q^j

Proof. First, prove that $\{\gamma^j\}_{j=1}^{\infty}$ is a bounded and non-increasing sequence of real numbers. It is actually sufficient to prove for an arbitrary j that $\gamma^j \geq \gamma^{j+1}$. Let $\psi^{j+1} = q^j$, choose $q^{j+1} = q^j$ and $p^{j+1} = p^j$, then

$$|(Gq^j - p^j)q^j| \leq \gamma^{j+1}\text{Re}\,((q^j)^{\sim}q^j) = \gamma^{j+1}|q^j|^2 \quad \forall \omega \in [0, \pi]$$

are the constraints of the program at $(j+1)$-th step. By construction of the j-th step:

$$|Gq^j - p^j| \leq \gamma^j|q^j|$$

we can always choose γ^{j+1} such that $\gamma^j \geq \gamma^{j+1}$. Since any γ^j is a positive number, the sequence $\{\gamma^j\}_{j=1}^{\infty}$ is bounded.

Finally since $\{\gamma^j\}_{j=1}^{\infty}$ is bounded and non-increasing, then there exist a limit point γ^{∞}. $\qquad\square$

The choice of the stopping criterion is guided by the following argument. If $\psi^j = \psi^{j-1}$ for some j, then it means that $\gamma^j = \gamma^{j+1}$ due to the definition of ψ^j. It also implies, that $\gamma^j = \gamma^{\infty}$. On the other hand, if convergence is checked only with respect to γ^j, that is, checking if $\gamma^j = \gamma^{j+1}$, then it can not be guaranteed that $\gamma^j = \gamma^{\infty}$.

The properties of the limit function p^{∞}/q^{∞} is an intriguing topic, however, such analysis is difficult since the behaviour of the limit is highly dependent on the initial point. If minimized over ψ as well, the program (2.13) is non-convex, non-smooth optimization and can have numerous local minima. Any iterative approach to solve the problem does not, generally, converge to globally optimal solution. Therefore, in order to evaluate if a solution is globally optimal, it is required to evaluate the properties of the $\mathbb{H}_{\infty}$ optimal solution. These are unknown, to the author's best knowledge. It can be only claimed that p^{∞}/q^{∞} is an order k stable transfer function. Due to these reasons, the approach is coupled with SHMR Algorithm 5, where a starting point is located close in $\mathbb{H}_{\infty}$ to a decent suboptimal solution.

2.4 Model Reduction Extensions

Many details are simply sketched in this section due to resemblance to the other cases. The derivation of the iterative algorithms for positive-real and frequency-weighted extension is not presented for the same reason.

Multi-Variable Extensions

As mentioned above, the multi-input-multi-output (MIMO) extension is different in some aspects from its SISO counterpart. However, the main attribute is preserved: optimization over anti-stable transfer functions.

Assume $G \in \mathbb{H}_{\infty}^{v \times m}$, where v is the number of outputs and m is the number of inputs, and the problem is formulated as:

$$\min_{P,Q} \left\| G - PQ^{-1} \right\|_{\mathbb{H}_{\infty}}$$

where $P = \sum_{i=0}^{k} P_i z^{-i} \in \mathbb{H}_{\infty}^{v \times m}$, $Q = \sum_{i=0}^{k} Q_i z^{-i} \in \mathbb{H}_{\infty}^{m \times m}$, and Q^{-1} is a stable transfer function. In this case, the degree of the reduced order transfer function PQ^{-1} can not be set freely. It may be shown that the order of PQ^{-1} is generally equal to the degree of $\det(Q)$, which is equal to km ([Kailath, 1980]). Therefore, the order can be set only to a multiple of m. Moreover, the parameterization PQ^{-1} does not describe all the stable models of order km, and it can happen that the "best" reduced order models can not be parameterized in this manner.

We start similarly to the SISO case and introduce into the program a new frequency dependent function $\Phi = \sum_{i=0}^{k} \Phi_i z^{-i} \in \mathbb{H}_{\infty}^{m \times m}$, which has a stable inverse, as:

$$\min_{\gamma > 0, P, Q, \Phi} \gamma \quad \text{subject to}$$
$$\Phi(GQ - P)^{\sim}(GQ - P)\Phi^{\sim} < \gamma^2 \Phi Q^{\sim} Q \Phi^{\sim} \quad \forall \omega \in [0, \pi]$$
$$Q^{-1}, \ \Phi^{-1} \text{ are stable}$$

This program is equivalent to the original one, since it is possible to cancel out Φ from both sides of inequalities and eliminate Φ from the program. In the SISO case the next step was replacing $|q\varphi^{\sim}|$ with $\mathrm{Re}\,(q\varphi^{\sim})$. This replacement had an interpretation of a rotation of the vector q such that the phase of q would be cancelled by φ. The concept of phase is not uniquely defined for the MIMO case and such replacement appears to be impossible. Therefore, first, the relaxation is performed and after that a tractable program is obtained. Introduce new variables $A :\triangleq Q\Phi^{\sim}$ and $B :\triangleq P\Phi^{\sim}$ and obtain:

$$\gamma_{\mathrm{mr}} = \min_{\gamma > 0, A, B} \gamma \quad \text{subject to}$$
$$(GA - B)^{\sim}(GA - B) < \gamma^2 A^{\sim} A \quad \forall \omega \in [0, \pi]$$
$$A \text{ has } km \text{ stable and } km \text{ unstable zeros}$$

To address the quadratic constraint the following relation is used (e.g., [Sootla and Sou, 2010], [Tobenkin *et al.*, 2010]), which is valid for arbitrary matrices X and Y:

$$X^{\sim} Y + Y^{\sim} X - Y^{\sim} Y \leq X^{\sim} X \tag{2.14}$$

It is obtained from $(X - Y)^\sim (X - Y) \geq 0$. Using this inequality means that yet another variable R is introduced, which is an initial guess on A:

$$\gamma_{\mathrm{mb}} = \min_{\gamma > 0, A, B, R} \gamma \quad \text{subject to}$$

$$(GA - B)^\sim (GA - B) < \gamma^2 \left(R^\sim A + A^\sim R - R^\sim R \right) \quad \forall \omega \in [0, \pi]$$

$$A \text{ have } km \text{ stable and } km \text{ unstable zeros}$$

The relaxed programs are equivalent, i.e., $\gamma_{\mathrm{mb}} = \gamma_{\mathrm{mr}}$. It is also comforting, that if G, A, B are all scalar valued and $R = (A + A^\sim)/2$ then γ_{mb} is equal to γ_{shmr}^c from (2.4). Unlike the SISO case, the MIMO programs are non-convex, and finding the global minimum is not guaranteed. In order to obtain a tractable program, let $R = rI$, where r is a positive, scalar-valued function. In this case, the program is quasi-convex and the LMI formulation can be obtained using the Schur complement as follows.

$$\gamma_{\mathrm{mimo}} = \min_{\gamma > 0, A, B, r} \gamma \quad \text{subject to} \tag{2.15}$$

$$\begin{pmatrix} \gamma^2 rI & GA - B \\ (GA - B)^\sim & A + A^\sim - rI \end{pmatrix} > 0 \quad \forall \omega \in [0, \pi]$$

where the constraint $r > 0$ is now implied in the LMI. The program (2.15) is the MIMO version of (2.4). To guarantee the condition on zeros of A, a general version of Lemma 2.3 can be formulated, where Cauchy's argument principle is applied to the determinant of A.

LEMMA 2.5

Consider an m by m transfer function $A = \sum_{i=-k}^{k} a_i z^i$, a unit disc $\mathbb{D}$ and a unit circle $\partial \mathbb{D}$. If $\mathrm{Re}\left(\det(A(\partial \mathbb{D}))\right) > 0$ then the pseudo-polynomial A has at most km zeros in $\mathbb{D}$ and no zeros on $\partial \mathbb{D}$. $\qquad\square$

A setting with matrix-valued R can be tackled in the same manner. In this case, the condition $R^\sim A + A^\sim R > 0$ implies that A has km stable zeros, if R has km stable zeros. An iterative procedure can be also applied to the relaxed problem by setting $R = A$ on the next iteration step. However, it seems more reasonable to iterate not in the space of the relaxed variables, but in the space of the actual variables using the iterative approach. The MIMO approach is summarized in Algorithm 7.

Regarding the error bound the results for the MIMO case are not as impressive as for the SISO case. Since a restriction is applied to obtain a quasi-convex program, it can not be stated that the obtained approximation level γ_{mimo} is equal to γ_{mb} and it is a lower bound on the model reduction problem.

THEOREM 2.2
Consider a reduced order model $P_* Q_*^{-1}$ is obtained from Algorithm 7 with $\Omega = [0, \pi]$, i.e., all the constraints enforced for all the frequencies in $[0, \pi]$. Then the following error bounds hold

$$\sigma_{km+1}(G) \leq \gamma_{\text{mimo}} \leq \|G - P_* Q_*^{-1}\|_\infty \leq (km + 1)\gamma_{\text{mimo}}$$

where $\sigma_i(G)$ is the i-th largest Hankel singular value of G. $\square$

Similar frequency griding remarks are valid for the MIMO version of SHMR. The computational cost is somewhat similar to the SISO case:

$$O(k^4 N^{2.5} m^{6.5} + m^{3.5} N^{3.5})$$

However, the cost grows not only with the order, but also with the number of inputs. Numerical examples show, that if the number of inputs is large, it can affect significantly the cost of the program. Note that the reduced

---

Algorithm 7 SHMR. Multiple-Input-Multiple-Output Case

---

1. Compute $G(e^{j\omega})$ for given all ω in $\Omega = \{\omega_i\}_{i=1}^N \in [0, \pi]$. Define

$$A = \sum_{i=-k}^{k} A_i e^{-ij\omega} \quad \text{and} \quad B = \sum_{i=-k}^{k} B_i e^{-ij\omega}$$

2. Solve the following quasi-convex program

$$\gamma_{\text{mimo}} = \min_{\gamma > 0, A, B, r} \gamma \quad \text{subject to:} \quad A + A^\sim > 0 \quad \forall \omega \in [0, \pi]$$
$$\begin{pmatrix} \gamma^2 r_i I & G(e^{j\omega})A(e^{j\omega}) - B(e^{j\omega}) \\ * & A(e^{j\omega}) + A^\sim(e^{j\omega}) - r_i I \end{pmatrix} > 0 \quad \forall \omega \in \Omega$$

   where asterisk stands for $(G(e^{j\omega})A(e^{j\omega}) - B(e^{j\omega}))^\sim$.

3. Given a solution A_*, perform the spectral factorization $A_* = Q_* \Phi_*^\sim$, where Φ_*, Q_* have only stable zeros and poles

4. Solve the following semidefinite program

$$P_* = \underset{P}{\text{argmin}} \max_{\omega \in \Omega} \overline{\sigma}(G(e^{j\omega}) - P(e^{j\omega})Q_*^{-1}(e^{j\omega})) \tag{2.16}$$

5. Construct the reduced order model as $P_* Q_*^{-1}$

---

model can be parameterized as $\widehat{G} = Q^{-1}P$ and a similar algorithm may be derived as well. In this case, m is the number of outputs in the complexity estimate.

To derive the iterative algorithm for multi-variable models similar to Algorithm 6, it is required to approach the problem from a different side as well. The general formulation of model reduction problem reads as:

$$\min_{\gamma>0,P,Q} \gamma \quad \text{subject to:} \quad Q^{-1} \text{ is stable}$$

$$(GQ - P)^{\sim}(GQ - P) < \gamma^2 Q^{\sim}Q \quad \forall \omega \in [0, \pi]$$

And it is required only to tackle the quadratic constraint, which is done again using (2.14), i.e., using the inequality

$$R^{\sim}Q + Q^{\sim}R - R^{\sim}R \leq Q^{\sim}Q$$

The MIMO version of the iterative Algorithm 6 is obtained by pre-computing R and fixing it in the optimization program as in Algorithm 8. The condition $Q^{\sim}R + (Q^{\sim}R)^{\sim} > 0$ is equivalent to $QR^{-1} + (QR^{-1})^{\sim} > 0$. Therefore, if R^{-1} is fixed and it is stable, then QR^{-1} is strictly positive real and Q^{-1} is stable. Similar convergence analysis may be applied to MIMO models as in SISO iterative Algorithm 6.

REMARK 2.3
The presented algorithms may be applied to the SISO models as well.

Algorithm 8 Iterative Approach to Model Reduction. Multiple-Input-Multiple-Output Case

---

Compute $G(e^{j\omega})$ for all ω in $\Omega = \{\omega_i\}_{i=1}^N$ and R^0 with a stable inverse
Let $j = 1$, $R^j = R^0$, $Q = \sum_{i=0}^k Q_i e^{-ij\omega}$ $\quad$ and $\quad$ $P = \sum_{i=0}^k P_i e^{-ij\omega}$
repeat
  Given R^j, solve

$$\min_{\gamma>0,P,Q} \gamma \quad \text{subject to:} \quad Q^{\sim}R^j + (Q^{\sim}R^j)^{\sim} > 0 \quad \forall \omega \in [0, \pi]$$

$$(GQ - P)^{\sim}(GQ - P) < \gamma^2 \left((R^j)^{\sim}Q + Q^{\sim}R^j - (R^j)^{\sim}R^j\right) \quad \forall \omega \in \Omega$$

  Denote by Q^j and P^j the solution of the program and by R_c^j the coefficients of R^j stacked in matrix. Set $R^{j+1} = Q^j$ and $j = j + 1$
until $\|R_c^j - R_c^{j-1}\|_F \leq \varepsilon$ for all i
Construct the reduced order model as $P^j(Q^j)^{-1}$

Indeed, in the case when Algorithm 8 applied to a SISO model:

$$\min_{\gamma > 0, q, p} \gamma \quad \text{subject to:}$$

$$|Gq - p|^2 < \gamma^2 \left(2\text{Re} \left(\psi^\sim q \right) - \psi^\sim \psi \right) \ \forall \omega \in [0, \pi]$$

In order to make the notation consistent, R is replaced with ψ. To compare the right-hand side to the SISO program (2.13), it is required to multiply with $|\psi|^2$ from both sides. Now, it becomes straightforward to verify that

$$|\psi|^2 (2\text{Re} \left(\psi^\sim q \right) - \psi^\sim \psi) \le (\text{Re} \left(\psi^\sim q \right))^2$$

for any ψ. The last inequality is equivalent to

$$(\text{Re} \left(\psi^\sim q \right))^2 - 2\text{Re} \left(\psi^\sim q \right) |\psi|^2 + |\psi|^4 = (\text{Re} \left(\psi^\sim q \right) - |\psi|^2)^2 \ge 0$$

The MIMO techniques are more restrictive than the SISO ones. However, a generalization of the SISO method directly to the MIMO case appears to be impossible. ☐

Frequency-Weighted Model Order Reduction

The frequency-weighted model reduction is formulated as:

$$\min_{p, q} \left\| w \left(G - \frac{p}{q} \right) \right\|_{\mathbb{H}_\infty}$$

where $p = \sum_{i=0}^{k} p_i z^{-i}$, $q = \sum_{i=0}^{k} q_i z^{-i}$ and q and w are stable with stable inverses. The conditions on the weight w are not restrictive. If a transfer function y^{-1} has an unstable pole, then the required w can be obtained by the spectral factorization $yy^\sim$, which does not affect the $\mathbb{H}_\infty$ norm. The derivation of the algorithm in the SISO case is very similar to the original one and results in a program:

$$\min_{\gamma > 0, a, b} \gamma \quad \text{subject to} \tag{2.17}$$

$$|w(e^{j\omega}) \left(G(e^{j\omega}) a(e^{j\omega}) - b(e^{j\omega}) \right)| \le \gamma \text{Re} \left(a(e^{j\omega}) \right) \quad \forall \omega \in [0, \pi]$$

In the multiple-input-multiple-output case, one sided weights are handled the same way. Therefore, consider the two sided case:

$$\min_{p, q} \left\| W_o \left(G - PQ^{-1} \right) W_i \right\|_{\mathbb{H}_\infty}$$

The weights W_i and W_o are asymptotically stable with asymptotically stable inverses W_i^{-1} and W_o^{-1}. First, rewrite the constraints in a more useful form using a series of equivalent transformations:

$$(W_o(GQ-P)Q^{-1}W_i)^\sim W_o(GQ-P)Q^{-1}W_i \leq \gamma^2 I$$
$$(W_o(GQ-P))^\sim W_o(GQ-P) \leq \gamma^2 Q^\sim W_i^{-\sim} W_i^{-1} Q$$

multiply from the right with $\Phi^\sim$ and from the left with Φ both sides of the inequality:

$$(W_o(GQ-P)\Phi^\sim)^\sim W_o(GQ-P)\Phi^\sim \leq \gamma^2 \Phi Q^\sim W_i^{-\sim} W_i^{-1} Q \Phi^\sim$$

Now employing the relaxation yields:

$$(W_o(GA-B))^\sim W_o(GA-B) \leq \gamma^2 A^\sim W_i^{-\sim} W_i^{-1} A$$

Apply a similar series of inequalities as in the unweighted MIMO case, namely:

$$(X-Y)^\sim Z(X-Y) \geq 0 \Leftrightarrow X^\sim ZX \geq Y^\sim ZX + X^\sim ZY - Y^\sim ZY$$

In our case Z is equal to $W_i W_i^\sim$, and X is equal to A. Let Y be a positive scalar-valued frequency dependent variable r, and obtain a program:

$$\min_{\gamma>0,A,B,r} \quad \gamma \quad \text{subject to for all } \omega \text{ in } [0,\pi]:$$
$$(W_o(GA-B)W_i)^\sim W_o(GA-B)W_i \leq \gamma^2(r(W_i^{-1}AW_i + W_i^\sim A^\sim W_i^{-\sim}) - r^2 I)$$

The quasi-convex program is obtained using the Schur's complement. The iterative approach can be obtained using the same techniques. Theoretical results similar to ones proposed in [Sandberg and Murray, 2007] and [Sootla and Rantzer, 2009], where the frequency-weighted versions of the QCO algorithms were reported, can be shown in the same manner.

