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Physical Bounds for Antennas Above a Ground Plane

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Abstract

Physical limitations of antennas above infinite perfect electric conductor (PEC) ground planes are determined using the stored electromagnetic energy. Stored energies are computed with the method of moments (MoM) and the image theory. Convex optimization is used to derive the G/Q ratio and Q-factor for a reference geometry and the results are compared for different antenna types.

1 Introduction

The proximity of a conducting ground plane is both a blessing and a curse for antenna designers. In some cases it limits the bandwidth, while in others this additional structure can improve antenna performance significantly [22, 27].

The Q-factor, which is inversely proportional to the fractional bandwidth, is the main and traditional figure of merit for physical bounds [25]. For a system, the Q-factor is defined as the ratio of its stored energy to the dissipated energy per cycle. Similarly, the Q-factor for an antenna is the ratio of stored energy to the radiated fields and antenna losses [5, 12, 13, 25, 26].

Early pioneers in antenna theory, such as Chu [5], have investigated the physical limits of antennas since the late 1940s. Chu calculated the stored energies analytically using spherical mode expansions, excluding a small spherical region near the antenna. Although his calculations are restricted to antennas circumscribed by a sphere, they were a significant breakthrough for its time.

Today progresses in both theory and computing tools in the field allow the analysis of antennas with different geometries. Physical bounds for antennas of arbitrary shape and size were computed using the forward scattering sum rule in [13]; whereas antenna current optimization is used to compute the physical bounds of arbitrary metallic structures using method of moments (MoM) software for antennas in free-space as well as antennas with finite ground planes in [6, 12, 14]. The comparison of these different techniques with planar and cylindrical antennas can be found in [2, 10, 13, 20, 21]

This paper extends the physical bounds calculation to cover a new class of antennas, antennas above an infinite ground plane. Previous research on this topic has been conducted for low-frequency limit using spherical harmonics in [23], vertically polarized antennas [13], and infinite antenna arrays [4, 7, 19]. Our approach is based on the application of classical image theory to the calculation of antenna stored energies using the expressions for the stored energy derived by Vandenbosch [24]. The proposed method can calculate the upper bound on G/Q, for any geometry and radiation polarization.

To demonstrate the physical bounds a reference rectangular patch geometry above a ground plane is presented and compared with different patch antenna designs from literature [8]. The selected antennas are simulated using the commercial electromagnetic solver FEKO [1]. The Q-factor of the simulated antennas are computed using both the MoM impedance matrix and the differentiated input impedance [28].

Simulation results show that the Q-factor of the antennas conform to the physical bounds of the reference geometry.

2 Stored Energies and Physical Bounds

While a specific definition is established on energy dissipated by an antenna, the same cannot be claimed for the stored energy. This paper uses the stored electric energy defined by Vandenbosch [24] which is equivalent to the coordinate independent part of the stored energy expression [17]

$$W_{\mathrm{e}} = \frac{\epsilon_0}{4}\int_{\mathbb{R}^3} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2}\mathrm{dV}. \tag{2.1}$$

In (2.1) $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{F}(\hat{\boldsymbol{r}})$ are respectively the electric field and the far-field with, $r = |\boldsymbol{r}|$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$, and the integration is over an infinite sphere. The magnetic stored energy is similarly obtained by replacing the electric field with the magnetic field in (2.1).

In (2.1), the stored energies are calculated by subtracting the far-field energy density from the electric energy density. It has been shown in [11, 28] that the resulting expression (2.1) is the sum of two terms; one of them is coordinate dependent and the other coordinate independent. The coordinate independent term may yield the stored energy to have negative values [12], an indication that the model is not exact, since the result is unphysical. The coordinate independent term is very accurate for antennas smaller than $ka \leq 1$ in dimension, where k is the wavenumber and a is the radius of the smallest sphere enclosing the antenna [11, 17].

