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In memory of my best friend Alexander
Светлой памяти моего лучшего друга Александра

Abstract

This work presents an extension, simplification and application of a design procedure for dynamic output feedback design for systems with nonlinearities satisfying quadratic constraints (QC). Our method was motivated by the challenges of output feedback control design for the three-state Moore-Greitzer (MG) compressor model. The classical three-state MG model is a nonlinear dynamical system that is widely used in stall/surge analysis and control design.

First, we find the parameter set of the stabilizing dynamic output feedback controllers for the surge subsystem by using conditions for stability of a transformed system and the associated matching conditions.

Second, we choose the optimal control parameters from the stabilizing set with respect to different desired criteria.

We show the set of parameters of the stabilizing controllers for the surge subsystem and the set of parameters of the stabilizing controllers with extended integral part for MG compressor.

We present simplified sufficient conditions for stabilization, new constraints for the corresponding parameters and examples of optimal problem for the surge subsystem of the Moore-Greitzer compressor model.

We discuss the degree of robustness and clarify an alternative proof of stability of the closed-loop system with the surge subsystem and the stabilizing dynamic output feedback controller without an integral state. In addition, we show the derivation of a quadratic function by using CVX.

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First of all, I would like to thank my supervisors Prof. Rolf Johansson and Prof. Anders Robertsson for their continuous guidance, care, patience and encouragement. I have been extremely lucky to have the opportunity to do my Ph.D studies in Lund University and to work with people with such a strong technical knowledge. I also express my thanks to Prof. Anton Shiriaev and Dr. Leonid Freidovich for helping me in the theoretical part of the work.

I want to thank all my colleagues at the Department of Automatic Control. Over the years, you became a family to me and without your guidance this thesis would not have been possible. A special thanks to the technical and administrative staff at the department. I will never forget your kindness, care and support.

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The author is a member of the LCCC Linnaeus Center and the eLLIIT Excellence Center at Lund University.

This publication is dedicated to the memory of my best friend Alexander. He was a very brave, fair and kind young man who always supported his friends. He was a special person with a great and unique sense of humor. Alexander will always live in my heart with the beautiful smile on his face.

Эта публикация посвящается памяти моего лучшего друга Александра. Он был особенной личностью с уникальным чувством юмора. Смелый, справедливый и добрый молодой человек, готовый с любую минуту поддержать своих друзей. Александр всегда будет жить в моем сердце с прекрасной улыбкой на лице.

Alexander Rybakov (1985.06.20-2013.04.20)

Contents

List of symbols

The equation numbers refer to where the given symbol was introduced.

Symbols	
$a(j\omega)$	Denominator of the rational function (3.21)
A_c	Flow area (2.2)
a_s	Speed of sound (2.2)
a_t	Time-lag parameter of the blade passage (2.3)
$a_\phi,\ a_\psi,\ a_u$	Non-negative design weights (3.27)
$b(j\omega)$	Numerator of the rational function (3.21)
$\mathcal{A}_{cl},\ \mathcal{A}_{cl_{MG}},$	
$\mathcal{B}_{cl},\ \mathcal{B}_{cl_{MG}},$	
$\mathcal{B}_{cl_1},\ \mathcal{B}_{cl_2},$	
$\mathcal{C}_{cl_1},\ \mathcal{C}_{cl_2}$	Matrices of the closed-loop systems (3.12) and (4.4)
$A_{11},\ A_{12},\ A_{22},$	
$B_1,\ B_2,\ C$	Constant matrices of appropriate dimensions (3.3)
B	Greitzer's B-parameter (2.4)
$D,\ D_2$	Constants in Eqs. (4.10, 4.22)
e	Transformed state vector (3.3)
$G(s)$	Transfer function (3.18)
H	Compressor characteristic semi height (2.4)
$J(\phi, \psi, u)$	Optimization problem cost function (3.27)
$J_1,\ J_2$	Jacobian matrices (2.15, 2.22)
$k_2,\ k_1$	Coefficients in (5.17)
l_c	Effective flow passage length of the compressor (2.4)
$l_0,\ l_1,\ l_2$	Coefficients in the transfer function (3.18, 4.26)
L_c	Length of compressor (2.2)
m	Coefficient in the parabola (2.33)
m_{cd}	Compressor-duct flow parameter (2.3)
$p_0,\ p_1$	Coefficients in the transfer function (3.18, 4.26)
R	Squared amplitude of rotating stall (1.1)

$t_\psi,\ t_z$	Parameters of the nonlinear part of the controller (3.11)
t_{z_1}	Parameters of the nonlinear part of the controller (4.2)
$T^{\{x\}},\ T^{\{e\}},$ $T_\psi^{\{x\}},\ T_z^{\{x\}},$ $T_\phi^{\{e\}},\ T_\psi^{\{e\}},\ T_z^{\{e\}}$	Constant matrices (3.5)
u	Control variable (1.1)
U	Constant compressor tangential speed (2.2)
V_p	Compressor plenum volume (2.2)
W	Compressor characteristic semi width (2.4)
$W^{\{\phi\}}(\phi)$	MG original nonlinearity (2.29)
$W^{\{u\}}(\psi, z),$ $W^{\{z\}}(\psi, z),$ $W^{\{x\}}(Cx),$ $W^{\{e\}}(x, e)$	Static nonlinearities that resemble the nonlinearity present in the original dynamics in MG (3.2, 3.3)
x	Transformed state vector (3.3)
y	Measurement (1.1)
z	Dynamic state of the controller (3.2)

Greek letters

α	Coefficient in Eq. (3.16)
α_1, α_2	Coefficients in Eq. (2.31)
β	Positive constant $\in \mathbb{R}$ (1.1)
γ_T	Throttle gain (2.5)
$\Lambda^{\{u\}},\ \Lambda^{\{z\}}$	Constant matrices of appropriate dimensions (3.2)
$\Lambda_\psi^{\{u\}},\ \Lambda_z^{\{u\}},$ $\Lambda_\psi^{\{z\}},\ \Lambda_z^{\{z\}}$	Parameters of the linear part of the controller for the surge subsystem (3.2)
$\Lambda_{z_1}^{\{u\}},\ \Lambda_{z_2}^{\{u\}},$ $\Lambda_\psi^{\{z_1\}},\ \Lambda_{z_1}^{\{z_1\}},$ $\Lambda_{z_2}^{\{z_1\}}$	Parameters of the linear part of the controller (4.2)
σ	Positive constant $\in \mathbb{R}$ (1.1)
$\tau_1,\ \tau_2,\ \tau_3$	Coefficients in Eqs. (5.13)
ϕ	Deviation of the averaged flow from its nominal value (1.1)
$\phi_T(\psi)$	Throttle mass flow coefficient (2.4)
ψ	Deviation of the averaged pressure from its nominal value (1.1)
$\Psi(s)$	Factor of the frequency condition (3.24)

ψ_{c_0}	Shut-off value of the compressor characteristic (2.4)
$\psi_c(\phi)$	Compressor characteristic (2.4)
$\psi_s(\phi)$	Stall characteristic (2.6)
$\psi_T(\phi)$	Inverse throttle characteristic (1.2)
$\omega_u,\ \omega_z$	Parameters of the nonlinear part of the controller (3.11)
ω_{z_1}	Parameter of the nonlinear part of the controller (4.2)

Acronyms	
CC	Circle Criterion
KYP	Kalman–Yakubovich–Popov lemma
LS	Least-squares
MG	Moore-Greitzer Compressor Model
QC	Quadratic Constraint

1

Introduction

A gas compressor is a mechanical device for compressing and supplying air or other gas under a certain pressure. As an example, in this work we used an axial compressor model. In fact, the axial compressor is a multi-axial fan and it basically consists of a hull and a rotor. Rotor blades rotate between the blades fixed on the hull, which direct the gas from one stage to another [Planovsky and Nikolaev, 1990]. This kind of compressors are important components of gas turbines, aircraft jet engines, high-speed ship engines and small-scale power stations. They are also widely used in high-voltage installations in blast furnaces, in chemical and petroleum industries [Vedernikov, 1974]. In compressor operation, it is important to understand the aerodynamic response resulting from an inlet flow deviation.

1.1 Multi-Stage Axial-Flow Compressors

Compression systems exhibit aerodynamic instabilities known as a surge (or axi-symmetric stall) and rotating stall which limit the range of operation and may cause serious loss of power [Greitzer, 1976], [Moore, 1984b; Moore, 1984a]. This is unacceptable for normal operation of a compressor [Paduano et al., 2001]. A simplified model of the axial compressor and the rotating stall effect is shown in Fig. 1.1.

Dynamic stall is the local aerodynamic effect when the airfoil of the compressor blade changes the angle of attack (the angle between the flow and the leading edge of the airfoil). It can provoke a strong vortex with high velocity airflows [McCroskey, 1982], [de Jager, 1995].

The second type of instability, surge, is a non-axisymmetric overall oscillation of the flow in the machine, which can result in high levels of vibration or even total destruction of the compressor [Kerrebrock, 1992]. The surge in axial compressors is not yet fully understood. It is difficult to distinguish between stall and surge because one instability can lead to the other [Saravanamuttoo et al., 2009].

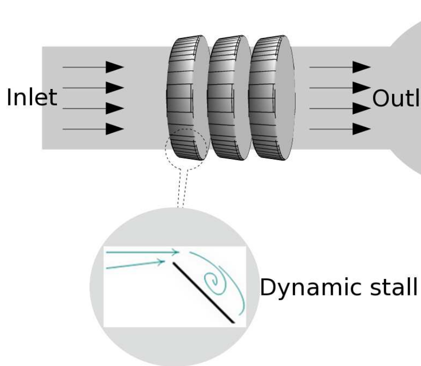

Figure 1.1 The simplified model of a compressor. On the zoomed part of the picture the convoluted blue lines represent the airflow with a strong vortex.

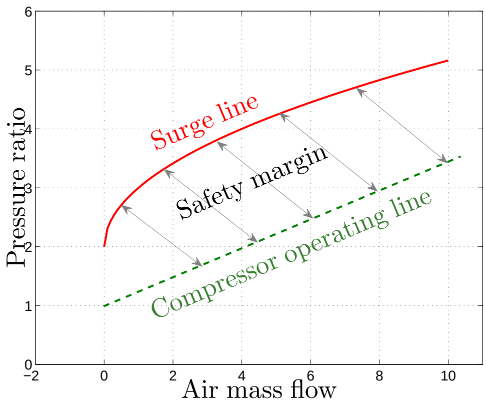

Figure 1.2 Compressor operating map. In normal conditions, the compressor is operated below the surge line.

An inlet flow distortion may cause failure of the blades to produce required duty. This increases the risk to have rotating stall or even surge, or a combination of them [Moore, 1984b; Moore, 1984a], [Greitzer, 1976], [Ng et al., 2007].

Both of these phenomena can be considered as separate aerodynamic effects (rotating stall is local to the blades and surge involves the entire compression system) but they cannot be tolerated during operation.

The instabilities are usually evaded by running the compressor below the surge line with a large enough safety margin (or surge margin), as schematically shown in Fig. 1.2. Unfortunately, this surge avoidance method can reduce the effectiveness of the machine [Garnier et al., 1991].

There are a lot of active surge/stall control techniques (e.g., air injection, bleed valves, throttle valves, close-coupled valve etc.) [Gravdahl, 1998], [Bianchi et al., 2013]. Additionally, one can find nonlinear control experiments results and a list of advices for a compressor design that are based on experiences to avoid surge [Japikse and Baines, 1997], [Fontaine et al., 2004]. The design methods and their applications are different but they all have disturbance rejection capabilities and robustness [Gravdahl, 1998], [Hagen et al., 2004], [Chaturvedi and Bhat, 2006]. However, studies of aerodynamic instabilities still have remaining challenges.

The search for a large family of robust globally stabilizing controllers for Eq. (1.1) has led to the methodology from [Shiriaev et al., 2010; Shiriaev et al., 2009; Shiriaev et al., 2005] which is an alternative to the classical observer-based design.

1.2 The Moore-Greitzer Compressor Model

Many previous contributions are devoted to the mathematical description of surge in a compressor [Tsabenko et al., 2010], [Abed et al., 1990], [Gravdahl, 1998], [Rasmussen and Jakobsen, 2000]:

- 1976 - the compression system model of Greitzer (for axial compressor) [Greitzer, 1976];
- 1981 - Hansen showed that the Greitzer mathematical model for axial compressors is also applicable to centrifugal compressors [Hansen et al., 1981];
- 1986 - The Moore-Greitzer extended compressor model was derived [Moore and Greitzer, 1986];
- 1992 - The Greitzer model was improved for variable speed centrifugal compressors by Fink [Fink et al., 1992].

There are many other models of different types of compressors described in the literature, but in this work we will focus on the Moore-Greitzer (MG) model.

In 1986 Moore and Greitzer published a differential equation model describing the airflow through the compression system in turbomachines (such as gas turbines, fans, etc.). In this work we are considering the three-state Moore-Greitzer compressor model obtained by the PDEs approximation using the Galerkin procedure [Moore and Greitzer, 1986], [Gravdahl, 1998].

The MG model includes the differential equations:

$$
\begin{aligned}
\frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1-(1+\phi)^3}{2} - 3R(1+\phi) \\
\frac{d}{dt}\psi &= \frac{1}{\beta^2}(\phi - u) \\
\frac{d}{dt}R &= -\sigma R^2 - \sigma R(2\phi + \phi^2),\ R(0) > 0 \\
y &= \psi
\end{aligned}
\tag{1.1}
$$

where:

- u is the control variable. In the MG compressor model, it is assumed that the flow is controlled by a throttle at the plenum exit [Moore and Greitzer, 1986]. Or, more precisely, the control signal u is a deviation of the coefficient of the inverse throttle characteristic from the nominal value. The inverse throttle characteristic is [Gravdahl, 1998]

  $$\psi_T(\phi) = \phi_T^{-1}(\phi) = \frac{1}{\gamma_T^2}\phi^2 \tag{1.2}$$

  where $\gamma_T \geq 0$ is the throttle gain.

  Usually, the physical interpretation of the control signal is dependent upon the design of the compressor, the chosen type of control strategy and the compressor application. For high speed, throttle control is often used for compressors and stage choke control, temperature or fuel control for single shaft turbines. For low speed it may be interpreted as a variable geometry in combination with bleed valves. We assume that u is a deviation of the coefficient of the inverse throttle characteristic from the nominal value.

  Since we only consider the mathematical model of an axial compressor, the real throttle configuration is unknown. Thus, in order to link the control variable to the specific operational characteristics of a particular compressor system, an additional dedicated study should be performed;

- y is the measurement (we consider that only pressure is available);

- ϕ is the deviation of the averaged flow from its known nominal value;

- ψ is the deviation of the averaged pressure from its known nominal value;

- the parameters $\{\beta; \sigma\} \in \mathbb{R}$ are positive constants;

- R is the Galerkin approximation of rotating stall squared amplitude. The fully developed surge is not viewed as an unsteady situation but as a set of equilibria along which R is nonzero [Paduano et al., 2001]. The term "unsteady" is equivalent to the term "transient". The only difference is that the term "transient" implies that the solutions will reach the steady state while the term "unsteady" expects their constant variation.

  It was shown in [Shiriaev et al., 2004] and [Shiriaev et al., 2005] that for any solutions to Eq. (1.1) the positive variable $R(t)$ never leaves the range between zero and one.

The Moore-Greitzer compressor model of Eq. (1.1) with the stall dynamics $R \equiv 0$ is called surge subsystem

$$\begin{aligned} \frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1-(1+\phi)^3}{2} \\ \frac{d}{dt}\psi &= \frac{1}{\beta^2}(\phi - u) \\ y &= \psi \end{aligned} \tag{1.3}$$

This model of Eq. (1.3) is also known as the Greitzer model [Moore and Greitzer, 1986]. In this thesis, we will consider controller design for the surge subsystem as an important subproblem for the full MG case. We also show simulation results of the closed-loop system with the same controller when applied to the MG compressor model.

In the sequel, we want to control the compressor dynamics to the desired set-point $(\phi,\ \psi,\ R) = (0,\ 0,\ 0)$.

As it can be seen in Fig. 1.3 and Fig. 1.4, already the simplified second order system (i.e., without considering stall dynamics, $R = 0$) shows considerable oscillatory behavior. The convergence of the output signals is too slow to be useful for a real compressor operation.

The Moore-Greitzer model is a model for flow through blade passages (or, roughly speaking, the air path through the blades) in the compressor [Greitzer, 1976], [Moore and Greitzer, 1986], [Birnir et al., 2007]. The model of Eq. (1.1) has been successful at predicting experimental data [Willems and Jager, 1999], [Garnier et al., 1991].

The Moore-Greitzer compressor model is a so-called "Gray-box" model [Rasmussen and Jakobsen, 2000]. That means that the system representation contains equations describing physical phenomena from fundamental equations but with parameters to be identified. The models of the top category ("White-box") require the highest level of knowledge and they were not yet developed for compressors.

All existing compressor models have certain limitations. Some limitations of the Moore-Greitzer compressor model are [Tsabenko et al., 2010]:

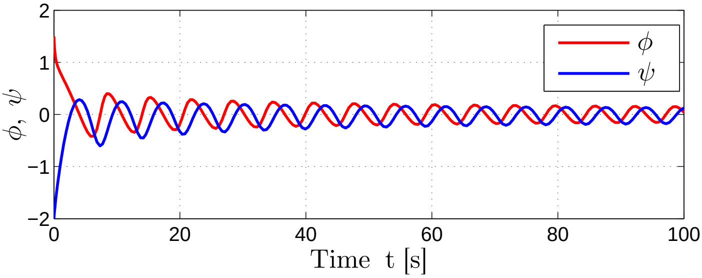

Figure 1.3 Simulation results of the simplified second order system of Eq. (1.3) (i.e., without considering stall dynamics, $R = 0$) shows considerable oscillatory behavior during deep surge. The control signal u is a deviation of the coefficient of the inverse throttle characteristic from the nominal value. For the simulation the throttle gain γ_T is chosen equal to 0.01 or, in other words, the throttle is almost closed that results in a compressor surge.

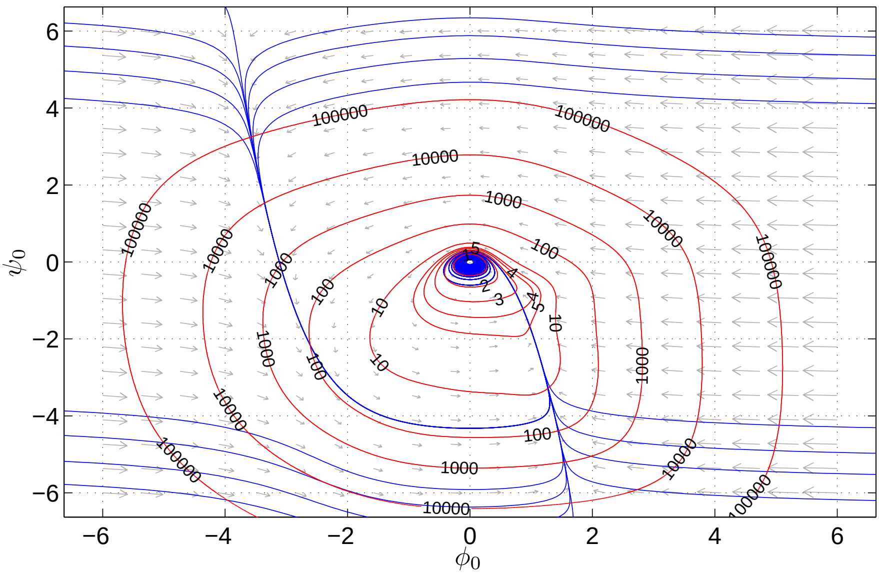

Figure 1.4 The phase plane of the surge subsystem of Eq. (1.3) shows the spiral sink. Hence, the decaying oscillation in the system is expected. For the simulation the throttle gain γ_T is chosen equal to one.