Positive Real Constraint

In order to enforce positive-realness, a well-known relationship will be used, i.e., a transfer function G is strictly positive real if and only if $(G-I)(I+G)^{-1}$ has the $\mathbb{H}_\infty$ norm less than one. Substituting p/q for G yields:

$$\text{Re}\left(\frac{p(e^{j\omega})}{q(e^{j\omega})}\right) > 0 \quad \forall \omega \in [0,\pi] \qquad \Longleftrightarrow \qquad \left\|\frac{p-q}{p+q}\right\|_{\mathbb{H}_\infty} < 1$$

In the presented framework, the second condition is easier to model by introducing an FIR filter φ with a stable inverse as follows:

$$\left| \frac{(p-q)\varphi^{\sim}}{(p+q)\varphi^{\sim}} \right| < 1 \quad \forall \omega \in [0, \pi]$$

For any such φ, such that $\mathrm{Re}\,((p+q)\varphi^{\sim})$ is positive for all the frequencies in $[0, \pi]$, the constraint presented below will ensure the positive realness of p/q :

$$|(p-q)\varphi^{\sim}| < \mathrm{Re}\,((p+q)\varphi^{\sim}) \quad \forall \omega \in [0, \pi]$$

Using the relaxation as in the previous sections yields a positive-real imposing condition:

$$|a - b| < \mathrm{Re}\,(a + b) \quad \forall \omega \in [0, \pi] \tag{2.18}$$

In the multiple-input-multiple-output case similar ideas are used. Another scalar frequency-dependent variable r is introduced as:

$$\begin{pmatrix} rI & A - B \\ * & A + B + (A+B)^{\sim} - rI \end{pmatrix} > 0 \quad \forall \omega \in [0, \pi]$$

2.5 Examples

These examples are set to estimate the actual relaxation gap in the Hankel-type approximation and a possible improvement by the iterative approach to any reduction procedure. The Hankel model reduction is implemented by a MATLAB™ routine HANKELMR. The SISO version of SHMR Algorithm 5 and QCO Algorithm 4 are implemented using a cutting plane algorithm (for more details, see [Sou, 2008]), the MIMO and positive-real versions of SHMR are implemented using the interior point solver SEDUMI [Sturm, 1999] and the parser YALMIP [Löfberg, 2004].

Data Pre-Processing and Frequency Griding

If a continuous-time model is approximated, it should be first discretized in order to apply the proposed approaches (as well as the QCO method). The discretization is performed, while warping around a particular frequency ω_0 in the process, using the formula:

$$s = \mu \frac{z - 1}{z + 1}, \quad \text{where} \quad \mu = \frac{\omega_0}{\tan(\omega_0 T_s / 2)}$$

and T_s is the Nyquist sampling time and parameter ω_0 is a tuning parameter for numerical conditioning. For example, if the biggest resonant peak occurs around a frequency ω_0, then pre-warping around this frequency will create a better numerically conditioned problem.

Choosing a frequency grid is a crucial part of the method. For some models, a uniform on $[0, \pi]$ grid with a large number of sample is sufficient. For others (e.g., transmission line Example 2.2), a uniform grid provides bad results. This is because, the semidefinite program tries to compensate for the quality resulting in over-fit. Over-fit is more likely to take place around peaks in magnitude, since there are poles near the unit circle. A slight change in the pole location can provide a better fit to data, however, a worse $\mathbb{H}_\infty$ approximation error, i.e., over-fit between the chosen frequency points. Therefore, the grid must be denser, around such peaks. In this chapter, a grid is manually adjusted to give the desired result - a small $\mathbb{H}_\infty$ norm. In practice, one can use randomized methods to adjust the grid, e.g., by assigning a higher probability to generate a sample to a frequency with a large transfer function magnitude. However, these methods are beyond the scope of this thesis.

EXAMPLE 2.1—RECONSTRUCTION OF AN ALL-PASS SYSTEM
This toy example is created to show a better numerical robustness of the proposed Hankel-type approximation approach (Algorithm 5) in comparison with the QCO method (Algorithm 4). Specify two all-pass models as:

$$H_i = \prod_{j=1}^{12} \frac{1 - z\xi_j^\sim}{z - \xi_j}, \text{ where } |\xi_j| = 0.96 \quad \forall i, j$$

and the arguments for the complex conjugate poles are chosen as:

$$H_1 : \arg\left(\xi_j\right) = \pm[0.11, 0.13, 0.14, 3.1, 3.11, 3.14]$$
$$H_2 : \arg\left(\xi_j\right) = \pm[0.11, 0.13, 0.14, 1.57, 1.57, 1.57]$$

Our goal here is to reconstruct the models H_1 and H_2 from the frequency response data using the QCO and SHMR algorithms. Theoretically both methods are able to do this with approximation error 0. The SHMR algorithm provided reduced order models with approximation error of less than 1%. Where 100% is the norm of H_i. However, the QCO method failed in both cases providing approximation error around 10%. One of the reasons is a fast phase variation of the models. Indeed, QCO can affect the phase of the reduced order model only through b pseudo-polynomial, since a has to be positive and thus the phase of a is equal to zero. The objective

of a variable is to match the magnitude of b, since H_i are all-pass, i.e., $\|H_i\|_\infty \equiv 1$. It results in ratio between maximal and minimal coefficients of a larger than 10^4. The SHMR approach does not restrict a to be positive and phase can be affected both from b and a variables.

Another way to see how SHMR can avoid numerical problems is the ability to choose φ freely in relationship $a = q\varphi^\sim$, the QCO approach is obliged to preserve the symmetry of a. Therefore in case of large ratio of coefficients of q, the SHMR method does not necessarily square this ratio in comparison to the QCO. $\square$

EXAMPLE 2.2—TRANSMISSION LINE MODELING
This example is described in [Chahlaoui and Dooren, 2002] and the references therein. A transmission line is a 2-input-2-output model, which is described by the following system of differential equations:

$$E\dot{x} = Ax + Bu$$
$$y = Cx$$

where matrix E is invertible, but badly conditioned. The order of the original model is 256. The comparison is performed in percent relative to the $\mathbb{H}_\infty$ norm of the full order model G.

Scalar-valued Model Reduction. In this example, only the $\{2, 2\}$-entry of the transfer function is being reduced. The $\mathbb{H}_\infty$ norms of the approximation errors are given in Table 2.1. Here, HMR stands for Hankel model reduction. The quality of Hankel approximations is generally worse than the quality of optimization approaches. However, applying Algorithms 2 and 6 reduces this conservatism. Recall, that in Algorithm 2 the denominator is fixed and is obtained from the Hankel approximation, in Algorithm 6,

Table 2.1   Approximation errors in percent in Example 2.2

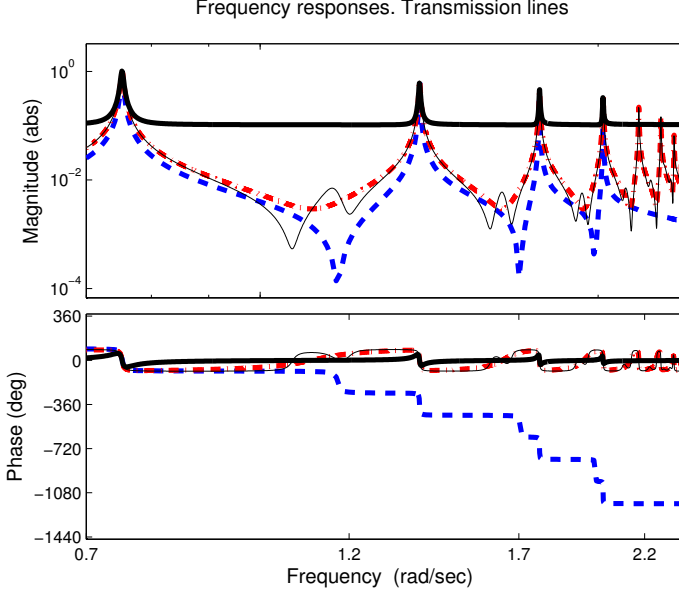| Reduction order k	8	13	16	18
$\sigma_{k+1}(G)$	10.83%	3.16%	0.43%	0.31%
lower bound of QCO Alg. 4	10.87%	3.32%	0.90%	0.41%
HMR	21.70%	6.35%	0.95%	0.65%
HMR+Algorithm 2	15.70%	4.02%	0.71%	0.47%
HMR+Algorithm 6	13.70%	3.95%	0.69%	0.43%
QCO Algorithm 4	11.61%	3.92%	1.16%	0.43%
SHMR Algorithm 5	11.27%	3.47%	0.55%	0.42%

Figure 2.1 Frequency responses for transmission line modeling. Thin black line - the full order model, red dash-dotted - SHMR approximation of order 18, thick black line - SHMR approximation of order 8, blue dashed - HMR approximation of order 8.

on the other hand, both the numerator and denominator are the decision variables. The QCO and SHMR algorithms usually deliver approximation quality very close to the lower bounds. For the order 16 the QCO algorithm provided conservative results due to numerical reasons. The frequency responses of the approximations are depicted in Figure 2.1. The 8-th order SHMR approximation looks like a horizontal line with peaks reflecting the peaks of the full order model. This is a typical behaviour of the $\mathbb{H}_\infty$ approximation, since the error spreads among all the frequencies equally. The Hankel 8-th order approximation on the other hand admits larger errors for higher frequencies.

Positive Real Model Order Reduction. The influence of the positive real restriction is studied as well. In this example for every order the positive real preserving SHMR showed almost identical behaviour to the original SHMR (see Table 2.2). Here, PR SHMR stands for Algorithm 5 with a positive real constraint. Table 2.2 also shows that both algorithms provide approximations, which are very close to the lower error bound on the Hankel model reduction $\sigma_{k+1}(G)$.

Table 2.2 Approximation errors in percent in Example 2.2. Positive Real Model Reduction

Reduction order k	1	2	3	4	5
$\sigma_{k+1}(G)$	51.19%	33.50%	33.50%	21.16%	21.16%
SHMR Alg. 5	51.20%	33.51%	33.51%	21.18%	21.18%
PR SHMR	51.20%	33.51%	33.52%	21.18%	22.02%

Table 2.3 Approximation errors in percent in Example 2.2. Multi-Variable Model Reduction

Reduction order km	2	4	6	8	10
$\sigma_{km+1}(G)$	33.50%	21.17%	16.78%	10.84%	7.29%
MIMO SHMR Alg. 7	51.19%	33.50%	33.50%	21.16%	21.16%
MIMO SHMR+Alg. 8	51.17%	33.50%	21.17%	16.91%	14.55%
HMR	65.45%	41.35%	32.80%	21.18%	14.65%

Multi-Variable Model Reduction. Here, the entire two-input-two-output model of transmission lines is considered. The numerical computations show that the iterative approach considerably improves the quality of the initial SHMR methods quality (see Table 2.3). Here, HMR stands for Hankel model reduction. In this example, at most 15 iterations of Algorithm 8 were used. MIMO SHMR Algorithm 7 presents conservatism for orders 6 to 10. Algorithm 8 is able to reduce this conservatism for higher orders. The quality of the iterative Algorithm 8 applied to the SHMR method is more consistent with Hankel model reduction quality. However, none of the methods are able to reach the lower bound $\sigma_{km+1}(G)$. □

2.6 Conclusion and Discussion

In this chapter, an approach to model reduction of linear time invariant systems has been presented. The method requires only the frequency response samples to obtain an approximation and guarantees stability in the $\mathbb{H}_\infty$ sense. The minimization is performed in a Hankel-type norm and the accuracy of the algorithm is expected to be close to the optimal Hankel one. The computational complexity of the algorithm is lower than the one of optimal Hankel approximation method for certain applications. Therefore, by sacrificing some quality, efficiency is gained.

Although the presented method is different from its predecessor, QCO, and provides a better guarantee on approximation quality (the upper bound on the error), there is no guarantee that the actual approximation quality is better. In fact, in many examples considered in preparation of this manuscript the approximation errors match or there is only an epsilon improvement. Nevertheless, aforementioned advantages outweigh possible drawbacks.

Although the lack of symmetry in the denominator seems to be a minor advantage, it provides a better numerical robustness. Moreover, in parameterized extension, this relaxation delivers significant improvement. A detailed discussion about the parameterized model reduction is available in Chapter 3.

The convergence properties of the iterative approach have to be evaluated as well. The method does not generally converge to the global minimum, since the model reduction is a non-convex problem. Therefore, an estimate of approximation quality improvement would be a decent result, especially given SHMR as an initial point. These qualities are also important since the iterative approach can be easily extended to the models with a structure as shown in Chapter 5.

3

Parameterized Model Order Reduction

Parameterized model reduction is arguably a more important tool in modeling and design than $\mathbb{H}_\infty$ model reduction. Parameterized model reduction deals with linear time-invariant models, which depend on certain design parameters. Since parameters of the models are subject to change or fine-tuning over time, it is often required to obtain a family of models which describe a particular system in various settings. The models in this family should preferably have the same orders, therefore, parameterized model reduction is used to model such families in a more convenient way. Extensive research using the Krylov techniques was performed for various applications, such as micro-electro-mechanical systems or MEMS ([Rudnyi *et al.*, 2006]), radio-frequency (RF) inductors ([Daniel and White, 2003]), interconnects ([Daniel *et al.*, 2002; Daniel *et al.*, 2004; Li *et al.*, 2005b; Li *et al.*, 2005a]), general linear systems ([Farle *et al.*, 2008; Feng *et al.*, 2009; Li *et al.*, 2007]) and non-linear systems ([Bond and Daniel, 2005]). Another interesting approach is being developed in [Lefteriu *et al.*, 2010; Lefteriu and Mohring, 2010]. In this approach, reduced models are obtained by interpolating the frequency response, which is calculated (or identified from a real system) for a finite number of frequencies.

The main focus of this chapter is extension to parameterized LTI systems of the algorithms from Chapter 2. Therein, the original model reduction problem was reformulated by adding an extra frequency-dependent variable φ. Performing a semidefinite relaxation to the problem yielded the semidefinite Hankel-type model reduction (SHMR Algorithm 5 on page 44). Whereas, applying an iterative approach to the new formulation formed Algorithm 6 on page 48. Note that when an extra constraint to SHMR Algorithm 5 is added, then a method from [Sou *et al.*, 2008] is obtained. The approach from [Sou *et al.*, 2008] is sketched in the thesis as QCO Algorithm 4 on page 37.

The extension to parameterized model order reduction itself is not the only result of this chapter. A theoretical statement regarding the approximation quality of reduced order models with respect to parameters is also derived. The error bounds are also valid for the parameterized QCO method, although such bounds were not shown in the original work [Sou *et al.*, 2008].

Numerical simulations show that the restriction, which separates the SHMR and QCO algorithms, results in a worse approximation quality of the parameterized QCO method. Performance improvement of such magnitude was not expected since non-parameterized LTI model reduction did not exhibit similar results.

Each of the methods in the framework has its benefits and drawbacks. For example, both parameterized QCO and parameterized SHMR deliver bounds on approximation quality, but can not provide explicit parameter-dependent models. Whereas, the parameterized extension of iterative Algorithm 6 can provide such models. However, the quality of approximation may be worse than SHMR and QCO have.

The chapter is organized as follows: the problem is formulated and the non-parameterized method from Chapter 2 is revised in Section 3.1. The parameterized extension is described in Section 3.2, where the main theoretical result shown as well. In Section 3.3 the problem of reconstructing explicit parameter dependent models is discussed. The most promising approach in this direction is the iterative approach to parameterized model reduction, which is also presented. The relationship between QCO, SHMR and the iterative approach is outlined therein, which explains why a certain method can provide a better match for certain models. The framework is illustrated on numerical examples in Section 3.4.

3.1 Preliminaries and Problem Formulation

Given a family of linear time-invariant (LTI) models $G(z, \theta)$, which are associated with a parameter θ, the following assumptions are made:

- The parameter θ is constant over time

- The parameter θ is a real scalar and belongs to the interval $[0, \pi]$

- The coefficients of the transfer function G depend smoothly on θ

- For every fixed θ the model $G(z, \theta)$ is an asymptotically stable rational transfer function with a frequency variable z

- The transfer function G is scalar-valued

The framework is still valid for a vector-valued θ. The restriction to the interval $[0, \pi]$ can always be achieved by a linear variable change if θ belongs to any other bounded interval. The extension to matrix-valued functions in the parameterized case is achieved in the same way as in Chapter 2 once the framework for the scalar-valued functions is obtained. The conservatism of the matrix-valued extension in Chapter 2 manifests itself in the same way for the parameterized case.

Given a family of models $G(z, \theta)$, our main goal is to find another family $\widehat{G}(z, \theta)$ that is close to G for all possible θ. Such a family can be found using a min-max criterion:

$$\min_{\widehat{G} \text{ is stable}} \max_{\theta \in [0, \pi]} \|G(z, \theta) - \widehat{G}(z, \theta)\|_{\infty}$$

The $\mathbb{L}_{\infty}$ norm is a maximization itself with respect to the frequency ω in the interval $[0, \pi]$. Therefore, this formulation is similar to the non-parameterized case.

$$\gamma_{\text{pmor}} = \min_{p_{ij}, q_{ij}} \max_{\omega, \theta \in [0, \pi]} |G(e^{j\omega}, \theta) - p(\omega, \theta)/q(\omega, \theta)| \qquad (3.1)$$

where p/q is stable

For parameterized models, stability is defined as stability for every parameter θ. The functions p and q are expressed as follows:

$$p(\omega, \theta) = \sum_{i=0}^{k_0} \sum_{j=0}^{k_1} p_{ij} F_i(\omega) H_j(\theta)$$

$$q(\omega, \theta) = \sum_{i=0}^{k_0} \sum_{j=0}^{k_1} q_{ij} F_i(\omega) H_j(\theta)$$

The basis functions F_n describe the dependence on the frequency variable ω. Therefore it is reasonable to choose $F_n = e^{-nj\omega}$. Still, if required, it is possible to choose a different basis. If extra information about a system is known, then a few possibilities are described in [Heuberger *et al.*, 2005]. The basis functions H_n describe the dependence on the parameter θ. In the thesis $H_n = \cos(n\theta)$ was chosen to use. However, in this case as well, different choices are possible. For example, one can set $H_n = \theta^n$.