The Q-factor, the main parameter in quantifying the physical limits of antennas, can also be used in optimization problems. For a lossless antenna the Q-factor is usually defined as [16, 28];

$$Q = \frac{2\omega \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}}{P_{\mathrm{d}}}, \tag{2.2}$$

where ω is the angular frequency, W_{e} is the stored electric energy, W_{m} stored magnetic energy, and P_{d} is the dissipated power. A useful approximation of the Q-factor is the derivative of antenna input impedance, Z_{in}, tuned to resonance [28],

$$Q_{\mathrm{Z}'_{\mathrm{in}}} = \frac{\omega|Z'_{\mathrm{in}}|}{2R_{\mathrm{in}}}, \tag{2.3}$$

where the terms Z'_{in} and R_{in} are the derivative of the tuned input impedance with respect to angular frequency, and the real part of the input impedance, respectively. It is assumed that the antenna is tuned to resonance with a single circuit element, which can be either capacitive or inductive (2.3).

The Q-factor can be calculated from the MoM impedance matrix and its frequency derivative [15]. The impedance matrix is written as the sum of its real and imaginary parts, $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$. The difference of stored energies are derived as

$$W_\mathrm{m} + W_\mathrm{e} = \frac{1}{4}\mathbf{I}^\mathrm{H}\mathbf{X}'\mathbf{I}, \tag{2.4}$$

$$W_\mathrm{m} - W_\mathrm{e} = \frac{1}{4\omega}\mathbf{I}^\mathrm{H}\mathbf{X}\mathbf{I}, \tag{2.5}$$

here $\mathbf{X}'$ is the derivative of the imaginary part of the MoM impedance matrix with respect to the radial frequency, $\mathbf{I}$ is a single column matrix representing surface currents on the antenna structure, and $\mathbf{I}^\mathrm{H}$ is the Hermitian transpose of the surface currents. By substituting one equation to the other, the stored energy expressions are found as

$$W_\mathrm{m} = \frac{1}{8}\mathbf{I}^\mathrm{H}\left(\frac{\partial\mathbf{X}}{\partial\omega} + \frac{\mathbf{X}}{\omega}\right)\mathbf{I} = \frac{1}{4\omega}\mathbf{I}^\mathrm{H}\mathbf{X}_\mathrm{m}\mathbf{I}, \tag{2.6}$$

$$W_\mathrm{e} = \frac{1}{8}\mathbf{I}^\mathrm{H}\left(\frac{\partial\mathbf{X}}{\partial\omega} - \frac{\mathbf{X}}{\omega}\right)\mathbf{I} = \frac{1}{4\omega}\mathbf{I}^\mathrm{H}\mathbf{X}_\mathrm{e}\mathbf{I}, \tag{2.7}$$

both (2.6) and (2.7) can simply be calculated from the MoM impedance matrix. The tuned Q-factor can then be expressed as

$$Q = \frac{\omega\mathbf{I}^\mathrm{H}\mathbf{X}'\mathbf{I} + |\mathbf{I}^\mathrm{H}\mathbf{X}\mathbf{I}|}{2\mathbf{I}^\mathrm{H}\mathbf{R}\mathbf{I}}. \tag{2.8}$$

The partial radiation intensity of an antenna with a unit polarization vector $\hat{\boldsymbol{e}}$ and direction $\hat{\boldsymbol{k}}$, is proportional to the far-field $\boldsymbol{F}(\hat{\boldsymbol{k}})$, and is expressed as

$$\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F}(\hat{\boldsymbol{k}}) = \frac{-\mathrm{j}\omega\eta_0}{4\pi}\int_S \hat{\boldsymbol{e}}^* \cdot \boldsymbol{J}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{dS}, \tag{2.9}$$

where η_0 is impedance of free space and k is the wavenumber. The partial gain is defined as

$$G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}}) = \frac{2\pi}{\eta_0}\frac{|\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F}(\hat{\boldsymbol{k}})|^2}{P_\mathrm{d}}, \tag{2.10}$$

the G/Q ratio is determined from (2.2) and (2.10)

$$\frac{G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}})}{Q} = \frac{\pi|\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F}(\hat{\boldsymbol{k}})|^2}{\omega\eta_0\max\{W_\mathrm{e}, W_\mathrm{m}\}}. \tag{2.11}$$