- the presence of an anti-surge valve is not taking into account;
- the dynamics of the compressor when bypassing a part of the compressed air from the compressor discharge (outlet) to the compressor suction (inlet) are not included. But many types of air compressors required such information, for example, rotary screw compressors.

Despite the simplicity of the MG model, it captures several interesting nonlinear effects and offers challenges for dynamic output feedback design.

To summarize the challenges to stabilize the origin of the three-state Moore-Greitzer model, we have [Shiriaev et al., 2009], [Shiriaev et al., 2010], [Planovsky and Nikolaev, 1990] [Rubanova et al., 2013]:

- the linearized dynamics are not stabilizable;
- the fact that ϕ and R cannot be measured or used for feedback design;
- the presence of a non-globally Lipschitz cubic nonlinearity;
- the nonlinearity in ϕ-dynamics is known only approximately (for example, the third order of the nonlinearity is an approximation).

Although static feedback does not stabilize the system, it was shown that dynamic output feedback control could stabilize the Moore-Greitzer model [Shiriaev et al., 2005]. The description of a recently developed procedure for dynamic output feedback design of systems with nonlinearities satisfying quadratic constraints is presented in this work [Shiriaev et al., 2009], [Shiriaev et al., 2010] and [Shiriaev et al., 2005].

1.3 Problem Statement

Conditions for synthesis of the stabilizing controllers presented previously have to be simplified and applied. The design of robust stable control is still an open problem. Robustness to stall dynamics and further characterization of the control parameterization are necessary.

1.4 Contributions

The main contributions of this work are:

- simplified conditions for the controller synthesis;
- a numerical search procedure for the controller parameter search;
- a large set of stabilizing controllers is presented;

- a feasibility study of optimization over the control parameter set;
- heuristic robustness of controller parameters;
- the existence of the unique stable equilibrium in origin for two different types of controllers is proved.

1.5 Publications

These papers[1] were mainly used in Chapters 3, 5:

Andersson, A., A. Robertsson, A. Shiriaev, and R. Johansson (2014). "Robustness of the Moore-Greitzer compressor model's surge subsystem with new dynamic output feedback controllers". In: *19th IFAC World Congress (IFAC2014).* Cape Town, South Africa, 24-29 August 2014, pp. 3690–3695.

Rubanova, A., A. Robertsson, A. Shiriaev, L. Freidovich, and R. Johansson (2013). "Analytic parameterization of stabilizing controllers for the surge subsystem of the Moore-Greitzer compressor model". In: *American Control Conference (ACC2013).* Washington, D.C., USA, 17-19 June 2013, pp. 5257–5262.

Related publication:

Shiriaev, A, L Freidovich, R Johansson, A Robertsson, and A Andersson (2015). "IQC arguments for analysis of the 3-state moore-greitzer compressor system". In: *1st IFAC Conference on Modelling, Identification and Control of Nonlinear Systems (MICNON 2015).* Saint-Petersburg, Russia, June 23-26, 2015. Accepted for publication.

[1] Rubanova is the author's maiden name.

2

Background

This work is based on control design for systems with nonlinearities satisfying quadratic constraints, see e.g., [Yakubovich et al., 2004], [Megretski and Rantzer, 1997], [Shiriaev et al., 2010]. The stability criteria rely on structural properties of nonlinearities of the system of Eq. (1.1). Quadratic constraints are presented in the form of sector conditions [Khalil, 2002].

In this chapter we will also determine equilibria of the MG compressor model. One of the important parts is the description of some physical characteristics of the compressor in order to understand some connections between the mathematical model and the real compressor.

2.1 Equilibrium of the Moore-Greitzer Compressor Model

The stability of the Moore-Greitzer compressor model equilibria was described in the doctoral thesis [Gravdahl, 1998]. Here we will introduce only the diagram with some compressor characteristics. The constants β, σ from the 3-state MG compressor model of Eq. (1.1) originally represent many physical characteristics of compressors. In this work we will use the simplified version of the MG compressor model and the fact that those constants are positive. For more details we refer to Gravdahl's thesis. Let us show how the MG model represents the compressor physics. Another version of the MG model of Eq. (1.1) is given by [Moore and Greitzer, 1986]

$$
\begin{aligned}
\frac{d}{dt}\psi &= \frac{W/H}{4B^2}\left(\frac{\phi}{W} - \frac{1}{W}\phi_T(\psi)\right)\frac{H}{l_c} \\
\frac{d}{dt}\phi &= \frac{H}{l_c}\left(-\frac{\psi-\psi_{c_0}}{H} - \frac{1}{2}\left(\frac{\phi}{W}-1\right)^3 + 1 + \frac{3}{2}\left(\frac{\phi}{W}-1\right)\left(1-\frac{R}{2}\right)\right) \\
\frac{d}{dt}R &= R\sigma\left(1-\left(\frac{\phi}{W}-1\right)^2 - \frac{R}{4}\right)
\end{aligned}
\tag{2.1}
$$

where

- $\phi_T(\psi)$ is the throttle mass flow coefficient;
- l_c is the effective flow passage length of the compressor;
- H is the height and W is the width of the cubic characteristic as shown in Fig. 2.1;
- B is Greitzer's B-parameter defined in [Greitzer, 1976] as

$$B = \frac{U}{2a_s}\sqrt{\frac{V_p}{A_c L_c}} \tag{2.2}$$

  where

  - U is the constant compressor tangential speed (mean rotor velocity);
  - a_s is the speed of sound;
  - V_p is the compressor plenum volume;
  - A_c is the flow area;
  - L_c is the length of compressor.

  The parameter β in the MG compressor model of Eq. (1.1) represents the simplified version of Greitzer's B-parameter and is assumed to be equal to one;

- σ is the positive constant defined as

$$\sigma = \frac{3a_t H}{(1 + m_{cd} a_t)W} \tag{2.3}$$

  where m_{cd} is the compressor-duct flow parameter and a_t is the time-lag parameter of the blade passage;

- $\phi,\ \psi,\ R$ are the same as in the MG compressor model of Eq. (1.1).

Basically, the models of Eq. (1.1) and Eq. (2.1) are similar, only the presence of the parameter l_c and the factor 4 are a result of a different method of normalizing pressure and time during the MG model derivation [Willems, 1997].

The pressure rise is included in the second equation $d\phi/dt$ of Eq. (2.1) and presented as a nonlinear function of the mass flow. This function is also known as the compressor characteristic

$$\psi_c(\phi) = \psi_{c_0} + H\left(1 + \frac{3}{2}\left(\frac{\phi}{W} - 1\right) - \frac{1}{2}\left(\frac{\phi}{W} - 1\right)^3\right) \tag{2.4}$$

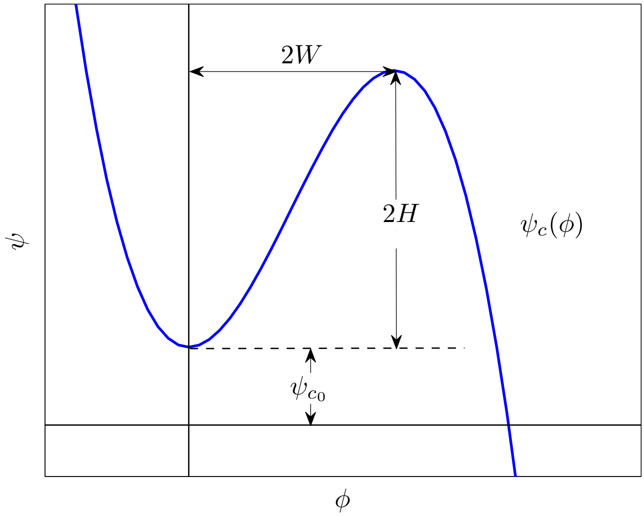

Figure 2.1 The cubic compressor characteristic in [Moore and Greitzer, 1986]. Here H is the height and W is the width of the cubic characteristic, ψ_{c_0} is the shut-off value of the compressor characteristic, ϕ is the averaged mass flow coefficient and ψ is the plenum pressure. The scale is not given because we are not working with some specific compressor, where the individual parameters H, W, ψ_{c_0} are known.

where the constant $\psi_{c_0} > 0$ being the shut-off value of the compressor characteristic. For more explanation of the constants we refer to [Gravdahl, 1998], [Gravdahl and Egeland, 1999].

The model input $\phi_T(\psi)$ is regulated by the throttle area $\gamma_T \geq 0$ (or throttle gain). The throttle gain is proportional to throttle opening, in other words, the throttle gain adjusts the operating point of the system (so-called throttle control) [Giarré et al., 2006], [Gravdahl, 1998], [Willems and Jager, 1999]. The compressor throttle characteristic that describes the throttle mass flow is

$$\phi_T(\psi) = \gamma_T \sqrt{\psi} \tag{2.5}$$

and this means that the throttle is closed ($\phi_T(\psi) = 0$) if $\gamma_T = 0$.

The stability of the stationary solutions for $R = 0$ of the Eq. (2.1) depends of the γ_T and the Greitzer surge parameter B (for larger values of the throttle gain these solutions are stable) [Humbert and Krener, 1998].

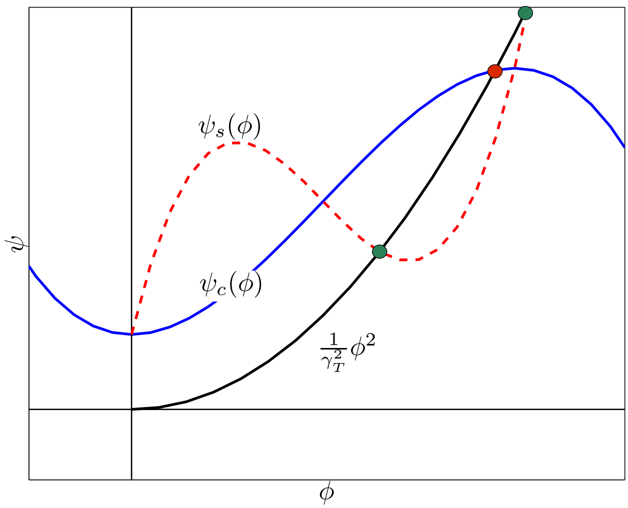

Figure 2.2 The inverse throttle characteristic ϕ^2/γ_T^2 intersects the cubic characteristic of the pressure rise $\psi_c(\phi)$ of Eq. (2.4) in a part of the positive slope (unstable area of the compressor map). That means that the compressor may enter into rotating stall or surge. The equilibrium values ϕ_0 and ψ_0 are defined by the intersection of the throttle characteristic and the stall characteristic $\psi_s(\phi)$. The scale is not given because we do not have a concrete compressor, where the individual parameters H, W, ψ_{c0}, γ_T are given by design.

The stall characteristic is found by analyzing the stall equation dR/dt of Eq. (2.1) and shown in [Gravdahl, 1998] as

$$\psi_s(\phi) = \psi_{c_0} + H\left(1 - \frac{3}{2}\left(\frac{\phi}{W} - 1\right) + \frac{5}{2}\left(\frac{\phi}{W} - 1\right)^3\right) \tag{2.6}$$

where the constant $\psi_{c_0} > 0$ is the same as in the Eq. (2.4) and represents the shut-off value of the compressor characteristic.

Let us show the standard chart of the operating area of compressors. As it can be seen in Fig. 2.2, the inverse throttle characteristic $\phi_T^{-1}(\phi)$ intersects the cubic characteristic of the pressure rise $\psi_c(\phi)$ of Eq. (2.4) in a part of the positive slope (unstable area of the compressor map). It was also shown in [Gravdahl and Egeland, 1999] that operating points situated at the positive compressor characteristic slope are unstable. That means that the compressor may enter into rotating stall or surge. The equilibrium values ϕ_0 and ψ_0 are defined by the intersection of the throttle characteristic and the stall characteristic $\psi_s(\phi)$.

We will find the equilibria of the Moore-Greitzer compressor model of Eq. (1.1) for constant $u = u_0$:

$$\begin{aligned} 0 &= -\psi_0 + \frac{3}{2}\phi_0 + \frac{1-(1+\phi_0)^3}{2} - 3R_0(1+\phi_0) \\ 0 &= \frac{1}{\beta^2}(\phi_0 - u_0) \\ 0 &= -\sigma R_0^2 - \sigma R_0(2\phi_0 + \phi_0^2) \end{aligned} \tag{2.7}$$

then equilibria expressed in terms of u_0 are:

$$\begin{aligned} \psi_0 &= \frac{3}{2}u_0 + \frac{1-(1+u_0)^3}{2} - 3R_0(1+u_0) \\ \phi_0 &= u_0 \\ R_{0_1} &= 0 \end{aligned} \tag{2.8}$$

or

$$\begin{aligned} \psi_0 &= \frac{3}{2}u_0 + \frac{1-(1+u_0)^3}{2} - 3R_0(1+u_0) \\ \phi_0 &= u_0 \\ R_{0_2} &= 1-(u_0+1)^2 \end{aligned} \tag{2.9}$$

Simplification gives the equilibria:

$$\begin{cases} \phi_0 = u_0 \\ \psi_0 = -\dfrac{1}{2}u_0^2(u_0+3) \\ R_0 = 0 \end{cases} \quad \text{or} \quad \begin{cases} \phi_0 = u_0 \\ \psi_0 = \dfrac{5}{2}u_0^3 + \dfrac{15}{2}u_0^2 + 6u_0 \\ R_0 = 1-(u_0+1)^2 \end{cases}$$

The condition $0 \leq R \leq 1$ restricts the second equilibrium to exist only for $-2 < u_0 < 0$.

We will now show the derivation procedure and stability information of the equilibria.

For the Moore-Greitzer model of Eq. (1.1), the stall equilibrium equation gives

$$0 = -\sigma R_0^2 - \sigma R_0(2\phi_0 + \phi_0^2) \tag{2.10}$$

that gives two solutions for R_0

$$\begin{aligned} R_{0_1} &= 0 \\ R_{0_2} &= 1-(\phi_0+1)^2 \end{aligned} \tag{2.11}$$

Since $0 \leq R \leq 1$, the second equation in the system of equations of Eq. (2.11) is valid for all $-2 < \phi_0 < 0$. Then the solutions for other equations are:

1. for $R_{0_1} = 0$:
   - The equilibrium equation for the deviation of the flow from respective nominal value is

$$\begin{aligned} 0 &= -\psi_0 + \frac{3}{2}\phi_0 + \frac{1-(1+\phi_0)^3}{2} \\ \psi_0 &= -\frac{1}{2}\phi_0^2(\phi_0+3) \end{aligned} \tag{2.12}$$

   - The equilibrium equation for the deviation of the pressure from respective nominal value is:

$$\begin{aligned} 0 &= \frac{1}{\beta^2}(\phi_0 - u_0) \\ \phi_0 &= u_0 \end{aligned} \tag{2.13}$$

     where u_0 is a constant.

   Then the equilibrium for the surge subsystem of the Moore-Greitzer compressor model is

$$\begin{aligned} \phi_0 &= u_0 \\ \psi_0 &= -\frac{1}{2}u_0^2(u_0+3) \\ R_0 &= 0 \end{aligned} \tag{2.14}$$

   The Jacobian matrix with the solution to Eq. (2.14) is then

$$J_1 = \begin{bmatrix} -\frac{3}{2}u_0^2 - 3u_0 & -1 & -3u_0 - 3 \\ 1 & 0 & 0 \\ 0 & 0 & -u_0^2 - 2u_0 \end{bmatrix} \tag{2.15}$$

   with the characteristic polynomial

$$det(sI_3 - J_1) = s^3 + \frac{5}{2}(u_0^2 + 2u_0)s^2 + (\frac{3}{2}u_0^4 + 6u_0^3 + 6u_0^2 + 1)s + u_0^2 + 2u_0 \tag{2.16}$$

   According to the Routh–Hurwitz criterion for third-order polynomials the system with characteristic equation of Eq. (2.16) is stable (or all the roots are in the left half plane) if all the coefficients satisfy [Khalil, 2002, p.52]:

$$\begin{aligned} 0 &< u_0^2 + 2u_0 \\ 0 &< \frac{3}{2}u_0^4 + 6u_0^3 + 6u_0^2 + 1 \\ 0 &< \frac{5}{2}(u_0^2 + 2u_0)(\frac{3}{2}u_0^4 + 6u_0^3 + 6u_0^2 + 1) - u_0^2 - 2u_0 \end{aligned} \tag{2.17}$$

which is valid for

$$u_0 > 0, \quad u_0 < -2 \tag{2.18}$$

The characteristics of the equilibria of the surge subsystem of the MG compressor model are as follows:

- $u_0 > 0$ or $u_0 < -2$ the equilibrium is asymptotically stable;
- $-2 < u_0 < 0$ the equilibrium is unstable;
- $u_0 = 0$ or $u_0 = -2$ is the saddle-node bifurcation (or non-hyperbolic equilibrium). Small perturbations can cause a bifurcation of the equilibrium (the equilibrium point can disappear or split into many equilibria) [Khalil, 2002].

2. for $R_{0_2} = 1 - (\phi_0 + 1)^2$:

   - The equilibrium equation for the deviation of the flow from respective nominal value is

$$\begin{aligned} 0 &= -\psi_0 + \frac{3}{2}\phi_0 + \frac{1-(1+\phi_0)^3}{2} - 3R(1+\phi) \\ \psi_0 &= \frac{5}{2}\phi_0^3 + \frac{15}{2}\phi_0^2 + 6\phi_0 \end{aligned} \tag{2.19}$$

   - The equilibrium equation for the deviation of the pressure from respective nominal value is the same as for $R_{0_1} = 0$:

$$\phi_0 = u_0 \tag{2.20}$$

   where u_0 is a constant.

Then the equilibrium for the surge subsystem of the Moore-Greitzer compressor model is

$$\begin{aligned} \phi_0 &= u_0 \\ \psi_0 &= \frac{5}{2}u_0^3 + \frac{15}{2}u_0^2 + 6u_0 \\ R_0 &= 1 - (u_0 + 1)^2 \end{aligned} \tag{2.21}$$

The Jacobian matrix with the solution to Eq. (2.21) is then

$$J_2 = \begin{bmatrix} \frac{3}{2}u_0^2 + 3u_0 & -1 & -3u_0 - 3 \\ 1 & 0 & 0 \\ 2u_0^3 + 6u_0^2 + 4u_0 & 0 & u_0^2 + 2u_0 \end{bmatrix} \tag{2.22}$$

with the characteristic polynomial

$$\begin{aligned}\det(sI_3 - J_2) = s^3 + \frac{5}{2}\left(-u_0^2 - 2u_0\right) s^2 \\ + \left(\frac{15}{2}u_0^4 + 30u_0^3 + 36u_0^2 + 12u_0 + 1\right) s - u_0^2 - 2u_0\end{aligned} \tag{2.23}$$

The third-order system with characteristic polynomial of Eq. (2.23) is stable if all the coefficients satisfy:

$$\begin{aligned}&0 < -u_0^2 - 2u_0 \\ &0 < \frac{15}{2}u_0^4 + 30u_0^3 + 36u_0^2 + 12u_0 + 1 \\ &0 < \frac{5}{2}\left(-u_0^2 - 2u_0\right)\left(\frac{15}{2}u_0^4 + 30u_0^3 + 36u_0^2 + 12u_0 + 1\right) + u_0^2 + 2u_0\end{aligned} \tag{2.24}$$

which is valid for

$$\begin{aligned}-2 <&u_0 < -1.9395 \\ -1.5631 <&u_0 < -0.4368 \\ -0.0604 <&u_0 < 0\end{aligned} \tag{2.25}$$

The characteristics of the equilibria of the MG compressor model are as follows:

- the equilibrium is asymptotically stable $\forall u_0$ in:

$$\begin{aligned}-2 <&u_0 < -1.9395 \\ -1.5631 <&u_0 < -0.4368 \\ -0.0604 <&u_0 < 0\end{aligned} \tag{2.26}$$

- the equilibrium is unstable $\forall u_0$ in:

$$\begin{aligned}-1.9395 <&u_0 < -1.5631 \\ -0.4368 <&u_0 < -0.0604 \\ -0.0604 <&u_0 < 0 \\ &u_0 < -2 \\ &u_0 > 0\end{aligned} \tag{2.27}$$

- $u_0 = 0$ or $u_0 = -2$ is a saddle-node bifurcation and

$$u_0 = \begin{cases} -1.9395 \\ -1.5631 \\ -0.4368 \\ -0.0604 \end{cases} \tag{2.28}$$

  are saddle points.

