Optimizing Positively Dominated Systems

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Published in:
[Host publication title missing]

2012

51st IEEE Conference on Decision and Control
December 10-13, 2012. Maui, Hawaii, USA

Optimizing Positively Dominated Systems

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***Abstract*— It has recently been shown that several classical open problems in linear system theory, such as optimization of decentralized output feedback controllers, can be readily solved for positive systems using linear programming. In particular, optimal solutions can be verified for large-scale systems using computations that scale linearly with the number of interconnections. Hence two fundamental advantages are achieved compared to classical methods for multivariable control: Distributed implementations and scalable computations. This paper extends these ideas to the class of *positively dominated systems*. The results are illustrated by computation of optimal spring constants for a network of point-masses connected by springs.**

Classical methods for multi-variable control, such as LQG and H_∞-optimization, suffer from a lack of scalability that make them hard to use for large-scale systems. The difficulties are partly due to computational complexity, but also absence of distributed structure in the resulting controllers. Complexity growth can be traced back to the fact that stability verification of a linear system with n states generally requires a Lyapunov function involving n^2 quadratic terms, even if the system matrices are sparse. In this paper we will see that the situation improves drastically if we restrict attention to closed loop dynamics described by system matrices with nonnegative off-diagonal entries. Then stability and performance can be verified using a Lyapunov function with only n linear terms. Sparsity can be exploited in performance verification and even synthesis of distributed controllers can be done with a complexity that grows linearly with the number of nonzero entries in the system matrices. These observations have far-reaching implications for control engineering:

1) The conditions that enable scalable solutions hold naturally in many application areas, such as stochastic systems, economics, transportation networks, chemical reactions and power systems.
2) In this paper, the essential mathematical property is extended to frequency domain models that are "positively dominated".
3) A large-scale control system can often be structured into local control loops that give positive dominance, thus enabling scalable methods for optimization of the global performance.

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I. Background

The study of matrices with nonnegative coefficients has a long history, dating back to the Perron-Frobenius Theorem in 1912. A classic book on the topic is [2]. The theory is used in Leontief economics [11], where the states denote nonnegative quantities of commodities. Systems defined by nonnegative matrices (so called positive systems) appear in the study of Markov chains [17], where the states denote nonnegative probabilities and in compartment models [7], where the states could denote populations of species. A nice introduction to the subject is given in [12].

A fundamental property of linear maps described by a positive matrix is that they are contractive in Hilbert's projective metric [3], [9]. This metric is closely related to the Lyapunov function $\max\{x_1, \dots, x_n\} - \min\{x_1, \dots, x_n\}$, used in analysis of consensus algorithms [17], [20]. See also [13], [18].

A nonlinear counterpart to positive systems is monotone systems, characterized by the property that a partial ordering of initial states is preserved by the dynamics. Such systems were studied by Hirsch [5], [6], showing that monotonicity generally implies convergence almost everywhere. Positive systems have also gained increasing attention in the control literature during the last decade. See for example [21], [4], [8]. Basic control theory for nonlinear monotone systems was developed in [1]. Feedback stabilization of positive linear systems was studied in [10], [15]. Stabilizing static output feedback controllers were parameterized in [14] using linear programming. A recent result by Tanaka and Langbort [19] also shows that decentralized controllers can be optimized for positive systems using semi-definite programming.

II. Notation

The inequality $X > 0$ ($X \geq 0$) means that all elements of the matrix (or vector) X are positive (nonnegative). For a symmetric matrix X, the inequality $X \succ 0$ means that the matrix is positive definite. A square matrix is said to be *Hurwitz* if all eigenvalues have positive real part. It is *Schur* if all eigenvalues are strictly inside the unit circle. Finally, the matrix is said to be *Metzler* if all off-diagonal elements are nonnegative. The notation $\mathbb{CH}_\infty^{n\times m}$ represents the set of $n \times m$ matrices whose entries are analytic in the right half plane and continuous on the imaginary axis (including infinity) .