To avoid any confusion in terminology in the various fields of engineering, the term "multi-variate pseudo-polynomial" will refer to a pseudo-polynomial of two or more variables (i.e. ω_1, ω_2 and so forth). The term "multi-variable" will be used for matrix-valued transfer functions $G(\cdot)$.

Hankel-type Model Reduction

The methods from Chapter 2 are the foundation of the entire framework. Therefore, to simplify the presentation of the parameterized extension, they are revised. A non-parameterized formulation of (3.1) is as follows:

$$\gamma_{\mathrm{mor}} = \min_{\gamma > 0, p, q} \gamma \quad \text{subject to} \tag{3.2}$$

$$|G(e^{j\omega})q(e^{j\omega}) - p(e^{j\omega})| < \gamma |q(e^{j\omega})| \quad \forall \omega \in [0, \pi]$$

$$p(e^{j\omega}) = \sum_{i=0}^{k} p_i e^{-i j \omega} \quad q(e^{j\omega}) = \sum_{i=0}^{k} q_i e^{-i j \omega}$$

where $q(z)$ has a stable inverse. In Chapter 2 it was shown that the latter program is equivalent to:

$$\gamma_{\mathrm{htf}} = \min_{p, q, \varphi} \gamma \quad \text{subject to} \tag{3.3}$$

$$|G(e^{j\omega})q(e^{j\omega})\varphi^{\sim}(e^{j\omega}) - p(e^{j\omega})\varphi^{\sim}(e^{j\omega})| < \gamma \mathrm{Re}\,(q(e^{j\omega})\varphi^{\sim}(e^{j\omega})) \quad \forall \omega \in [0, \pi]$$

where $\varphi^{-1}(z)$ is a stable transfer function, $\varphi^{\sim}(z) = \varphi(1/z)$ and

$$\varphi = \sum_{i=0}^{k} \varphi_i e^{-i j \omega} \quad p = \sum_{i=0}^{k} p_i e^{-i j \omega} \quad q = \sum_{i=0}^{k} q_i e^{-i j \omega}$$

The equivalence is understood as $\gamma_{\mathrm{mor}} = \gamma_{\mathrm{htf}}$. Moreover, for every solution $(p_{\mathrm{mor}}, q_{\mathrm{mor}}, \gamma_{\mathrm{mor}})$ of the problem (3.2), there exists φ such that the quadruple $(p_{\mathrm{mor}}, q_{\mathrm{mor}}, \gamma_{\mathrm{mor}}, \varphi)$ satisfies the conditions of (3.3). And if the quadruple $(\gamma_{\mathrm{htf}}, p_{\mathrm{htf}}, q_{\mathrm{htf}}, \varphi_{\mathrm{htf}})$ is an optimal solution to (3.3), then the triple $(\gamma_{\mathrm{htf}}, p_{\mathrm{htf}}, q_{\mathrm{htf}})$ is an optimal solution to (3.2). And finally, it may be shown that if φ has a stable inverse then so does q due to the positivity of $\mathrm{Re}\,(q\varphi^{\sim})$ for all the frequencies in $[0, \pi]$.

The iterative approach is obtained by setting the variable φ equal to 1 in the optimization. On the next iteration φ is set equal to the solution on the current iteration q until the solution q becomes equal to φ. The relaxation (the SHMR algorithm) is obtained by introducing new variables $a :\triangleq q\varphi^{\sim}$ and $b :\triangleq p\varphi^{\sim}$. The SHMR approach is summarized in Algorithm 9.

3.2 Parameterized Semidefinite Hankel-type Model Reduction

Introduce the notation $\boldsymbol{\omega} = \{\omega, \theta\} \in [0, \pi]^2$, where θ is a parameter, ω is

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Algorithm 9 Semidefinite Hankel-type Model Reduction

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1. Compute the frequency response $G(e^{j\omega})$ for every ω in $\Omega = \{\omega_i\}_{i=1}^{N}$

2. Solve the relaxed problem:

$$\min_{a,b} \gamma \quad \text{subject to} \ \ \text{Re}\,(a) > 0 \quad \forall \omega \in [0, \pi] \quad\quad\quad (3.4)$$

$$|G(e^{j\omega})a(e^{j\omega}) - b(e^{j\omega})| < \gamma\text{Re}\,(a(e^{j\omega})) \quad \forall \omega \in \Omega$$

3. Compute the denominator q from the spectral factorization $a = q\varphi^{\sim}$

4. For a given q, solve

$$\min_{p} \max_{\omega \in \Omega} |G(e^{j\omega}) - p(e^{j\omega})/q(e^{j\omega})|$$

5. Set the reduced order model $\widehat{G} = p/q$

---

a frequency domain variable. Introduce the pseudo-polynomials

$$a(\boldsymbol{\omega}) = \sum_{i=-k_0}^{k_0} \sum_{j=0}^{k_1} a_{ij} e^{-iJ\omega} \cos(j\theta) \qquad b(\boldsymbol{\omega}) = \sum_{i=-k_0}^{k_0} \sum_{j=0}^{k_1} b_{ij} e^{-iJ\omega} \cos(j\theta)$$

and substitute them into (3.4) which results in:

$$\min_{a,b} \gamma \quad \text{subject to} \ \ \text{Re}\,(a) \geq \mu \quad \forall \boldsymbol{\omega} \in [0, \pi]^2 \quad\quad (3.5)$$

$$|G(\boldsymbol{\omega})a(\boldsymbol{\omega}) - b(\boldsymbol{\omega})| < \gamma\text{Re}\,(a(\boldsymbol{\omega})) \quad \forall \boldsymbol{\omega} \in \Omega \quad\quad (3.6)$$

where μ is a positive pre-defined scalar and needed as a technical requirement, $\Omega = \{\boldsymbol{\omega}_i\}_{i=1}^{N}$ is a pre-defined finite grid and with a slight abuse of notation $G(\boldsymbol{\omega})$ stands for $G(e^{j\omega}, \theta)$. The non-negative constraint $\text{Re}\,(a) - \mu \geq 0$ is convex, but not tractable for a multi-variate a. Therefore, a semidefinite Sum-Of-Squares (SOS) constraint will be used instead, i.e., $\text{Re}\,(a) - \mu$ is an SOS. It is a standard way to approach the problem and is called an "SOS relaxation" (see, Theorem 1.5 for the actual algebraic constraints). As before, the positivity constraint $\text{Re}\,(a) > 0$ has the interpretation of a stability ensuring constraint. It can be shown that if $\text{Re}\,(a) > 0$, then for every fixed θ the pseudo-polynomial a has k stable zeros and k unstable ones. Given a, a stable model can be reconstructed.

In the non-parameterized case, the next step after solving (3.5-3.6) would be the non-symmetric spectral factorization of a. However, if there

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Algorithm 10 Parameterized SHMR

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1. Compute the frequency response $G(\boldsymbol{\omega})$ for every $\boldsymbol{\omega} = \{\omega, \theta\}$ in $\Omega = \{\boldsymbol{\omega}_i\}_{i=1}^N$

2. Solve
$$\gamma_{\text{pshmr}}^N = \min_{a,b} \gamma \quad \text{subject to}$$
$$\text{Re}\,(a) \geq \mu \quad \forall \boldsymbol{\omega} \in [0,\pi]^2$$
$$|G(\boldsymbol{\omega})a(\boldsymbol{\omega}) - b(\boldsymbol{\omega})| < \gamma \text{Re}\,(a(\boldsymbol{\omega})) \quad \forall \boldsymbol{\omega} \in \Omega$$

3. For a particular θ, compute the denominator q_θ from $a_\theta = q_\theta \varphi_{\tilde{\theta}}$ (e.g., [Fairman *et al.*, 1992])

4. For a particular θ, compute the numerator p_θ from

$$p_\theta = \operatorname*{argmin}_{p} \max_{\boldsymbol{\omega} \in \Omega} |b_\theta a_\theta^{-1} - p q_\theta^{-1}|$$

where p is parameterized as $p = p_0 + p_1 e^{-J\omega} + \cdots + p_k e^{-k_0 J\omega}$

---

are two or more variables, even the symmetric spectral factorization has much stricter conditions. For a general case, such conditions are described in [Dritschel and Woerdeman, 2005] and for the two dimensional case (one parameter in our notation), refer to [Geronimo and Lai, 2006]. The conditions are not convex in the chosen variables (the coefficients of a) and are equivalent to a rank constraint. Given such a complexity, obtaining a reasonable relaxation seems to impossible. An approximate solution to the multi-variate symmetric spectral factorization may be found in, for example, [Ekstrom and Woods, 1976]. In this section, the multi-variate spectral factorization problem is avoided. For a fixed θ, let $a_\theta = a(\cdot, \theta)$ and $b_\theta = b(\cdot, \theta)$ and the method can be summarized in Algorithm 10.

As the output of the algorithm, two mappings are obtained. In realization mappings p_θ, q_θ can be stored as look-up tables for required values of θ. Or one can store pseudo-polynomials a and b, and obtain the required p_θ, q_θ when needed. The latter approach is tractable, since a and b have low order and the factorization can be done cheaply. The final optimization problem is a second order cone program and is also much cheaper than the relaxed program.

Error Bounds

If in Algorithm 10 all the constraints are enforced for all the frequencies $\boldsymbol{\omega} \in [0,\pi]^2$, then γ_{pshmr}^N is denoted as γ_{pshmr}^c. The main theoretical result

of this chapter is outlined in the next statement:

THEOREM 3.1
Consider Algorithm 10 with the full sampling (the constraints are enforced for all the frequencies $\boldsymbol{\omega} \in [0, \pi]^2$) with the solution γ^c_{pshmr}, b_θ, a_θ and p_θ, q_θ. The following inequalities hold:

1. $\max\limits_{\theta \in [0,\pi]} \sigma_{k_0+1}(G(\cdot, \theta)) \leq \gamma^c_{\text{pshmr}}$

2. $\gamma^c_{\text{pshmr}} \leq \max\limits_{\theta \in [0,\pi]} \|G - p_\theta/q_\theta\|_\infty$

3. $\max\limits_{\theta \in [0,\pi]} \|G - p_\theta/q_\theta\|_\infty \leq (k_0 + 1)\gamma^c_{\text{pshmr}}$

where $\sigma_{k_0+1}(G(\cdot, \theta))$ is the $(k_0 + 1)$-th largest Hankel singular value of $G(\cdot, \theta)$, k_0 is the order of pseudo-polynomials p_θ, q_θ with respect to z. ☐

Proof.
1. For every θ, Algorithm 10 is essentially Hankel model reduction with extra restrictions, as it was shown for the non-parameterized case in Chapter 2. Therefore, for every θ, the inequality $\sigma_{k_0+1}(G(\cdot, \theta)) \leq \gamma^c_{\text{pshmr}}$ is satisfied. By taking the maximum over all θ, the result is achieved.

2. Consider two optimization programs:

$$\min_b \max_{\theta \in [0,\pi]} \|G - b/a_\theta\|_\infty \qquad \min_p \max_{\theta \in [0,\pi]} \|G - p/q_\theta\|_\infty$$

where q_θ and a_θ are defined as above. Since $a_\theta = q_\theta \widetilde{\varphi_\theta}$ the variable b can be chosen such that, the stable part of b/a_θ is equal to p/q_θ and the anti-stable part equal to zero. Clearly, the left-most problem has more freedom and its solution is equal to γ^c_{pshmr}. The right-most program has the solution equal to $\max\limits_{\theta \in [0,\pi]} \|G - p_\theta/q_\theta\|_\infty$. Thus the inequality is obtained.

3. Let the anti-stable part of b_θ/a_θ be $\hat{G}_-$. Since $\|G(\cdot, \theta) - b_\theta/a_\theta\|_\infty \leq \gamma^c_{\text{pshmr}}$, by the celebrated AAK theorem, there exists a matrix $K(\theta)$ such that the following bound is satisfied.

$$\|\hat{G}_-\|_H \leq \gamma^c_{\text{pshmr}} \Rightarrow \|\hat{G}_- + K\|_\infty \leq k_0 \gamma^c_{\text{pshmr}}$$

By the triangular inequality we have:

$$\|G(\cdot, \theta) - p_\theta/q_\theta\|_\infty \leq \|G(\cdot, \theta) - b_\theta/a_\theta\|_\infty + \\ + \|b_\theta/a_\theta - p_\theta/q_\theta\|_\infty \leq (k_0 + 1)\gamma^c_{\text{pshmr}}$$

and since it is always possible to pick p_θ such that

$$\|b_\theta/a_\theta - p_\theta/q_\theta\|_\infty \leq \|\hat{G}_- + K\|_\infty \leq k_0\gamma^c_{\text{pshmr}}$$

After taking the maximum over θ the result follows. □

It is a direct generalization of the non-parameterized error bounds. However, in the parameterized case, it is not clear if γ^c_{pshmr} is smaller than γ_{pmor} from (3.1). An SOS relaxation is employed, i.e., $\text{Re}\,(a) - \mu$ is an SOS and rather than $\text{Re}\,(a) - \mu$ is non-negative. Despite the name, the SOS relaxation is actually a restriction on decision variables compared to the non-negative condition. Hence, as opposed to Algorithm 9, it can not be guaranteed that γ^c_{pshmr} is a lower error bound on the parameterized model reduction program (3.1).

REMARK 3.1
The extension to the multi-variable (MIMO) systems is done by employing the techniques from Chapter 2 to obtain a tractable algorithm. The error bounds are essentially the same

1. $\displaystyle\max_{\theta \in [0,\pi]} \sigma_{k+1}(G(\cdot, \theta)) \leq \gamma^c_{\text{pshmr}}$

2. $\gamma^c_{\text{pshmr}} \leq \displaystyle\max_{\theta \in [0,\pi]} \|G - p_\theta q_\theta^{-1}\|_\infty$

3. $\displaystyle\max_{\theta \in [0,\pi]} \|G - p_\theta q_\theta^{-1}\|_\infty \leq (k+1)\gamma^c_{\text{pshmr}}$

However, now p_θ and q_θ are matrix valued functions and k is the order of $p_\theta q_\theta^{-1}$, not the order of p_θ and q_θ with respect to z. If the order of p_θ and q_θ with respect to z is set to k_0 and $q_{ij} \in \mathbb{R}^{m \times m}$, then the order k of $p_\theta q_\theta^{-1}$ is generally equal to $k_0 m$. □

REMARK 3.2
The Quasi-Convex Optimization (QCO) approach by [Sou *et al.*, 2008] is a specific case of the algorithm described in the this section. It requires an additional constraint on a, namely $a^\sim(z) = a(z)$, or $a^T(1/z) = a(z)$, then $\text{Im}\,(a) = 0$, and $\text{Re}\,(a) = a$. All the theoretical results of SHMR approach are still valid for the QCO method. □

3.3 Computation of Explicit Parameter Dependent Models

The output of SHMR Algorithm 10 is a model b/a, where a and b are pseudo-polynomials. For every value of the parameter θ, the model b/a

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Algorithm 11 Dominant part heuristic

1. For every parameter θ compute the dominant part of $G(\cdot, \theta)$

2. Build a transfer function $G_d(z, \theta)$ by interpolation of dominant part with respect to θ

3. Solve the following approximation problem using one of the presented algorithms:
$$\min_{p,q} \|G - G_d p/q\|_\infty$$

   REMARK. At the final step, one can solve instead:

   $$\min_{p,q} \|G G_d^{-1} - p/q\|_\infty$$

   The choice of the program is left to the designer

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contains stable and unstable poles. The actual approximations p/q are, in fact, obtained for every given θ, which is not the desired output. The desired output would be a model p/q, where p and q are pseudo-polynomials in both ω and θ. Such models significantly simplify the analysis and speed up the simulations as well. Moreover, in some applications it may be **required** to compute such models. In this section, a few techniques are presented, which can produce parameter dependent models with explicit dependence on the parameter.

Dominant Part Heuristics

Extraction of the dominant poles is a simple, but an important heuristic in the parameterized reduction. By itself, it does not provide an explicit parameter-dependent model, however, coupled with the iterative approach presented in the sequel it delivers very competitive solutions. This approach can considerably speed up the computation of SHMR solutions as well. The dominant poles are usually defined as the poles near the unit circle which contribute to the Bode magnitude plot in the form of peaks. The dominant part is a collection of such poles. The approach is outlined in Algorithm 11.

An argument can be made that after computation of the dominant part most of the work (if not all of it) has been done. However, in the parameterized case the dependence of the rest of the model (without the dominant part) may be complicated as well due to the parameter variations. Therefore, a simple interpolation may not give a good result in terms of accuracy on the validation stage.

Algorithm 12 Parameterized Iterative Approach to Model Reduction

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Compute the frequency response samples $G(\boldsymbol{\omega})$ for every $\boldsymbol{\omega} = \{\omega, \theta\}$ in $\Omega = \{\boldsymbol{\omega}_i\}_{i=1}^N$

Given an initial point φ^0, set $j = 1$, $\varphi^j = \varphi^0$

repeat

  Solve

$$\gamma_{\text{prd}}^N = \min_{p,q} \gamma \quad \text{subject to}$$

$$\text{Re}\,(q^{\sim}\varphi^j) \geq \mu \quad \forall \boldsymbol{\omega} \in [0, \pi]^2$$

$$|(G(\boldsymbol{\omega})q(\boldsymbol{\omega}) - p(\boldsymbol{\omega}))\varphi^j(\boldsymbol{\omega})| < \gamma \text{Re}\,(q^{\sim}(\boldsymbol{\omega})\varphi^j(\boldsymbol{\omega})) \quad \forall \boldsymbol{\omega} \in \Omega$$

  Denote φ_c^j all the coefficients of φ^j stacked in a vector. Set $j = j + 1$ and $\varphi^j = q$

until $\|\varphi_c^j - \varphi_c^{j-1}\|_2 \leq \varepsilon$

Construct the reduced order model as p/q

Iterative Approach to Parameterized Model Reduction

In the non-parameterized framework, the iterative approach is obtained by fixing φ in (3.3). Similarly, Algorithm 12 is derived. If it is possible to estimate $\varphi(\boldsymbol{\omega})$ in advance, then an explicit parameter-dependent model p/q can be computed.