2.2 Structural Properties of Nonlinearities of the System

The stability criteria used in this work rely on properties of nonlinearities [Shiriaev et al., 2010]. In this section we will show structural properties of nonlinearities of the system of Eq. (1.1) by quadratic constraints in the form of sector conditions. A procedure for the derivation of the QC is also presented. We introduce the following notation for the nonlinearity in Eq. (1.3)

$$W^{\{\phi\}}(\phi) := 1 - (1+\phi)^3 \tag{2.29}$$

The graphical interpretation of the nonlinearities of the Moore-Greitzer compressor model is shown in Fig. 2.3.

The QC for the nonlinearity of Eq. (2.29) is

$$-\phi(W^{\{\phi\}}(\phi)) \geq \frac{3}{4}\phi^2 \tag{2.30}$$

We will now show that the inequality of Eq. (2.30) holds.

- We will use the Circle criterion [Khalil, 2002]

$$\begin{aligned} \alpha_1\phi^2 \leq \phi W^{\{\phi\}}(\phi) \leq \alpha_2\phi^2 \\ (\alpha_2\phi - W^{\{\phi\}}(\phi))(W^{\{\phi\}}(\phi) - \alpha_1\phi) \geq 0 \end{aligned} \tag{2.31}$$

  or

$$-\alpha_1\alpha_2\phi^2 - (W^{\{\phi\}}(\phi))^2 \geq -(\alpha_1+\alpha_2)\phi W^{\{\phi\}}(\phi) \tag{2.32}$$

  where the coefficients $\alpha_1,\ \alpha_2$ represent the slopes of two lines forming the boundary sector for the original nonlinearity of Eq. (2.29).

- The right side of the expression of Eq. (2.32) is the 4th-order function that can be bounded from below by a second-order function (or parabola) of a form $m\phi^2$, where $m \in \mathbb{R}$ is the parabola coefficient. We need to find the contact points with the parabola.

$$\begin{aligned} -(\alpha_1+\alpha_2)\phi W^{\{\phi\}}(\phi) = m\phi^2 \\ -\phi(1-(1+\phi)^3) = \frac{m}{(\alpha_1+\alpha_2)}\phi^2 \\ \phi^2 + 3\phi + \left(3 - \frac{m}{(\alpha_1+\alpha_2)}\right) = 0 \end{aligned} \tag{2.33}$$

  The real roots of this expression are obtained when

$$\sqrt{9 - 4(3 - m/(\alpha_1+\alpha_2))} = 0 \tag{2.34}$$

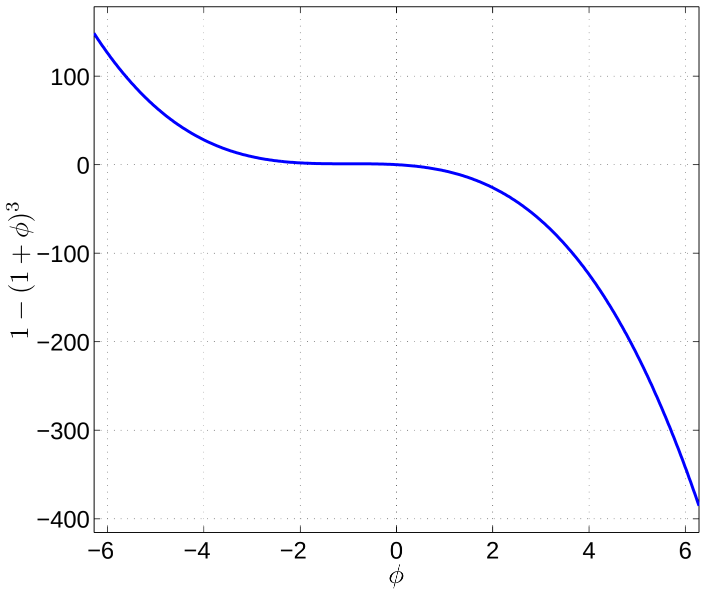

Figure 2.3 The nonlinearity $W^{\{\phi\}}(\phi)$ of the Moore-Greitzer compressor model of Eq. (1.1).

which is possible if $m/(\alpha_1+\alpha_2) \geq 3/4$. Then the constraint for the nonlinearity of Eq. (2.29) is

$$-\phi(W^{\{\phi\}}(\phi)) \geq \frac{3}{4}\phi^2 \tag{2.35}$$

such that

$$-\phi(W^{\{\phi\}}(\phi)) - \frac{3}{4}\phi^2 = \phi^2(\phi+\frac{3}{2})^2 \geq 0 \tag{2.36}$$

The graphical interpretation of such a constraint is shown in Fig. 2.4.

As a result, we know that the static nonlinearity of Eq. (2.29) satisfies the quadratic constraint of Eq. (2.30). This QC is relatively simple and closest to the quadratic form of Eq. (2.32) of the given nonlinearity.

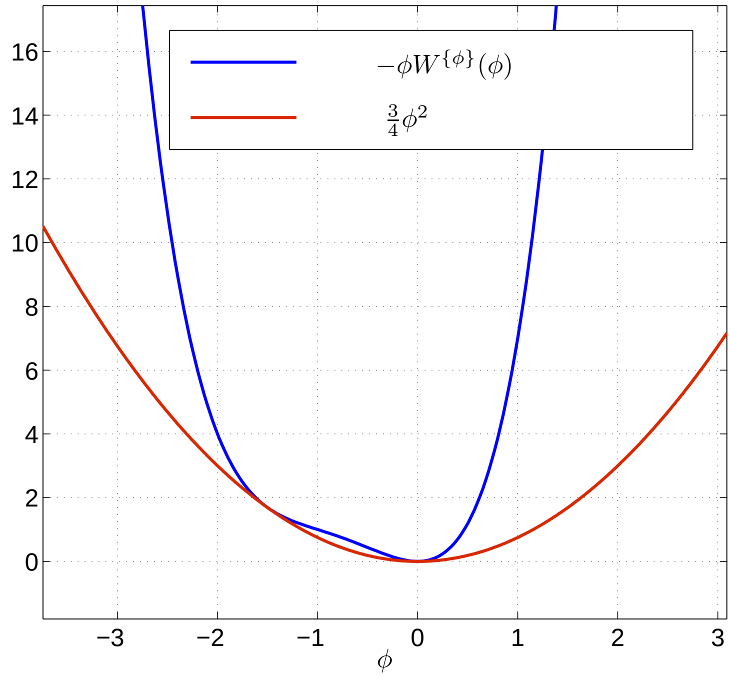

Figure 2.4 The right side of the QC of Eq. (2.30) (or parabola) is presented by the red line and the left side of the QC of Eq. (2.30) is presented by the blue line.

3

Analytical Parameterization of Stabilizing Controllers: Design I

The search for a large family of robust globally stabilizing controllers for Eq. (1.1) has led to the methodology presented in [Shiriaev et al., 2010; Shiriaev et al., 2009; Shiriaev et al., 2005]. In this chapter we introduce a short description of the control method. First, we show the parametric set of controllers and then we will rewrite the closed-loop system into a specific block-form that will simplify the determination of the controller parameters. Second, we show the application of the method and some important simplifications of the procedure.

3.1 Parametric Set of Controllers

First we will introduce and apply the control method to the surge subsystem of Eq. (1.3) of the Moore-Greitzer compressor model of Eq. (1.1).

Consider the general form of a dynamic output feedback control law

$$u = \mathcal{U}(z, y), \quad \dot{z} = \mathcal{F}(z, y) \tag{3.1}$$

where $\mathcal{U}(\cdot)$ and $\mathcal{F}(\cdot)$ are smooth functions of appropriate dimensions and $z \in \mathbb{R}^n$. The linear and nonlinear parts in the control law will be separated.

A class of output feedback controllers on the form of Eq. (3.1) for the systems of Eqs. (1.1–1.3) was introduced in [Shiriaev et al., 2009]. The family of output feedback controllers has the following structure

$$
\begin{aligned}
u &= \Lambda^{\{u\}} \begin{bmatrix} \psi \\ z \end{bmatrix} + W^{\{u\}}(\psi, z) \\
\frac{d}{dt} z &= \Lambda^{\{z\}} \begin{bmatrix} \psi \\ z \end{bmatrix} + W^{\{z\}}(\psi, z) \\
\Lambda^{\{u\}} &= \begin{bmatrix} \Lambda_{\psi}^{\{u\}} & \Lambda_{z}^{\{u\}} \end{bmatrix} \\
\Lambda^{\{z\}} &= \begin{bmatrix} \Lambda_{\psi}^{\{z\}} & \Lambda_{z}^{\{z\}} \end{bmatrix}
\end{aligned}
\tag{3.2}
$$

where we will later choose $\Lambda^{\{u\}}$ and $\Lambda^{\{z\}}$ as constant matrices of appropriate dimensions and $W^{\{u\}}(\cdot,\cdot)$ and $W^{\{z\}}(\cdot,\cdot)$ as static nonlinearities that should resemble the nonlinearity present in the original dynamics of Eq. (1.1).

3.2 A Class of Dynamical Systems with Dynamic Output Feedback Controllers

The closed-loop system will be transformed into a block form. After changing the coordinates according to [Shiriaev et al., 2010], the closed-loop system can be rewritten as

$$
\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \\
&\quad + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} W^{\{x\}}(Cx) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} W^{\{e\}}(x, e)
\end{aligned}
\tag{3.3}
$$

where the combined components of x and e define the new state vector, $A_{11},\ A_{12},\ A_{22},\ B_1,\ B_2$ and C are constant matrices of appropriate dimensions, $W^{\{x\}}(\cdot)$ and the $W^{\{e\}}(\cdot)$ are static nonlinearities that resemble the nonlinearity present in the original dynamics [Shiriaev et al., 2010]. The detailed description of these components of the block form Eq. (3.3) will be presented in the following chapter. The discussion about the matching conditions that make the closed-loop system equivalent to a dynamical system of the suitable block form in Eq. (3.3) is presented in [Shiriaev et al., 2009] and will be briefly shown in the next chapter.

The motivation for the decomposition of the state of the original closed-loop system into the block form in Eq. (3.3) is the following [Shiriaev et al., 2010]:

- the vector x (or known subsystem) is composed from the measured state ψ and some known and transformed states of the dynamic feedback controller. In our case with $z \in \mathbb{R}^n$ of Eq. (3.2) this vector x will be related to $[\psi; z_n]$. The corresponding relation will be shown below.

- the vector e includes the same known states, the unknown state and some components that may be interpreted as a state of the error dynamics of a reduced-order observer for the unmeasured state ϕ (deviation of the averaged flow). Therefore this state vector will be described by a transformation of $[\phi; \psi; z_n]$. As for the vector x, the corresponding relation will also be shown below.

The nonlinear system of Eq. (3.3) is quadratically stable if there are matrices $\mathcal{P} = \mathcal{P}^T > 0$ and $\mathcal{Q} = \mathcal{Q}^T > 0$ such that along any solution $[x(t); e(t)]$ of the given nonlinear system we have [Gelig et al., 1978], [Shiriaev et al., 2010]

$$\frac{d}{dt}\left(\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}^T \mathcal{P} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}\right) < -\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}^T \mathcal{Q} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{3.4}$$

As presented in [Shiriaev et al., 2010] sufficient conditions for stabilization for the surge subsystem of Eq. (1.3) are relying on quadratic stability of the x- and e-subsystems of Eq. (3.8).

The conditions that allow rewriting the closed-loop system in the special block form Eq. (3.3) follow from the transformations

$$\begin{aligned} x &= T^{\{x\}} \begin{bmatrix} \psi \\ z_n \end{bmatrix} = T^{\{x\}}_{\psi} \psi + T^{\{x\}}_{z} z_n \\ e &= T^{\{e\}} \begin{bmatrix} \phi \\ \psi \\ z_n \end{bmatrix} = T^{\{e\}}_{\phi} \phi + T^{\{e\}}_{\psi} \psi + T^{\{e\}}_{z} z_n \end{aligned} \tag{3.5}$$

where $T^{\{x\}}$ and $T^{\{e\}}$ are constant matrices of appropriate dimensions. We will show the explicit transformation of the closed-loop system and the corresponding necessary matching conditions. These parameterizations will be the basis of calculation of stabilizing controllers.

For design we will first consider the two subsystems

$$\frac{d}{dt}x = A_{11}x + B_1 W^{\{x\}}(Cx) + A_{12}e(t) \tag{3.6}$$

where $e(t)$ is assumed $\equiv 0$, and separately,

$$\frac{d}{dt}e = A_{22}e + B_2 W^{\{e\}}(x(t), e) \tag{3.7}$$

with $x(t)$ being considered an unknown signal. We will use the following notation for the static nonlinearity from the vector x that satisfies the above given system decomposition: $W^{\{x\}}(Cx) = W(\psi, z)$. The nonlinearity entering the differential equation for the vector e is $W^{\{e\}}(x, e) = W^{\{\phi\}}(\phi) - W(\psi, z)$. More detailed calculations were presented in [Shiriaev et al., 2010]. The form of the nonlinearity $W(\psi, z)$ depends of the transformation chosen. In this chapter we show the analytical parameterization of stabilizing controllers of the form of Eq. (3.2) with scalar dynamic z i.e., when $z \in \mathbb{R}$. The transformations of Eq. (3.5) are then defined by

$$
\begin{aligned}
x &= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=T^{\{x\}}} \begin{bmatrix} \psi \\ z \end{bmatrix} \in \mathbb{R}^2 \\
e &= \underbrace{\begin{bmatrix} 1 & -t_\psi & -t_z \end{bmatrix}}_{=T^{\{e\}}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \in \mathbb{R}
\end{aligned}
\tag{3.8}
$$

with the constant matrices $T^{\{x\}}$ and $T^{\{e\}}$. Parameters of the matrix $T^{\{e\}}$ are to be defined too.

The outline is thus to first derive parameters of the controller to guarantee stability of the two subsystems of Eq. (3.6) and of Eq. (3.7) separately and after that consider the conditions to ensure stability of the whole interconnected system of Eq. (3.3).

3.3 Parameterization of Controllers

Suppose that the nonlinearities $W^{\{u\}}(\psi, z)$, $W^{\{z\}}(\psi, z)$ in the controller of Eq. (3.2) are static nonlinearities defined as

$$W^{\{u\}}(\psi, z) = \omega_u \cdot W(\psi, z), \quad W^{\{z\}}(\psi, z) = \omega_z \cdot W(\psi, z) \tag{3.9}$$

where $\omega_u, \omega_z \in \mathbb{R}$ and

$$W(\psi, z) = 1 - (1 + t_\psi \psi + t_z z)^3 \tag{3.10}$$

where t_ψ, $t_z \in \mathbb{R}$ are constants from the vector $T^{\{e\}}$ from the linear transformation of Eq. (3.8). The form of the nonlinearity of Eq. (3.10) resembles the form of the nonlinearity present in the original dynamics of Eq. (1.1). With such a choice, the controller of the form of Eq. (3.2) is

$$
\begin{aligned}
u &= \Lambda_\psi^{\{u\}} \psi + \Lambda_z^{\{u\}} z + \omega_u \cdot W(\psi, z) \\
\frac{d}{dt} z &= \Lambda_\psi^{\{z\}} \psi + \Lambda_z^{\{z\}} z + \omega_z \cdot W(\psi, z)
\end{aligned}
\tag{3.11}
$$

with $\Lambda_{\psi}^{\{u\}}$, $\Lambda_{z}^{\{u\}}$, $\Lambda_{\psi}^{\{z\}}$, $\Lambda_{z}^{\{z\}}$, $z \in \mathbb{R}$.
Another related observer design for systems with monotone time-varying sector nonlinearities in the unmeasured states was presented in [Arcak and Kokotović, 2001]. Their approach represented the observer error system as a feedback interconnection of a nonlinearity and a linear system.

The closed-loop system (with the surge subsystem of Eq. (1.3) and the controller of Eq. (3.11)) transformed into a specific block form of Eq. (3.3) takes the form

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{\beta^2} & -\frac{\Lambda_{\psi}^{\{u\}}}{\beta^2} & -\frac{\Lambda_{z}^{\{u\}}}{\beta^2} \\ 0 & \Lambda_{\psi}^{\{z\}} & \Lambda_{z}^{\{z\}} \end{bmatrix}}_{=\mathcal{A}_{cl}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{\omega_u}{\beta^2} \\ 0 & \omega_z \end{bmatrix}}_{\mathcal{B}_{cl}=[\mathcal{B}_{cl_1},\, \mathcal{B}_{cl_2}]} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W(\psi, z) \end{bmatrix} \tag{3.12}$$

with $z \in \mathbb{R}$ and the output matrices $C_{cl_1} = [1\ 0\ 0]$ and $C_{cl_2} = [0\ t_\psi\ t_z]$.

We will now find the parameters that will provide the set of stabilizing controllers by using sufficient conditions for stabilization and matching conditions between the original and the transformed closed-loop systems.

3.4 Matching Conditions

By using the previous results and the information that the original nonlinearity satisfies certain quadratic constraints one can obtain some identities between matrices of the original closed-loop system, transformed closed-loop system of Eq. (3.12) and matrices from the new transformed states of Eq. (3.8). The following matching conditions are based on calculations that were suggested in [Shiriaev et al., 2010].

For example, to transform Eq. (3.12) into the form of Eq. (3.3) we obtain the linear matrix equation (or Sylvester equation)

$$[1 - t_\psi - t_z] \begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{\beta^2} & -\frac{1}{\beta^2}\Lambda_{\psi}^{\{u\}} & -\frac{1}{\beta^2}\Lambda_{z}^{\{u\}} \\ 0 & \Lambda_{\psi}^{\{z\}} & \Lambda_{z}^{\{z\}} \end{bmatrix} = A_{22}\,[1 - t_\psi - t_z] \tag{3.13}$$

where $[1 - t_\psi - t_z]$ is an appropriate left eigenvector of the closed-loop matrix $\mathcal{A}_{cl}$ in Eq. (3.12) and A_{22} is the corresponding eigenvalue. The equality of Eq. (3.13) is equivalent to

$$\begin{aligned} t_\psi &= \left[\frac{3}{2} - A_{22}\right]\beta^2 \\ t_z &= \frac{A_{22}t_\psi - 1}{\Lambda_\psi^{\{z\}}} + \frac{\Lambda_\psi^{\{u\}} t_\psi}{\beta^2 \Lambda_\psi^{\{z\}}} \end{aligned} \tag{3.14}$$

The variables $\omega_u,\ \omega_z$ belong to the line described by

$$\frac{1}{\beta^2} t_\psi \omega_u - t_z \omega_z = -\frac{1}{2} \tag{3.15}$$

or equivalently

$$\left[\begin{array}{c} \omega_u \\ \omega_z \end{array}\right] = \left[\begin{array}{c} 0 \\ \dfrac{1}{2t_z} \end{array}\right] + \alpha \left[\begin{array}{c} t_z \\ \dfrac{1}{\beta^2} t_\psi \end{array}\right] \tag{3.16}$$

where $t_z \neq 0$ and α is a constant. With the above choices the rest of the matrices in Eq. (3.3) become

$$\begin{aligned} A_{11} &= \left[\begin{array}{cc} -\dfrac{1}{\beta^2}(\Lambda_\psi^{\{u\}} - t_\psi) & -\dfrac{1}{\beta^2}(\Lambda_z^{\{u\}} - t_z) \\ \Lambda_\psi^{\{z\}} & \Lambda_z^{\{z\}} \end{array}\right] \\ A_{12} &= \left[\begin{array}{c} \dfrac{1}{\beta^2} \\ 0 \end{array}\right];\ B_1 = \left[\begin{array}{c} -\dfrac{1}{\beta^2}\omega_u \\ \omega_z \end{array}\right];\ B_2 = \frac{1}{2}; \\ C &= \left[\begin{array}{cc} t_\psi & t_z \end{array}\right] \end{aligned} \tag{3.17}$$

More detailed calculations were presented in [Shiriaev et al., 2010].