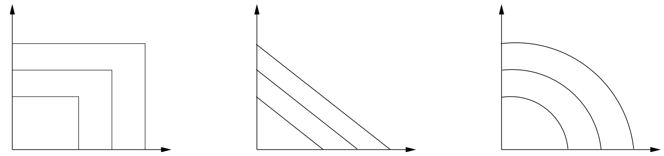

Fig. 1. Level curves of Lyapunov functions corresponding to the conditions (1.2), (1.3) and (1.4) in Proposition 1: If $A\xi < 0$, then $V(x) = \max_i(x_i/\xi_i)$ is a Lyapunov function with rectangular level curves. If $z^T A < 0$, then $V(x) = z^T x$ is a linear Lyapunov function. Finally if $A^T P + PA \prec 0$ and $P \succ 0$, then $V(x) = x^T P x$ is a quadratic Lyapunov function for the system $\dot{x} = Ax$.

III. Distributed Stability Verification

We start be reviewing some properties of positive systems. For further details see [16].

Proposition 1: Let $A \in \mathbb{R}^{n\times n}$ be Metzler. Then the following are equivalent:

(1.1) The matrix A is Hurwitz.
(1.2) There exists $\xi \in \mathbb{R}^n$ with $\xi > 0$ and $A\xi < 0$.
(1.3) There exists $z \in \mathbb{R}^n$ with $z > 0$ and $z^T A < 0$.
(1.4) There exists a *diagonal* matrix $P \succ 0$ such that $A^T P + PA \prec 0$.
(1.5) The matrix $-A^{-1}$ exists and $-A^{-1} \geq 0$.

Moreover, if $\xi = (\xi_1, \dots, \xi_n)$ and $z = (z_1, \dots, z_n)$ satisfy the conditions of (1.2) and (1.3) respectively, then the matrix $P = \operatorname{diag}(z_1/\xi_1, \dots, z_n/\xi_n)$ satisfies (1.4).

Remark 1. Each of the conditions (1.2), (1.3) and (1.4) corresponds to a Lyapunov function of a specific form. See Figure 1.

A discrete time counterpart to Proposition 1 can be stated as follows:

Proposition 2: Let $B \in \mathbb{R}_+^{n\times n}$. Then the following statements are equivalent:

(2.1) The matrix B is Schur stable.
(2.2) There exists $\xi \in \mathbb{R}^n$ with $\xi > 0$ and $B\xi < \xi$.
(2.3) There exists $z \in \mathbb{R}^n$ with $z > 0$ and $B^T z < z$.
(2.4) There exists a *diagonal* matrix $P \succ 0$ such that $B^T P B \prec P$.
(2.5) The matrix $(I-B)^{-1}$ exists and $(I-B)^{-1} \geq 0$.

Moreover, if $\xi = (\xi_1, \dots, \xi_n)$ and $z = (z_1, \dots, z_n)$ satisfy the conditions of (2.2) and (2.3) respectively, then the matrix $P = \operatorname{diag}(z_1/\xi_1, \dots, z_n/\xi_n)$ satisfies (2.4).

One of the main observations of [16] was that verification and synthesis of positive control systems can be done with methods that scale linearly with the number of interconnections. For stability, this claim follows directly from Proposition 1: Given ξ, verification of the inequality $A\xi < 0$ requires a number of scalar additions and multiplications that is directly proportional to the number of nonzero elements in the matrix A. In fact, the search for a feasible ξ also scales linearly, since integration of the differential equation $\dot{\xi} = A\xi$ with $\xi(0) = \xi_0$ for an arbitrary $\xi_0 > 0$ generates a feasible $\xi(t)$ in finite time provided that A is Metzler and Hurwitz. Two examples are illustrative:

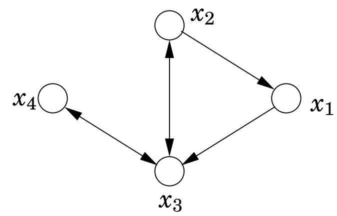

Fig. 2. A graph of interconnected systems. In Example 1 the interpretation is a transportation network and each arrow indicates a transportation link. In Example 2 the interpretation is instead a vehicle formation and each arrow indicates the use of a distance measurement.