In order to enforce the condition $\text{Re}\,(q\varphi^{\sim}) \geq \mu$ for all the frequencies in $[0, \pi]^2$, an explicit parameter dependence of φ is also required. In the parameterized case, the SHMR algorithm can not be used for these purposes, since it does not provide this type of solution. The obvious choice is iterating, while starting with φ equal to one. However, it can be very costly if the number of iterations is large.

The main conservatism of setting $\varphi \equiv 1$ is inability of tracking the dominant poles, which corresponds to fast phase variations (this case is discussed in [Sootla and Sou, 2010]). If $\text{Re}\,(q)$ is positive, then the phase is confined to the interval between $[-\pi/2, \pi/2]$. Note, by extracting the dominant part, the phase variations are reduced. Therefore, the dominant part heuristic described above can be used to significantly reduce such conservatism, and thus reduce the number of iterations.

Besides delivering explicit parameter-dependent functions, there is another advantage of the iterative approach in comparison to QCO. It is a more efficient parameterization, which may result in a lower computational cost. Indeed, consider a transfer function G, the numerator and the denominator of which depend as (trigonometric) polynomials of order r on a parameter. To approximate it with the QCO approach, it is required to

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create pseudo-polynomials a and b of order $2r$ in parameter dependence, since $a = qq^{\sim}$. The iterative approach, on the other hand, requires polynomials of order r to parameterize the reduced order model. If $\varphi \equiv 1$, then the SOS constraints for the iterative method will be much smaller than the ones for QCO method. Then again, there is no extra advantage, in terms of accuracy, of the iterative method in comparison to the SHMR. It occurs due to the fact that SHMR is a relaxation of the iterative approach. Therefore, the SHMR algorithm will always deliver the most accurate approximation among all the methods in the framework.

Interpolation of an SHMR solution

After SHMR Algorithm 10 is solved, transfer functions $p_\theta(z, \theta_i)/q_\theta(z, \theta_i)$ for some i are obtained. If a transfer function with an explicit dependence on θ is required, it can be obtained by interpolation of coefficients of $p_\theta(z, \theta_i)/q_\theta(z, \theta_i)$. However, in this case, stability for every value of θ_i can not be guaranteed. Such an approach is less accurate than the dominant part heuristic. In the dominant part heuristic only the dominant part is interpolated. The rest of the model is computed with stability and norm constraints. It can be done with iterative Algorithm 12, which also counteracts possibly occurring interpolation errors. Although, the interpolation of the SHMR solution is still a valid approach, it is not considered in the current thesis due to stability problems.

3.4 Implementation and Examples

The methods were implemented using YALMIP ([Löfberg, 2004]) and SeDuMi ([Sturm, 1999]). Recall that the described method addresses the reduction of discrete-time systems, therefore in order to reduce a continuous time model it is discretized using a bi-linear transformation (as in Chapter 2). Regarding the positivity constraint a few options are described in [Dumitrescu, 2007, Chapter 3].

Computational Complexity

There are two main contributors to the complexity of the methods. The first one is computing the frequency samples which is $O(l^3)$ for each frequency point, with l being the order of the original model. In many cases, it can be lowered to $O(l \log(l))$ even for dense models (for example, using [Kamon *et al.*, 1997; Moselhy *et al.*, 2007; Zhu *et al.*, 2003]). The second one is the cost of the optimization algorithm. It does not exceed $O(N_1^2 N_2^{2.5} + N_2^{3.5})$ with N_1 being the number of decision variables and N_2 the number of rows in the LMI constraints, when the method is implemented with SeDuMi (see [Peaucelle *et al.*, 2002]). In parameterized

reduction, N_1 will be $O((k_0 + 1) \cdots (k_n + 1))^2$, if the parameter θ is in $\mathbb{R}^n$ and $N_2 = O(N + (k_0 + 1) \cdots (k_n + 1))$. The reduced model orders $k_i \ll l$ and N is the number of points in the grid. Note that number of frequency samples N generally does not depend on the order of the system. Typically, the larger number N is required, when some poles are close to the unit circle. The total cost will be

$$O(l \log(l) N) + O(((k_0 + 1) \cdots (k_n + 1))^4 N_2^{2.5} + N_2^{3.5})$$

Based on the numerical simulations, the semidefinite program now is much more costly than in the non-parameterized case. The main reason for such a cost is the positivity constraint, which can create intractable programs for a large number of the parameters and large orders of the pseudo-polynomials. The quadratic term $((k_0 + 1) \cdots (k_n + 1))^2$ comes from the positivity constraint enforced for all the frequencies in $[0, \pi]^2$. If another strategy is used, for example enforcing the positivity constraint on a parameter grid, then $N_1 = O((k_0 + 1) \cdots (k_n + 1) + (k_0 + 1)^2)$ and $N_2 = O(N + k_0)$. Which significantly reduces the cost, however, for some parameter values an unstable model can be obtained.

Possible Applications

Consider a linear parameter-varying (LPV) model, which depends on a parameter θ. For these models it is typically assumed that θ is a time dependent function. A very simple example of a linear parameter-varying model is a time-varying one, with time t as a parameter. Modeling of such systems can be approached by assuming the parameter is constant. After that, the proposed model reduction framework can be used. This approach is typically called a local approach to linear parameter-varying system modeling (see, [Petersson and Löfberg, 2009; Tóth *et al.*, 2009] and the references therein for other approaches). The positivity condition in the method is equivalent to stability of the function $G(\cdot, \theta) \in \mathbb{H}_\infty$ for any fixed θ, which is a necessary condition for the global LPV stability.

It is also quite tempting to apply the presented approach to n-D system modeling (see [Beck *et al.*, 1995; Wu and Lam, 2008] for existing approaches). In this case, the basis functions should be modified. Now both variables are varying on the unit circle

$$F_s = e^{-s j \omega} \quad H_s = e^{-s j \theta}$$

The basis can be analytically extended to the complex plane (z_1, z_2), where z_1 corresponds to $e^{j\omega}$ and z_2 corresponds to $e^{j\theta}$. At the same time, the positivity constraint $\mathrm{Re}\,(a) > 0$ guarantees that the zeros of a lie in

$$\mathbb{D} = \{(z_1, z_2) \big| |z_1| \leq 1, |z_2| = 1\} \cup \{(z_1, z_2) \big| \ z_1 = 0, |z_2| \leq 1\}$$

which is a stability domain for the 2-D transfer functions ([Jury, 1978]). Using the iterative approach, it is possible to obtain explicit dependence on variables ω and θ, and thus z_1 and z_2. In the case, for 2-D transfer functions the order of a state-space realization of the system may be higher than $r_1 + r_2$, where r_1 is the number of basis functions F_s and r_2 is the number of basis function H_s. Furthermore, obtaining a general minimal realization is still an open question in the theory. However, it is shown that there are systems that do not have a real valued realization of order $r_1 + r_2$. A counter example may be found in [Kung *et al.*, 1977]. Where an algorithm to obtain a state-space realization of order $\min(2r_1 + r_2, r_1 + 2r_2)$ is described.

Both applications are merely possibilities of the further developments of the framework. In both cases, there are some difficulties, which are unavoidable at the moment.

EXAMPLE 3.1—EFFICIENCY OF PARAMETERIZATION
To illustrate the parameterization issues, consider this toy example. Consider a transfer function:

$$G(z, \theta) = \frac{\omega_1^2}{s^2 + 2\zeta_1\omega_1 s + \omega_1^2} \frac{\omega_2^2}{s^2 + 2\zeta_2\omega_2 s + \omega_2^2}$$

where $\omega_1 = 1$, $\omega_2 = 3$, $\zeta_1 = 0.1 + 0.9\,\theta$, $\zeta_1 = 0.1 + 0.9(1 - \theta)$ and $\theta \in [0, 1]$. If a state-space modeling approach is used, for example [Petersson and Löfberg, 2009], then the required parameterization is a linear one with $A(\theta) = A_0 + A_1\theta$ and B, C, D matrices constant. Using the described framework, the required parameterization is of order 2 for iterative Algorithm 12 and SHMR Algorithm 10, while the QCO method requires parameterization of order 4. However, [Petersson and Löfberg, 2009] is not a semidefinite approach, which means that the computational efficiency is lower than in the presented framework. The approximation errors are given in Table 3.1. The notation $(4, 1)$ means that the frequency component $e^{-j\omega}$ has the order 4 and parameter θ has a basis of order 1. □

EXAMPLE 3.2—DEFORMABLE MIRROR MODELING
The following model was studied in [Giselsson, 2006] and obtained by means of a finite element modeling approach that resulted in a system of second-order differential equations:

$$I\ddot{x} + \alpha\Lambda^2\dot{x} + \Lambda = Fu$$
$$y = F^T x$$

where a model has 420 sensors and actuators, 2000 states, and the friction coefficient α is chosen as a modeling parameter. The comparison of the
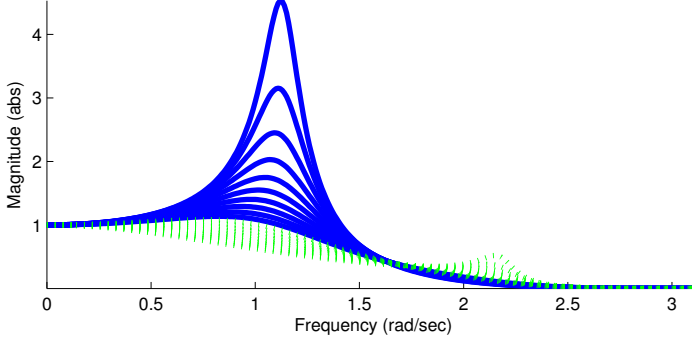
Figure 3.1 Frequency response of models in Example 3.1 for different values of parameters. The curves are solid blue for $\theta < 0.5$ and dotted green for $\theta > 0.5$

approximation error is performed in percent, where 100% is the maximum $\mathbb{H}_\infty$ norm of the models on the training grid.

A single entry of the corresponding transfer matrix is considered. The objective is to obtain a family of low-order models that depend on α which belongs to the interval $[0.01, 0.06]$. The models are trained (computed) on the parameter $\alpha = 0.01, 0.02, 0.03, 0.04, 0.05, 0.06$ (the training grid). The validation is performed on another grid $\alpha = 0.015, 0.025, 0.035, 0.045, 0.055$. For every value of the parameter, 300 frequency response samples are computed. For iterative Algorithm 12, only one iteration is computed with $\varphi \equiv 1$.

The results of reduction for different orders is found in Tables 3.2 and 3.3. The QCO reduced order models had almost the same approximation quality for orders $(10, 1)$, $(13, 1)$ and $(16, 1)$. At the same time, significantly improving if the order of the parameter is raised to 2. This behaviour signals that the order equal 1 for the parameter is not sufficiently large in this example. On the other hand, using Algorithm 12

Table 3.1 Approximation errors in Example 3.1

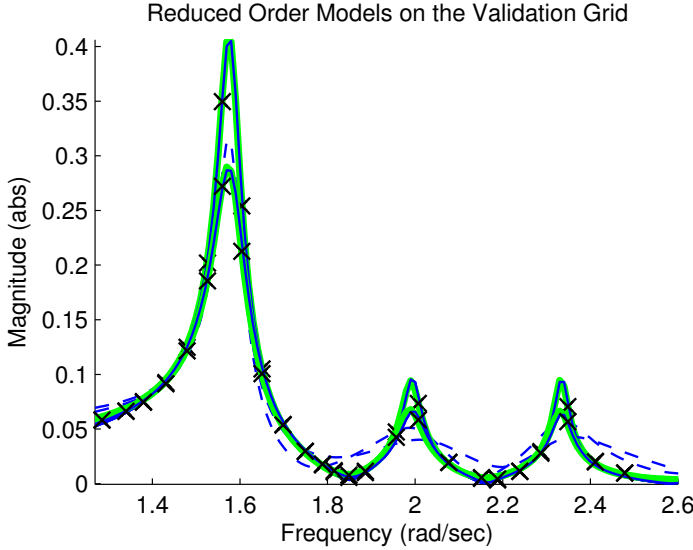| Identification orders	$(4, 1)$	$(4, 2)$	$(4, 4)$
[Petersson and Löfberg, 2009] (error in $\mathbb{H}_2$)	≈ 0	≈ 0	≈ 0
QCO (error in $\mathbb{H}_\infty$)	0.441	0.025	≈ 0
Iter. Alg. 12 (error in $\mathbb{H}_\infty$)	0.0557	≈ 0	≈ 0
SHMR Alg. 10 (error in $\mathbb{H}_\infty$)	0.0337	≈ 0	≈ 0

Figure 3.2   Frequency responses for $\alpha = 0.025$, 0.045 (values belong to the validation grid) of the SHMR reduced order (thick green lines), the reduced order model obtained with iterative Algorithm 12 (blue solid lines), the QCO reduced order (dashed thin blue lines) and the full order (black crosses) models. Order of reduction is set to $(16, 1)$.

(with $\varphi \equiv 1$) it is possible to achieve a decent quality if the order is set to $(16, 1)$, but the approximations are very conservative with lower orders of frequency variable z. The SHMR method is consistently better than iterative Algorithm 12 and the QCO, both on the training and the validation grids. Except for the order $(10, 2)$ where the approximation error of SHMR is larger than the one of QCO on the validation grid. In fact, even $(10, 1)$ SHMR approximation is better than $(10, 2)$ SHMR approximation. It means that over-fit has occurred and there is not sufficiently many constraints for this order. Over-fit can be cured in this case by set-

Table 3.2   Approximation errors on the training grid (in %) in Example 3.2.

Orders	$(10, 1)$	$(10, 2)$	$(13, 1)$	$(13, 2)$	$(16, 1)$	$(16, 2)$
SHMR Alg. 10	0.99	1.24	1.11	0.93	0.99	1.03
QCO	11.19	3.81	8.46	1.23	9.75	1.11
Iter. Alg. 12	14.74	14.70	12.41	12.39	1.42	1.07

Table 3.3 Approximation errors on the validation grid (in %) in Example 3.2.

Orders	$(10, 1)$	$(10, 2)$	$(13, 1)$	$(13, 2)$	$(16, 1)$	$(16, 2)$
SHMR Alg. 10	1.52	5.69	1.37	1.83	1.19	1.19
QCO	9.25	5.09	7.64	2.42	7.77	2.58
Iter. Alg. 12	13.42	11.08	10.36	9.25	1.58	1.53

ting tolerance of bisection higher. Then the semidefinite program will not try to compensate for the better fit on the validation grid. This example validates that the iterative approach may be conservative with respect to the order of the frequency variable and the QCO method is conservative with respect to the order of parameters.

Note also that quality of reduced order model on the validation grid is sometimes better than the quality on the training grid. One of the possible reasons for this is that the tolerance for bisection is too high and the approximations are not close enough to the optimal value. Therefore the program does not compensate by over-fitting on the validation grid. The frequency responses of the full order model and the approximations for particular parameters are depicted in Figure 3.2.

Dominant Part Heuristic. For the dominant part heuristic three pairs of poles and zeros were identified forming a dominant part G_d. The system gain, every zero and pole were identified separately with first order trigonometric polynomials. Iterative Algorithm 12 was applied to this problem with orders $(4, 1)$, giving a resulting approximation of with the order of frequency variable 10. The approximation error on both training and validation grids is around 4.5%, which gives improvement by a factor of 3 in comparison to non-modified iterative Algorithm 12. The frequency responses for chosen parameters is depicted in Figure 3.3. Recall that the iterative approach also delivers an explicit dependence on the parameter α. Therefore, iterative Algorithm 12 coupled with the extraction of the dominant part is a very powerful heuristics. □

3.5 Conclusion and Discussion

The main contribution of this chapter is a framework for modeling parameterized LTI systems. It is based on the semidefinite programming approach and matching of frequency response samples. Two methods have been presented, which extend the ones proposed in Chapter 2. One method delivers more accurate solutions than the predecessor, while the other de-
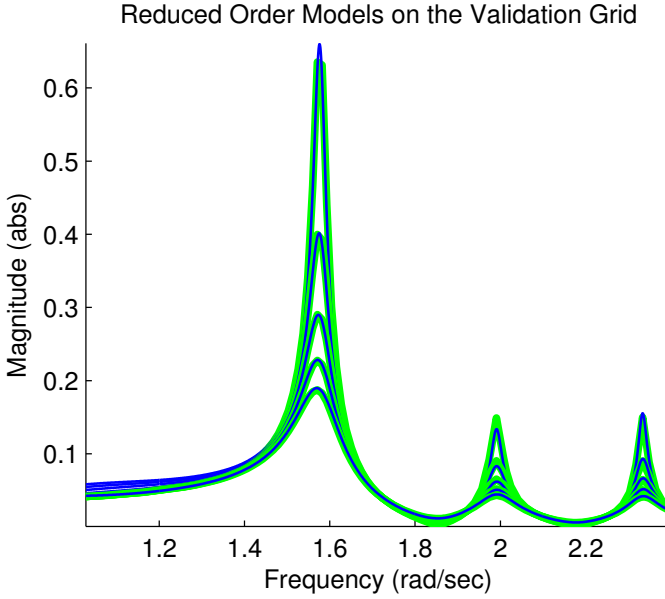
Reduced Order Models on the Validation Grid

Figure 3.3 Frequency responses for $\alpha = 0.015, 0.025, 0.035$ and 0.045 (values belong to the validation grid) of the full order (thick green lines) and the reduced order models obtained using Algorithm 12 (blue solid lines). Order of reduction is set to $(4, 1)$ with a dominant part which had three pairs of poles and zeros.

livers explicit parameter-dependent models.