At this stage we have

- the family of the controllers is chosen in Eq. (3.11);
- the closed-loop system is transformed into a special block form of Eq. (3.3) and shown in Eq. (3.12);
- matching conditions between the original closed-loop system and the transformed closed-loop system have been presented (see Eqs. (3.13–3.17)).

We will now analyze and simplify sufficient conditions for stabilization for the surge subsystem of Eq. (1.3) following the method in [Shiriaev et al., 2009], [Shiriaev et al., 2010].

3.5 Sufficient Conditions for Stabilization

It was also shown that the closed-loop system of Eq. (3.12) is quadratically stable if the parameters satisfy the following conditions [Shiriaev et al., 2010]:

1. The static nonlinearity of Eq. (2.29) satisfies the QC of Eq. (2.30). By using this fact and the frequency condition presented in [Yakubovich et al., 2004], [Shiriaev et al., 2009], [Shiriaev et al., 2010] we can show a new frequency condition, that is

$$\mathrm{Re}\{G(j\omega)\} - \frac{3}{4}|G(j\omega)|^2 < 0 \tag{3.18}$$

   is valid for all $\omega \geq 0$, where

$$G(s) = -C(sI_2 - A_{11})^{-1}B_1 = \frac{-\frac{1}{2}s + p_0}{s^2 + l_1 s + l_0} \tag{3.19}$$

   and with the coefficients

$$\begin{aligned} l_1 &= \frac{\Lambda_\psi^{\{u\}} - t_\psi}{\beta^2} - \Lambda_z^{\{z\}} \\ l_0 &= \frac{1}{\beta^2}(\Lambda_\psi^{\{z\}}(\Lambda_z^{\{u\}} - t_z) - \Lambda_z^{\{z\}}(\Lambda_\psi^{\{u\}} - t_\psi)) \\ p_0 &= -\frac{1}{\beta^2}(\omega_u(t_\psi\Lambda_z^{\{z\}} - t_z\Lambda_\psi^{\{z\}}) - \omega_z(t_\psi\Lambda_z^{\{u\}} - t_z\Lambda_\psi^{\{u\}})) \end{aligned} \tag{3.20}$$

2. The 2×2 matrix $(A_{11} - \frac{3}{4}B_1C)$ is Hurwitz.

3. The scalar A_{22} is negative.

For any chosen numerical values of the parameters $\Lambda_\psi^{\{u\}}$, $\Lambda_z^{\{u\}}$, $\Lambda_\psi^{\{z\}}$, $\Lambda_z^{\{z\}}$ belonging to the linear part of the controller of Eq. (3.2) one can evaluate the rest of the numerical values of the parameters t_ψ, t_z, ω_u, ω_z by using the matching conditions of Eqs. (3.13–3.17).

The next step is to verify that the parameters are satisfying the given sufficient conditions 1-3 above. The result will be a set/data-base of stabilizing controllers.

3.6 Exhaustive Parameters Search Method

Sufficient conditions for stabilization of the surge subsystem require an additional search procedure for five coefficients of the controller in

Eq. (3.11). We will use the exhaustive search methods to obtain coefficients: $\Lambda_{\psi}^{\{u\}}$, $\Lambda_{z}^{\{u\}}$, $\Lambda_{\psi}^{\{z\}}$, $\Lambda_{z}^{\{z\}}$ from the linear part of the controller and for ω_u from the nonlinear part that satisfy the given conditions for stabilization in Section 3.5. The rest of the parameters from the nonlinear part of the controller: t_z, t_ψ, ω_z are calculated by using the matching conditions from Section 3.4 [Rubanova et al., 2013].

To use this method we need to find some analytical constraints for the parameters that will simplify the original conditions for stabilization. To do this we will revisit the frequency constraint of Eq. (3.18). Introducing the notation for the transfer function of Eq. (3.19) as

$$G(j\omega) = \frac{-\frac{1}{2}j\omega + p_0}{(j\omega)^2 + l_1(j\omega) + l_0} \tag{3.21}$$

The numerator of Eq. (3.21) will be denoted by $b(j\omega)$ and the denominator will be denoted by $a(j\omega)$.

Then the frequency condition of Eq. (3.18) becomes

$$\frac{\mathrm{Re}(b(j\omega)\bar{a}(j\omega)) - \frac{3}{4}b(j\omega)\bar{b}(j\omega)}{a(j\omega)\bar{a}(j\omega)} < 0, \ \forall\omega \geq 0 \tag{3.22}$$

Substituting polynomials $b(j\omega)$ and $a(j\omega)$ from (3.22) we have

$$\begin{aligned}
&\frac{1}{|a(j\omega)|^2} \cdot \mathrm{Re}\{\frac{1}{2}j\omega^3 - (\frac{1}{2}l_1 + p_0 + \frac{3}{16})\omega^2 + (-l_1p_0 - \frac{1}{2}l_0)j\omega + (p_0l_0 - \frac{3}{4}p_0^2)\} \\
&\qquad = -\frac{1}{|a(j\omega)|^2} \cdot ((\frac{1}{2}l_1 + p_0 + \frac{3}{16})\omega^2 + (-p_0l_0 + \frac{3}{4}p_0^2)) \\
&\qquad = -\Psi(j\omega) \cdot \overline{\Psi(j\omega)}
\end{aligned} \tag{3.23}$$

The identity (3.23) is the so called *spectral factorization* of the rational function [Anderson and Moore, 2012, Ch. 9]. Here we are interested to find a rational function $\Psi(s)$ with real-valued coefficients.

For the case of Eq. (3.23) the choice of the rational function is straightforward

$$\Psi(s) = \frac{s\sqrt{\frac{1}{2}l_1 + p_0 + \frac{3}{16}} + \sqrt{\frac{3}{4}p_0^2 - p_0l_0}}{s^2 + l_1s + l_0} \tag{3.24}$$

Then the inequality of Eq. (3.22) is equivalent to

$$\begin{aligned}
&0 < \frac{1}{2}l_1 + p_0 + \frac{3}{16} \\
&0 < -p_0l_0 + \frac{3}{4}p_0^2 = p_0(\frac{3}{4}p_0 - l_0)
\end{aligned} \tag{3.25}$$

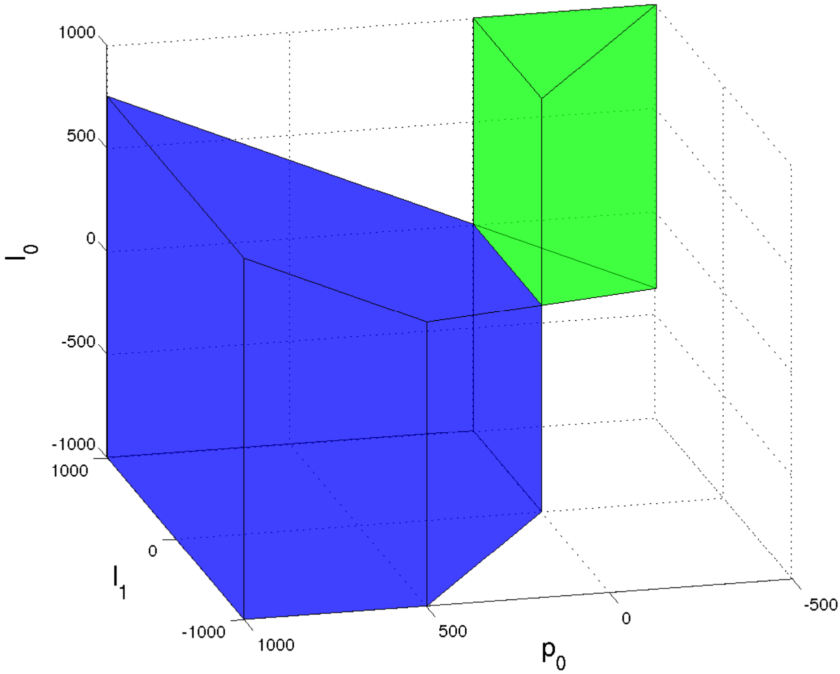

Figure 3.1 Solutions of the system of inequalities of Eq. (3.25). The left blue area corresponds to the first option ($p_0 < 0$) and the right green area corresponds to the second option ($p_0 > 0$). All coefficients of the transfer function of Eq. (3.19) that are constructed by the expressions of Eq. (3.20) and that appear inside those shapes satisfy the frequency condition of Eq. (3.18). (Note: These two spaces are not bounded by the box of axes presented in the diagram.)

One can see the graphical interpretation of the solutions (two spaces) from Eq. (3.25) in Fig. 3.1.

The first step to the analytic parameterization of stabilizing controllers for the Moore-Greitzer compressor model is using the formulas of Eqs. (3.15, 3.14) and of Eq. (3.25). By using those simplified and updated versions of the matching conditions and the frequency condition we will solve the main problem and find the numerical values for stabilizing controllers.

Exhaustive search methods for numerical values for the linear part of the controller have been implemented. The program contains all the matching conditions and the sufficient conditions for the stabilization and investigates

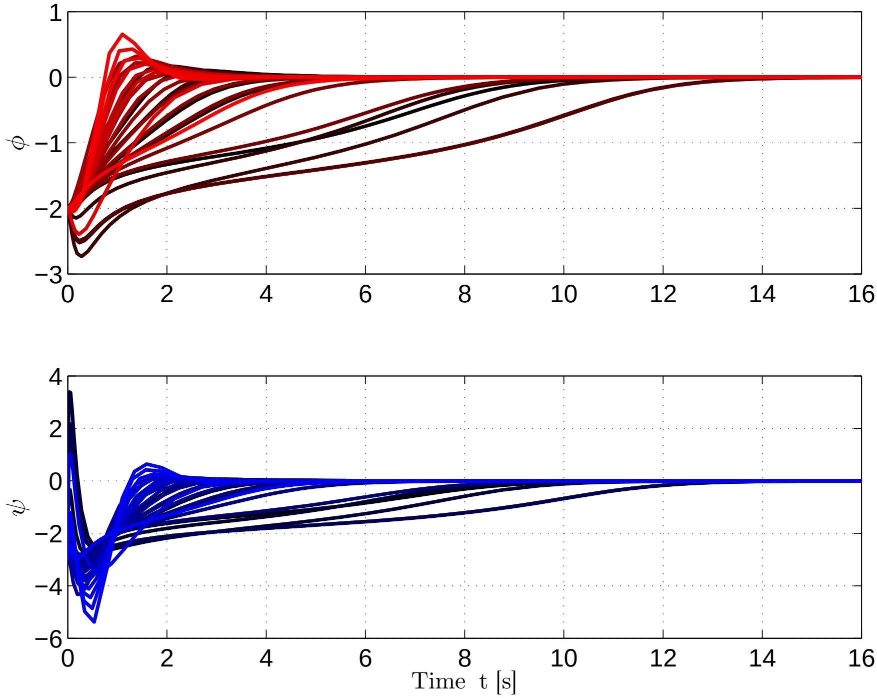

Figure 3.2 Simulation results illustrating the responses of the closed-loop surge subsystem of Eq. (1.3) ($R(0) = 0$). Here we use some randomly chosen stabilizing controllers from the data set. All the simulations have the same fixed non-zero initial conditions for deviations of flow and pressure.

a large range of the numerical values of the parameters to find the set of the controllers of the given form of Eq. (3.11).

The response of the closed-loop system without the stall dynamics (surge subsystem) of Eq. (1.3) is shown in Fig. 3.2. All of these randomly chosen controllers have been found by the main program and simulations confirm the result. For the case described and visualized in Fig. 3.2 we have identified a set of controllers that guarantee the stability of the model of Eq. (1.3).

The desired characteristics for the final stabilizing controller were not specified a priori (required settling time, rise time, etc.). That is why the initial evaluation process for the large ranges may require a great deal of time to complete. By knowing required characteristics of the controller and physical limits of some given compressor model, one can search the parameters in the correspondingly smaller range, as our data give us the information about the parameter dispersion and about the behavior of the corresponding controller.

Since we have an infinite number of stabilizing controllers it is a challenge to find some concrete required controller. However, these data give us the possibility to solve optimization problems based on some real model requirements.

By using the data sets of the controllers and a desired cost function one can quickly find the required numerical values, as shown in the following example.

3.7 Example I

From Fig. 3.2 we see that all the controllers are stabilizing but they all give different qualitative behaviour of the closed-loop system. If we need, for example, the controller with least oscillations we can use the data and solve the corresponding optimization problem.

Without the specific task and knowledge about the physical and design limitations it is not easy to describe the optimization problem. On the other hand, we have the flexibility to choose the task in order to simplify the future work and the method applied.

Let us choose the task when the output of the system will have minimum oscillations. We also would like to have as small amplitude for the control signal as possible. In the beginning, we can analyze the integrals of the outputs $\phi,\ \psi,\ u$

$$\underset{u}{\text{minimize}} \quad J(\phi, \psi, u) \tag{3.26}$$

where

$$J(\phi, \psi, u) = a_\phi \cdot \int \phi^2 dt + a_\psi \cdot \int \psi^2 dt + a_u \cdot \int u^2 dt \tag{3.27}$$

where $a_\phi,\ a_\psi,\ a_u$ are non-negative design weights for the integrals of the squared states and squared control signal. In this case, we use squares to show that the direction of the deviations is allowed to be both positive and negative. The weights can be chosen based on the specific task, here we used equal weights $a_\phi = a_\psi = a_u = 1$ in order to make an example. The resulting controller is providing the non-oscillating output with relatively small rise time and small amplitude for the control signal. Controllers were chosen randomly and tested with different initial condition on the states.

The numerical values for the linear part for the controller of Eq. (3.11) that were chosen from the data set are

$$\Lambda_\psi^{\{u\}} = -19,\ \Lambda_z^{\{u\}} = -7,\ \Lambda_\psi^{\{z\}} = -73,\ \Lambda_z^{\{z\}} = -26 \tag{3.28}$$

and then we can find the parameters of the nonlinear part of the controller by using the matching conditions

$$\begin{aligned} A_{22} &= -3.2168; & & \\ t_\psi &= 4.7168, & t_z &= 1.4492; \\ \omega_u &= -1, & \omega_z &= -2.9027; \end{aligned} \tag{3.29}$$

The controller is thus given by

$$\begin{aligned} u &= -19\psi - 7z - W(\psi, z) \\ \frac{d}{dt}z &= -73\psi - 26z - 2.9027W(\psi, z) \end{aligned} \tag{3.30}$$

where

$$W(\psi, z) = 1 - (1 + 4.7168\psi + 1.4492z)^3 \tag{3.31}$$

1. From the condition in Eq. (3.25) (i.e., sufficient conditions for stabilization for the surge subsystem of Eq. (3.5)) we have that

   $$\begin{aligned} -\frac{1}{2}l_1 - p_0 - \frac{3}{16} &= -0.3988 \\ p_0 l_0 - \frac{3}{4}p_0^2 &= -0.7931 \end{aligned} \tag{3.32}$$

   with $p_0 = -0.9303$, $l_0 = 0.1548$, $l_1 = 2.2832$ satisfy the given inequalities.

   The frequency condition of Eq. (3.18) is equivalent to the inequality of a circle

   $$(G(s) - \frac{2}{3})(\overline{G(s)} - \frac{2}{3}) > \frac{4}{9} \tag{3.33}$$

   That means that the hodograph (i.e., Nyquist diagram) of the corresponding transfer function

   $$G(s) = \frac{-0.5s - 0.9303}{s^2 + 2.2832s + 0.1548} \tag{3.34}$$

   should be outside of the circle with radius $r = 2/3$ and centered in $[2/3, 0j]$ which is shown in Fig. 3.3.

2. The eigenvalues of the matrix $(A_{11} - \frac{3}{4}B_1C)$ are $(-0.875,\ -7.873)$; hence, it is Hurwitz.

3. A_{22} is negative.

We want to control the compressor dynamics to the desired set-points $(\phi,\ \psi) = (0,\ 0)$ for the surge subsystem and $(\phi,\ \psi,\ R) = (0,\ 0,\ 0)$ for the MG compressor model. Running the simulation model with the controller of Eq. (3.30) shows that the closed-loop system of Eq. (3.12) is quadratically stable. This controller, shown in Fig. 3.4, is also the optimal one in a least-squares (LS) sense.

Simulation results showing the behavior of the original nonlinearity $W^{\{\phi\}}(\phi)$ and the estimated one $W(\psi, z)$ are presented in Fig. 3.5. One can see fast convergence of both nonlinearities to each other.

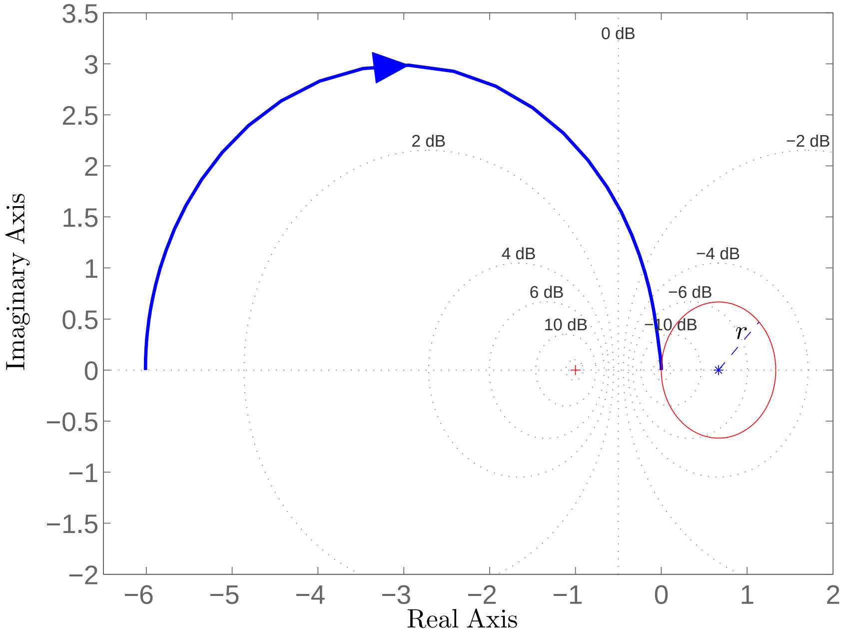

Figure 3.3 The frequency condition of Eq. (3.18) is equivalent to the inequality of a circle of Eq. (3.33). Nyquist diagram (blue line) of the transfer function of Eq. (3.34) is lying outside the red circle with radius $r = 2/3$ and center in $[2/3, 0j]$.

However, by running the simulation model of the closed-loop system with the same dynamic output feedback controller of Eq. (3.30) and the Moore-Greitzer compressor model of Eq. (1.1) with $\beta = 1,\ \sigma = 1$ we can see that the compressor dynamics are not converging to the desired set-point (see Fig. 3.6). The required extended stabilizing dynamic output feedback controller is presented in the next chapter.