Example 1. Linear transportation network. Consider a dynamical system interconnected according to the graph illustrated in Figure 2:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0\\ 0 & 2-\ell_{12}-\ell_{32} & \ell_{23} & 0\\ \ell_{31} & \ell_{32} & 3-\ell_{23}-\ell_{43} & \ell_{34}\\ 0 & 0 & \ell_{43} & -4-\ell_{34}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \tag{1}$$

The model could for example be used to describe an transportation network connecting four buffers. The states x_1, x_2, x_3, x_4 represent the contents of the buffers and the parameter ℓ_{ij} determines the rate of transfer from buffer j to buffer i. Without such transfer the content of the second and third buffer would grow exponentially due to the diagonal elements 2 and 3, corresponding to unstable internal dynamics of those buffers.

Notice that the dynamics can be written as $\dot{x} = Ax$ where A is a Metzler matrix provided that every ℓ_{ij} is nonnegative. Hence, by Proposition 1, stability is equivalent to existence of numbers $\xi_1, \dots, \xi_4 > 0$ such that

$$\begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0\\ 0 & 2-\ell_{12}-\ell_{32} & \ell_{23} & 0\\ \ell_{31} & \ell_{32} & 3-\ell_{23}-\ell_{43} & \ell_{34}\\ 0 & 0 & \ell_{43} & -4-\ell_{34}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\end{bmatrix} < \begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

Given these numbers, stability can be verified by a distributed test where the first buffer verifies the first inequality, the second buffer verifies the second and so on. In particular, the relevant test for each buffer only involves parameter values at the local node and the neighboring nodes, so a global model is not needed anywhere. □

Example 2. Vehicle formation (or distributed Kalman filter). Another system structure, which can be viewed as a dual of the previous one, is the following:

$$\begin{cases} \dot{x}_1 = -x_1 + \ell_{13}(x_3 - x_1)\\ \dot{x}_2 = \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2)\\ \dot{x}_3 = \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3)\\ \dot{x}_4 = -4x_4 + \ell_{43}(x_3 - x_4) \end{cases} \tag{2}$$

This model could for example be used to describe a platoon of four vehicles. The parameters ℓ_{ij} represent position adjustments based on distance measurements between the vehicles. The terms $-x_1$ and $-4x_4$ reflect that the first and fourth vehicle are equipped to maintain stable positions on their own, but the second and third vehicle rely on the distance measurements for stabilization. Again, stability can be verified by a distributed test where the first vehicle verifies the first inequality, the second vehicle verifies the second inequality and so on. □

IV. Input-Output Performance of Positive Systems

A connection between stability and performance is established by the following extension of a result in [16]:

Proposition 3: Suppose $\mathbf{G}(s) = C(sI - A)^{-1}B + D$ with $A \in \mathbb{R}^{n\times n}$ Metzler and $B \in \mathbb{R}_+^{n\times 1}$, $C \in \mathbb{R}_+^{1\times n}$, $D \in \mathbb{R}_+$. Define $\|\mathbf{G}\|_\infty = \sup_\omega |G(i\omega)|$. Then the following statements are equivalent:

(3.1) The matrix A is Hurwitz and $\|\mathbf{G}\|_\infty < \gamma$.

(3.2) The matrix $\begin{bmatrix} A & B \\ C & D-\gamma \end{bmatrix}$ is Hurwitz.

(3.3) There exists a diagonal $P \succ 0$ such that all solutions to $\dot{x} = Ax + Bw$ satisfy

$$\frac{d}{dt} x(t)^T P x(t) + |Cx(t) + Dw(t)|^2 \le \gamma^2 |w(t)|^2 \tag{3}$$

with strict inequality when $(x, w) \neq (0, 0)$.

