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Balanced Truncation for Discrete Time Markov Jump Linear Systems.

Georgios Kotsalis, Anders Rantzer

Abstract

This paper investigates the model reduction problem for mean square stable discrete time Markov jump linear systems. For this class of systems a balanced truncation algorithm is developed. The reduced order model is suboptimal, however the approximation error, which is captured by means of the stochastic L_2 gain, is bounded from above by twice the sum of singular numbers associated to the truncated states of each mode. Such a result allows rigorous simplification of the dynamics of each mode in an independent manner with respect to a metric which is relevant from a robust control point of view.

I. INTRODUCTION

Jump linear systems (JLS's) form an important class of hybrid systems, which combine continuous and discrete dynamics. They present an extension of linear time invariant (LTI) systems, in the sense that they use state update laws, which are linear with respect to the analog state, with matrix coefficient depending on a quantized auxiliary input, frequently referred to as the switching signal. The transition between the different modes of operation is controlled by this exogenous parametric input. In this work it is assumed that the switching signal takes values in a finite set and that it follows an unconstrained evolution, modeled by a finite memory stochastic process.

There is a large body of literature in the fields of econometrics and system theory pertaining to the class of JLS's with randomly varying parameters. Various analysis and synthesis results applicable to linear time-invariant systems have been extended to the class of Markov jump linear systems (MJLS's). A comprehensive review of this material can be found in [1] and the references therein.

A major question associated with MJLS's is that of complexity reduction. The work in [2] investigates the problem of obtaining an optimal in terms of the stochastic L_2 gain reduced model of fixed order. The formulation in [2] leads to a non convex optimization problem and the proposed algorithms do not guarantee convergence to the global optimum. In contrast to [2] the search of a reduced model in the current paper is based on a convex programming formulation, the obtained reduced model is suboptimal in terms of the stochastic L_2 gain, however it

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is accompanied by an a priori computable upper bound to the approximation error. The reduction algorithm in this work can be interpreted as an extension of the well known balanced truncation algorithm for linear time invariant systems (LTI) to the wider class of MJLS's.

Balanced realizations were originally proposed in the controls literature in [3]. Their utilization for model reduction purposes of LTI systems and associated error bounds in continuous and discrete-time settings can be found in [4], [5] and [6]. Balanced truncation has been investigated also outside the realm of LTI systems. In [7] a generalization to multidimensional and uncertain systems in the linear-fractional framework is presented. The case of linear parameter-varying systems is the subject of [8] and linear time-varying systems are handled in [9] and [10]. Balanced Truncation of JLS's with independent identically distributed parameters is investigated in [11].

Approximation algorithms for various classes of stochastic hybrid systems based on the concept of approximate bisimulation are developed in [12].