3 Antennas Above Ground Planes

Assuming an infinite perfect electric conductor (PEC) plane in the xy-plane Fig. 1, the current density can be decomposed into horizontal and vertical components. The current density is expressed as $\boldsymbol{J}(\boldsymbol{r}) = \boldsymbol{J}_\mathrm{v}(\boldsymbol{r}) + \boldsymbol{J}_\mathrm{h}(\boldsymbol{r})$ above the PEC plane and its image current is written as [18]

$$\boldsymbol{J}^\mathrm{im}(\boldsymbol{r}) = \boldsymbol{J}_\mathrm{v}(\boldsymbol{r}_\mathrm{i}) - \boldsymbol{J}_\mathrm{h}(\boldsymbol{r}_\mathrm{i}) = \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \cdot \boldsymbol{J}(\boldsymbol{r}_\mathrm{i}) - [\boldsymbol{J}(\boldsymbol{r}_\mathrm{i}) - \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \cdot \boldsymbol{J}(\boldsymbol{r}_\mathrm{i})], \tag{3.1}$$

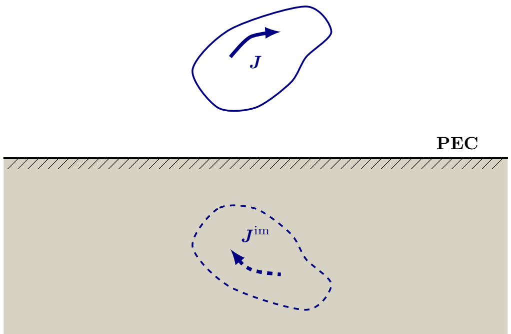

Figure 1: The image current density $\boldsymbol{J}^{\mathrm{im}}$ of an arbitrarily placed current density $\boldsymbol{J}$

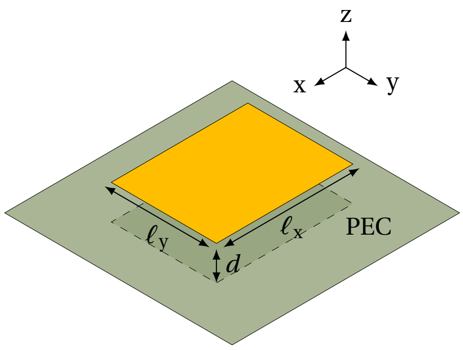

Figure 2: The reference patch geometry of dimensions ℓ_{x}, $\ell_y = 0.77\ell_{\mathrm{x}}$ with a height of d above an infinite PEC ground plane.

where $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ and $\boldsymbol{r}_i = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} - z\hat{\boldsymbol{z}}$ are the positions of the current density and its image respectively. The dyadic Green's function for solving this problem can be written as [18]

$$\mathbf{G}\left(\boldsymbol{r}_1, \boldsymbol{r}_2\right) = \left[G(\boldsymbol{r}_1, \boldsymbol{r}_2) - G(\boldsymbol{r}_1, \boldsymbol{r}_{2\mathrm{i}})\right]\left(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\right) + \left[G(\boldsymbol{r}_1, \boldsymbol{r}_2) + G(\boldsymbol{r}_1, \boldsymbol{r}_{2\mathrm{i}})\right]\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} = \mathbf{G}_{\parallel} + \mathbf{G}_{\perp} \quad (3.2)$$

here $\boldsymbol{r}_2$ is the source position, $\boldsymbol{r}_1$ is the observation point, $\mathbf{G}_{\parallel}$ is the parallel Green's dyadic, $\mathbf{G}_{\perp}$ is the orthogonal Green's dyadic, and G is the free-space Green's function. For horizontal currents ($\hat{\boldsymbol{x}}$ or $\hat{\boldsymbol{y}}$ directed) the source current and image current are subtracted, while for vertical currents ($\hat{\boldsymbol{z}}$ directed) the source and image currents are added.