(3.4) There exists $p \in \mathbb{R}^n$ such that $p > 0$ and all solutions to $\dot{x} = Ax + Bw$ satisfy

$$\frac{d}{dt}\left(p^T |x(t)|\right) + |Cx(t) + Dw(t)| \le \gamma |w(t)| \tag{4}$$

with strict inequality when $(x, w) \neq (0, 0)$.

A discrete time version can be stated as follows:

Proposition 4: Let $\mathbf{G}(z) = C(zI - A)^{-1}B + D$ where $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times 1}$, $C \in \mathbb{R}_+^{1\times n}$ and $D \in \mathbb{R}_+$. Define $\|\mathbf{G}\|_\infty = \sup_\omega |\mathbf{G}(e^{i\omega})|$. Then the following conditions are equivalent:

(4.1) The matrix A is Schur and $\|\mathbf{G}\|_\infty < \gamma$.

(4.2) The matrix $\begin{bmatrix} A & B \\ \gamma^{-1}C & \gamma^{-1}D \end{bmatrix}$ is Schur.

(4.3) There exists a diagonal matrix $P \succ 0$ such that

$$|x(t+1)|_P^2 + |Cx(t) + Dw(t)|^2 \le |x(t)|_P^2 + \gamma^2 |w(t)|^2$$

for all solutions to $x(t+1) = Ax(t) + Bw(t)$.

(4.4) There exists a $p \in \mathbb{R}^n$ such that $p > 0$ and

$$p^T|x(t+1)| + |Cx(t) + Dw(t)| \le p^T|x(t)| + \gamma|w(t)|$$

for all solutions to $x(t+1) = Ax(t) + Bw(t)$.

V. Distributed Control Synthesis by Linear Programming

Equipped with scalable analysis methods for stability and performance, we are now ready to consider synthesis of controllers by distributed optimization. We will start by re-visiting an example of section III.

Example 3. Consider again the transportation network (1), this time with the flow parameters $\ell_{31} = 2$, $\ell_{34} = 1$ and $\ell_{43} = 2$ fixed:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -3 & \ell_{12} & 0 & 0 \\ 0 & 2-\ell_{12}-\ell_{32} & \ell_{23} & 0 \\ 2 & \ell_{32} & 1-\ell_{23} & 1 \\ 0 & 0 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{5}$$

We will ask the question how to find the remaining parameters ℓ_{12}, ℓ_{23} and ℓ_{32} in the interval $[0, 10]$ such that the closed loop system (5) becomes stable. According to Proposition 1, stability is equivalent to existence of $\xi_1, \ldots, \xi_4 > 0$ such that

$$\begin{bmatrix} -3 & \ell_{12} & 0 & 0 \\ 0 & 2-\ell_{12}-\ell_{32} & \ell_{23} & 0 \\ 2 & \ell_{32} & 1-\ell_{23} & 1 \\ 0 & 0 & 2 & -5 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} < 0$$

At first sight, this looks like a difficult problem due to multiplications between the two categories of parameters. However, a closer look suggests the introduction of $\mu_{12} := \ell_{12}\xi_2$, $\mu_{32} := \ell_{32}\xi_2$ and $\mu_{23} := \ell_{23}\xi_3$. The problem then reduces to linear programming: Find $\xi_1, \xi_2, \xi_3, \xi_4 > 0$ and $\mu_{12}, \mu_{32}, \mu_{23} \ge 0$ such that

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & -5 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_{12} \\ \mu_{32} \\ \mu_{23} \end{bmatrix} < 0$$

$$\mu_{12} \le 10\xi_2 \qquad \mu_{32} \le 10\xi_2 \qquad \mu_{23} \le 10\xi_3$$

with the solution $(\xi_1, \xi_2, \xi_3, \xi_4) = (43, 12.8, 10.1, 4.2)$ and $(\mu_{12}, \mu_{32}, \mu_{23}) = (128, 0, 101)$. The corresponding stabilizing gains can then be computed as

$$\ell_{12} = \mu_{12}/\xi_2 = 10 \quad \ell_{32} = \mu_{32}/\xi_2 = 0 \quad \ell_{23} = \mu_{23}/\xi_3 = 10$$

□

The idea can be generalized into the following:

Theorem 5: Let the matrices $A \in \mathbb{R}^{n\times n}, E \in \mathbb{R}^{n\times m}, F \in \mathbb{R}^{m\times n}, K \in \mathbb{R}^{m\times m}$ be given and let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices with entries in $[0, 1]$. Suppose that $(I - LK)^{-1}$ exists and $A + E(I - LK)^{-1}LF$ is Metzler for all $L \in \mathcal{D}$. If F and K have nonnegative coefficients, then the following two conditions are equivalent:

(5.1) There exists $L \in \mathcal{D}$ such that $A + E(I - LK)^{-1}LF$ is Hurwitz.

(5.2) There exist $\xi \in \mathbb{R}_+^n$, $\mu \in \mathbb{R}_+^m$ with $\mu \le F\xi + K\mu$ and $A\xi + E\mu < 0$.

Alternatively, if E and K have nonnegative coefficients, then (5.1) is equivalent to

(5.3) There exist $p \in \mathbb{R}_+^n$, $q \in \mathbb{R}_+^m$ with $q \le E^T p + K^T q$ and $A^T p + F^T q < 0$.

Remark 2. If the diagonal elements of $\mathcal{D}$ are restricted to $\mathbb{R}_+$ instead of $[0,1]$, then the condition $\mu \le F\xi + K\mu$ is replaced by $0 \le F\xi + K\mu$.

Remark 3. Each row of the vector inequalities can be verified separately to get a distributed test.

Remark 4. It is natural to compare the expression $A + E(I-LK)^{-1}LF$ with the "state feedback" expression $A + BL$ of standard linear quadratic optimal control. A major difference is the presence of F and K which make the optimization into a problem of "static output feedback" rather than state feedback. Another difference is the diagonally structured L instead of a full matrix. The diagonal structure gives a much higher degree of flexibility, particularly in the specification of distributed controllers.

Proof of Theorem 5. Suppose (5.1) holds. Let $A + E(I-LK)^{-1}LF$ be Hurwitz and define $\xi \in \mathbb{R}_+^n$ with $[A + E(I-LK)^{-1}LF]\xi < 0$. Let $\mu = (I-LK)^{-1}LF\xi$. Then $\mu = L(F\xi + K\mu)$ and $A\xi + E\mu = (A + E(I-LK)^{-1}LF)\xi < 0$.

Conversely, suppose that (5.2) holds. Choose $L \in \mathcal{D}$ to get $\mu = (I-LK)^{-1}LF\xi$. Then

$$[A + E(I-LK)^{-1}LF]\xi = A\xi + E\mu < 0$$

so $A+E(I-LK)^{-1}LF$ is Hurwitz. The equivalence between (5.1) and (5.3) follows immediately by replacing $A + E(I-LK)^{-1}LF$ with its transpose. □

Combining Proposition 3 with Theorem 5 also gives a linear programming formulation of the problem to minimize input-output gain:

Corollary 6: Let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices with entries in $[0,1]$. Suppose that D is scalar and that $A + ELF$ is Metzler for all $L \in \mathcal{D}$.

If the matrices B, C, D and F have nonnegative coefficients, then the following two conditions are equivalent:

(6.1) There exists $L \in \mathcal{D}$ such that $A + ELF$ is Hurwitz and

$$\|C[sI - (A+ELF)]^{-1}B + D\|_\infty < \gamma. \tag{6}$$

(6.2) There exist $\xi \in \mathbb{R}_+^n$, $\mu \in \mathbb{R}_+^m$ with

$$A\xi + E\mu + B < 0 \quad C\xi + D < \gamma \quad \mu \le F\xi$$

If ξ, μ satisfy (6.2), then (6.1) holds for every L such that $\mu = LF\xi$.

Alternatively, if B, C, D and E are nonnegative, then (6.1) is equivalent to

(6.3) There exist $p \in \mathbb{R}_+^n$, $q \in \mathbb{R}_+^m$ with

$$A^T p + F^T q + C^T < 0 \quad B^T p + D < \gamma \quad q \le E^T p$$

If p, q satisfy (6.3), then (6.1) holds for every L such that $q = LE^T p$.