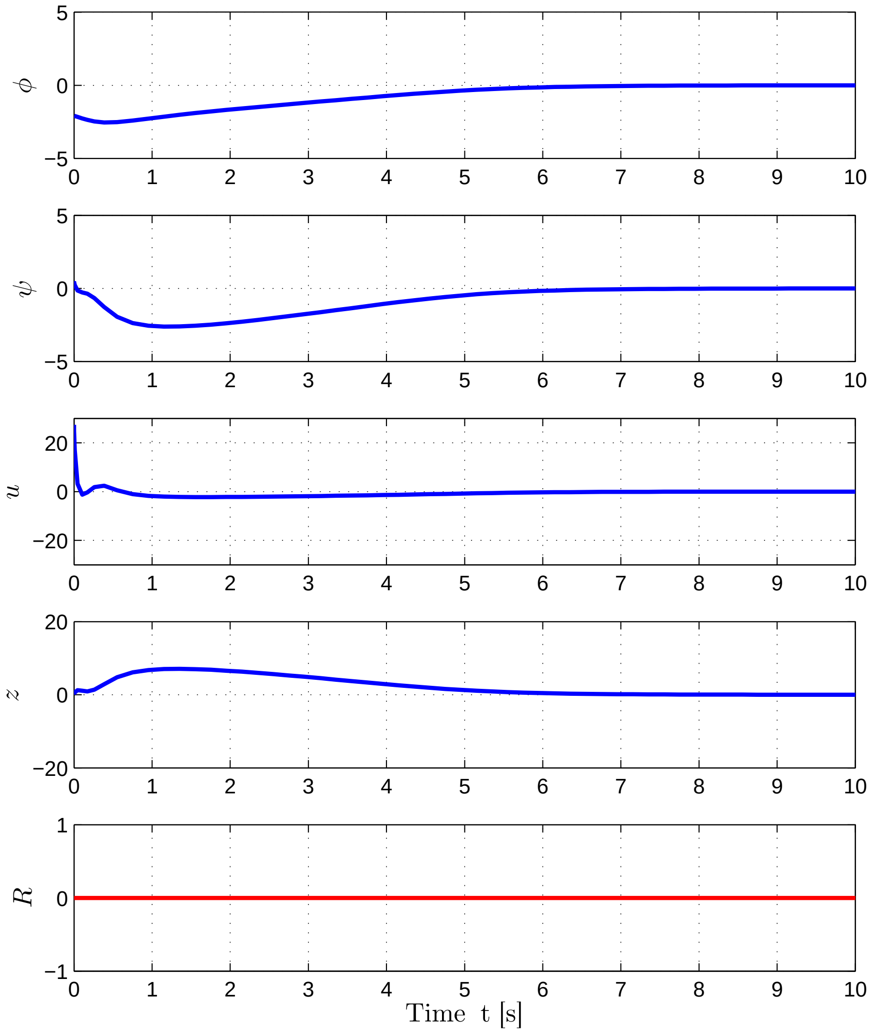

Figure 3.4 Simulation results of the closed-loop system with the dynamic output feedback controller of Eq. (3.30) and the surge subsystem of Eq. (1.3) with $\beta = 1$.

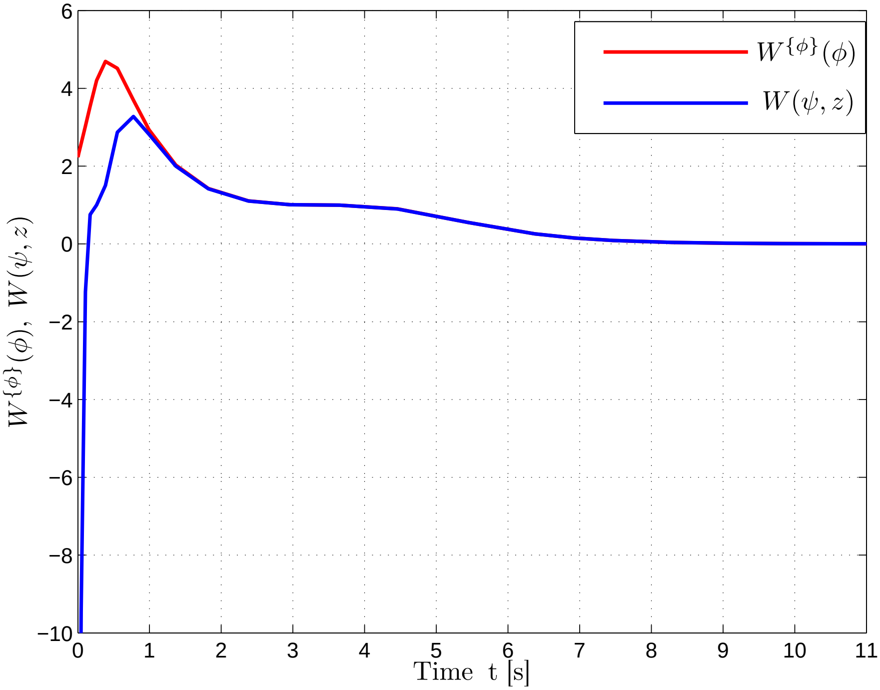

Figure 3.5 The simulation results showing the behavior of the original nonlinearity $W^{\{\phi\}}(\phi)$ and the estimated one $W(\psi, z)$.

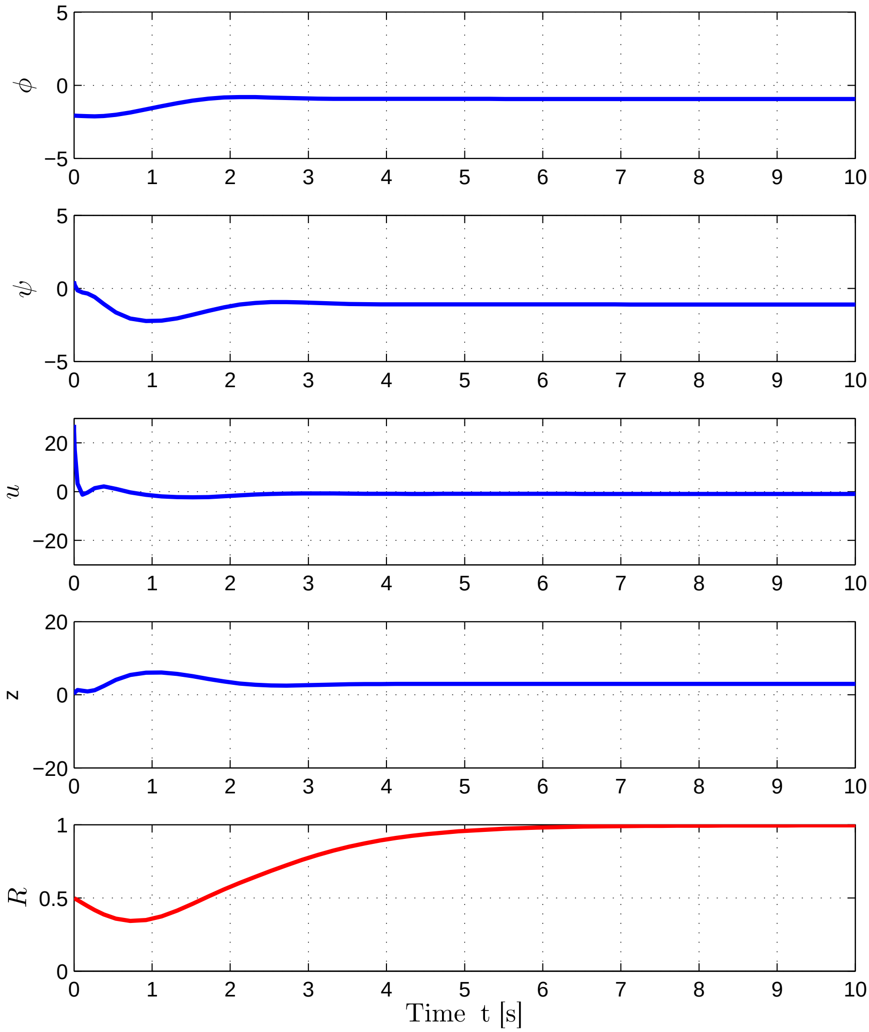

Figure 3.6 Simulation results of the closed-loop system with the dynamic output feedback controller of Eq. (3.30) and the Moore-Greitzer compressor model of Eq. (1.1) with $\beta = 1$, $\sigma = 1$ and the nonzero stall dynamics with $R_0 = 0.5$.

4

Analytical Parameterization of Stabilizing Controllers: Design II

According to Proposition 2 from the paper [Shiriaev et al., 2010] we are guarantied a global asymptotic and local exponential stability with the dynamic output feedback controller defined by the parameters that satisfy certain conditions. In this chapter we introduce the possibility to increase the order of the dynamics z of the controller of Eq. (3.2). Then we use the same method as in Design I in Chapter 3, although the computation complexity increases.

4.1 Parameterization of Controllers

We can modify the family of output feedback controllers (3.2) by adding a new integral state. The new linear transformations are defined by

$$
\begin{aligned}
x &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=T^{\{x\}}} \begin{bmatrix} \psi \\ z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^3 \\
e &= \underbrace{\begin{bmatrix} 1 & -t_\psi & -t_{z_1} & 0 \end{bmatrix}}_{=T^{\{e\}}} \begin{bmatrix} \phi \\ \psi \\ z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}
\end{aligned}
\tag{4.1}
$$

where $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^2$ and the new output feedback controller is:

$$
\begin{aligned}
u &= \underbrace{\begin{bmatrix} \Lambda_{\psi}^{\{u\}} & \Lambda_{z_1}^{\{u\}} & \Lambda_{z_2}^{\{u\}} \end{bmatrix}}_{=\Lambda^{\{u\}}} \begin{bmatrix} \psi \\ z_1 \\ z_2 \end{bmatrix} + \underbrace{\omega_u \cdot W(\psi, z_1)}_{=W^{\{u\}}(\cdot)} \\
\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} \Lambda_{\psi}^{\{z_1\}} & \Lambda_{z_1}^{\{z_1\}} & \Lambda_{z_2}^{\{z_1\}} \\ -t_{\psi} & -t_{z_1} & 0 \end{bmatrix}}_{=\Lambda^{\{z\}}} \begin{bmatrix} \psi \\ z_1 \\ z_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \omega_{z_1} \\ 0 \end{bmatrix} W(\psi, z_1)}_{=W^{\{z\}}(\cdot)}
\end{aligned}
\tag{4.2}
$$

where the nonlinearity $W(\psi, z_1)$ is similar to the original nonlinearity and defined as

$$W(\psi, z_1) = 1 - (1 + t_{\psi}\psi + t_{z_1}z_1)^3 \tag{4.3}$$

The closed-loop system modified by a special block form of Eq. (3.3) with the surge subsystem of Eq. (1.3) and with the new controller (4.2) takes the form:

$$
\begin{aligned}
\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z_1} \\ \dot{z_2} \end{bmatrix} &= \underbrace{\begin{bmatrix} \frac{3}{2} & -1 & 0 & 0 \\ \frac{1}{\beta^2} & -\frac{\Lambda_{\psi}^{\{u\}}}{\beta^2} & -\frac{\Lambda_{z_1}^{\{u\}}}{\beta^2} & -\frac{\Lambda_{z_2}^{\{u\}}}{\beta^2} \\ 0 & \Lambda_{\psi}^{\{z_1\}} & \Lambda_{z_1}^{\{z_1\}} & \Lambda_{z_2}^{\{z_1\}} \\ 0 & -t_{\psi} & -t_{z_1} & 0 \end{bmatrix}}_{=\mathcal{A}_{cl_{MG}}} \begin{bmatrix} \phi \\ \psi \\ z_1 \\ z_2 \end{bmatrix} \\
&\quad + \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{\omega_u}{\beta^2} & 0 \\ 0 & 0 & \omega_{z_1} \\ 0 & 0 & 0 \end{bmatrix}}_{=\mathcal{B}_{cl_{MG}}} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W(\psi, z_1) \\ W(\psi, z_1) \end{bmatrix}
\end{aligned}
\tag{4.4}
$$

The new task is to find the parameters that will provide stabilizing controllers. First, we will show necessary matching conditions between the original the transformed closed-loop systems.

4.2 Matching Conditions

As in Design I, we will use identities between matrices of the original closed-loop system, transformed closed-loop system of Eq. (3.12) and matrices from the transformations of Eq. (3.8). The following matching conditions were also suggested in [Shiriaev et al., 2010]. The calculations are more complicated due to the increasing order of the transformation matrices of Eq. (4.1).

For example, to transform the closed-loop system with the surge subsystem of Eq. (1.3) and the controller of Eq. (4.2) into the form of Eq. (3.3) we use the linear matrix equation (or Sylvester equation)

$$\begin{bmatrix} 1 \\ -t_\psi \\ -t_{z_1} \\ 0 \end{bmatrix}^T \begin{bmatrix} \frac{3}{2} & -1 & 0 & 0 \\ \frac{1}{\beta^2} & -\frac{\Lambda_\psi^{\{u\}}}{\beta^2} & -\frac{\Lambda_{z_1}^{\{u\}}}{\beta^2} & -\frac{\Lambda_{z_2}^{\{u\}}}{\beta^2} \\ 0 & \Lambda_\psi^{\{z_1\}} & \Lambda_{z_1}^{\{z_1\}} & \Lambda_{z_2}^{\{z_1\}} \\ 0 & -t_\psi & -t_{z_1} & 0 \end{bmatrix} = A_{22} \begin{bmatrix} 1 \\ -t_\psi \\ -t_{z_1} \\ 0 \end{bmatrix}^T \tag{4.5}$$

where $[1 \;\; -t_\psi \;\; -t_{z_1} \;\; 0]$ is an appropriate left eigenvector of the closed-loop matrix $\mathcal{A}_{cl_{MG}}$ in Eq. (4.4) and A_{22} is the corresponding eigenvalue. Since we are using this equation in the program, we have to note at least three possible problems that can appear: existence of complex solutions, sensitivity of the solution to errors in the data and roundoff errors in the computation [Moler, 2004].

In this work we will show identities that we will use in our calculations, for more details we refer to [Shiriaev et al., 2010].

In order to check that the matrix equation of Eq. (4.5) has a solution we need the following additional identity

$$\Lambda_{z_2}^{\{u\}} = \frac{\Lambda_{z_2}^{\{z_1\}} t_{z_1} \beta^2}{t_\psi} \tag{4.6}$$

With the above choices, the matrices in Eq. (3.3) become

$$\begin{aligned} A_{11} &= \begin{bmatrix} -\frac{1}{\beta^2}(\Lambda_\psi^{\{u\}} - t_\psi) & -\frac{1}{\beta^2}(\Lambda_{z_1}^{\{u\}} - t_{z_1}) & -\frac{\Lambda_{z_2}^{\{u\}}}{\beta^2} \\ \Lambda_\psi^{\{z_1\}} & \Lambda_{z_1}^{\{z_1\}} & \Lambda_{z_2}^{\{z_1\}} \\ -t_\psi & -t_{z_1} & 0 \end{bmatrix} \\ A_{12} &= \begin{bmatrix} \frac{1}{\beta^2} \\ 0 \\ 0 \end{bmatrix}; \; B_1 = \begin{bmatrix} -\frac{1}{\beta^2}\omega_u \\ \omega_{z_1} \\ 0 \end{bmatrix}; \; B_2 = \frac{1}{2}; \\ C &= \begin{bmatrix} t_\psi & t_{z_1} & 0 \end{bmatrix} \end{aligned} \tag{4.7}$$

At this stage we have:

- the family of the controllers is chosen in Eq. (4.2);
- the closed-loop system is transformed into a special block form of Eq. (3.3) and shown in Eq. (4.4);
- matching conditions between the original closed-loop system and the transformed closed-loop system have been presented (see Eqs. (4.5–4.7)).

We will now discuss equilibria of two closed-loop systems with the same controller of Eq. (4.2): first, that for the surge subsystem of Eq. (1.3), second, that for the MG compressor model of Eq. (1.1).

4.3 Equilibrium of the Closed-Loop System

Equilibrium of the Closed-Loop System with the Surge Subsystem
Proposition 1: The unique and stable equilibrium of the closed-loop system with the surge subsystem of Eq. (1.3) and the dynamic output feedback controller of Eq. (4.2) for constant $\phi = \phi_0$ is:

$$\begin{cases} \phi_0 = 0 \\ \psi_0 = 0 \\ z_{1_0} = 0 \\ z_{2_0} = 0 \end{cases} \tag{4.8}$$

for

$$\begin{aligned} 0 &= -\psi_0 + \frac{3}{2}\phi_0 + \frac{1-(1+\phi_0)^3}{2} \\ 0 &= \frac{1}{\beta^2}(\phi_0 - (\Lambda_{\psi}^{\{u\}}\psi + \Lambda_{z_1}^{\{u\}}z_1 + \Lambda_{z_2}^{\{u\}}))z_2 + \omega_u \cdot W(\psi, z_1) \\ 0 &= \Lambda_{\psi}^{\{z_1\}}\psi + \Lambda_{z_1}^{\{z_1\}}z_1 + \Lambda_{z_2}^{\{z_1\}}z_2 + \omega_{z_1}W(\psi, z_1) \\ 0 &= -t_{\psi}\psi - t_{z_1}z_1 \end{aligned} \tag{4.9}$$

if there is a constant

$$-\frac{8}{9} < D < 0 \tag{4.10}$$

such that

$$D = \frac{-\Lambda_{\psi}^{\{u\}}\Lambda_{z_2}^{\{z_1\}}t_{z_1} + \Lambda_{z_1}^{\{u\}}t_{\psi}\Lambda_{z_2}^{\{z_1\}} + \Lambda_{z_2}^{\{u\}}\left(\Lambda_{\psi}^{\{z_1\}}t_{z_1} - \Lambda_{z_1}^{\{z_1\}}t_{\psi}\right)}{\Lambda_{z_2}^{\{z_1\}}t_{z_1}} \tag{4.11}$$

Proof: In the following itemization we will now show the derivation procedure of the equilibrium.

- for $d\phi/dt = 0$:

$$\begin{aligned} 0 &= -\psi_0 + \frac{3}{2}\phi_0 + \frac{1-(1+\phi_0)^3}{2} \\ \psi_0 &= -\frac{1}{2}\phi_0^2(\phi_0+3) \end{aligned} \tag{4.12}$$

- for $dz_2/dt = 0$:

$$\begin{aligned} 0 &= -t_\psi\psi_0 - t_{z_1}z_{1_0} \\ z_{1_0} &= -\frac{t_\psi\psi_0}{t_{z_1}} = \frac{1}{2}\phi_0^2(\phi_0+3)\frac{t_\psi}{t_{z_1}} \end{aligned} \tag{4.13}$$

  It is easy to notice from Eq. (4.13) that the nonlinearity of Eq. (4.3) $W(\psi_0, z_{1_0}) = 0$.

- for $dz_1/dt = 0$

$$\begin{aligned} 0 &= \Lambda_\psi^{\{z_1\}}\psi_0 + \Lambda_{z_1}^{\{z_1\}}z_{1_0} + \Lambda_{z_2}^{\{z_1\}}z_{2_0} \\ z_{2_0} &= \frac{\Lambda_\psi^{\{z_1\}}t_{z_1}\frac{1}{2}\phi_0^2(\phi_0+3) - \Lambda_{z_1}^{\{z_1\}}t_\psi\frac{1}{2}\phi_0^2(\phi_0+3)}{\Lambda_{z_2}^{\{z_1\}}t_{z_1}} \\ &= \frac{1}{2}\phi_0^2(\phi_0+3)\left(\frac{\Lambda_\psi^{\{z_1\}}t_{z_1} - \Lambda_{z_1}^{\{z_1\}}t_\psi}{\Lambda_{z_2}^{\{z_1\}}t_{z_1}}\right) \end{aligned} \tag{4.14}$$

- for $d\psi/dt = 0$:

$$\begin{aligned} 0 &= \frac{1}{\beta^2}(\phi_0 - \Lambda_\psi^{\{u\}}\psi_0 - \Lambda_{z_1}^{\{u\}}z_{1_0} - \Lambda_{z_{2_0}}^{\{u\}}z_{2_0}) \\ \phi_0 &= \Lambda_\psi^{\{u\}}\psi_0 + \Lambda_{z_1}^{\{u\}}z_{1_0} + \Lambda_{z_2}^{\{u\}}z_{2_0} = \frac{1}{2}\phi_0^2(\phi_0+3)\cdot D \end{aligned} \tag{4.15}$$

  where

$$D = \frac{-\Lambda_\psi^{\{u\}}\Lambda_{z_2}^{\{z_1\}}t_{z_1} + \Lambda_{z_1}^{\{u\}}t_\psi\Lambda_{z_2}^{\{z_1\}} + \Lambda_{z_2}^{\{u\}}\left(\Lambda_\psi^{\{z_1\}}t_{z_1} - \Lambda_{z_1}^{\{z_1\}}t_\psi\right)}{\Lambda_{z_2}^{\{z_1\}}t_{z_1}} \tag{4.16}$$

  is a constant.