A few applications of the framework can be proposed. The most intriguing one is linear parameter-varying system modeling. In order to apply the framework, it is required to assume that parameters are constant with respect to time. The question, which should be asked is: "is it possible to relax this assumption?". If it is possible, for example with frequency and parameter dependent weights, then what are the mechanisms to determine such weights? The stability condition at the moment is necessary, but not sufficient. To ensure sufficiency, the parameter dependence on time should be taken into account. If the parameter variations are known, how does one incorporate such information into the semidefinite program? At the moment, the author does not see a straightforward answer to these questions, making them are topics of future research.

4

Model Order Reduction in the ν-gap metric

All the standard model reduction methods measure the approximation error in $\mathbb{H}_\infty$ or $\mathbb{H}_2$ spaces, meaning that the models are open loop stable. These norms are measures on the distance in the open loop setting. An open loop stable model can create instabilities in a closed loop settings. Therefore in the closed loop setting, these norms do not, generally, reflect the distance adequately. An early attempt to create a more reliable metric in the closed loop setup was the introduction of the gap metric in [Zames and El Sakkary, 1980], followed by many papers including [Vidyasagar, 1984] and [Vinnicombe, 1993b; Vinnicombe, 1993a]. In the last reference the ν-gap metric was introduced and it is the only metric for which *"... any plant at a distance less than β from the nominal will be stabilized by any compensator stabilizing the nominal with a stability margin β. Furthermore, any plant at a distance* **greater** *than β from the nominal will be* **destabilized** *by some compensator that stabilizes the nominal with a stability margin of at least β"* ([Vinnicombe, 1993a]). Therein the stability margin is defined. Moreover, the ν-gap induces the weakest topology in the space of controllers in which closed loop stability is a robust property. To some extent, the stability of a closed loop can be evaluated without considering the other plants in this loop. Therefore, the ν-gap metric may be a crucial tool in distributed system modeling, where the evaluation of the entire system can be computationally overwhelming.

Early work in ν-gap model reduction includes [Cantoni, 2001] and [Buskes and Cantoni, 2007]. However, only in [Buskes and Cantoni, 2008] was a tractable algorithm obtained, which uses a state-space representation of a system. The solution is computed step-wise in their algorithm. If the order of the full order model is n, then, first, an optimal in the ν-gap metric $n-1$-st order approximation is obtained. After that an optimal $n-2$-nd order approximation to $n-1$-st order model is calculated, and so

on down to the required reduction order.

As opposed to [Buskes and Cantoni, 2008], the algorithm presented in this chapter uses the frequency domain data to obtain a reduced model. The method employs semidefinite programming as a tool and coprime factors of the original system. The algorithm was also extended to account for the overall performance in controller-plant loops, which is presented in Section 4.3. Although the algorithm was initially derived for scalar valued transfer functions, with extra restrictions, it is extended to matrix valued transfer functions using the techniques described in [Sootla and Sou, 2010; Tobenkin *et al.*, 2010].

4.1 Preliminaries

First, it may be useful to illustrate why and in which situations the *v*-gap metric is employable. Consider the following toy example:

EXAMPLE 4.1—[ÅSTRÖM AND MURRAY, 2008, PP. 349-350]
Given the systems below, their step responses are investigated in Figure 4.1.

$$P_1 = \frac{100}{s+1} \qquad P_2 = \frac{100}{(s+1)(0.0025s+1)^2} \qquad P_3 = \frac{100}{s-1}$$

Two settings are considered: open loop and closed loop. The last one is a simple negative feedback with the closed loop transfer functions $P_i(1 - P_i)^{-1}$.

The distance in the open loop setting (the $\mathbb{H}_\infty$ norm) between the plants P_1 and P_2 is small. Since the plant P_3 is open loop unstable, its open loop distance (in the $\mathbb{H}_\infty$ norm) to P_1 and P_3 is infinite. The upper plots in Figure 4.1 confirm this discussion.

On the other hand, the *v*-gap metric between the plants P_1 and P_3 is small. The step responses of $P_1(1 - P_1)^{-1}$ and $P_3(1 - P_3)^{-1}$ are similar as well. Since the function $P_2(1 - P_2)^{-1}$ is unstable, the distance in *v*-gap between P_2 and P_3, P_2 and P_1 is very large.

To summarize, the step responses in Figure 4.1 confirm that *v*-gap is a better measure on the distance in this particular closed loop setting, than the $\mathbb{H}_\infty$ norm. □

There are several equivalent definitions of the metric, but the chosen one is more convenient for our goal. Let b, a be a left normalized coprime factorization (NCF) of G_1, that is, $G_1 = a^{-1}b$. Let also p, q be a right coprime factorization, not necessarily normalized, of $G_2 = pq^{-1}$.
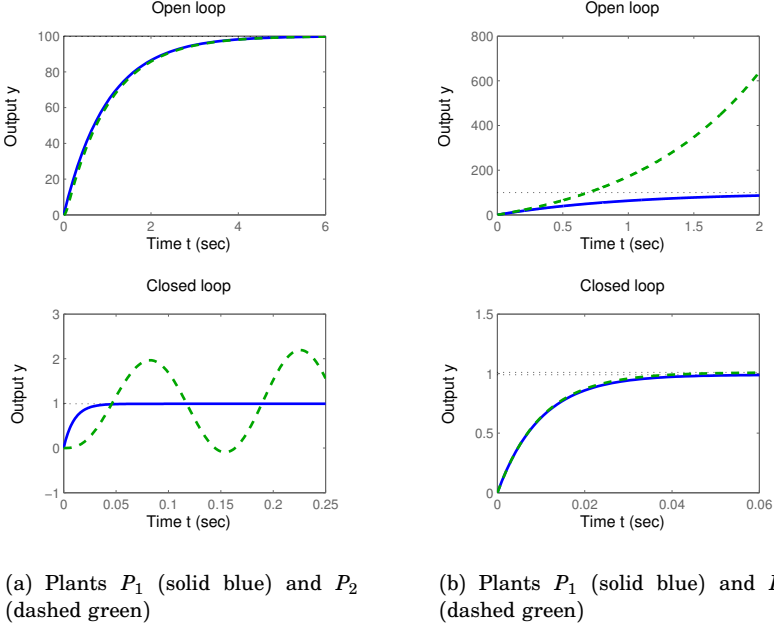
(a) Plants P_1 (solid blue) and P_2 (dashed green)

(b) Plants P_1 (solid blue) and P_3 (dashed green)

Figure 4.1 Open and closed loop step responses of the plants. The figures in (a) depict the situation, when the $\mathbb{H}_\infty$ norm between the plants P_1 and P_2 is small, however the ν-gap is large. The figures in (b) depict the opposite situation, when the ν-gap metric between the plants P_1 and P_3 is small, but the $\mathbb{H}_\infty$ norm is infinite.

DEFINITION 4.1—ν-GAP METRIC
Define a function $\delta_\nu(\cdot,\cdot) : \mathcal{R}^{m_1 \times m_2} \times \mathcal{R}^{m_1 \times m_2} \to \mathbb{R}$ as follows

$$\delta_\nu(G_1, G_2) = \begin{cases} \delta_{\mathbb{L}_2}(G_1, G_2) & \text{if } \eta\left([G_2, -G_1^\sim]\right) = \eta\left([G_1, -G_1^\sim]\right) \\ 1 & \text{otherwise} \end{cases}$$

where

$$\delta_{\mathbb{L}_2}(G_1, G_2) = \sqrt{1 - \left\| \begin{pmatrix} p \\ q \end{pmatrix} (a^\sim q + b^\sim p)^{-1} \right\|_\infty^{-2}} \qquad \square$$

In this definition, symbol $\cdot^\sim$, means $G^\sim(z) = G^T(1/z)$. The constraint $\eta\left([G_2, -G_1^\sim]\right) = \eta\left([G_1, -G_1^\sim]\right)$ is usually called a "winding number condition" in the control literature. This is a necessary condition to preserve stability in a closed loop, if G_1 is replaced with G_2. If the condition is satisfied, there is a certain robustness guarantee on a closed loop transfer function as well. Throughout the paper, if not stated otherwise, it is

assumed that G_1 and G_2 are scalar, not necessarily stable, transfer functions. Therefore, a, b, p and q are scalar transfer functions as well.

Finally, we are ready to formulate the ν-gap model reduction problem as an optimization one. Given asymptotically stable a and b, such that $b^\sim b + a^\sim a = 1$ and b/a is not necessarily stable, solve

$$\gamma_{\text{opt}} = \min_{\gamma > 0, p, q} \quad \gamma \tag{4.1}$$

$$\left\| \begin{pmatrix} p \\ q \end{pmatrix} (a^\sim q + b^\sim p)^{-1} \right\|_\infty \leq \gamma \tag{4.2}$$

$$\eta \left(\left[\frac{p}{q}, -\frac{b^\sim}{a^\sim} \right] \right) = \eta \left(\left[\frac{b}{a}, -\frac{b^\sim}{a^\sim} \right] \right) \tag{4.3}$$

$$\deg(p/q) = k$$

Note that $\delta_\nu (b/a, p/q) \leq \sqrt{1 - 1/\gamma_{\text{opt}}^2}$ by construction. In this formulation, b/a is a full order model, and p/q parameterizes its k-th order approximation. However, the obtained program is not generally convex even for the scalar-valued functions due to the winding number condition and the computation of $\mathbb{L}_\infty$ norm. Therefore, a convexification is required.

4.2 Model Reduction in the ν-gap Metric

The program is addressed only for the discrete-time transfer functions. The continuous-time transfer functions can be discretized as in Chapter 2.

There are two contributors of non-convexity in the presented formulation. The first one is the $\mathbb{L}_\infty$ norm optimization with a degree constraint, which creates a non-smooth optimization problem. Nevertheless, an efficient and accurate suboptimal solution may be derived using a method from Chapter 2, i.e., iterative Algorithm 6 (page 48). The second one is the winding number condition, which is much harder to address. Luckily, the techniques from Chapter 2 address the stability condition in a manner that can be exploited here. Therefore, only the norm constraint will be addressed first. Then it will be shown that all the solutions of such a problem satisfy the winding number condition.

The usual approach to $\mathbb{L}_\infty$ minimization is rewriting the constraint (4.2) as infinite number of constraints for every frequency ω in $[0, \pi]$:

$$\gamma_1 = \min_{\gamma > 0, p_i, q_i} \quad \gamma \tag{4.4}$$

$$\forall \, \omega \in [0, \pi] : \left\| \begin{pmatrix} p \\ q \end{pmatrix} \right\|_2 < \gamma |q a^\sim + p b^\sim|$$

where $p=\sum\limits_{i=0}^{k} p_i e^{-i\cdot J\omega}$, $q=\sum\limits_{i=0}^{k} q_i e^{-i\cdot J\omega}$, and q_i, p_i are the actual decision variables. Introduce a new variable ϕ into the program as:

$$\gamma_2 = \min_{\gamma>0, p_i, q_i, \phi} \quad \gamma \tag{4.5}$$

$$\forall\, \omega \in [0,\pi] : \left\| \begin{pmatrix} p \\ q \end{pmatrix} \phi^\sim \right\|_2 < \gamma \mathrm{Re}\left((qa^\sim + pb^\sim)\phi^\sim \right)$$

The programs (4.4) and (4.5) are equivalent, meaning that $\gamma_1 = \gamma_2$. The proof of this fact is identical to the one of Lemma 2.2. It can be also shown that an optimal choice of ϕ is $qa^\sim + pb^\sim$. Introducing an extra variable ϕ does not solve all the problems. However, by computing ϕ in advance and fixing it in the minimization, the program becomes quasi-convex. Given an initial point p^0 and q^0, the algorithm can be iterated, with $\phi = q^0 a^\sim + p^0 b^\sim$. The choice of the initial point is described in the sequel.

Is the described set big enough to provide any improvement at all? This question was studied in [Henrion *et al.*, 2003] and [Yang *et al.*, 2007], besides this thesis. The results for low orders as 2 and 3 are colorfully illustrated in [Henrion *et al.*, 2003]. Given two polynomials θ and ξ this condition describes all positive real transfer functions with a fixed denominator θ. The set of all possible ξ is shown in numerical examples to be significantly big in comparison to the set of all stable ξ. However, no theoretical results were provided in any work.

By convexifying (4.2) the constraint (4.3) was incorporated into the program. A proof of this fact is summarized in the following lemma.

LEMMA 4.1
Assume that p^0 and q^0 are FIR filters of the same order, and b and a are asymptotically stable functions with identical poles. Assume also that p^0/q^0 satisfies the winding number condition for b/a. Suppose p and q are coprime FIR filters of the same order as p^0 and q^0. Let $\phi = q^0 a^\sim + p^0 b^\sim$. If $\mathrm{Re}\left((qa^\sim + pb^\sim)\phi^\sim \right) > 0$ for all the frequencies ω in $[0,\pi]$ then the winding number condition $\eta\left([p/q, -b^\sim/a^\sim]\right) = \eta\left([b/a, -b^\sim/a^\sim]\right)$ is satisfied.   □

Proof.   The proof is based on the proof of Lemma 2.3.
Since p^0/q^0 satisfies the winding number condition, it is only required to show that $\eta\left([p^0/q^0, -b^\sim/a^\sim]\right) = \eta\left([p/q, -b^\sim/a^\sim]\right)$. This is shown in a straightforward manner from the condition $\mathrm{Re}\left((qa^\sim + pb^\sim)\phi^\sim\right) > 0$. Let $c = qa^\sim + pb^\sim$, the inequality $\mathrm{Re}\left(c\phi^\sim\right) > 0$ on the unit circle is equivalent to $\mathrm{Re}\left(c/\phi\right) > 0$ on the unit circle. Note that the number of zeros of c and ϕ is equal (since the order of p, q, p^0 and q^0 is the same), and the poles of c and ϕ are identical. Furthermore, if $\mathrm{Re}\left(c/\phi\right)$ is positive on the unit circle, then the function $c(e^{J\omega})/\phi(e^{J\omega})$ does not encircle the origin.

Therefore, by Cauchy's argument principle, the number of zeros and poles of c/ϕ inside the unit circle is equal. Since the number of zeros of c and ϕ is the same, it also implies that c has the same number of unstable zeros as ϕ. Therefore, $qa^\sim + pb^\sim$ has the same number of unstable poles as $q^0a^\sim + p^0b^\sim$, and thus $\eta\left(\left[p^0/q^0, -b^\sim/a^\sim\right]\right) = \eta\left(\left[p/q, -b^\sim/a^\sim\right]\right)$ can be verified. Finally, $\eta\left(\left[p/q, -b^\sim/a^\sim\right]\right) = \eta\left(\left[b/a, -b^\sim/a^\sim\right]\right)$. $\qquad\square$

REMARK 4.1
It is assumed that the order of q, p, q^0 and p^0 is the same and it is equal to k in (4.5). It is theoretically possible that the order of the obtained p and q is less than k. On the other hand in semidefinite programming, obtaining coefficients q_k and p_k equal to zero (which corresponds to order $k-1$) is equivalent to obtaining a matrix which is rank-deficient. The set of rank-deficient matrices is a null measure subset of the space of full rank matrices. Therefore, it is almost improbable to obtain a rank-deficient matrix in semidefinite programming. With a similar reasoning, it can be stated that p and q are, in fact, coprime. $\qquad\square$

Finally, a quasi-convex semidefinite program for a given ϕ may be deduced:

$$\gamma_v^c = \min_{p_i, q_i} \gamma \quad \text{subject to} \tag{4.6}$$

$$\forall\, \omega \in [0, \pi]:$$

$$\left\|\begin{pmatrix} p(e^{j\omega}) \\ q(e^{j\omega}) \end{pmatrix} \phi^\sim(e^{j\omega})\right\|_2 < \gamma \operatorname{Re}\left((q(e^{j\omega})a^\sim(e^{j\omega}) + p(e^{j\omega})b^\sim(e^{j\omega}))\phi^\sim(e^{j\omega})\right)$$

THEOREM 4.1
Consider the program (4.6) with a full sampling, i.e., the constraints are enforced for all the frequencies ω in $[0, \pi]$, where γ_v^c, p, and q is the output of the algorithm. Then $\delta_v(G, p/q) \leq \sqrt{1 - (1/\gamma_v^c)^2}$. $\qquad\square$

Proof. Shown by construction using Lemma 4.1. $\qquad\square$

Tractable Algorithm and Implementation

The choice of the starting point is the most important part of the algorithm. A reasonable guideline to choose the initial point is to use an approximation of the right NCF stacked in a vector (or the normalized right graph symbol). A known result is used to justify such an action:

LEMMA 4.2— [VINNICOMBE, 2000]
Given a plant P_0 of order n

$$\sigma_k \leq \inf_{P_1 \in \mathscr{P}(k)} \delta_v(P_0, P_1)$$

Algorithm 13 Initial point computation

1. Compute an approximation $(\,n/\theta \quad m/\theta\,)$ of $(\,b \quad a\,)$, which is a normalized right graph symbol of the full order model G. Note that m, n and θ are FIR filters

2. Set $\phi = nb^\sim + ma^\sim$

---

where $\mathcal{P}(k)$ denotes the transfer functions P_1, which have a minimal realization of order k. σ_k is a k-th largest Hankel singular value of the right normalized graph symbol. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Using Hankel-type reduction Algorithm 5 of the normalized right graph symbol, it is possible to approach the optimal level of approximation. This provides a reasonable way of choosing ϕ, which is outlined in Algorithm 13. There are a number of other ways to choose ϕ (e.g., the choice $\phi = \theta$ also provided excellent numerical results). Although a rigorous proof, that this particular choice of ϕ will always deliver results, was not provided, the intuition behind the choice described above, is reasonable and provides reasonable results in numerical experiments. The program (4.6) is quasi-convex and can be solved using standard tools as in Algorithm 14.