Remark 5. It is interesting to compare Corollary 6 with the corresponding synthesis method proposed in [19], which exploits (1.4) of Proposition 1 as criterion for stability rather than (1.2) and (1.3). An advantage of the approach in [19] is that the Metzler property of the closed loop system matrix can be enforced in the synthesis procedure as a constraint, rather than being verified a priori for all $L \in \mathcal{D}$. On the other hand, the linear programming approach proposed here has a simpler structure, where the distributed and scalable nature of the conditions is apparent.

Proof. According to Proposition 3, condition (6.1) holds if and only if there exists $\xi \in \mathbb{R}_+^n$ with

$$\begin{bmatrix} A+ELF & B \\ C & D-\gamma \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} < 0 \tag{7}$$

Given (7), the inequalities of (6.2) hold with $\mu = LF\xi$. Conversely, given (6.2), the inequalities of (7) follow provided that $\mu = LF\xi$. This proves the desired equivalence between (6.1) and (6.2). The equivalence between (6.1) and (6.3) follows immediately by replacing $G(s)$ with its transpose. □

Example 4. Disturbance rejection in vehicle formation. Consider the vehicle formation model

$$\begin{cases} \dot{x}_1 = -x_1 + \ell_{13}(x_3 - x_1) + w_1 \\ \dot{x}_2 = \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) + w_2 \\ \dot{x}_3 = \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) + w_3 \\ \dot{x}_4 = -4x_4 + \ell_{43}(x_3 - x_4) + w_4 \end{cases} \tag{8}$$

where $w_1, \dots, w_4$ are external disturbances acting on the vehicles. Our problem is to find feedback gains gains $\ell_{ij} \in [0,1]$ that stabilize the formation and minimize the gain from w to x. The problem can be solved by applying condition (6.2) with

$$A = \operatorname{diag}\{-1, 0, 0, -4\} \qquad D = 0$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \qquad K = 0$$

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$L = \operatorname{diag}\{\ell_{13}, \ell_{21}, \ell_{23}, \ell_{32}, \ell_{34}, \ell_{43}\}$$

$$F = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The optimal $L = \operatorname{diag}\{0,1,1,0,1,0\}$ gives $\gamma = 4.125$. If instead $B = \begin{pmatrix} 10 & 10 & 1 & 1 \end{pmatrix}^T$, the minimal value $\gamma = 15.562$ is attained with $L = \operatorname{diag}\{1,1,1,0,1,0\}$. Conversely, $B = \begin{pmatrix} 1 & 1 & 10 & 10 \end{pmatrix}^T$ gives the minimum $\gamma = 12.750$ for $L = \operatorname{diag}\{0,1,0,1,1,0\}$. □

There is one important limitation that becomes apparent in the example of a vehicle formation: Only single integrators can be used as vehicle models. With double integrators, the Metzler structure breaks down and none of the results above can be applied. This motivates the introduction a new concept in the next section.

VI. Positively Dominated Systems

$\mathbf{G} \in \mathbb{CH}_\infty^{m\times n}$ is called *positively dominated* if every matrix entry satisfies $|\mathbf{G}_{jk}(i\omega)| \le \mathbf{G}_{jk}(0)$ for $\omega \in \mathbb{R}$. The set of all such matrices is denoted $\mathbb{PH}_\infty^{m\times n}$. Some properties follow immediately:

Proposition 7: Let $\mathbf{G}, \mathbf{H} \in \mathbb{PH}_\infty^{n\times n}$. Then $\mathbf{GH} \in \mathbb{PH}_\infty^{n\times n}$ and $a\mathbf{G} + b\mathbf{H} \in \mathbb{PH}_\infty^{n\times n}$ when $a, b \in \mathbb{R}_+$. Moreover $\|\mathbf{G}\|_\infty = \|\mathbf{G}(0)\|$.