A. Notation

The set of nonnegative integers is denoted by \mathbf{N} , the set of positive integers by \mathbf{Z}_+ and the set of real numbers by \mathbf{R} . For $n \in \mathbf{Z}_+$ let \mathbf{R}^n denote the Euclidean n -space. The transpose of a column vector $x \in \mathbf{R}^n$ is x' . For $x \in \mathbf{R}^n$ let $|x|^2 = x'x$ denote the square of the Euclidean norm. For $P \in \mathbf{R}^{n \times n}$ let $P > 0$ indicate that it is a positive definite matrix and the notation $|x|_P^2$ stands for $x'Px$, the square of the weighted norm of $x \in \mathbf{R}^n$. The positive definite square root of P is denoted by $P^{\frac{1}{2}}$. The identity matrix in $\mathbf{R}^{n \times n}$ is written as I_n . For $A \in \mathbf{R}^{n \times n}$ let $r_\sigma[A]$ denote the spectral radius of A . For $P, Q \in \mathbf{R}^{n \times n}$, the inner product of these two matrices is defined as $\langle P, Q \rangle = \text{Tr}[P'Q]$. For $f : \mathbf{N} \rightarrow \mathbf{R}^n$, the notation f and $\{f(k)\}_{k=0}^\infty$ will be used interchangeably. The space of square summable vector sequences with elements in \mathbf{R}^n is denoted by l_2^n . For $f \in l_2^n$ let $\|f\|_2^2$ stand for $\sum_{k=0}^\infty |f(k)|^2$. The unit sphere in l_2^n is denoted by $S_2^n = \{f \in l_2^n : \|f\|_2 = 1\}$. The expected value of the random variable x is denoted by $\mathbf{E}[x]$. Given any time domain signal $x(t)$ its Laplace transform is denoted by $X(s)$. For $N \in \mathbf{Z}_+$, $n : \{1, \dots, N\} \rightarrow \mathbf{Z}_+$, define the space $\mathbf{H}^n = \mathbf{R}^{n[1] \times n[1]} \times \dots \times \mathbf{R}^{n[N] \times n[N]}$ and its subset \mathbf{H}_+^n as $\mathbf{H}_+^n = \{U \in \mathbf{H}^n \mid U[i] > 0, i \in \{1, \dots, N\}\}$. For $N \in \mathbf{Z}_+$, $n : \{1, \dots, N\} \rightarrow \mathbf{Z}_+$, $r : \{1, \dots, N\} \rightarrow \mathbf{N}$ with $r[i] < n[i], \forall i \in \{1, \dots, N\}$. The subspace of $\mathbf{R}^{n[i]}$ whose elements have the last $r[i]$ coordinates identically zero is $VC_{n[i]-r[i]} = \{x \in \mathbf{R}^{n[i]} \mid x[j] = 0, j > n[i] - r[i]\}$. Let $A[i] \in \mathbf{R}^{n[i] \times n[i]}, i \in \{1, \dots, N\}$, the block diagonal matrix

$$\begin{bmatrix} A[1] & 0 & 0 \\ & \ddots & \\ 0 & 0 & A[N] \end{bmatrix}$$

is denoted by $\text{diag}\{A[1], \dots, A[N]\}$.

II. PRELIMINARIES

A. System Model

Let $m, q, N \in \mathbf{Z}_+$, $\Theta = \{1, \dots, N\}$ and $n : \Theta \rightarrow \mathbf{Z}_+$. Define a MJLS \mathbf{L} as a dynamical system with input, $f(k) \in \mathbf{R}^m$, discrete valued state variable $\theta(k) \in \Theta$, also referred to as the system mode, mode transition signal $\phi(k) = (\theta(k-1), \theta(k))$, $k \in \mathbf{Z}_+$, continuous valued state variable $x(k) \in \mathbf{R}^{n[\theta(k)]}$, and output $y(k) \in \mathbf{R}^q$, related by the state space equations

$$\begin{aligned} x(k+1) &= A[\phi(k+1)]x(k) + B[\phi(k+1)]f(k), \\ y(k) &= C[\theta(k)]x(k), \quad k \in \mathbf{N}. \end{aligned} \tag{1}$$

The system mode θ can also be interpreted as an exogenous parametric input and follows an unconstrained stochastic evolution, modeled as a Markov process on Θ . The transition probability matrix of the Markov chain is denoted by $P = [p_{ij}]$, $i, j \in \Theta$, where $p_{ij} = \mathbf{P}[\theta(k+1) = j | \theta(k) = i]$. The input f is assumed to be deterministic, i.e. each $f(k)$ is a random variable with zero variance. The state space matrices have compatible dimensions, in particular

$$\begin{aligned} A[\phi(k+1)] &\in \mathbf{R}^{n[\theta(k+1)] \times n[\theta(k)]}, \\ B[\phi(k+1)] &\in \mathbf{R}^{n[\theta(k+1)] \times m}, \\ C[\theta(k)] &\in \mathbf{R}^{q \times n[\theta(k)]}, \quad \theta(k), \theta(k+1) \in \Theta. \end{aligned}$$

By having the matrices in the state space recursion depend on the mode transition rather than the discrete mode itself as is the case in the standard MJLS model [1], one can allow the dimension of the continuous valued part of the state variable to vary depending on which discrete mode the system resides in. Similar type of MJLS's as in (34) were considered in [13] and [14]. Apart from that, the model class in this paper is representative of MJLS's where the state space matrices in the state recursion exhibit stochastic dependence on the current mode. The latter model class has been used extensively in the study of networked control systems in a probabilistic framework, a review paper in that area is [15].