Algorithm 14 v-gap model reduction

Compute the normalized coprime factors a and b of G, where $G = ba^{-1}$
Compute the function ϕ^1 using Algorithm 13. Set $j = 1$ and

$$p(e^{J\omega}) = \sum_{i=0}^{k} p_i e^{-iJ\omega} \quad q(e^{J\omega}) = \sum_{i=0}^{k} q_i e^{-iJ\omega}$$

repeat
    Solve the following quasi-convex program:

$$\gamma_v^N = \min_{p_i, q_i} \gamma \quad \text{subject to for all } \omega \text{ in } [0, \pi]:$$

$$\left\| \begin{pmatrix} p \\ q \end{pmatrix} \phi^j \right\|_2 < \gamma \mathrm{Re}\left((qa^\sim + pb^\sim)^\sim \phi^j\right)$$

Compute an NCF m/θ, n/θ of p/q. Set $\phi^{j+1} = nb^\sim + ma^\sim$, $j = j + 1$.
until $\|\phi^j - \phi^{j-1}\|_\infty \leq \varepsilon$
The reduced order plant $\widehat{G}$ is computed as p/q

The second order cone constraint can be easily transformed into a linear matrix inequality (LMI) using the Schur complement, providing a semidefinite constraint instead. A frequency dependent semidefinite constraint may be imposed for all the frequencies in $[0, \pi]$ at ones using the Kalman-Yakubovitch-Popov lemma (see, Lemma 1.3). To provide a computationally cheaper program the constraints may be enforced on a frequency grid $\{\omega_i\}_{i=1}^N$, where N is big enough to avoid over-fit. The algorithm is implemented using the interior-point solvers SeDuMi ([Sturm, 1999]) and SDPT^3 ([Tütüncü *et al.*, 1999]) and the parser YALMIP ([Löfberg, 2004])

Computational Complexity

There are two main contributors to computational complexity: computation of normalized coprime factors and the optimization problem. Computation of NCFs is done using Riccati equations and, therefore, complexity is $O(l^3)$ floating point operations (flops), where l is the order of the equation (of the full order model G).

   The optimization cost of a semidefinite program differs depending on the tolerance level, the number of decision variables and if the constraints are enforced for all the frequencies or just on a grid. The total cost when solved with SeDuMi does not exceed $O(N_1^2 N_2^{2.5} + N_2^{3.5})$ flops, where N_1 is the number of decision variables and N_2 the number of rows in the LMI constraints ([Peaucelle *et al.*, 2002]). If constraints are enforced on a frequency grid $\{\omega_i\}_{i=1}^N$, then $N_1 = O(k)$ and $N_2 = N$. Here, k is the order of the approximation. In general, computing the frequency samples costs $O(l^3)$ and can be lowered to $O(l \log(l))$ in certain cases ([Moselhy *et al.*, 2007; Zhu *et al.*, 2003]). If constraints are enforced using the KYP lemma for all the frequencies in $[0, \pi]$, then $N_1 = O(l^2)$ and $N_2 = O(l)$.

   Since the program is quasi-convex, it is solved using bisection. The tolerance of bisection should be higher than in the similar model reduction methods. Low values of δ_ν correspond to values of γ very close to 1, for instance if $\delta_\nu = 0.05$ then $\gamma = 1.00125$. Therefore, the tolerance of approximation has to be modified accordingly depending on an application.

Multi-input-multi-output (MIMO) case

In the SISO case, a second order cone program was used, which simply does not exist for matrix constraints. Therefore, another technique is employed. Suppose a plant with left NCF $G = A^{-1}B \in \mathcal{R}^{m_1 \times m_2}$ with $A \in \mathcal{R}^{m_1 \times m_1}$ and $B \in \mathcal{R}^{m_1 \times m_2}$. Assume P, Q are the decision variables and matrix-valued FIR filters of degree k, such that $Q \in \mathcal{R}^{m_2 \times m_2}$ and $P \in \mathcal{R}^{m_1 \times m_2}$. Again, G is a full order model, and PQ^{-1} is a reduced order one.

$$\gamma_{\text{opt}} = \min_{\gamma > 0, P_i, Q_i} \quad \gamma \tag{4.7}$$

$$\left\| \begin{pmatrix} P \\ Q \end{pmatrix} (A^\sim Q + B^\sim P)^{-1} \right\|_\infty \leq \gamma \qquad (4.8)$$

$$\eta\left([PQ^{-1}, -A^{-\sim}B^\sim]\right) = \eta\left([A^{-1}B, -A^{-\sim}B^\sim]\right) \qquad (4.9)$$

Similarly, the condition (4.9) comes essentially for free. However, the LMI, which is obtained by enforcing the constraint for all the frequencies in $[0, \pi]$, is now more involved

$$P^\sim P + Q^\sim Q \leq \gamma^2 (A^\sim Q + B^\sim P)^\sim (A^\sim Q + B^\sim P)$$

To address the problem, a technique introduced in [Sootla and Sou, 2010; Tobenkin *et al.*, 2010] is used. Consider a matrix inequality $(X - Y)^\sim (X - Y) \geq 0$, where X and Y are complex valued matrices. From this inequality it is easy to obtain

$$X^\sim Y + Y^\sim X - Y^\sim Y \leq X^\sim X$$

which is valid for arbitrary X and Y. By introducing Y - an estimate on X - the quadratic function is bounded from below and linearized. Applying this inequality yields the constraint

$$P^\sim P + Q^\sim Q \leq \gamma^2 ((A^\sim Q + B^\sim P)^\sim R + R^\sim (A^\sim Q + B^\sim P) - R^\sim R)$$

which is convex and a semidefinite one is obtained using the Schur's complement. In our notation, a variable R is an estimate on $A^\sim Q + B^\sim P$, which allows a similar iteration as in the SISO case.

The winding number condition is satisfied by the same argument. Given an appropriate frequency dependent R we have:

$$\gamma^2 ((A^\sim Q + B^\sim P)^\sim R + R^\sim (A^\sim Q + B^\sim P) - R^\sim R) \geq P^\sim P + Q^\sim Q \geq 0$$

and thus $(A^\sim Q + B^\sim P)^\sim R + R^\sim (A^\sim Q + B^\sim P) \geq 0$ for all the frequencies ω in $[0, \pi]$. Now, a similar statement to Lemma 4.1 can be formulated and proved for $\mathrm{Re}\left(\det(A^\sim Q + B^\sim P)^\sim R\right)$.

To compute the initial point R_0 in the MIMO case, it is required to factorize the initial approximation $G_0 = P_0 Q_0^{-1}$, where P_0 and Q_0 are matrix-valued FIR filters of the required order and dimensions. However, such a factorization is not always possible, to the author's best knowledge. As a remedy, one can use directly the method from Chapter 2 to approximate the normalized left graph symbol, and P_0 and Q_0 will be parameterized as required by default.

4.3 Examples

The continuous-time models in the examples are discretized as in Chapter 2. Therein other implementation details are also described. Throughout the section METHOD 1 will denote approximation in the ν-gap metric. METHOD 2 will denote approximation in the ν-gap with a fixed closed loop performance degradation level, which will be introduced shortly.

EXAMPLE 4.2—APPROXIMATION OF A FLEXIBLE BEAM MODEL.
A continuous time model of a flexible beam is described in [Chahlaoui and Dooren, 2002]. The full order model has 398 states. As shown in Table 4.1, METHOD 1 always provides a better ν-gap match than Hankel approximations (including the one of a graph symbol). For orders 2 and 4 a considerable improvement was not achieved, since any method can match the peaks in the frequency response (every peak corresponds to a pair of complex conjugate poles). For orders 1 and 3 there is extra freedom in the choice of poles of the system, which is exploited by METHOD 1. A similar behavior is exhibited by method [Buskes and Cantoni, 2008], however, it was not employed directly due to computational constraints. First, the model was reduced to order 50 in order to apply the method. The accuracy loss during such an approximation was negligible, however, significantly improved the speed of the method. For orders 2 and 4 the accuracy of [Buskes and Cantoni, 2008] is slightly lower than the accuracy of Hankel reduction of the left graph symbol. This loss happens since [Buskes and Cantoni, 2008] is a step-wise procedure and we do not have an explicit control over the resulting ν-gap distance. The parameter Γ for [Buskes and Cantoni, 2008] was chosen as identity. □

Application. Controller Reduction
The controller reduction is a very complicated problem since a designer must keep in mind both robustness and performance criteria to obtain

Table 4.1 Approximation errors in $\delta_\nu \cdot 10^{-2}$ of various methods in Example 4.2

Reduction Orders	1	2	3	4
Lower bound	36.0	6.0	5.8	0.98
HMR	67.2	11.4	11.5	1.88
HMR of NCFs	60.7	11.3	11.5	1.88
[Buskes and Cantoni, 2008]	39.7	11.8	6.2	2.01
METHOD 1	37.9	11.2	6.2	1.87

a reasonable controller. Good surveys of methods using coprime factorization and frequency weighted approaches, which tackle both criteria are given in [Zhou *et al.*, 1996] and [Obinata and Anderson, 2001]. Even though the performance is a robust property in the ν-gap, the metric does not account explicitly for the performance of the closed loop system. Therefore, it may be desirable to include the constraints on the performance level into the optimization problem. Using the semidefinite program as a tool makes it possible. As a basis the same idea as for the ν-gap metric optimization is employed. As an example, consider a closed loop transfer function $H(G, K)$ where G is a plant and K is a controller and H is a so called gang-of-four:

$$H(G, K) = \begin{pmatrix} \dfrac{G}{1 + GK} & \dfrac{-KG}{1 + GK} \\ \dfrac{1}{1 + GK} & \dfrac{-K}{1 + GK} \end{pmatrix}$$

Let b/a be an NCF of K and c/d be an NCF of G.

$$\min_{\gamma_r > 0, p_i, q_i} \gamma_r \quad \text{subject to for all } \omega \text{ in } [0, \pi]:$$

$$\overline{\sigma}\left(\left(H(qd + pc) - \begin{pmatrix} qc & -pc \\ qd & -pd \end{pmatrix}\right)\psi^\sim\right) < \gamma_p \mathrm{Re}\left((qd + pc)\psi^\sim\right)$$

$$\left\| \begin{pmatrix} p \\ q \end{pmatrix} \phi^\sim \right\|_2 < \gamma_r \mathrm{Re}\left((qa^\sim + pb^\sim)\phi^\sim\right)$$

where γ_p is a pre-determined performance degradation level and ϕ is an initial guess on $qa^\sim + pb^\sim$, and ψ is an initial guess on $qd + pc$. Alternatively, one can include a constraint on the ν-gap metric while minimizing the performance level in the same manner.

EXAMPLE 4.3—APPROXIMATING A YOULA CONTROLLER.
Consider gang-of-four $H(G, K)$, where the plant G is controlled in a robust manner by a controller K. The 152-nd order controller K was obtained in [Garpinger, 2009] using Youla parameterization. The controller itself is stable and so is the third order plant G. For every order, the level γ_p will be fixed to 75% of the performance obtained by METHOD 1. The results are presented in Table 4.3. The parameter Γ for [Buskes and Cantoni, 2008] was chosen as identity. The method from [Buskes and Cantoni, 2008] performs slightly better than METHOD 1 for some orders and worse for others. However, a further extension of the approach is not easily obtainable. But using the presented approach it is simply a matter of adding a corresponding constraint. □

Table 4.2 Approximation errors of various methods in the v-gap metric and effect on the closed loop performance in Example 4.3

Distance between K and $\widehat{K}$ in the v-gap metric ($\cdot 10^{-2}$)

Reduction Orders	2	3	4	5	6
HMR	72.9	83.14	24.58	18.88	12.67
HMR of NCF	67.01	70.48	13.47	4.96	6.29
Lower bound	33.80	6.82	2.99	2.95	1.15
[Buskes and Cantoni, 2008]	36.68	11.54	4.69	3.25	1.70
METHOD 1	37.93	7.49	3.35	3.23	3.20
METHOD 2	44.56	7.84	4.63	4.05	4.27

Distance between $H(G, K)$ and $H(G, \widehat{K})$ in $\mathbb{H}_\infty$

Reduction Orders	2	3	4	5	6
HMR	3.64	26.02	1.09	0.84	0.35
HMR of NCF	2.37	2.34	0.92	0.34	0.38
[Buskes and Cantoni, 2008]	6.94	1.31	0.48	0.26	0.08
METHOD 1	1.92	0.58	0.27	0.26	0.26
METHOD 2	1.53	0.43	0.21	0.19	0.19

4.4 Conclusion and Discussion

A linear system approximation method in the v-gap metric was presented in this chapter. Such a method may be very useful for modeling of structured or multi-agent systems. Approximation is obtained using semidefinite programming and a normalized coprime factorization of the original model. This method can be applied to controller reduction by taking into account the performance of a closed loop.

The current algorithm has some advantages in comparison to [Buskes and Cantoni, 2008]. Incorporating extra constraints, adding frequency-depended weights and/or restricting the objective to a specific frequency interval is straightforward using the frequency domain representation. However, using [Buskes and Cantoni, 2008] such extensions are not easy to obtain.

5

Structured Model Order Reduction

One of the first steps to incorporate a structure into the model reduction problem was made by [Enns, 1984]. The method extended the established balanced truncation algorithm ([Moore, 1981]) to a frequency-weighted problem. Essentially, a cascade interconnection of three systems was approximated. A number of approaches to address this problem have been proposed (e.g., [Zhou, 1995], [Sandberg and Murray, 2007]). A method reported by [Sandberg and Murray, 2008] can be also seen as a generalized version of frequency-weighted model reduction problems. An important subclass of systems with a structure is a controller-plant interconnection. A number of methods address the controller reduction problem including those described in [Zhou *et al.*, 1996; Li and Paganini, 2005; Yang *et al.*, 2007; Burke *et al.*, 2006] and recently proposed by [Sou and Rantzer, 2010b; Sou and Rantzer, 2010a].

In this chapter, the focus is on an interconnection of *"subsystems"* in a typical block diagram. In order to avoid any confusion the interconnected system will be named a *"supersystem"*, as opposed to subsystems. Interactions of subsystems define a certain topology of interconnections or a *"structure"* in a supersystem. Clearly, a modeling procedure should account for such a structure. Therefore, the goal of model order reduction in this setting is to reduce certain subsystems, while preserving the overall structure of the supersystem. This concept can be formalized in a concrete example. Consider two subsystems N and G interacting in a linear fractional transformation (or LFT) loop (see, [Zhou *et al.*, 1996]). The supersystem $\mathcal{F}_l(N, G)$ is depicted in Figure 5.1. Assume N should be preserved in the reduction procedure and G should be approximated by some $\widehat{G}$. It is also required that the resulting LFT is similar to the original one, which entails that the error $\|\mathcal{F}_l(N, G) - \mathcal{F}_l(N, \widehat{G})\|_{\mathbb{H}_\infty}$ has to be minimized. However, to author's best knowledge, there is no such
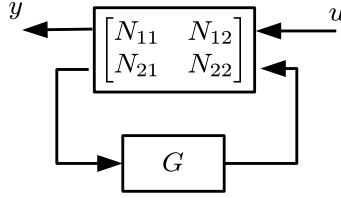
Figure 5.1   An LFT loop with two subsystems G and N.

method, that can guarantee finding a solution for an arbitrary order of $\widehat{G}$. Therefore, in order to simplify this problem, auxiliary input and output signals are introduced. By adding these signals, the transfer function $\mathcal{F}_l\,(N,G)$ will be replaced by an extended one $\mathcal{S}_e(N,G)$, which will be, in fact, reduced in the optimization problem. The extended supersystem $\mathcal{S}_e$ depicts the input-output mapping in Figure 5.2. Finally, the structured model reduction problem is cast as a minimization one.

$$\min_{\text{low-order } \widehat{G}} \|\mathcal{S}_e(N,G) - \mathcal{S}_e(N,\widehat{G})\|_{\mathbb{H}_\infty} \tag{5.1}$$

This minimization problem is approached by rewriting the system $\mathcal{S}_e(N,\widehat{G})$ in a coprime factor form. These coprime factors will also represent the structure of interconnections. The optimization is addressed using a two-step procedure employed in different fixed order controller design methods (see, [Apkarian and Noll, 2006; Arzelier *et al.*, 2010] and the references therein). In such a procedure, first a stabilizing plant is obtained and then the performance is minimized. The LFT case is discussed in detail in Section 5.1.

In Section 5.2, the reduction of multiple subsystems is discussed. An LFT loop is replaced by an arbitrary block diagram, with multiple subsystems interacting with each other. Finally, numerical examples are presented in Section 5.3.
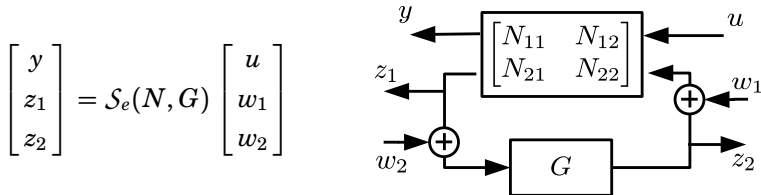
$$\begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} = \mathcal{S}_e(N,G) \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix}$$

Figure 5.2   An extended supersystem $\mathcal{S}_e$ with two subsystems G and N.