Then, the solutions to Eq. (4.15)

$$\begin{aligned} \frac{1}{2}D\phi_0^3 + \frac{3}{2}D\phi_0^2 - \phi_0 &= \phi_0(\frac{1}{2}D\phi_0^2 + \frac{3}{2}D\phi_0 - 1) = 0 \\ \phi_{0_{(1)}} &= 0 \\ \phi_{0_{(2,3)}} &= \frac{-\frac{3}{2}D \pm \sqrt{\left(\frac{3D}{2}\right)^2 + 2D}}{D} \end{aligned} \tag{4.17}$$

that means we have at least three solutions for the closed-loop system of Eq. (4.4).

Since the multiple equilibrium for the closed-loop system refutes the possibility of the global stability we need additional constraints on the coefficients D of Eq. (4.15). Complex numerical values will not make sense in physics, thus the constraint that will help us to avoid the existence of the solutions $\phi_{0_{(2,3)}}$ can be chosen as

$$-\frac{8}{9} < D < 0 \tag{4.18}$$

then we would only expect unique stable equilibrium

$$\begin{cases} \phi_0 = 0 \\ \psi_0 = 0 \\ z_{1_0} = 0 \\ z_{2_0} = 0 \end{cases} \tag{4.19}$$

■

Equilibrium of the Closed-Loop System with the MG Compressor Model

Proposition 2: The unique stable equilibrium of the closed-loop system with the MG compressor model of Eq. (1.1) and the dynamic output feedback controller of Eq. (4.2) for constant $\phi = \phi_0$ is

$$\begin{cases} \phi_0 = 0 \\ \psi_0 = 0 \\ R_{0_1} = 0 \\ z_{1_0} = 0 \\ z_{2_0} = 0 \end{cases} \tag{4.20}$$

for

$$
\begin{aligned}
0 &= -\psi_0 + \frac{3}{2}\phi_0 + \frac{1-(1+\phi_0)^3}{2} - 3R_0(1+\phi_0) \\
0 &= \frac{1}{\beta^2}(\phi_0 - (\Lambda_{\psi}^{\{u\}}\psi_0 + \Lambda_{z_1}^{\{u\}}z_{1_0} + \Lambda_{z_2}^{\{u\}}))z_{2_0} + \omega_u \cdot W(\psi_0, z_{1_0}) \\
0 &= -\sigma R_0^2 - \sigma R_0(2\phi_0 + \phi_0^2) \\
0 &= \Lambda_{\psi}^{\{z_1\}}\psi_0 + \Lambda_{z_1}^{\{z_1\}}z_{1_0} + \Lambda_{z_2}^{\{z_1\}}z_{2_0} + \omega_{z_1}W(\psi_0, z_{1_0}) \\
0 &= -t_{\psi}\psi_0 - t_{z_1}z_{1_0}
\end{aligned}
\tag{4.21}
$$

if there is a constant

$$D_2 < 0 \tag{4.22}$$

such that

$$
\begin{aligned}
\Lambda_1 =& \Lambda_{\psi}^{\{u\}} \\
\Lambda_2 =& \Lambda_{z_1}^{\{u\}} \\
\Lambda_3 =& \Lambda_{\psi}^{\{z_1\}} \\
\Lambda_4 =& \Lambda_{z_1}^{\{z_1\}} \\
\Lambda_5 =& \Lambda_{z_2}^{\{u\}} \\
\Lambda_6 =& \Lambda_{z_2}^{\{z_1\}} \\
D_2 =& -15\Lambda_2^2 t_{\psi}^2 \Lambda_6^2 + 30\Lambda_1 t_{z_1}\Lambda_6^2\Lambda_2 t_{\psi} - 30\Lambda_2 t_{\psi}\Lambda_6\Lambda_5\Lambda_3 t_{z_1} \\
& - 30\Lambda_2 t_{\psi}^2\Lambda_6\Lambda_5\Lambda_4 - 15\Lambda_1^2 t_{z_1}^2\Lambda_6^2 + 30\Lambda_1 t_{z_1}^2\Lambda_6\Lambda_5\Lambda_3 \\
& + 30\Lambda_1 t_{z_1}\Lambda_6\Lambda_5\Lambda_4 t_{\psi} - 15\Lambda_5^2\Lambda_3^2 tz1^2 - 30\Lambda_5^2\Lambda_4 t_{\psi}\Lambda_3 t_{z_1} \\
& - 15\Lambda_5^2\Lambda_4^2 t_{\psi}^2 + 40\Lambda_1 t_{z_1}^2\Lambda_6^2 - 40\Lambda_2 t_{\psi}\Lambda_6^2 t_{z_1} - 40\Lambda_5\Lambda_4 t_{\psi} t_{z_1}\Lambda_6 \\
& - 40\Lambda_5\Lambda_3 t_{z_1}^2\Lambda_6
\end{aligned}
\tag{4.23}
$$

Proof: There are other equilibria of the closed-loop system with the MG compressor model of Eq. (1.1) and the dynamic output feedback controller of Eq. (4.2) for constant $\phi = \phi_0$

$$
\begin{aligned}
\phi_0 &= \Lambda_{\psi}^{\{u\}}\psi_0 + \Lambda_{z_{1_0}}^{\{u\}}z_{1_0} + \Lambda_{z_2}^{\{u\}}z_{2_0} \\
\psi_0 &= 6\phi_0 + \frac{5}{2}{\phi_0}^3 + \frac{15}{2}{\phi_0}^2 \\
R_{0_2} &= 1 - (\phi_0 + 1)^2 \\
z_{1_0} &= -\frac{t_{\psi}(6\phi_0 + \frac{5}{2}{\phi_0}^3 + \frac{15}{2}{\phi_0}^2)}{t_{z_1}} \\
z_{2_0} &= -\frac{\phi_0(12 + 5\phi_0^2 + 15\phi_0)(\Lambda_{\psi}^{\{z_1\}}t_{z_1} + \Lambda_{z_1}^{\{z_1\}}t_{\psi})}{2(t_{z_1}\Lambda_{z_2}^{\{z_1\}})}
\end{aligned}
\tag{4.24}
$$

For the solutions to the first equation ϕ_0 of Eq. (4.24) we need an additional constraint to make sure, that we will have only one equilibrium at the origin and all others will be complex. The expressions are very long so we will only present the result in Eq. (4.22). If the constant $D_2 < 0$ then all solutions to Eq. (4.24) will be complex.

Then we would also expect a unique stable equilibrium of the closed-loop system with the MG compressor model of Eq. (1.1) and the dynamic output feedback controller of Eq. (4.2)

$$\begin{cases} \phi_0 = 0 \\ \psi_0 = 0 \\ R_{0_1} = 0 \\ z_{1_0} = 0 \\ z_{2_0} = 0 \end{cases} \tag{4.25}$$

■

We will now analyze and simplify sufficient conditions for stabilization for the surge subsystem of Eq. (1.3) following the method in [Shiriaev et al., 2009], [Shiriaev et al., 2010].

4.4 Sufficient Conditions for Stabilization

In [Shiriaev et al., 2010] it is also shown that the closed-loop system of Eq. (4.4) is quadratically stable if the parameters satisfy certain conditions. As a summary of this chapter we present the updated version of these sufficient conditions:

1. The inequality

$$\mathrm{Re}\{G(j\omega)\} - \frac{3}{4}|G(j\omega)|^2 \leq 0 \tag{4.26}$$

is valid for all $\omega \geq 0$, where

$$G(s) = -C(sI_3 - A_{11})^{-1}B_1 = \frac{-\frac{1}{2}s^2 + p_1 s}{s^3 + l_2 s^2 + l_1 s + l_0} \tag{4.27}$$

with

$$\begin{aligned} p_1 &= -(\omega_{z_1} t_{z1} \Lambda_{\psi}^{\{u\}} - \omega_u \Lambda_{\psi}^{\{z_1\}} t_{z_1} - \omega_{z_1} t_{\psi} \Lambda_{z_1}^{\{u\}} + \omega_u t_{\psi} \Lambda_{z_1}^{\{z_1\}}) \\ l_2 &= -\beta^2 \Lambda_{z_1}^{\{z_1\}} - t_{\psi} + \Lambda_{\psi}^{\{u\}} \\ l_1 &= -\Lambda_{\psi}^{\{u\}} \Lambda_{z_1}^{\{z_1\}} - \Lambda_{\psi}^{\{z_1\}} t_{z_1} + \beta^2 \Lambda_{z_2}^{\{z_1\}} t_{z_1} \\ &\qquad + t_{\psi} \Lambda_{z_1}^{\{z_1\}} + \Lambda_{\psi}^{\{z_1\}} \Lambda_{z_1}^{\{u\}} - \Lambda_{z_2}^{\{u\}} t_{\psi} \\ l_0 &= t_{\psi} \Lambda_{z_2}^{\{u\}} \Lambda_{z_1}^{\{z_1\}} + \Lambda_{\psi}^{\{u\}} \Lambda_{z_2}^{\{z_1\}} t_{z_1} - \Lambda_{\psi}^{\{z_1\}} \Lambda_{z_2}^{\{u\}} t_{z_1} - t_{\psi} \Lambda_{z_2}^{\{z_1\}} \Lambda_{z_1}^{\{u\}} \end{aligned} \tag{4.28}$$

The equivalent condition is:

$$
\begin{aligned}
0 &\le 2l_2 + 4p_1 + \frac{3}{4} \\
0 &\le -2l_0 - 4l_1 p_1 + 3p_1^2
\end{aligned}
\tag{4.29}
$$

2. The 2 x 2 matrix $(A_{11} - 3B_1 C/4)$ is Hurwitz [Moore and Greitzer, 1986].
3. The pair $[C, A_{11}]$ is observable and the pair $[A_{11}, B_1]$ is controllable.
4. The scalar A_{22} is negative.
5. - For the surge subsystem of Eq. (1.3): By using the expression for l_0 of Eq. (4.28) we simplify Eq. (4.16) to

   $$D = -\frac{l_0 \beta^2}{\Lambda_{z_2}^{\{z_1\}} t_{z_1}} \tag{4.30}$$

   The inequality

   $$-\frac{8}{9} < D < 0 \tag{4.31}$$

   is valid with D calculated from Eq. (4.30)
   - For the MG compressor model of Eq. (1.1):

   $$D_2 < 0 \tag{4.32}$$

   is valid with D_2 calculated from Eq. (4.22)

The next step is to verify that the parameters are satisfying the given sufficient conditions 1-5.

4.5 Example II

By using the exhaustive parameter search method from Section 3.6 we developed a new program based on the sufficient conditions for stabilization from Section 4.4. As in Design I, we also have a set of stabilizing controllers and here we present one of them.

For example, if we choose

$$
\begin{aligned}
\Lambda_{\psi}^{\{u\}} &= -19; \quad \Lambda_{z_1}^{\{u\}} = -7 \\
\Lambda_{\psi}^{\{z_1\}} &= -73; \quad \Lambda_{z_1}^{\{z_1\}} = -26
\end{aligned}
\tag{4.33}
$$

then we find parameters of the nonlinear part of the controller by using the similar matching conditions from Section 4.2.

First, we solve the Sylvester equation of Eq. (4.5) by finding eigenvalues and left eigenvectors of the state matrix A_{cl_2} of Eq. (4.4). The results are:

$$\begin{aligned} A_{22} &= -3.2168 \\ t_\psi &= 4.7168; \quad t_{z_1} = 1.4492 \\ \omega_u &= -1; \quad\quad \omega_{z_1} = -2.9027 \end{aligned} \tag{4.34}$$

By using the equation of Eq. (4.18) we can derive additional parameters for Design II:

$$\Lambda_{z_2}^{\{z_1\}} = 4; \quad \Lambda_{z_2}^{\{u\}} = 1.229 \tag{4.35}$$

Then the controller is

$$\begin{aligned} u &= \begin{bmatrix} -19 & -7 & 1.229 \end{bmatrix} \begin{bmatrix} \psi \\ z_1 \\ z_2 \end{bmatrix} - W(\psi, z_1) \\ \frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} -73 & -26 & 4 \\ -4.7168 & -1.4492 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ z_1 \\ z_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} -2.9027 \\ 0 \end{bmatrix} W(\psi, z_1) \end{aligned} \tag{4.36}$$

where the nonlinearity $W(\psi, z_1)$ is similar to the original nonlinearity and defined as

$$W(\psi, z_1) = 1 - (1 + 4.7168\psi + 1.4492 z_1)^3 \tag{4.37}$$

Now we check if all these parameters satisfy given conditions and constraints:

1. From item one in Section 4.4 (Frequency constraint) we have the following coefficients for the transfer function of Eq. (4.27)

$$\begin{aligned} p_1 &= -0.9302 \\ l_2 &= 2.2832 \\ l_1 &= 0.1548 \\ l_0 &= 1.2289 \\ G(s) &= \frac{-0.5s^2 - 0.9302s}{s^3 + 2.2832s^2 + 0.1548s + 1.2289} \end{aligned} \tag{4.38}$$

   and the analog of the frequency condition of Eq. (4.29) is then

$$\begin{aligned} 2l_2 + 4p_1 + \frac{3}{4} &= 1.5952 > 0 \\ -2l_0 - 4l_1 p_1 + 3p_1^2 &= 0.7143 > 0 \end{aligned} \tag{4.39}$$

   which is true and hence Frequency condition of Eq. (4.26) is satisfied;

2. The additional constraints of Eqs. (4.31, 4.22) that exclude multiple equilibria are
$$\begin{aligned} D &= -0.21209 \\ D_2 &= -1.3038 \cdot 10^6 \end{aligned} \tag{4.40}$$

3. The eigenvalues of the matrix $(A_{11} - 3/4B_1C)$ are $(-2.5211, -0.0724 + 0.6944i, -0.0724 - 0.6944i)$; hence, it is Hurwitz;

4. The pair $[C, A_{11}]$ represented by the matrix
$$W_o = \begin{bmatrix} 4.7168 & 1.4492 & 0 \\ 6.0758 & 2.1739 & 0 \\ -14.6024 & -5.1879 & 1.2289 \end{bmatrix} \tag{4.41}$$
is observable and the pair $[A_{11}, B_1]$ represented by the matrix
$$W_c = \begin{bmatrix} 1 & -0.8086 & 2.3186 \\ -2.9027 & 2.4701 & -7.2314 \\ 0 & -0.5102 & 0.2346 \end{bmatrix} \tag{4.42}$$
is controllable;

5. A_{22} is negative and equal to -3.2168.

The simulation results are shown in Fig. 4.1 for the closed-loop system with the surge subsystem and in Fig. 4.2 for the closed-loop system with the MG compressor model. Both systems were controlled by the same controller of Eq. (4.36). The desired set-points $(\phi,\ \psi) = (0,\ 0)$ and $(\phi,\ \psi, R) = (0,\ 0,\ 0)$ are achieved.

In Fig. 4.3 we compare Design I and Design II. The dashed blue line in the figure represents the output of the closed-loop system with the controller of Eq. (3.30) and the MG compressor model of Eq. (1.1). The solid lines represent the output of the closed-loop system with the same system and the controller with additional dynamics of Eq. (4.36).

As seen, the dynamic output feedback controller of Eq. (4.2) designed for the surge subsystem of Eq. (1.3) does stabilize the whole Moore-Greitzer compressor model of Eq. (1.1) with the stall dynamics included. The controller design excludes the appearance of other equilibria besides the origin. In Subsection 4.3 we proved the presence of the unique stable equilibrium of the closed-loop system with stall dynamics.

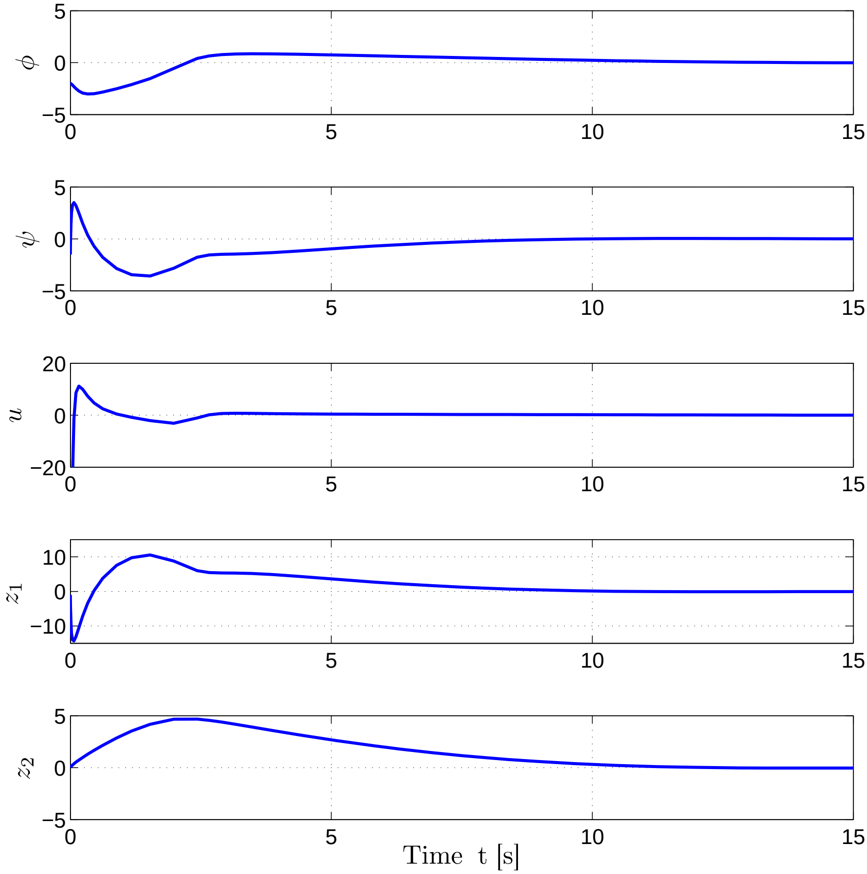

Figure 4.1 Simulation results of the closed-loop system with the controller of Eq. (4.36) and the surge subsystem of Eq. (1.3) with $\beta = 1$.