In the current setting a MJLS is defined on a directed graph, with Θ being the set of nodes. The results of this work generalize to the case where the MJLS is defined on a directed multigraph, namely a graph that has multiple directed edges emanating from a node and terminating to some other node. This direction is not going to be pursued in this paper, solely for purposes of achieving enhanced clarity in the exposition.

B. Stability

There are several stability notions for stochastic systems. The relevant concept to this work is that of mean square stability.

Definition 2.1: [16] The MJLS \mathbf{L} with $\{f(k)\} = \{0\}$ is mean square stable, if for every initial condition $\theta(0) \in \Theta$, $x(0) \in \mathbf{R}^{n[\theta(0)]}$

$$\mathbf{E}[|x(k)|^2] \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Theorem 2.1: Consider a MJLS \mathbf{L} . Let $F \in \mathbf{H}_+^n$. The MJLS \mathbf{L} is mean square stable iff there exists a unique $G \in \mathbf{H}_+^n$ such that :

$$G[i] - \sum_{j \in \Theta} p_{ij} A[i, j]' G[j] A[i, j] = F[i], \quad \forall i \in \Theta \quad (2)$$

Proof: "←" The above relation implies that $V[x, \theta] = |x|_{G[\theta]}^2$ acts as a stochastic Lyapunov function for the system \mathbf{L} . Consider the unforced evolution of the system \mathbf{L}

$$x(k+1) = A[\phi(k+1)]x(k),$$

and let

$$\hat{G}[i] = \sum_{j \in \Theta} p_{ij} A[i, j]' G[j] A[i, j],$$

then one has

$$\mathbf{E}[V[x(k+1), \theta(k+1)] | x(k), \theta(k)] = x(k)' \hat{G}[\theta(k)] x(k).$$

Relation (2) implies,

$$\begin{aligned} \mathbf{E}[V[x(k+1), \theta(k+1)] | x(k), \theta(k)] - V[x(k), \theta(k)] &< 0 \\ \forall x(k) \in \mathbf{R}^{n[\theta(k)]}, \forall \theta(k) \in \Theta. \end{aligned}$$

By making use of the law of iterated expectations, namely for any two random variables, x, y , $\mathbf{E}[\mathbf{E}[x|y]] = \mathbf{E}[x]$, one gets from the above relation

$$\mathbf{E}[V[x(k+1), \theta(k+1)] - V[x(k), \theta(k)]] < 0.$$

Thus

$$\mathbf{E}[V[x(k), \theta(k)]] \rightarrow 0 \text{ as } k \rightarrow \infty$$

and since $G[\theta(k)] > 0, \forall \theta(k) \in \Theta$ mean square stability follows.

"→" Define the mode indicator function

$$Z_i(k) = \begin{cases} 1 & \text{if } \theta(k) = i \\ 0 & \text{otherwise,} \end{cases}$$

and note that

$$\mathbf{E}[|x(k)|^2] = \mathbf{E}\left[\sum_{i \in \Theta} |x(k)|^2 Z_i(k)\right].$$

Let $X_i(k) = \mathbf{E}[x(k)x(k)'] Z_i(k)$. The mean square stability assumption implies that $X_i(k) \rightarrow 0$ as $k \rightarrow \infty$. The dynamics of $X_i(k)$ are given by

$$X_j(k+1) = \sum_{i \in \Theta} p_{ij} A[i, j] X_i(k) A[i, j]', \quad \forall j \in \Theta. \quad (3)$$

Define the linear operator $\mathcal{T} : \mathbf{H}^n \rightarrow \mathbf{H}^n$,

$$\mathcal{T}[V] = W, \quad W[j] = \sum_{i \in \Theta} p_{ij} A[i, j] V[i] A[i, j]', \quad j \in \Theta. \quad (4)$$