5.1 Model Reduction in an LFT loop

Consider an LFT loop in Figure 5.2 and the minimization criterion (5.1). The biggest advantage of introducing the auxiliary signals w_i and z_i is the ability to create a convenient coprime factor representation. This representation is computed by a direct calculation using the coprime factorization of N and G. Indeed, the extended system $\mathcal{S}_e$ can be described by the following set of equations:

$$\begin{bmatrix} y \\ z_1 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} u \\ z_2 + w_1 \end{bmatrix}$$

$$z_2 = G(w_2 + z_1)$$

Factorize N and G using a left coprime factorization, i.e.:

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} Q_{N11} & Q_{N12} \\ Q_{N21} & Q_{N22} \end{bmatrix}^{-1} \begin{bmatrix} P_{N11} & P_{N12} \\ P_{N21} & P_{N22} \end{bmatrix}$$

$$G = Q_G^{-1} P_G$$

and substitute them into the equations above:

$$\begin{bmatrix} Q_{N11} & Q_{N12} \\ Q_{N21} & Q_{N22} \end{bmatrix} \begin{bmatrix} y \\ z_1 \end{bmatrix} = \begin{bmatrix} P_{N11} & P_{N12} \\ P_{N21} & P_{N22} \end{bmatrix} \begin{bmatrix} u \\ z_2 + w_1 \end{bmatrix}$$

$$Q_G z_2 = P_G (w_2 + z_1)$$

Since z_1 and z_2 are treated as outputs, they are moved to the left hand side of the equations

$$\begin{bmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \end{bmatrix} \begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P_{N11} & P_{N12} \\ P_{N21} & P_{N22} \end{bmatrix} \begin{bmatrix} u \\ w_1 \end{bmatrix}$$

$$-P_G z_1 + Q_G z_2 = P_G w_2$$

Both equations can be united into the following matrix equation

$$\begin{pmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \\ 0 & -P_G & Q_G \end{pmatrix} \begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} = \begin{pmatrix} P_{N11} & P_{N12} & 0 \\ P_{N21} & P_{N22} & 0 \\ 0 & 0 & P_G \end{pmatrix} \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix}$$

Finally, define the transfer matrices $\mathcal{P}$ and $\mathcal{Q}$ as follows.

$$\mathcal{P} = \begin{pmatrix} P_{N11} & P_{N12} & 0 \\ P_{N21} & P_{N22} & 0 \\ 0 & 0 & P_G \end{pmatrix} \qquad \mathcal{Q} = \begin{pmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \\ 0 & -P_G & Q_G \end{pmatrix}$$

LEMMA 5.1
The transfer matrices $\mathcal{P}$ and Q are left coprime over $\mathbb{H}_\infty$. ☐

Proof. To show coprimeness, the definition from [Zhou *et al.*, 1996] is used: transfer matrices $\mathcal{P}$ and Q are left coprime in $\mathbb{H}_\infty$ if there exist rational transfer matrices X and Y in $\mathbb{H}_\infty$ such that

$$\mathcal{P} \cdot X + Q \cdot Y = I$$

Due to coprimeness of P_N and Q_N, there exist such X_{Nii} and Y_{Nii} that

$$\begin{pmatrix} P_{N11} & P_{N12} \\ P_{N21} & P_{N22} \end{pmatrix} \begin{pmatrix} X_{N11} & X_{N12} \\ X_{N21} & X_{N22} \end{pmatrix} + \begin{pmatrix} Q_{N11} & Q_{N12} \\ Q_{N21} & Q_{N22} \end{pmatrix} \begin{pmatrix} Y_{N11} & Y_{N12} \\ Y_{N21} & Y_{N22} \end{pmatrix} = I$$

similarly X_G and Y_G are defined trough P_G and Q_G

$$P_G X_G + Q_G Y_G = I$$

To prove coprimeness of $\mathcal{P}$ and Q, the transfer matrices X and Y can be chosen as

$$X = \begin{pmatrix} X_{N11} & X_{N12} & 0 \\ X_{N21} & X_{N22} & Y_G \\ Y_{N21} & Y_{N22} & X_G \end{pmatrix} \quad Y = \begin{pmatrix} Y_{N11} & Y_{N12} & 0 \\ Y_{N21} & Y_{N22} & 0 \\ 0 & 0 & Y_G \end{pmatrix}$$

Finally, the relation $\mathcal{P}X + QY = I$ is verified by direct computation. ☐

Now examine the transfer matrices $\mathcal{P}$ and Q closely. Every block-row of each transfer matrix depends either on a coprime factor of N, either on a coprime factor of G. It can not be called a "sparsity structure", since some entries are repeated in $\mathcal{P}$ and Q, however, this kind of structure can be exploited by iterative Algorithm 6 on page 48. Note also that S_e is stable if and only if Q has a stable inverse.

REMARK 5.1
The state-space representation also manifests the structure in a convenient manner. Assume the space-space representations of N and G are given as follows.

$$N = \left[\begin{array}{c|cc} A_N & B_{N1} & B_{N2} \\ \hline C_{N1} & D_{N11} & D_{N21} \\ C_{N2} & D_{N21} & D_{N22} \end{array} \right] \quad G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array} \right] \tag{5.2}$$

where D_G is set to 0, which is a common assumption in control theory. To shorten the notation, additionally define

$$B_N = [\, B_{N1} \quad B_{N2} \,] \qquad D_N = \begin{bmatrix} D_{N11} & D_{N12} \\ D_{N21} & D_{N22} \end{bmatrix} \qquad C_N = \begin{bmatrix} C_{N1} \\ C_{N2} \end{bmatrix}$$

A state space representation can be derived for $\mathcal{P}$ and $\mathcal{Q}$

$$[\mathcal{P} \quad \mathcal{Q}] =$$

$$\left[\begin{array}{cc|ccccc} A_N + L_N C_N & 0 & B_N + L_N D_N & 0 & L_N & -B_{N2} \\ 0 & A_G + L_G C_G & 0 & B_G & [\,0 \;\; -B_G\,] & L_G \\ \hline C_N & 0 & D_N & 0 & I & 0 \\ 0 & C_G & 0 & 0 & 0 & I \end{array} \right]$$

where L_N and L_G are free parameters, which are used to stabilize the coprime factors. The dynamics of the subsystems N and G are completely decoupled. The structure manifests itself only in the input and output matrices. If model reduction in the time domain is considered, the transfer function $[\mathcal{P} \quad \mathcal{Q}]$ can be approximated using a structured Gramian framework from [Sandberg and Murray, 2008]. However, stability of $\mathcal{S}_e(N, \widehat{G})$ is achieved if the reduced order $\mathcal{Q}$ has a stable inverse. This property is generally hard to accommodate. Also in the presented examples, this approach reduces to known coprime factor reduction techniques. Due to these reasons, this approach is not currently investigated. □

Analysis of the Problem Formulation

Above, instead of solving

$$\underset{\text{low-order } \widehat{G}}{\text{argmin}} \; \|\mathcal{F}_l\,(N, G) - \mathcal{F}_l\,(N, \widehat{G})\|_{\mathbb{H}_\infty} \tag{5.3}$$

it is proposed to address the following optimization problem

$$\underset{\text{low-order } \widehat{G}}{\text{argmin}} \; \|\mathcal{S}_e(N, G) - \mathcal{S}_e(N, \widehat{G})\|_{\mathbb{H}_\infty} \tag{5.4}$$

The major feature in (5.4) is the ability of tracking signals w_i and z_i. It means that not only the behaviour of $\mathcal{F}_l$ is approximated as in (5.3), but also the interaction between N and G.

Now let us try to understand what kind of problem is being addressed. The transfer function $\mathcal{S}_e$ reads as

$$
\mathcal{S}_e = \begin{pmatrix}
\mathcal{F}_l\,(N,G) & N_{12}\Xi & N_{12}\Xi G \\
\Theta N_{21} & N_{22}\Xi & N_{22}\Xi G \\
G\Theta N_{21} & G\Theta N_{22} & G\Theta
\end{pmatrix}
$$

where $\Theta = (I - N_{22}G)^{-1}$ and $\Xi = (I - GN_{22})^{-1}$. Due to the structure of $\mathcal{S}_e$, it can be shown that the program (5.4) is equivalent to:

$$
\min_{\text{low-order } \widehat{G}} \left\| \begin{bmatrix} N_{12} \\ N_{22} \\ I \end{bmatrix} ((I - GN_{22})^{-1}G - (I - \widehat{G}N_{22})^{-1}\widehat{G})\begin{bmatrix} N_{21} & N_{22} & I \end{bmatrix} \right\|_{\mathbb{H}_\infty}
$$

where the LFT loop $\mathcal{F}_l\,(N,G)$ is approximated at the block entry $\{1,1\}$. Therefore (5.4) provides an estimate on (5.3).

If only the transfer matrix generated by signals w_1, w_2, z_1 and z_2 is considered (the lower two by two block of $\mathcal{S}_e$), it can be shown, that the loop depicted in Figure 5.3 is approximated. It is a so called "gang-of-four" applied to G and N_{22}. It is also known that this loop admits a very convenient coprime factorization parametrization, which is exploited in $\mathbb{H}_\infty$ loop shaping (see, [Glover and McFarlane, 1989]). Adding these transfer functions into the objective takes also care of robust stability of the LFT loop.

Given these insights, the problem (5.4) can be modified as follows:

$$
\operatorname*{argmin}_{\text{low-order } \widehat{G}} \|W_o(\mathcal{S}_e(N,G) - \mathcal{S}_e(N,\widehat{G}))W_i\|_{\mathbb{H}_\infty}
$$

where the weights W_o and W_i regulate the trade-off between performance (block-entry $\{1,1\}$) and robustness of the loop (block-entries $\{2,2\}$–$\{3,3\}$).

Structured Model Reduction
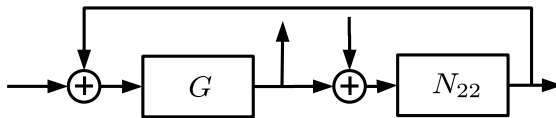Assume $\mathcal{S}_e(N,G)$ is an asymptotically stable discrete-time transfer func-

Figure 5.3   Gang of Four Loop

tion. The $\mathbb{H}_\infty$ optimization problem being addressed reads as

$$\min_{\mathcal{P},\mathcal{Q}} \|\mathcal{S}_e(N,G) - \mathcal{Q}^{-1}\mathcal{P}\|_{\mathbb{H}_\infty}$$

subject to: $\mathcal{Q}$ has a stable inverse

(5.5)

where $\mathcal{Q}$ and $\mathcal{P}$ correspond to the left coprime factorization of $\mathcal{S}_e(N,\widehat{G}) = \mathcal{Q}^{-1}\mathcal{P}$ and $\widehat{G}$ is a low order approximation of G. The reduction procedure will be decoupled into two problems: finding a stability preserving low-order approximation of G by some $\widehat{G}_0$ without considering the quality of the loop $\mathcal{S}_e(N,\widehat{G}_0)$. There are quite a few reduction techniques of structured models which guarantee stability under certain conditions. However, none of those can guarantee finding a solution, if one exists. Given this initial point $\widehat{G}_0$, the approximation quality of (5.5) is minimized.

Let us return to the problem at hand: (5.5). The transfer matrices $\mathcal{Q}$ and $\mathcal{P}$ admit the following structure:

$$\mathcal{P} = \begin{pmatrix} P_{N11} & P_{N12} & 0 \\ P_{N21} & P_{N22} & 0 \\ 0 & 0 & Y \end{pmatrix} \quad \mathcal{Q} = \begin{pmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \\ 0 & -Y & X \end{pmatrix}$$

where

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} Q_{N11} & Q_{N12} \\ Q_{N21} & Q_{N22} \end{bmatrix}^{-1} \begin{bmatrix} P_{N11} & P_{N12} \\ P_{N21} & P_{N22} \end{bmatrix}$$

and unknown functions are parameterized as

$$X = \sum_{i=0}^{r} X_i z^{-i} \quad Y = \sum_{i=0}^{r} Y_i z^{-i}$$

with the real matrices X_i, Y_i being the decision variables. The transfer matrices X and Y constitute a left coprime factorization of $\widehat{G} = X^{-1}Y$.

To address this problem, rewrite it with an infinite number of constraints as in Chapter 2. For brevity, let $\mathcal{S}_e = \mathcal{S}_e(N,G)$, and assume that $\mathcal{Q}$ and $\mathcal{P}$ are our decision variables.

$$\min_{\mathcal{Q},\mathcal{P},\gamma} \gamma^2 \quad \text{subject to: } \mathcal{Q} \text{ has a stable inverse}$$

$$(\mathcal{S}_e - \mathcal{Q}^{-1}\mathcal{P})(\mathcal{S}_e - \mathcal{Q}^{-1}\mathcal{P})^\sim \le \gamma^2 I \quad \forall \omega \in [0,\pi]$$

where $\mathcal{Q}^\sim(z) = \mathcal{Q}^T(1/z)$. For the moment, consider only the norm constraint, which using algebraic manipulations yields:

$$(\mathcal{Q}\mathcal{S}_e - \mathcal{P})(\mathcal{Q}\mathcal{S}_e - \mathcal{P})^\sim \le \gamma^2 \mathcal{Q}\mathcal{Q}^\sim \quad \forall \omega \in [0,\pi]$$

This non-convex quadratic constraint is dealt with by replacing it with more restrictive inequalities

$$(QS_e - P)(QS_e - P)^\sim \leq \gamma^2 (QQ_0^\sim + Q_0 Q^\sim - Q_0 Q_0^\sim) \quad \forall \omega \in [0, \pi] \quad (5.6)$$

Note that $QQ_0^\sim + Q_0 Q^\sim - Q_0 Q_0^\sim \leq QQ^\sim$ for all Q_0 (see, Section 2.4). For every fixed Q_0 the constraint (5.6) is convex, which allows to solve the problem with respect to Q and P. The semidefinite constraint is obtained using the Schur complement.

The transfer matrix Q_0 is computed based on the initial point $\widehat{G}_0$. It is reasonable to assume that Q_0 should have the same structure as Q, therefore:

$$Q = \begin{pmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \\ 0 & -Y^0 & X^0 \end{pmatrix}$$

where X^0, Y^0 are FIR filters of order r, and $\widehat{G}_0 = (X^0)^{-1} Y^0$. Since, $S_e(N, \widehat{G}_0)$ is stable, Q_0 has a stable inverse, and so does Q (as shown in Section 2.4). Now, given an initial point Q_0, it is possible to find another

---

Algorithm 15 Structured Model Reduction

Compute P_N, Q_N as described above. Obtain Q_0, e.g., using a stability preserving heuristics from [Zhou *et al.*, 1996]. Set $Q_j = Q_0$ and $j = 1$

Introduce $X = \sum\limits_{i=0}^{r} X_i z^{-i}$, $Y = \sum\limits_{i=0}^{r} Y_i z^{-i}$ and

$$P = \begin{pmatrix} P_{N11} & P_{N12} & 0 \\ P_{N21} & P_{N22} & 0 \\ 0 & 0 & Y \end{pmatrix} \quad Q = \begin{pmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \\ 0 & -Y & X \end{pmatrix}$$

repeat

Solve a semidefinite problem

$$\min_{X_i, Y_i, \gamma} \gamma^2 \quad \text{subject to}$$

$$\begin{bmatrix} Q_j Q^\sim + QQ_j^\sim - Q_j Q_j^\sim & QG - P \\ (QG - P)^\sim & \gamma^2 I \end{bmatrix} \geq 0 \quad \forall \omega \in [0, \pi]$$

Set $Q_{j+1} = Q$ and $j = j + 1$
until $\|Q_{j+1} - Q_j\|_\infty \leq \varepsilon$
Compute the reduced model as $\widehat{G} = X^{-1} Y$

---

feasible point Q with an improved approximation quality. The procedure is concluded in Algorithm 15. Convergence of the algorithm can be treated as in Chapter 2.

5.2 Generalization to Multiple Subsystems

Let us start with an example, to see what kind of problems can occur, if a generalization is not performed carefully.

Consider a block diagram with three subsystems G_1, G_2 and G_3 in Figure 5.4. In order to obtain a coprime factorization as before, excite every subsystem G_i by an additional signal w_i and measure its output by an additional signal z_i. Note that three signals u, w_1 and w_3 are exciting only two subsystems G_1 and G_3. Additionally, the output of the supersystem y is a sum of z_1 and z_3. This way some degree of redundancy appears in the extended supersystem. These simple observations raise a question: how the is model reduction problem affected? Assume the extended transfer function is introduced as follows:

$$
\begin{bmatrix} y \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \mathcal{S}_e \begin{bmatrix} u \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}
$$

Note that the sum of last two rows of $\mathcal{S}_e$ will be equal to the first one, since $y = z_1 + z_3$. The sum of the second and fourth columns of $\mathcal{S}_e$ will be equal to the first one, since u excites the supersystem in the same manner
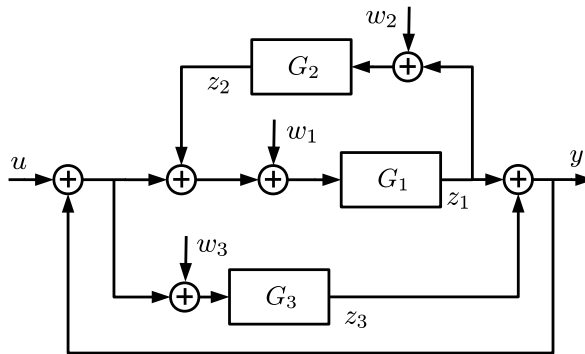

Figure 5.4 A block diagram with three subsystems

as the sum of w_1 and w_3. Thus, the transfer matrix S_e will have at least one zero singular value for all the frequencies ω in $[0,\pi]$. Solving a model reduction problem in this setting is problematic, since a rank-deficient matrix is approximated. On the other hand, all zero singular values can be eliminated while designing the extended supersystem. Therefore, some signals should be eliminated to provide a full rank S_e for all the frequencies. Here, signals z_1 and w_3 were chosen to be eliminated, providing the equations:

$$y = z_3 + G_1(u + y + w_1 + z_2)$$
$$z_2 = G_2(w_2 + y - z_3)$$
$$z_3 = G_3(u + y)$$

Factorize the subsystems G_i using left coprime factorizations $Q_i^{-1} P_i$. Substitute them into the equations above, while multiplying both sides with Q_i. Also, separate the outputs and inputs on the different sides of the equations:

$$(Q_1 - P_1)y + P_1 z_2 - Q_1 z_3 = P_1(u + w_1)$$
$$-P_2 y + Q_2 z_2 + P_2 z_3 = P_2 w_2$$
$$-P_3 y + Q_3 z_3 = P_3 u$$

The relationship between the signals can be computed as:

$$
\begin{bmatrix} Q_1 - P_1 & P_1 & -Q_1 \\ -P_2 & Q_2 & P_2 \\ -P_3 & 0 & Q_3 \end{bmatrix}
\begin{bmatrix} y \\ z_2 \\ z_3 \end{bmatrix}
=
\begin{bmatrix} P_1 & 0 & P_1 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}
\begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}
$$

This representation is left coprime, to show this let:

$$
\mathcal{P} = \begin{bmatrix} P_1 & 0 & P_1 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}
\qquad
\mathcal{Q} = \begin{bmatrix} Q_1 - P_1 & -P_2 & -Q_1 \\ -P_2 & Q_2 & P_2 \\ -P_3 & 0 & Q_3 \end{bmatrix}
$$

$$
\mathcal{P}_G = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}
\qquad
\mathcal{Q}_G = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix}
$$

Given the notations, it can be computed that

$$[\mathcal{P} \quad \mathcal{Q}] = [\mathcal{P}_G \quad \mathcal{Q}_G] T$$

where the transformation T is invertible. Note that $\mathcal{P}_G$ and $\mathcal{Q}_G$ are left coprime, since they are block diagonal with left coprime factors of the

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Algorithm 16 Coprime factorization of an arbitrary block diagram

---

- Given a block diagram with interconnected subsystems G_1, G_2, ..., G_n, introduce signals w_i, z_i. Every w_i excites the subsystem G_i, and every z_i measures its output

- If u excites the system S_e the same way as a linear combination of w_j, then one of the signals w_j is set to zero

- If a signal z_j is a linear combination of other output signals it is eliminated by $z_j = y - \sum_{j \neq i}^{n} \gamma_j z_j$, where γ_j are zeros or ones

- Write down the equations describing dependence of z_i on the signals w_j, u, y and z_j, while replacing G_i by its left coprime factorization $Q_i^{-1} P_i$

- Compute the left coprime factors $\mathcal{P}$ and Q

---

block diagonal and there exist X_G and Y_G such that:

$$[\mathcal{P}_G \quad Q_G] \begin{bmatrix} X_G \\ Y_G \end{bmatrix} = I$$

Thus, there exist stable X and Y such that:

$$[\mathcal{P} \quad Q] \begin{bmatrix} X \\ Y \end{bmatrix} = I \quad \text{where} \quad \begin{bmatrix} X \\ Y \end{bmatrix} = T^{-1} \begin{bmatrix} X_G \\ Y_G \end{bmatrix}$$

This proves that $\mathcal{P}$ and Q are left coprime if and only if $\mathcal{P}_G$ and Q_G are left coprime.