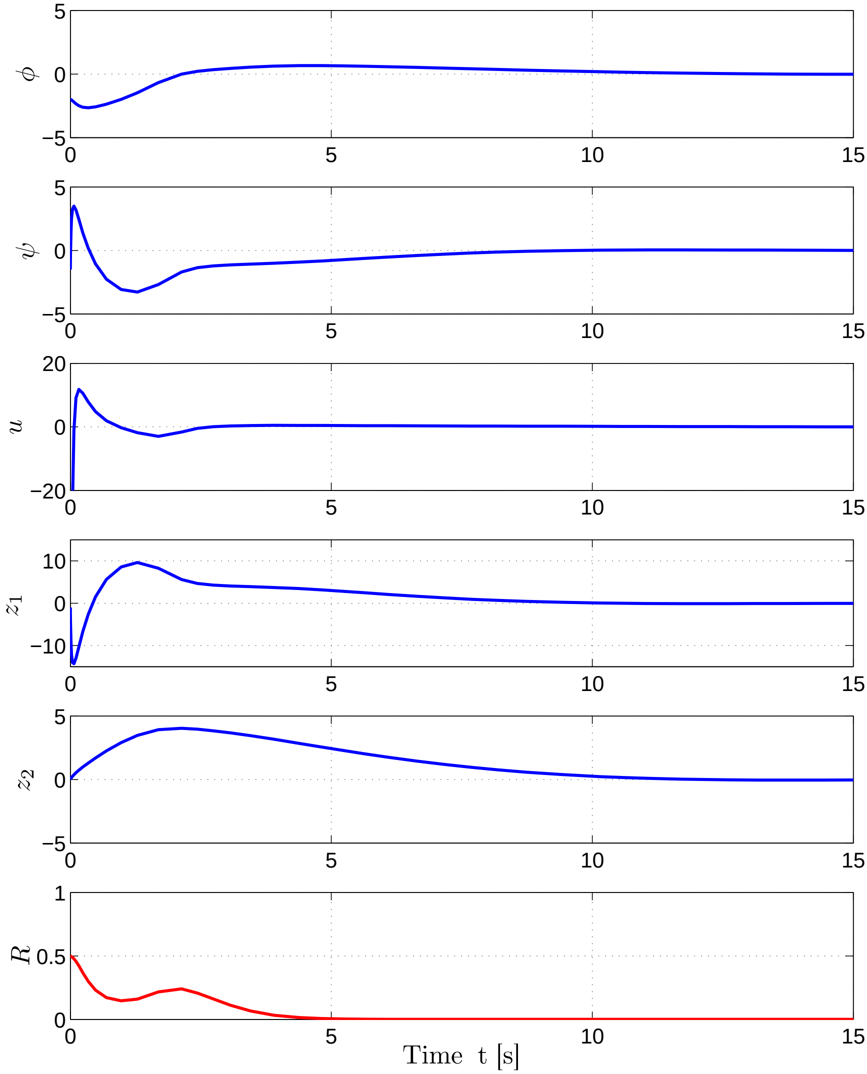

Figure 4.2 Simulation results of the closed-loop system with the controller of Eq. (4.36) and the MG compressor model of Eq. (1.1) with $\beta = 1,\ \sigma = 1$. Stall dynamics initial value $R_0 = 0.5$.

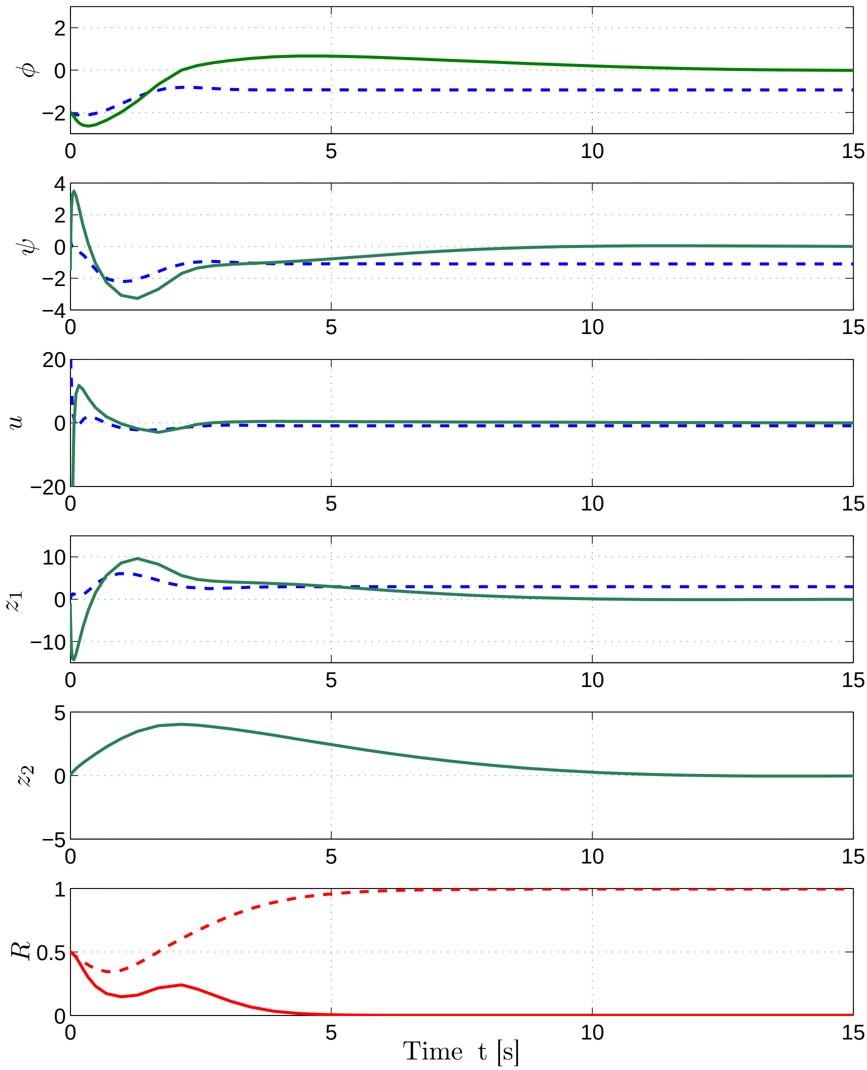

Figure 4.3 Simulation results of closed-loop systems with the different controllers. The dashed lines represent the output of the closed-loop system with the controller of Eq. (3.11) and the MG compressor model of Eq. (1.1). The solid lines represent the output of the closed-loop system with the same system and the controller with an additional dynamic of Eq. (4.2).

5

Stability Analysis

We have described the synthesis of a class of stabilizing controllers and their application to the MG compressor model and to its surge subsystem. This chapter is an extension of previous research based on the same procedure for dynamic output feedback design for systems with nonlinearities satisfying quadratic constraints [Shiriaev et al., 2009], [Shiriaev et al., 2010], [Rubanova et al., 2013], [Andersson et al., 2014].

In Chapters 3 and 4 we presented the exhaustive parameter search method for dynamic output feedback controllers. In this chapter we use the same method but with different constraints for the parameters that will simplify the controller choice based on a specific task [Andersson et al., 2014]. By this method and with the help of the results presented in Chapters 3-4 one can choose the optimal controller for the given model and analyze the quality of the controller design. We also present an alternative proof of stability of the closed-loop system of Eq. (3.12) which is based on the Circle criterion [Yakubovich et al., 2004], [Khalil, 2002].

In addition to this, we discuss the degree of robustness and present more general conditions for robustness with respect to parametric uncertainties.

We also present the search of a quadratic function by using the same QC of Eq. (2.30) for the nonlinearity of the MG compressor model.

5.1 Robustness of the Surge Subsystem with New Dynamic Output Feedback Controllers

Let us summarize the synthesis of stabilizing controllers as in [Shiriaev et al., 2010]. We will apply this method to the surge subsystem of the MG compressor model (1.3).

From Eq. (1.3), we know that the surge subsystem is:

$$\begin{aligned}\frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1-(1+\phi)^3}{2}\\ \frac{d}{dt}\psi &= \frac{1}{\beta^2}(\phi - u)\\ y &= \psi\end{aligned} \tag{5.1}$$

with the nonlinearity

$$W^{\{\phi\}}(\phi) = 1 - (1+\phi)^3 \tag{5.2}$$

We used the general form of a dynamic output feedback control law

$$u = \mathcal{U}(z, y), \quad \dot{z} = \mathcal{F}(z, y) \tag{5.3}$$

where $\mathcal{U}(\cdot)$ and $\mathcal{F}(\cdot)$ are smooth functions of appropriate dimensions. The family of stabilizing output feedback controllers has the following structure:

$$\begin{aligned}u &= \Lambda_\psi^{\{u\}}\psi + \Lambda_z^{\{u\}}z + \omega_u \cdot W(\psi, z)\\ \frac{d}{dt}z &= \Lambda_\psi^{\{z\}}\psi + \Lambda_z^{\{z\}}z + \omega_z \cdot W(\psi, z)\end{aligned} \tag{5.4}$$

with $z \in \mathbb{R}$, where $\Lambda_\psi^{\{u\}}$, $\Lambda_z^{\{u\}}$, $\Lambda_\psi^{\{z\}}$, $\Lambda_z^{\{z\}}$, ω_u, ω_z are constants to be defined.

The nonlinearities in the controller of Eq. (5.4) are static nonlinearities defined as

$$W(\psi, z) = 1 - (1 + t_\psi\psi + t_z z)^3 \tag{5.5}$$

where t_ψ, t_z are constants to be determined.

The closed-loop system with the surge subsystem of Eq. (5.1) and the controller of Eq. (5.4) takes the form:

$$\begin{aligned}\begin{bmatrix}\dot{\phi}\\ \dot{\psi}\\ \dot{z}\end{bmatrix} = &\underbrace{\begin{bmatrix}\frac{3}{2} & -1 & 0\\ \frac{1}{\beta^2} & -\frac{\Lambda_\psi^{\{u\}}}{\beta^2} & -\frac{\Lambda_z^{\{u\}}}{\beta^2}\\ 0 & \Lambda_\psi^{\{z\}} & \Lambda_z^{\{z\}}\end{bmatrix}}_{=\mathcal{A}_{cl}}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix}\\ &+ \underbrace{\begin{bmatrix}\frac{1}{2} & 0\\ 0 & -\frac{\omega_u}{\beta^2}\\ 0 & \omega_z\end{bmatrix}}_{\mathcal{B}_{cl}=[\mathcal{B}_{cl_1},\ \mathcal{B}_{cl_2}]}\begin{bmatrix}W^{\{\phi\}}(\phi)\\ W(\psi, z)\end{bmatrix}\end{aligned} \tag{5.6}$$

with $z \in \mathbb{R}$ and output matrix $\mathcal{C}_{cl_2} = [0,\ t_\psi,\ t_z]$.

The task is to analyze the quality of the set of the presented stabilizing dynamic output feedback controllers.

As we already know, in the closed-loop system of Eq. (5.6) there are two nonlinearities of Eqs. (5.2) and (5.5). We will simplify the notation

$$\begin{aligned} W^{\{\phi\}}(\upsilon_1) &= W^{\{\phi\}}(\phi) \\ W(\upsilon_2) &= W(\psi, z) \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} \upsilon_1 = \mathcal{C}_{cl_1}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \\ \upsilon_2 = \mathcal{C}_{cl_2}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} &= \begin{bmatrix} 0 & t_\psi & t_z \end{bmatrix}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \end{aligned} \tag{5.8}$$

One of the main parts of the constructive steps in the design is that the nonlinearities $W^{\{\phi\}}(\upsilon_1)$ and $W(\upsilon_2)$ have to satisfy the quadratic constraints of Eq. (2.30)

$$\begin{aligned} -\mathcal{C}_{cl_1}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} W^{\{\phi\}}(\phi) - \frac{3}{4}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix}^T \mathcal{C}_{cl_1}^T \mathcal{C}_{cl_1}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \geq 0 \\ -\mathcal{C}_{cl_2}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} W(\psi, z) - \frac{3}{4}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix}^T \mathcal{C}_{cl_2}^T \mathcal{C}_{cl_2}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \geq 0 \end{aligned} \tag{5.9}$$

By using the notation of Eqs. (5.7, 5.8) we can rewrite the quadratic constraints of Eq. (5.9)

$$\begin{aligned} -W^{\{\phi\}}(\upsilon_1)\upsilon_1 - \frac{3}{4}|\upsilon_1|^2 \geq 0 \\ -W(\upsilon_2)\upsilon_2 - \frac{3}{4}|\upsilon_2|^2 \geq 0 \end{aligned} \tag{5.10}$$

Since the static nonlinearity $W(\upsilon_2)$ is assumed to resemble the original nonlinearity $W^{\{\phi\}}(\upsilon_1)$ of the system of Eq. (5.1) we need to have one additional constraint that will be connected to both nonlinearities

$$-(W^{\{\phi\}}(\upsilon_1) - W(\upsilon_2))(\upsilon_1 - \upsilon_2) \geq 0 \tag{5.11}$$

The three quadratic constraints of Eqs. (5.10–5.11) should be satisfied $\forall \phi, \psi, z$.

To check the stability of the closed-loop system of Eq. (5.6) we will use the *Circle criterion* (CC) [Yakubovich et al., 2004], [Shiriaev et al., 2010]. In general we have

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z} \end{bmatrix} = \mathcal{A}_{cl} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + [\mathcal{B}_{cl_1},\ \mathcal{B}_{cl_2}] \begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{bmatrix} \tag{5.12}$$

where $\tilde{\omega}_1$ and $\tilde{\omega}_2$ represent the nonlinearities of the form $W^{\{\phi\}}(\phi)$ and $W(\psi, z)$ and satisfy the given conditions of Eqs. (5.10–5.11). The constrained problem can be reformulated:

1. There are constants $\tau_1 \geq 0,\ \tau_2 \geq 0,\ \tau_3 \geq 0$ such that $\tau_1 + \tau_2 + \tau_3 > 0$ and there are transfer functions

$$\begin{aligned} G_{11}(j\omega)\tilde{\omega}_1 &= \mathcal{C}_{cl_1}(j\omega I_n - \mathcal{A}_{cl})^{-1}\mathcal{B}_{cl_1}\tilde{\omega}_1 \\ G_{12}(j\omega)\tilde{\omega}_2 &= \mathcal{C}_{cl_1}(j\omega I_n - \mathcal{A}_{cl})^{-1}\mathcal{B}_{cl_2}\tilde{\omega}_2 \end{aligned} \tag{5.13}$$

   such that

$$\begin{aligned} &-\tau_1 \mathrm{Re}\{\tilde{\omega}_1^* \tilde{v}_1 + \frac{3}{4}|\tilde{v}_1|^2\} - \tau_2 \mathrm{Re}\{\tilde{\omega}_2^* \tilde{v}_2 + \frac{3}{4}|\tilde{v}_2|^2\} \\ &\qquad - \tau_3 \mathrm{Re}\{(\tilde{\omega}_1 - \tilde{\omega}_2)^*(\tilde{v}_1 - \tilde{v}_2)\} < 0 \end{aligned} \tag{5.14}$$

   holds $\forall \tilde{\omega}_1 \in \mathbb{C},\ \forall \tilde{\omega}_2 \in \mathbb{C},\ \forall \omega \in \mathbb{R}$, where

$$\begin{aligned} \tilde{v}_1 &= G_{11}(j\omega)\tilde{\omega}_1 + G_{12}(j\omega)\tilde{\omega}_2 \\ &= \mathcal{C}_{cl_1}(j\omega I_n - \mathcal{A}_{cl})^{-1}\left[\mathcal{B}_{cl_1}\tilde{\omega}_1 + \mathcal{B}_{cl_2}\tilde{\omega}_2\right] \\ \tilde{v}_2 &= G_{21}(j\omega)\tilde{\omega}_1 + G_{22}(j\omega)\tilde{\omega}_2 \\ &= \mathcal{C}_{cl_2}(j\omega I_n - \mathcal{A}_{cl})^{-1}\left[\mathcal{B}_{cl_1}\tilde{\omega}_1 + \mathcal{B}_{cl_2}\tilde{\omega}_2\right] \end{aligned} \tag{5.15}$$

2. The matrix

$$\left(\mathcal{A}_{cl} - \frac{3}{4}\mathcal{B}_{cl_1}\mathcal{C}_{cl_1} - \frac{3}{4}\mathcal{B}_{cl_2}\mathcal{C}_{cl_2}\right) \tag{5.16}$$

   is Hurwitz.

   In Fig. 5.1 sector conditions for some nonlinearity $\tilde{\Psi}(\tilde{x}, t)$ and a loop transformation are illustrated as an example [Khalil, 2002, p.255]. The given nonlinearity never leaves the sector area between two lines

$$k_2\tilde{x} \geq \tilde{\Psi}(\tilde{x}, t) \geq k_1\tilde{x} \tag{5.17}$$

   We can move the whole sector and the given nonlinearity clockwise on the same angle as the angle between zero and the line $k_1\tilde{x}$. As a result we

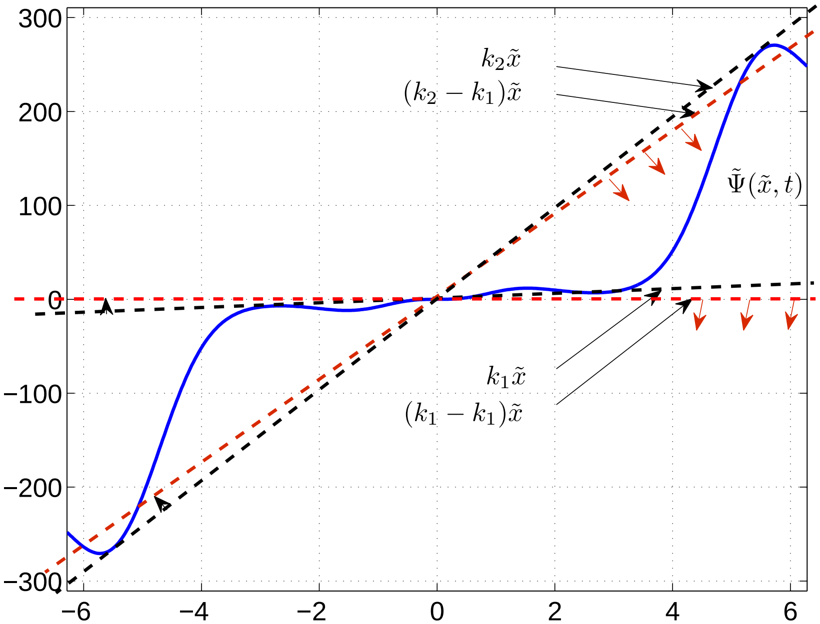

Figure 5.1 An example demonstrating how sector conditions change by loop transformation.

will get a new nonlinearity and new sector conditions for it to simplify the following calculations.

We will rewrite the closed-loop system of Eq. (5.6) and Eq. (5.12) as follows

$$\dot{x} = \mathcal{A}_{cl}x + \begin{bmatrix} \mathcal{B}_{cl_1} & \mathcal{B}_{cl_2} \end{bmatrix} \begin{bmatrix} \tilde{\omega}_1 + \frac{3}{4}\mathcal{C}_{cl_2}x \\ \tilde{\omega}_2 + \frac{3}{4}\mathcal{C}_{cl_2}x \end{bmatrix} - \begin{bmatrix} \mathcal{B}_{cl_1} & \mathcal{B}_{cl_2} \end{bmatrix} \begin{bmatrix} \frac{3}{4}\mathcal{C}_{cl_2}x \\ \frac{3}{4}\mathcal{C}_{cl_2}x \end{bmatrix} \quad (5.18)$$

where $x = [\phi, \psi, z]^T$ is the state vector.

That gives us

$$\begin{aligned}\dot{x} &= \mathcal{A}_{cl}x - [\mathcal{B}_{cl_1}\ \mathcal{B}_{cl_2}]\begin{bmatrix}\frac{3}{4}\mathcal{C}_{cl_2}x \\ \frac{3}{4}\mathcal{C}_{cl_2}x\end{bmatrix} + [\mathcal{B}_{cl_1}\ \mathcal{B}_{cl_2}]\begin{bmatrix}\hat{\omega}_1 \\ \hat{\omega}_2\end{bmatrix} \\ &= \underbrace{\left[\mathcal{A}_{cl} - \frac{3}{4}\mathcal{B}_{cl_1}\mathcal{C}_{cl_2} - \frac{3}{4}\mathcal{B}_{cl_2}\mathcal{C}_{cl_2}\right]}_{\hat{\mathcal{A}}_{cl}} x + [\mathcal{B}_{cl_1}\ \mathcal{B}_{cl_2}]\begin{bmatrix}\hat{\omega}_1 \\ \hat{\omega}_2\end{bmatrix}\end{aligned} \tag{5.19}$$

where $\hat{A}_{cl}$ is a new state matrix and $\hat{\omega}_{1,2}$ are new nonlinearities such that

$$\begin{aligned}\hat{\omega}_1 &= \tilde{\omega}_1 + \frac{3}{4}\mathcal{C}_{cl_2}x, \\ \hat{\omega}_2 &= \tilde{\omega}_2 + \frac{3}{4}\mathcal{C}_{cl_2}x\end{aligned} \tag{5.20}$$

and satisfy the similar conditions of Eqs. (5.10–5.11).