Let $X(k) = [X_1(k), \dots, X_N(k)]$, $k \in \mathbf{Z}_+$, one can then write relation (3) compactly

$$\mathcal{T}[X(k)] = X(k+1).$$

The mean square stability assumption implies that $r_\sigma[\mathcal{T}] < 1$. Define the linear operator $\mathcal{L} : \mathbf{H}^n \rightarrow \mathbf{H}^n$,

$$\mathcal{L}[V] = W, \quad W[i] = \sum_{j \in \Theta} p_{ij} A[i, j]' V[j] A[i, j], \quad i \in \Theta. \quad (5)$$

The set of equations in (2) can then be written as

$$\mathcal{L}[G] - G = -F.$$

The inner product of $V, S \in \mathbf{H}^n$ is given by

$$\langle V, S \rangle = \sum_{i \in \Theta} \text{Tr}[V[i]' S[i]].$$

The following calculation verifies that

$$\mathcal{L}' = \mathcal{T}.$$

$$\begin{aligned} \langle \mathcal{T}[V], S \rangle &= \sum_{j \in \Theta} \text{Tr}[\mathcal{T}_j[V]' S[j]] = \sum_{j \in \Theta} \text{Tr}[\mathcal{T}_j[V] S[j]] \\ &= \sum_{j \in \Theta} \sum_{i \in \Theta} p_{ij} \text{Tr}[A[i, j] V[i] A[i, j]' S[j]] \\ &= \sum_{j \in \Theta} \sum_{i \in \Theta} p_{ij} \text{Tr}[V[i] A[i, j]' S[j] A[i, j]] \\ &= \sum_{i \in \Theta} \sum_{j \in \Theta} p_{ij} \text{Tr}[V[i] A[i, j]' S[j] A[i, j]] \\ &= \sum_{i \in \Theta} \text{Tr}[V[i]' \mathcal{L}_i[S]] \\ &= \langle V, \mathcal{L}[S] \rangle. \end{aligned}$$

Since \mathcal{L} is the adjoint of \mathcal{T} one has $r_\sigma[\mathcal{T}] < 1 \rightarrow r_\sigma[\mathcal{L}] < 1$ and

$$G = \sum_{i=0}^{\infty} \mathcal{L}^i[F] > 0$$

is the unique positive definite solution to $\mathcal{L}[G] - G = -F$. ■

Definition 2.2: The stochastic L_2 gain for the MJLS \mathbf{L} is denoted by $\gamma_{\mathbf{L}}$ and is defined by

$$\gamma_{\mathbf{L}}^2 = \sup_{\theta(0) \in \Theta} \sup_{f \in S_2^m} \sum_{k=0}^{\infty} \mathbf{E}[|y(k)|^2]$$

under the assumption $x(0) = 0$, $\theta(0) \in \Theta$.

Lemma 2.1: Given is a MJLS \mathbf{L} and let $\gamma \in \mathbf{R}$, $\gamma > 0$. Consider a nonnegative, real valued, measurable function $V[x, \theta]$, $x \in \mathbf{R}^{n[\theta]}$, $\theta \in \Theta$, with $V[0, \theta] = 0$ and $\mathbf{E}[V[x(k), \theta(k)]] < \infty$ for all trajectories of \mathbf{L} . Suppose that

$$\begin{aligned} |y(k)|^2 + \mathbf{E}[V[x(k+1), \theta(k+1)] \mid x(k), \theta(k)] &\leq V[x(k), \theta(k)] + \gamma^2 |f(k)|^2 \\ \forall f(k) \in \mathbf{R}^m, \forall x(k) \in \mathbf{R}^{n[\theta(k)]}, \forall \theta(k) \in \Theta, \end{aligned} \quad (6)$$

then the stochastic L_2 gain of \mathbf{L} does not exceed γ .