An in-house MoM electric field integral equation (EFIE) implementation with Galerkin test functions has been used to compute the stored energies. The MoM impendence matrix elements for antennas above ground plane can be written by substituting (3.2) to the free-space Green's function

$$Z_{mn} = \mathrm{j}k\eta_0 \int_S\int_S \Big[\boldsymbol{\psi}_m(\boldsymbol{r}_1)\cdot\boldsymbol{\psi}_n(\boldsymbol{r}_2) - \frac{1}{k^2}(\nabla_1\cdot\boldsymbol{\psi}_m(\boldsymbol{r}_1)\nabla_2\cdot\boldsymbol{\psi}_n(\boldsymbol{r}_2)\Big]\tilde{G}\left(\boldsymbol{r}_1, \boldsymbol{r}_2\right)\mathrm{dS}_1\,\mathrm{dS}_2, \quad (3.3)$$

here Z_{mn} denotes the elements of the MoM impedance matrix, $\boldsymbol{\psi}_m$ and $\boldsymbol{\psi}_n$ are the expansion of source and test basis functions that are obtained by expanding the currents in basis functions, $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n(\boldsymbol{r})$. The $\tilde{G}$ Green's function is the parallel Green's dyadic or the orthogonal Green's dyadic, or their sums depending on the direction of the basis function.

The stored energy expressions (2.6), (2.7) are then used to optimize the antenna G/Q quotient. Formulating the physical bounds for the maximal G/Q quotient results in a convex optimization problem. The physical bounds are computed using convex optimization [3], where the G/Q quotient is optimized by finding the best arrangement of surface currents over the antenna geometry. This approach allows the optimization of the antenna bandwidth for different polarizations and/or radiation patterns. The convex optimization problem can be written as [12]

$$\begin{aligned} &\text{minimize} \quad \max\{\mathbf{I}^{\mathrm{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathrm{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ &\text{subject to} \quad \mathrm{Re}\{\mathbf{F}\mathbf{I}\} = 1 \end{aligned} \tag{3.4}$$

the $\mathbf{F}$ is a matrix which specifies the radiation polarization and direction (2.9). The convex optimization problem (3.4) can be readily solved using the Matlab toolbox CVX [9].

4 Numerical Examples

The physical bounds for a reference metallic rectangular geometry (Fig. 2) are calculated using the convex optimization formulation (3.4). The geometry has a dimension of $\ell_{\mathrm{y}} = 0.77\ell_{\mathrm{x}}$. It is placed above an infinite PEC ground plane and the bounds are computed for three different patch heights $d = \{0.01, 0.05, 0.1\}\ell_{\mathrm{x}}$ radiating in the $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$ direction, with either $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ polarization. It should be noted that computation of the physical bounds do not require a feed point. The physical bounds for the patch in Fig. 2 are illustrated in Fig. 3 and Fig. 4 with respect to $\ell_{\mathrm{x}}/\lambda$, the patch length normalized to the wavelength. As expected, the G/Q bound and hence the patch bandwidth deteriorates rapidly as the proximity to the ground plane increases.

The physical bounds of the reference rectangular region are also compared with different patch antennas with identical maximum dimensions, for a height of $d = 0.05\ell_{\mathrm{x}}$. The patch antennas are simulated in FEKO and matched to 50Ω. The Q-factors are computed with (2.3). The comparison of the physical bounds and antenna Q-factor is illustrated in Fig. 5. The simulated patch antennas match the physical bounds and demonstrate that the bound for a single rectangular patch cannot be exceeded using special geometries or configurations.