The following property is also fundamental:

Theorem 8: Let $\mathbf{G} \in \mathbb{PH}_\infty^{n\times n}$. Then $(I-\mathbf{G})^{-1} \in \mathbb{PH}_\infty^{n\times n}$ if and only if $\mathbf{G}(0)$ is Schur.

Proof. That $(I-\mathbf{G})^{-1}$ is stable and positively dominated implies that $[I-\mathbf{G}(0)]^{-1}$ exists and is nonnegative, so $\mathbf{G}(0)$ must be Schur according to Proposition 2. On the other hand, if $\mathbf{G}(0)$ is Schur we may choose $\xi \in \mathbb{R}_+$ and $\epsilon > 0$ with $\mathbf{G}(0)\xi < (1-\epsilon)\xi$. Then for every $z \in \mathbb{C}^n$ with $0 < |z| < \xi$ and $s \in \mathbb{C}$ with $\mathrm{Re}\, s \ge 0$ we have

$$|\mathbf{G}(s)^t z| \le \mathbf{G}(0)^t |z| < (1-\epsilon)^t |z| \quad \text{for } t = 1, 2, 3, \ldots$$

Hence $\sum_{k=0}^\infty \mathbf{G}(s)^t z$ is convergent and bounded above by $\sum_{k=0}^\infty \mathbf{G}(0)^t |z| = [I-\mathbf{G}(0)]^{-1}|z|$. The sum of the series solves the equation $[I-\mathbf{G}(s)]\sum_{k=0}^\infty \mathbf{G}(s)^t z = z$, so therefore $\sum_{k=0}^\infty \mathbf{G}(s)^t z = [I-\mathbf{G}(s)]^{-1} z$. This proves $(I-\mathbf{G})^{-1}$ is stable and positively dominated and the proof is complete. □

The ideas of the previous section can now be extended to positively dominated systems:

Theorem 9: Let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices with entries in $[0,1]$. Suppose that $\mathbf{B} \in \mathbb{PH}_\infty^{n\times 1}$, $\mathbf{C} \in \mathbb{PH}_\infty^{1\times n}$, $\mathbf{D} \in \mathbb{PH}_\infty$ and $\mathbf{A} + \mathbf{E}L\mathbf{F} \in \mathbb{PH}_\infty^{n\times n}$ for all $L \in \mathcal{D}$.

If $\mathbf{F} \in \mathbb{PH}_\infty^{m\times n}$, then the following are equivalent:

(9.1) Some $L \in \mathcal{D}$ gives $(I-\mathbf{A}-\mathbf{E}L\mathbf{F})^{-1} \in \mathbb{PH}_\infty^{n\times n}$ and $\|\mathbf{C}(I-\mathbf{A}-\mathbf{E}L\mathbf{F})^{-1}\mathbf{B}+\mathbf{D}\|_\infty < \gamma$

(9.2) There exist $\xi \in \mathbb{R}_+^n$, $\mu \in \mathbb{R}_+^m$ with

$$\begin{aligned} \mathbf{A}(0)\xi + \mathbf{E}(0)\mu + \mathbf{B}(0) &< \xi \\ \mathbf{C}(0)\xi + \mathbf{D}(0) &< \gamma \\ \mathbf{F}(0)\xi &\ge \mu \end{aligned}$$

If instead $\mathbf{E} \in \mathbb{PH}_\infty^{n\times m}$, then (9.1) is equivalent to

(9.3) There exist $p \in \mathbb{R}_+^n$, $q \in \mathbb{R}_+^m$ with

$$\begin{aligned} \mathbf{A}(0)^T p + \mathbf{F}(0)^T q + \mathbf{C}(0)^T &< p \\ \mathbf{B}(0)^T p + \mathbf{D}(0) &< \gamma \\ \mathbf{E}(0)^T p &\ge q \end{aligned}$$

If (9.2) holds and $\mu = L\mathbf{F}(0)\xi$, then (9.1) holds too. Similarly, if (9.3) holds and $q = L\mathbf{E}(0)^T p$, then (9.1) follows.