Proof: Use the law of iterated expectations and sum up the above relation from $k = 0$ to $k = T$ to obtain

$$\begin{aligned} \sum_{k=0}^T \mathbf{E}[|y(k)|^2] + \mathbf{E}[V[x(T+1), \theta(T+1)]] &\leq V[x(0), \theta(0)] + \gamma^2 \sum_{k=0}^T |f(k)|^2 \\ \forall f(k) \in \mathbf{R}^m, k \in \{0, \dots, T\}. \end{aligned}$$

Under the assumption $x(0) = 0$, $\theta(0) \in \Theta$ one has that $V[x(0), \theta(0)] = V[0, \theta(0)] = 0$ and since

$$\mathbf{E}[V[x(T+1), \theta(T+1)]] \geq 0,$$

one gets

$$\begin{aligned} \sum_{k=0}^T \mathbf{E}[|y(k)|^2] &\leq \gamma^2 \sum_{k=0}^T |f(k)|^2 \\ \forall f(k) \in \mathbf{R}^m, k \in \{0, \dots, T\}. \end{aligned}$$

Let $T \rightarrow \infty$ and $f \in S_2^m$ to get

$$\sum_{k=0}^{\infty} \mathbf{E}[|y(k)|^2] \leq \gamma^2$$

and the lemma is proved. ■

Lemma 2.2: If the MJLS \mathbf{L} is mean square stable, then its stochastic L_2 gain is finite.

Proof: Lemma 2.1 will be employed in this proof. Consider the quadratic function $V[x, \theta] = x'G[\theta]x$, where $G \in \mathbf{H}_+^n$ and $x \in \mathbf{R}^{n[\theta]}$, $\theta \in \Theta$. The dissipation inequality (6) can be written equivalently as

$$W[i] \leq \begin{bmatrix} G[i] & 0 \\ 0 & \gamma^2 I_m \end{bmatrix}, \quad \forall i \in \Theta, \quad (7)$$

where

$$\begin{aligned} W[i] &= \begin{bmatrix} W_{11}[i] & W_{12}[i] \\ W'_{12}[i] & W_{22}[i] \end{bmatrix}, S[i, j] = \begin{bmatrix} A[i, j] & B[i, j] \\ C[i] & 0 \end{bmatrix} \\ W[i] &= \sum_{j \in \Theta} p_{ij} S[i, j]' \begin{bmatrix} G[j] & 0 \\ 0 & I_p \end{bmatrix} S[i, j], \quad i \in \Theta. \end{aligned}$$

By taking the Schur complement one obtains the following set of sufficient conditions for (7) to hold

$$\begin{aligned} W_{11}[i] &< G[i], \\ W_{22}[i] - W'_{12}[i](W_{11}[i] - G[i])^{-1}W_{12}[i] &< \gamma^2 I_m, \end{aligned} \quad (8)$$

$\forall i \in \Theta$. Mean square stability implies existence of positive definite matrices $\tilde{P} \in \mathbf{H}_+^n$ such that

$$\sum_{j \in \Theta} p_{ij} A[i, j]' \tilde{P}[j] A[i, j] - \tilde{P}[i] < 0.$$

Set $G[i] = \alpha \tilde{P}[i]$, $i \in \Theta$ with $\alpha \in \mathbf{R}$, $\alpha \geq 1$. The first set of conditions in (8) can be satisfied by taking α large enough. For a fixed value of α the second set of conditions in (8) can always be satisfied by taking γ large enough. So it has been shown that mean square stability implies that the conditions in (7) are satisfied and by invoking lemma 2.1 the gain of \mathbf{L} is finite. ■

C. Reduced order model and state truncation

Let $\hat{n} : \Theta \rightarrow \mathbf{Z}_+$ and $\hat{n}[\theta] \leq n[\theta]$, $\forall \theta \in \Theta$ with the inequality being strict for at least one of the modes. A reduced order MJLS is denoted by $\hat{\mathbf{L}}$ and has state space representation