The simulated antennas include: a rectangular patch antenna, a slot loaded patch antenna, a H-shaped antenna. From these the rectangular patch and the slot loaded patch are fed in both $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ polarizations. The feed points for the patch antenna are $0.18\ell_{\mathrm{x}}$ for $\hat{\boldsymbol{x}}$ and $0.25\ell_{\mathrm{x}}$ for $\hat{\boldsymbol{y}}$ polarization from the center of the antenna. The slot size of the slot loaded antenna is $(0.2 \times 0.2)\ell_{\mathrm{x}}$ and the feed points are located at $0.1\ell_{\mathrm{x}}$ for $\hat{\boldsymbol{x}}$ and $0.2\ell_{\mathrm{x}}$ for $\hat{\boldsymbol{y}}$ polarization from the center. The H-shaped antenna feed

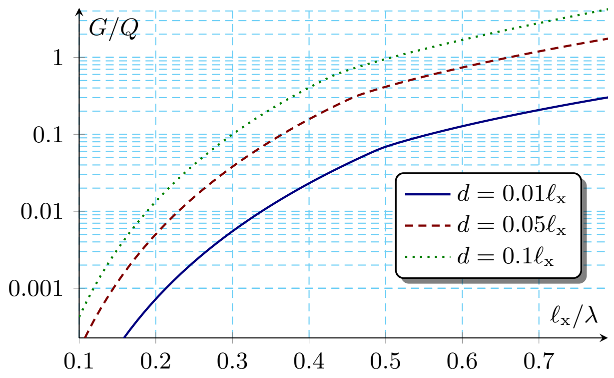

Figure 3: Physical bounds G/Q for a patch above a PEC ground plane for different patch heights $d = \{0.01, 0.05, 0.1\}\ell_x$.

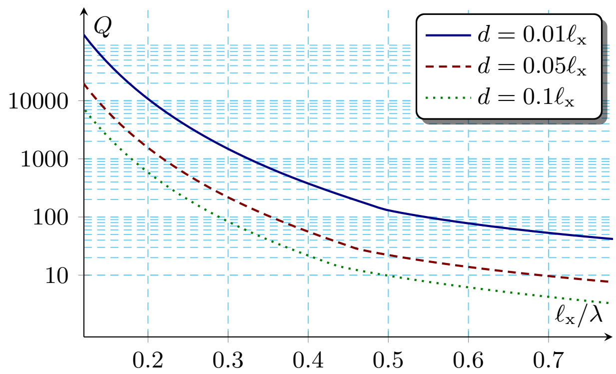

Figure 4: The resulting Q-factor from the physical bounds Fig. 3 for a patch above a PEC ground plane for different patch heights $d = \{0.01, 0.05, 0.1\}\ell_x$.

point is located at $0.06\ell_x$ from the center of the antenna. All of the antennas are matched to 50Ω and their Q-factor is calculated using (2.3) at the match frequency. The H-shaped antenna and slot antennas resonate at lower frequencies than the patch antenna as the effective length is increased by extending the current path in both antenna types.

5 Conclusion

The paper discusses the estimation of the physical bounds of antennas above a PEC ground plane, based on the classical image theory and the use of Vandenbosch‘s expressions for stored energy. The proposed method calculates the Q-factor for antennas of any geometry and radiation polarization. Numerical results performed for a metallic rectangular region match well with the simulation results using FEKO. Further work should include multi-layered dielectric patches.

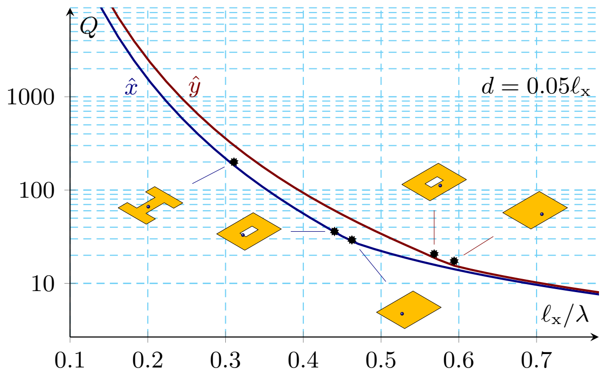

Figure 5: Comparison of the Q-factor from the physical bound on G/Q for the patch geometry in Fig. 2 with $d = 0.05\ell_x$ for six patch antennas.

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