The new quadratic constraints are

$$\begin{aligned}\hat{\omega}_2^* \upsilon_2 &\leq 0 \\ (\hat{\omega}_1 - \hat{\omega}_2)^*(\upsilon_1 - \upsilon_2) &\leq 0\end{aligned} \tag{5.21}$$

As presented in [Shiriaev et al., 2010], by choosing $\tau_1,\ \tau_2$ or τ_3 equal to zero we can remove one of the terms of Eq. (5.14). For example, with $\tau_1 = 0$ and $\tau_2 = 1$ we can reduce the inequality of Eq.(5.14) with the new nonlinearities of Eq. (5.20) to

$$-\mathrm{Re}\{\hat{\omega}_2^* \hat{\upsilon}_2\} - \tau_3 \mathrm{Re}\{(\hat{\omega}_1 - \hat{\omega}_2)^*(\hat{\upsilon}_1 - \hat{\upsilon}_2)\} < 0 \tag{5.22}$$

with

$$\begin{aligned}\upsilon_1 &= \hat{G}_{11}(j\omega)\hat{\omega}_1 + \hat{G}_{12}(j\omega)\hat{\omega}_2 \\ &= \mathcal{C}_{cl_1}(j\omega I_n - \hat{\mathcal{A}}_{cl})^{-1}\left[\mathcal{B}_{cl_1}\hat{\omega}_1 + \mathcal{B}_{cl_2}\hat{\omega}_2\right] \\ \upsilon_2 &= \hat{G}_{21}(j\omega)\hat{\omega}_1 + \hat{G}_{22}(j\omega)\hat{\omega}_2 \\ &= \mathcal{C}_{cl_2}(j\omega I_n - \hat{\mathcal{A}}_{cl})^{-1}\left[\mathcal{B}_{cl_1}\hat{\omega}_1 + \mathcal{B}_{cl_2}\hat{\omega}_2\right]\end{aligned} \tag{5.23}$$

which holds $\forall \hat{\omega}_1 \in \mathbb{C},\ \forall \hat{\omega}_2 \in \mathbb{C},\ \forall \omega \in \mathbb{R}$.

To present the inequality of Eq. (5.22) in matrix form we have to change the variables as follows:

1. From the first part of the inequality of Eq. (5.22) we have

$$\begin{aligned}\mathrm{Re}\{\hat{\omega}_2^*\hat{v}_2\} &= \mathrm{Re}\{\hat{\omega}_2^*(\hat{G}_{21}(j\omega)\hat{\omega}_1 + \hat{G}_{22}(j\omega)\hat{\omega}_2 \pm \hat{G}_{21}(j\omega)\hat{\omega}_2)\} \\ &= \mathrm{Re}\{\hat{\omega}_2^*(\hat{G}_{21}(j\omega)(\hat{\omega}_1 - \hat{\omega}_2) + \hat{\omega}_2 \underbrace{(\hat{G}_{22}(j\omega) + \hat{G}_{21}(j\omega))}_{G_1(j\omega)})\} \\ &= \frac{1}{2}\{\hat{\omega}_2^*\hat{G}_{21}(j\omega)(\hat{\omega}_1 - \hat{\omega}_2)\} + \frac{1}{2}\{(\hat{\omega}_1 - \hat{\omega}_2)^*\hat{G}_{21}^*(j\omega)\hat{\omega}_2\} \\ &\quad + \mathrm{Re}\{\hat{\omega}_2^* G_1(j\omega)\hat{\omega}_2\}\end{aligned} \tag{5.24}$$

with

$$G_1(j\omega) = \hat{G}_{22}(j\omega) + \hat{G}_{21}(j\omega) \tag{5.25}$$

2. From the second part of the inequality of Eq. (5.22) we have

$$\begin{aligned}&\tau_3\mathrm{Re}\{(\hat{\omega}_1 - \hat{\omega}_2)^*(\hat{v}_1 - \hat{v}_2)\} = \\ &\quad = \tau_3\mathrm{Re}\{(\hat{\omega}_1 - \hat{\omega}_2)^*(\hat{\omega}_1 \underbrace{(\hat{G}_{11}(j\omega) - \hat{G}_{21}(j\omega))}_{G_2(j\omega)})\} \\ &\qquad + \tau_3\mathrm{Re}\{(\hat{\omega}_1 - \hat{\omega}_2)^*(\hat{\omega}_2 \underbrace{(\hat{G}_{12}(j\omega) - \hat{G}_{22}(j\omega))}_{G_3(j\omega)})\} \\ &\quad = \tau_3\mathrm{Re}\{(\hat{\omega}_1 - \hat{\omega}_2)^*(G_2(j\omega)(\hat{\omega}_1 - \hat{\omega}_2) + \hat{\omega}_2 \underbrace{(G_3(j\omega) + G_2(j\omega))}_{G_4(j\omega)})\} \\ &\quad = \tau_3\mathrm{Re}\{(\hat{\omega}_1 - \hat{\omega}_2)^* G_2(j\omega)(\hat{\omega}_1 - \hat{\omega}_2)\} \\ &\qquad + \frac{1}{2}\{(\hat{\omega}_1 - \hat{\omega}_2)^*\tau_3 G_4(j\omega)\hat{\omega}_2\} + \frac{1}{2}\{\hat{\omega}_2^*\tau_3 G_4^*(j\omega)(\hat{\omega}_1 - \hat{\omega}_2)\}\end{aligned} \tag{5.26}$$

with

$$\begin{aligned}G_2(j\omega) &= \hat{G}_{11}(j\omega) - \hat{G}_{21}(j\omega) \\ G_3(j\omega) &= \hat{G}_{12}(j\omega) - \hat{G}_{22}(j\omega) \\ G_4(j\omega) &= G_3(j\omega) + G_2(j\omega)\end{aligned} \tag{5.27}$$

Now we are able to rewrite the inequality of Eq. (5.22) in the matrix form

$$\omega_a^*\Pi(j\omega)\omega_a > 0,\ \|\omega_a\| \neq 0 \tag{5.28}$$

with

$$\omega_a = \begin{bmatrix} \hat{\omega}_1 \\ \hat{\omega}_1 - \hat{\omega}_2 \end{bmatrix} \tag{5.29}$$

and with

$$\Pi(j\omega) = \begin{bmatrix} \mathrm{Re}\{G_1(j\omega)\} & 0.5\,(G_{21}(j\omega) + \tau_3 G_4^*(j\omega)) \\ 0.5\,(G_{21}^*(j\omega) + \tau_3 G_4(j\omega)) & \tau_3\mathrm{Re}\{G_2(j\omega)\} \end{bmatrix} \tag{5.30}$$

and it should be valid for some $\tau_3 > 0$, $\forall \tilde{\omega}_1, \tilde{\omega}_2 \in \mathbb{C}$ and $\forall \omega \in \mathbb{R}$. The inequality of Eq. (5.28) is positive if the matrix $\Pi(j\omega)$ is positive definite. Since in this case we have a 2×2 matrix it will be enough to show that its determinant and diagonal elements are positive.

The alternative proof of stability of the closed-loop system of Eq. (3.12) uses a smaller number of conditions than in Section 3.5. There is a possibility to include a new quadratic constraint in the analysis if necessary. But the sufficient conditions for stabilization in Section 3.5 are easier to use in application programming in terms of accuracy and the order of calculations.

5.2 Example III

To show the benefit of the alternative proof we choose the same numerical values for the linear part for the controller (5.4) as in Example 3.7:

$$\Lambda_{\psi}^{\{u\}} = -19,\ \Lambda_{z}^{\{u\}} = -7,\ \Lambda_{\psi}^{\{z\}} = -73,\ \Lambda_{z}^{\{z\}} = -26 \tag{5.31}$$

and the corresponding parameters of the nonlinear part of the controller are:

$$\begin{aligned} t_{\psi} &= 4.7168,\ t_z = 1.4492; \\ \omega_u &= -1,\ \omega_z = -2.9027; \end{aligned} \tag{5.32}$$

The controller is thus given by

$$\begin{aligned} u &= -19\psi - 7z - W(\psi, z) \\ \frac{d}{dt}z &= -73\psi - 26z - 2.9027W(\psi, z) \end{aligned} \tag{5.33}$$

where

$$W(\psi, z) = 1 - (1 + 4.7168\psi + 1.4492z)^3 \tag{5.34}$$

Now we are able to check the conditions of Eqs. (5.14, 5.16):

1. The transfer functions are

$$\begin{aligned} G_1(j\omega) &= \frac{0.5102s + 0.8708}{s^2 + 2.671s + 0.8143} \\ G_{21}(j\omega) &= \frac{2.358s + 8.423}{s^3 + 5.883s^2 + 9.393s + 2.615} \\ G_2(j\omega) &= \frac{0.5s + 0.1858}{s^2 + 3.563s + 1.127} \\ G_4(j\omega) &= \frac{-0.01021s - 7.644 \cdot 10^{-4}}{s^2 + 3.563s + 1.127} \end{aligned} \tag{5.35}$$

We already know that the inequality of Eq. (5.28) is positive if the matrix $\Pi(j\omega)$ of Eq. (5.30) is positive definite. In our case we have a 2×2 matrix and it will be sufficient to show the positiveness of its determinant and diagonal elements.

The determinant and the diagonal elements of the matrix of Eq. (5.30) are positive for the parameter

$$\tau_3 > 2.64 \tag{5.36}$$

For higher-order polynomials the derivation of the determinant of the matrix $\Pi(j\omega)$ will be more complicated. Hence, the approximation was chosen in order to simplify the method presentation in this work.

2. The eigenvalues of the system matrix of Eq. (5.16) are $(-0.1268,\ -0.8753,\ -7.8730)$, hence it is Hurwitz.

For analyzing results we have to refer to the condition for the nonlinearities of Eq. (5.11). The nonlinearity $W^{\{\phi\}}(\phi)$ is the original nonlinearity from the MG compressor model of Eq. (1.1). The nonlinearity $W(\psi, z)$ is from the controller of Eq. (5.4) that is defined during the parameterization of the dynamic output feedback controllers.

The parameter τ_3 belongs to the condition for the nonlinearities of Eq. (5.11) as it can be seen in the inequality of Eq. (5.14). In other words, the controller of Eq. (5.33) is resistant to some of the parametric uncertainty for some $\tau_3 > 2.64$ in the matrix $\Pi(j\omega)$ of Eq. (5.30).

5.3 The Matrix Search Method

A Lyapunov function is a scalar function that can be used to prove the stability of an equilibrium point. It is named in honor of the Russian mathematician Aleksandr Mikhailovich Lyapunov [Lyapunov, 1892]. In general the search of a Lyapunov function $V(\cdot)$ for nonlinear systems is not a simple procedure. In our case, the complication is in the unknown state ϕ in the equations describing the MG model and the approximation of the order of the nonlinearity (the third order of the nonlinearity is an approximation).

We know, that in the MG compressor model of Eq. (1.1) the deviation of the averaged flow ϕ is not available to measurements. We also know, that the control design is based on the system transformation into a specific block form of Eq. (3.3). Thereby, to simplify the calculations, we will find a quadratic function for the known subsystems from the system of Eq. (3.6)

$$\begin{bmatrix} \dot{\psi} \\ \dot{z} \end{bmatrix} = \mathcal{A}_1 \begin{bmatrix} \psi \\ z \end{bmatrix} + \mathcal{B}_{cl_2} W(\psi, z) \tag{5.37}$$

First, we choose the quadratic function as

$$V(\psi, z) = \frac{1}{2}\begin{bmatrix} \psi \\ z \end{bmatrix}^T P \begin{bmatrix} \psi \\ z \end{bmatrix} \tag{5.38}$$

with a matrix $P^T = P > 0$ such that the time-derivative of $V(\psi, z)$ is negative.

The quadratic constraint of Eq.(5.9) is valid for the static nonlinearity $W(\psi, z)$ from the closed-loop system with the subsystem of Eq. (5.37) and the controller of Eq. (5.4).

Then the time-derivative of the quadratic function of Eq. (5.38) is

$$\begin{aligned}
\frac{d}{dt}V(\psi, z) &= \frac{1}{2}\begin{bmatrix} \psi \\ z \end{bmatrix}^T [\mathcal{A}_1^T P + P\mathcal{A}_1]\begin{bmatrix} \psi \\ z \end{bmatrix} + \begin{bmatrix} \psi \\ z \end{bmatrix}^T P\mathcal{B}_{cl_2}W(\psi, z) \\
&\leq \frac{1}{2}\begin{bmatrix} \psi \\ z \end{bmatrix}^T [\mathcal{A}_1^T P + P\mathcal{A}_1]\begin{bmatrix} \psi \\ z \end{bmatrix} + \begin{bmatrix} \psi \\ z \end{bmatrix}^T P\mathcal{B}_{cl_2}W(\psi, z) \\
&\quad + \left[-\mathcal{C}_{cl_2}\begin{bmatrix} \psi \\ z \end{bmatrix} W(\psi, z) - \frac{3}{4}\begin{bmatrix} \psi \\ z \end{bmatrix}^T \mathcal{C}_{cl_2}^T\mathcal{C}_{cl_2}\begin{bmatrix} \psi \\ z \end{bmatrix}\right] \\
&= \frac{1}{2}\begin{bmatrix} \psi \\ z \end{bmatrix}^T \left[\mathcal{A}_1 P + P\mathcal{A}_1 - \frac{3}{2}\mathcal{C}_{cl_2}^T\mathcal{C}_{cl_2}\right]\begin{bmatrix} \psi \\ z \end{bmatrix} \\
&\quad + \begin{bmatrix} \psi \\ z \end{bmatrix}^T \left[P\mathcal{B}_{cl_2} - \mathcal{C}_{cl_2}^T\right] W(\psi, z)
\end{aligned} \tag{5.39}$$

where matrices $\mathcal{B}_{cl_2}$, $\mathcal{C}_{cl_2}$ are the same as in Eq. (5.6).

A sufficient condition for a negative time derivative of the quadratic function of Eq. (5.38) can be written as

$$\begin{cases} \mathcal{A}_1^T P + P\mathcal{A}_1 - \frac{3}{2}\mathcal{C}_{cl_2}^T\mathcal{C}_{cl_2} < 0 \\ P\mathcal{B}_{cl_2} = \mathcal{C}_{cl_2}^T \end{cases} \tag{5.40}$$

To find the matrix P for the expression of Eq. (5.40) we solve a convex optimization problem by using CVX - a package for specifying and solving convex programs [Boyd and Vandenberghe, 2004], [Grant and Boyd, 2008], [CVX Research, 2012].

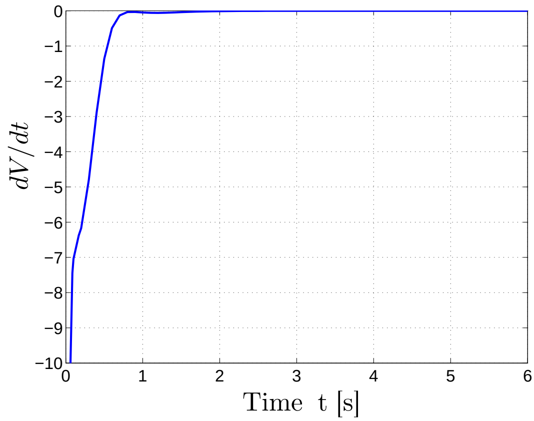

Figure 5.2 The time derivative of the quadratic function of Eq. (5.38).

The specification for CVX looks like:

```
cvx_begin sdp
variable P(2,2) symmetric
minimize(0)
subject to
P>0;
A1'*P+P*A1-3/2* C2'*C2<0;
P*B2==C2';
cvx_end
```

For the example in Subsection 5.2 we get

$$P = \begin{bmatrix} 31.5657 & 10.0621 \\ 10.0621 & 3.2168 \end{bmatrix} > 0 \tag{5.41}$$

In Fig. 5.2 the time derivative of the quadratic function of Eq. (5.38) is presented.

According to the CVX Users' Guide this software tool is working within a predefined tolerance and its computational methods for convex optimization are not exact. The presence of the equality in the expression of Eq. (5.40) is making this mathematical problem complicated because solvers treat it as a non-strict inequality. The matrix P of Eq. (5.41) is an approximate solution and it is depending on the chosen accuracy for the calculations. In addition, we were not able to find the solution for the similar condition as in Eq. (5.40) but for the closed-loop surge subsystem of Eq. (5.6) by using CVX solvers. That is why for the higher-order calculations we suggest to use the condition of Eq. (5.28) to investigate the robustness of the controller of Eq. (5.4).

5.4 Discussion

In control theory it is very important to investigate the robustness of the stabilizing controllers. We presented an extension of previous research based on a new procedure for dynamic output feedback design for systems with nonlinearities satisfying quadratic constraints [Shiriaev et al., 2009], [Shiriaev et al., 2010], [Rubanova et al., 2013]. The new constraint for the robustness investigation was presented as an inequality of Eq. (5.28).

The coefficient τ_3 belongs to the condition for the nonlinearities of Eq. (5.11) as it shown in the inequality of Eq. (5.14). It is possible to find some positive τ_3 for all the controller of Eq. (3.2). In other words, the controllers have a certain degree of robustness and there is the possibility to resist some of the parametric uncertainty.

The inequality of Eq. (5.28) is a more general stability condition than presented in the previous part of the research in [Rubanova et al., 2013]. The matrix $\Pi(j\omega)$ derivation (see Eq. (5.30)) is shown for the general formula of Eq. (5.28) that can be used for all controllers of the structure of Eq. (5.4) which parameters are satisfying all the conditions, presented in [Shiriaev et al., 2010], [Rubanova et al., 2013].

6

Conclusions

We presented an extension, simplification and application of a method in [Shiriaev et al., 2010] for dynamic output feedback design for systems with nonlinearities satisfying quadratic constraints applied to the surge subsystem and the Moore-Greitzer compressor model. Sufficient conditions for stabilization of the surge subsystem and of the MG compressor model required an additional search procedure for coefficients of the controller. An analytical parameterization of the family of controllers has been derived and verified by simulations. Moreover, we presented two different types of controllers: one for the surge subsystem and another (with an integral state) for both the full MG system and the surge subsystem. We proved the existence of the unique stable equilibrium in origin for both cases.

Sufficient conditions for stabilization were simplified, updated and used in the analytical parameterization of the family of controllers. Therefore, we were able to speed up the process of deriving numerical values of parameters for the stabilizing controllers and give the regions of the parameter search that is shown in Fig. 3.1. The set of parameters of the stabilizing controllers for the surge subsystem and the set of parameters of the stabilizing controllers with extended integral part for MG compressor model has been presented.

The optimal control parameters from the stabilizing set were chosen with respect to desired criteria. We decided to choose the task when the output of the system will have minimum oscillations and the control signal will have as small amplitude as possible. The results were presented in Example I (Chapter 3) and in Example II (Chapter 4).

The results extend to systems with nonlinearities satisfying quadratic constraints. The implementation of the procedure for dynamic output feedback design has been done for the surge subsystem and the MG compressor model. It is possible to adapt our controller parameter search method for similar systems and derive a set of stabilizing dynamic output feedback controllers.

We also discussed the degree of robustness and presented an alternative proof of the stability of the closed-loop system from the Design I (the surge subsystem with the stabilizing dynamic output feedback controller without an integral state). This proof uses less numbers of stability conditions, than in Sections 3.5-4.4. Also, there is a possibility to include a new quadratic constraint in the analysis. However, the sufficient conditions for stabilization are easier to apply. The derivation of a quadratic function for the known subsystem of Eq. (3.6) by using the Lyapunov function search method was presented.

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