Proof. Theorem 8 shows that (9.1) holds if and only if $\mathbf{A}(0) - \mathbf{E}(0)L\mathbf{F}(0)$ is Schur and $\mathbf{C}[I-\mathbf{A}(0) - \mathbf{E}(0)L\mathbf{F}(0)]^{-1}\mathbf{B}(0) + \mathbf{D}(0) < \gamma$. According to Proposition 4, this is true if and only if

$$\begin{bmatrix} \mathbf{A}(0) + \mathbf{E}(0)L\mathbf{F}(0) & \mathbf{B}(0) \\ \gamma^{-1}\mathbf{C}(0) & \gamma^{-1}\mathbf{D}(0) \end{bmatrix} \tag{9}$$

is Schur. By Proposition 2 this is equivalent to existence of $\xi \in \mathbb{R}_+^n$ such that

$$\begin{bmatrix} \mathbf{A}(0) + \mathbf{E}(0)L\mathbf{F}(0) & \mathbf{B}(0) \\ \gamma^{-1}\mathbf{C}(0) & \gamma^{-1}\mathbf{D}(0) \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} < \begin{bmatrix} \xi \\ 1 \end{bmatrix}$$

This is equivalent to (9.2) if we set $\mu = L\mathbf{F}(0)\xi$, so the desired equivalence between (9.1) and (9.2) in Theorem 9 follows. The equivalence between (9.1) and (9.3) is obtained by replacing $G(s)$ with its transpose. □

Example 5. With the new concept at hand, we can now return to the vehicle formation, but this time model each vehicle as a double integrator. Alternatively, the interconnection can be viewed as a mechanical structure consisting of N point-masses connected by springs. The dynamics is described by the equations

$$\ddot{x}_i = \sum_j \ell_{ij}(x_j - x_i) + u_i + w_i \qquad i = 1, \ldots, N$$

where u_i is a local control force, w_i is a disturbance and ℓ_{ij} is the spring constant between the point masses i and j. Suppose local control laws $u_i = -k_i x_i - d_i \dot{x}$ are given and consider the problem to find spring constants $\ell_{ij} \in [0, \bar{\ell}_{ij}]$ that minimize the gain from w_1 to x_1.

The closed loop system has the following frequency domain description

$$\begin{aligned} &\left(s^2 + d_i s + k_i + \sum_j \bar{\ell}_{ij}\right) X_i(s) \\ &= \sum_j \left(\ell_{ij} X_j(s) + (\bar{\ell}_{ij} - \ell_{ij}) X_i(s)\right) + W_i(s) \end{aligned}$$

After dividing both sides with $s^2 + d_i s + k_i + \sum_j \bar{\ell}_{ij}$, we write this on matrix form as

$$X = (\mathbf{A} + \mathbf{E}LF)X + \mathbf{B}W$$

The transfer matrices $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{A}+\mathbf{E}L\mathbf{F}$ are positively dominated for all $L \in \mathcal{D}$ provided that $d_i \ge k_i + \sum_j \bar{\ell}_{ij}$. Hence Theorem 9 can then be applied to find the optimal spring constants. Notice that ℓ_{ij} and ℓ_{ji} must be optimized separately, even though by symmetry they must be equal at optimum. □

VII. Conclusions

The results above indicate that the monotonicity properties of positive systems and positively dominated systems bring remarkable benefits to control theory. Most important is the opportunity for scalable verification and synthesis of H_∞ optimal performance. In particular, the optimal solution comes with a certificate (the numbers ξ_k, μ_k) that makes it possible to verify optimality locally, without access to a global model.

Many important problems remain open for future research. Here are two examples:

- How can the scalable methods for verification be extended to monotone nonlinear systems in a nonconservative way?
- How can local controllers be designed to get positively dominated interactions with optimal properties? (This would be in contrast with the mass-spring example where the local control parameters d_i and k_i were fixed a priori.)

VIII. Acknowledgment

The author is grateful for suggestions and comments by numerous colleagues. The work has been supported by the Swedish Research Council through the LCCC Linnaeus Center.

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