Inspired by this simple example, Algorithm 16 is formulated. The algorithm is constructive, however, it has to be shown that it always produces a left coprime factorization.

LEMMA 5.2
The transfer matrices $\mathcal{P}$ and Q obtained in Algorithm 16 are left coprime. $\square$

Proof. The proof is a generalization of the technique described for the block diagram investigated above. Introduce $\mathcal{P}_G$ and Q_G:

$$\mathcal{P}_G = \begin{bmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_n \end{bmatrix} \qquad Q_G = \begin{bmatrix} Q_1 & & 0 \\ & \ddots & \\ 0 & & Q_n \end{bmatrix}$$

where Q_i and P_i are left coprime and $G_i = Q_i^{-1} P_i$. The equations describing the relationships between the signals read as:

$$Q_i z_i = P_i(\alpha_0^i u + \alpha_i w_i + \beta_0^i y + \sum_{j=1}^{n} \beta_j^i z_j) \qquad \text{or}$$

$$Q_i(y - \sum_{j \neq i}^{n} \gamma_j z_j) = P_i(\alpha_0^i u + \alpha_i w_i + \beta_0^i y + \sum_{j=1}^{n} \beta_j^i z_j)$$

where $\alpha_0^i, \alpha_i, \beta_j^i$ and γ_j are equal to zero or one, depending on a particular block diagram. The lower case appears when a signal z_i is a linear combination of other outputs, i.e., $z_i = y - \sum_{j \neq i}^{n} \gamma_j z_j$, and it is eliminated. Note that $\mathcal{P}$ will not depend on Q_G since none of the signals u, w_i are multiplied with Q_i. The static transformation between the transfer matrices $\mathcal{P}_G$, Q_G and $\mathcal{P}$, Q are given as follows.

$$[\mathcal{P} \quad Q] = [\mathcal{P}_G \quad Q_G] \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

Since the transformation is block triangular, it remains to show that T_{11} and T_{22} are invertible. For most of the signals, we have

$$Q_i z_i = P_i(\alpha_0^i u + \alpha_i w_i + \beta_0^i y + \sum_{j=1}^{n} \beta_j^i z_j)$$

The transfer matrix $Q_G T_{22}$ will be affected only by the summands containing multiplication of Q_i and the outputs y or z_i. Therefore, most of the block rows of $Q_G T_{22}$ will have only one non-zero entry Q_i and it will appear on the block diagonal. For a single equation we have:

$$Q_i(y - \sum_{j \neq i}^{n} \gamma_j z_j) = P_i(\alpha_0^i u + \alpha_i w_i + \beta_0^i y + \sum_{j=2i}^{n} \beta_j^i z_j)$$

and therefore one block row of $Q_G T_{22}$ consists of multiple Q_i with different signs. Without loss of generality, we can assume that z_i with $i = 1$ is eliminated, therefore T_{22} is a block-triangular matrix with the identity matrices on the block diagonal. Only the first block row has non-zero off diagonal entries. Given these facts T_{22} is invertible.

It can be similarly shown that T_{11} is an invertible block-triangular matrix, where only one block-column has non-zero entries except for the block-diagonal elements. Since T_{11} and T_{22} are invertible so is the whole

transformation. Due to coprimeness of $\mathcal{P}_G$ and Q_G, there exist X_G and Y_G such that:

$$[\mathcal{P}_G \quad Q_G] \begin{bmatrix} X_G \\ Y_G \end{bmatrix} = I$$

Finally, there exist stable X and Y such that

$$[\mathcal{P} \quad Q] \begin{bmatrix} X \\ Y \end{bmatrix} = I \quad \text{where} \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}^{-1} \begin{bmatrix} X_G \\ Y_G \end{bmatrix}$$

Therefore $\mathcal{P}$ and Q are left coprime if and only if $\mathcal{P}_G$ and Q_G are left coprime, which is satisfied by construction. $\qquad\square$

The statement of this lemma is not unexpected. If there is just one subsystem with one input and one output, then the coprime factors $\mathcal{P}$, Q should have one input and one output. It stands to reason that the extended supersystem $\mathcal{S}_e$ should have n inputs and n outputs, if there is n subsystems in $\mathcal{S}_e$ with one input and one output each. Stability of the supersystem is equivalent to Q and Q^{-1} being stable, since the transfer matrices $\mathcal{P}$ and Q constitute a coprime factorization of $\mathcal{S}_e$.

Addressing the model reduction problem can be done similarly as was done before. Assume without loss of generality, that subsystems $G_{k+1}, \ldots, G_n$ have to be preserved during the approximation procedure and $G_1, \ldots, G_k$ are being reduced. All the subsystems $G_{k+1}, \ldots, G_n$ are treated as one subsystem N and the problem is cast a minimization one as follows.

$$\min_{\text{low order } \widehat{G}_1, \ldots, \widehat{G}_k} \| \mathcal{S}_e(N, G_1, \ldots, G_k) - \mathcal{S}_e(N, \widehat{G}_1, \ldots, \widehat{G}_k) \|_{\mathbb{H}_\infty}$$

After that, Algorithm 6 can be modified in a straightforward manner in order to address this problem.

5.3 Examples

All the algorithms are implemented using SeDuMi ([Sturm, 1999]) and YALMIP ([Löfberg, 2004]). The first example depicts a continuous time closed loop. Therefore, the model is discretized as in Chapter 2.

EXAMPLE 5.1—CONTROLLER REDUCTION IN THE LFT STRUCTURE
Consider an LFT structure $\mathcal{F}_l(G, K)$ as in Figure 5.5 with G being a plant and K a controller, which is desirable to reduce. The structured left coprime factorization is computed as follows.

$$\mathcal{P} = \begin{bmatrix} P_{G_{11}} & P_{G_{12}} & 0 \\ P_{G_{21}} & P_{G_{22}} & 0 \\ 0 & 0 & P_K \end{bmatrix} \quad Q = \begin{bmatrix} Q_{G_{11}} & Q_{G_{12}} & -P_{G_{12}} \\ Q_{G_{21}} & Q_{G_{22}} & -P_{G_{22}} \\ 0 & -P_K & Q_K \end{bmatrix}$$
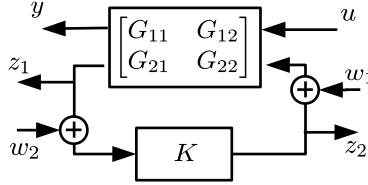
Figure 5.5   LFT structure

Table 5.1   Approximation errors $\|\mathcal{F}_l(G,K) - \mathcal{F}_l(G,\widehat{K})\|_{\mathbb{H}_\infty}$ in Example 5.1

Reduction Orders	2	3	4	5	6	7
SWA SHMR	2.75	0.69	0.42	0.34	0.19	0.22
ALG 15	0.98	0.68	0.34	0.31	0.17	0.14

The reduction techniques were tested on the famous example by Enns, which was a model reduction case study in [Zhou *et al.*, 1996]. The initial point computation is performed by using SWA method (see, [Zhou *et al.*, 1996]), which can be formulated as follows:

$$\min_{\widehat{K}} \left\| W_a(K - \widehat{K}) \right\|_{\mathbb{H}_\infty} \quad \text{where } W_a = (I - G_{22}K)^{-1}G_{22}$$

The problem was addressed by frequency-weighted semidefinite Hankel-type model reduction (see, Chapter 2). ALG 15 will denote Algorithm 15 with the result of SWA SHMR as a starting point.

Table 5.1 shows that the improvement in the approximation error between the original and the reduced models is noticeable almost for every order. Table 5.2 shows that the performance of ALG 15 is near the original one $\|\mathcal{F}_l(G,K)\|_{\mathbb{H}_\infty} = 1.2$ for orders $4-7$. For order 2 ALG 15 outperformed SWA considerably. In fact, even performance oriented methods PWRCF and PWA by [Zhou *et al.*, 1996] can not achieve such performance levels. □

EXAMPLE 5.2—MIXED SENSITIVITY CONTROL LOOP
Consider a structure as in Figure 5.6, where K is a controller and G is a plant. Notice that in this example there is no need to add extra inputs and outputs. The left coprime factorization of this supersystem is as follows:

$$\mathcal{P} = \begin{bmatrix} P_G & 0 \\ 0 & P_K \end{bmatrix} \quad Q = \begin{bmatrix} Q_G & -P_G \\ -P_K & Q_K \end{bmatrix}$$

Table 5.2 Performance of the reduced closed loop $\|\mathcal{F}_l(G,\widehat{K})\|_{\mathbb{H}_\infty}$ in Example 5.1

Reduction Orders	2	3	4	5	6	7
SWA SHMR	3.44	1.65	1.34	1.29	1.23	1.23
PWRCF H/O	1.98	2.03	1.19	1.19	1.19	1.19
PWA H/O	1.94	2.73	1.19	1.19	1.19	1.19
ALG 15	1.34	1.34	1.21	1.23	1.20	1.20

Table 5.3 Approximation errors $\|\mathcal{S}_e(G,K) - \mathcal{S}_e(G,\widehat{K})\|_{\mathbb{H}_\infty}$ in Example 5.2

Reduction Orders	2	3	4	5	6	7	8
SWA SHMR	0.91	0.39	0.13	0.14	0.04	0.04	0.04
ALG 15	0.63	0.07	0.06	0.04	0.01	0.01	0.01

where $K = Q_K^{-1}P_K$ and $G = Q_G^{-1}P_G$. In this subsection a similar notation and methods will be used. However, the weight in SWA is changed to:

$$W_a = \begin{bmatrix} G \\ I \end{bmatrix} (I + KG)^{-1}$$

ALG 15 will denote Algorithm 15 with a starting point from SWA SHMR.

The controller was designed by [Garpinger, 2009]. The plant is asymptotically stable with poles near the imaginary axis. This makes the stabilization almost redundant, requiring only the performance level to be optimized.

As seen from Table 5.3 the approximation errors are significantly improved by the proposed approach in comparison to SWA. However, it should be noted that, since the plant is stable, the problem for the fixed-order controller design is much easier than in Example 5.1, for instance. Table 5.4 shows, that the performance levels of ALG 15 is always within 10% of the nominal value $\|\mathcal{S}_e(G,K)\|_{\mathbb{H}_\infty} = 1$. The improvement in approximation error is quite remarkable when comparing ALG 15 to SWA. $\square$
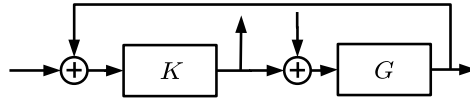

Figure 5.6 Mixed Sensitivity Control loop

Table 5.4 Performance of the closed loop $\|\mathcal{S}_e(G,\widehat{K})\|_{\mathbb{H}_\infty}$ in Example 5.2

Reduction Orders	2	3	4	5	6	7	8
SWA SHMR	1.38	1.15	0.87	0.87	0.99	0.98	0.98
ALG 15	1.11	0.93	0.94	0.95	1.00	1.01	1.01

5.4 Conclusion and Discussion

A few generalizations of known results are described in this chapter. For instance, a coprime factorization of structured systems is introduced, a known two-step procedure is generalized to an arbitrary supersystem and essentially any number of subsystems. There are a few issues which are still not addressed in this thesis and require further research.

- Is it possible to generalize the *normalized* coprime factorizations to the structured case? In the current setting it is not clear how this generalization may occur. Perhaps instead of a direct generalization to supersystems, one can normalize the factors of subsystems.

- The problem of computing an initial point is crucial for this approach. However, to the author's best knowledge, it is a common problem in all the methods unless a special type of system is being reduced, such as a positive, positive-real system. Nevertheless, a number of methods can obtain an initial point in a heuristic manner and using the presented framework it is possible to considerably improve the quality of approximation.

- There are similar two-step procedures, which can be used to address the model reduction problem. One such method addresses the non-smooth optimization problem directly ([Apkarian and Noll, 2006]). The comparison of the proposed approach to [Apkarian and Noll, 2006] is a topic of future research.

6

Conclusion and Discussion

6.1 Summary of Thesis

The main contribution of this thesis is the development of two model reduction methods. Both methods are based on matching of the frequency response samples of the full and reduced order models. This approach is computationally cheap for a certain class of models. Both methods also guarantee stability. If required, a number of extensions can be performed as well, such as frequency-weighting, enforcing passivity and so on.

The first method is based on a relaxation approach, which results in semidefinite Hankel-type model reduction (SHMR). Due to significant resemblance to optimal Hankel model reduction, the accuracy of SHMR is very similar to the Hankel model reduction one. It can be argued that SHMR is a numerically scalable approach to Hankel model reduction. The SHMR algorithm can be applied to $\mathbb{H}_\infty$ model reduction and parameterized model reduction problems, which are described in chapters 2 and 3. However, in the second case, explicit parameter-dependent models are not easily obtainable. In both $\mathbb{H}_\infty$ and parameterized model reduction problems the relaxation gap is estimated, which acts as lower and upper error bounds on the approximation error. Although the upper bounds are very conservative, estimation of a relaxation gap is required to motivate the relaxation.

The second method constitutes an iterative approach, where on every iteration a semidefinite program has to be solved. This method can be applied to all the problems described in the thesis: $\mathbb{H}_\infty$ model reduction, parameterized model reduction, ν-gap model reduction and, finally, structured model reduction. The major issue of the method is the computation of the initial point of the algorithm. In $\mathbb{H}_\infty$ model reduction computation of initial point is rather straightforward. One can also be computed based on

the approximation computed by another model reduction method. There-fore, the iterative approach can potentially improve the quality of any model reduction method. In parameterized and ν-gap model reduction the computation of an initial point is more involved, but always feasible. A few guidelines is provided for a reasonable computation of the initial point. However, in structured model reduction computation of the initial point is an open question. In this thesis, such computation was not the foremost goal and a few known heuristics were listed, which address this problem.

The iterative approach is, essentially, addressing a non-smooth, non-convex optimization problem. Therefore, it is hard to theoretically predict the properties of the resulting solution. Nonetheless, the numerical sim-ulations show promising results with respect to accuracy of the iterative approach.

6.2 Discussion on Future Work

There are a few open questions concerning the presented framework. Some of the questions are well known in the control literature. For example, computation of the initial point in structured model reduction is related to fixed order controller design and static output feedback problem. These problems are known to be hard and all the solutions are yet to be found. Other questions arose with the development of the methods in this thesis. These are discussed in detail.

Properties of the Iterative Approach

The iterative approach is a powerful tool to address the model reduc-tion problem. It was shown that the limit point always exists, however, the properties of this limit point seem hard to investigate. Due to non-convexity of the objective function, it is certainly possible to obtain a limit point, which is not equal to the global minimizer. The relaxation approach (the SHMR method) guarantees a certain sub-optimality and a solution that is relatively close to the global minimum in terms of the objective function. However, there is no guarantee that minimizer of the relaxed problem is also close to the global minimizer. The main obstacle in evalu-ation of the obtained minimizer is lack of properties of a globally optimal solution. To author's best knowledge, there is no known results concern-ing the optimal solutions in $\mathbb{H}_\infty$ model reduction, which is a non-smooth optimization problem.

A certain analogue to the iterative method can be obtained for the $\mathbb{H}_2$ model reduction problem for which conditions for local optimality are

known. Therefore, one of the future work directions will be evaluation of such a method. Such an evaluation can shed light on $\mathbb{H}_\infty$ problem as well.

Applications of Parameterized Model Order Reduction Framework

The presented parameterized model order reduction method performs extremely well when the parameters in the model are constant. When the LPV systems are considered, the picture is not as clear. One problem is stability, since only a necessary condition is guaranteed.

Even if stability can be addressed, a problem occurs in accuracy of approximation. Consider the variables ω and θ used in the presented algorithms. Here ω is a frequency variable, and θ is a parameter. Throughout the derivation of the algorithms, it is assumed that ω and θ are independent. However, in practice, the parameter θ does depend on time and states and, therefore, it depends on ω. It means that inactive modes of θ are kept, which creates conservatism. One possibility to reduce conservatism, is using frequency weights in order to get a better match along some frequencies. However, it is not clear at this point how determine these weights and if this approach is valid at all.

7

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