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}[\phi(k+1)]\hat{x}(k) + \hat{B}[\phi(k+1)]f(k), \\ \hat{y}(k) &= \hat{C}[\theta(k)]\hat{x}(k), \quad k \in \mathbf{N}, \end{aligned} \quad (9)$$

where $\hat{y}(k) \in \mathbf{R}^q$, $f(k) \in \mathbf{R}^m$, $\theta(k) \in \Theta$ and $\hat{x}(k) \in \mathbf{R}^{\hat{n}[\theta(k)]}$. In order to quantify the fidelity of $\hat{\mathbf{L}}$, an error system $\mathbf{E}_{\{\mathbf{L}, \hat{\mathbf{L}}\}}$ is introduced, whose inputs are the common inputs $f(k)$, $\theta(k)$ of \mathbf{L} and $\hat{\mathbf{L}}$ and whose output is the difference of their outputs, namely $e(k) = y(k) - \hat{y}(k)$, $k \in \mathbf{N}$. The state space representation of the reduced order system is given by

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} [\phi(k+1)] \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} + \begin{bmatrix} B \\ \hat{B} \end{bmatrix} [\phi(k+1)] f(k), \\ e(k) &= \begin{bmatrix} C & -\hat{C} \end{bmatrix} [\theta(k)] \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}, \quad k \in \mathbf{N}. \end{aligned} \quad (10)$$

The objective of model reduction is to find a reduced order model such that the stochastic L_2 gain of the error

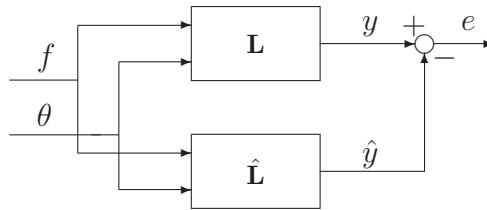


Fig. 1. Error System $\mathbf{E}_{\{\mathbf{L}, \hat{\mathbf{L}}\}}$

system $\mathbf{E}_{\{\mathbf{L}, \hat{\mathbf{L}}\}}$ is small. Reduced order models are obtained by means of truncation. The number of truncated states

at a particular mode is given by $r[\theta(k)] = n[\theta(k)] - \hat{n}[\theta(k)]$, $\theta(k) \in \Theta$. The following partitions are used :

$$\begin{aligned} A[\phi(k+1)] &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} [\phi(k+1)], \quad A_{11}[\phi(k+1)] \in \mathbf{R}^{\hat{n}[\theta(k+1)] \times \hat{n}[\theta(k)]}, \\ B[\phi(k+1)] &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [\phi(k+1)], \quad B_1[\phi(k+1)] \in \mathbf{R}^{\hat{n}[\theta(k)] \times m}, \\ C[\theta(k)] &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} [\theta(k)], \quad C_1[\theta(k)] \in \mathbf{R}^{q \times \hat{n}[\theta(k)]}, \\ x(k)' &= \begin{bmatrix} x_1(k)' & x_2(k)' \end{bmatrix}, \quad x_1(k) \in \mathbf{R}^{\hat{n}[\theta(k)]}. \end{aligned}$$

The state space matrices of the reduced order model are given by

$$\{\hat{A}[\phi(k+1)], \hat{B}[\phi(k+1)], \hat{C}[\theta(k)]\} = \{A_{11}[\phi(k+1)], B_1[\phi(k+1)], C_1[\theta(k)]\}.$$

In order to shorten subsequent notation, it will be convenient to think of the continuous part of the state variable of the reduced system submerged in the original state space. Let $\bar{x}(k)' = \begin{bmatrix} x_1(k)' & 0' \end{bmatrix} \in \mathbf{R}^{n[\theta(k)]}$. Consider the system $\bar{\mathbf{L}}$ with state space representation

$$\begin{aligned} \bar{x}(k+1) &= (I_{n[\theta(k+1)]} - E_{r[\theta(k+1)]}) \times (A[\phi(k)]\bar{x}(k) + B[\phi(k+1)]f(k)), \\ \bar{y}(k) &= C[\theta(k)]\bar{x}(k), \quad k \in \mathbf{N}, \end{aligned} \tag{11}$$

where

$$E_{r[\theta(k)]} = \begin{bmatrix} 0 & 0 \\ 0 & I_{r[\theta(k)]} \end{bmatrix} \in \mathbf{R}^{n[\theta(k)] \times n[\theta(k)]}.$$

Evidently one can identify $\bar{\mathbf{L}}$ with $\hat{\mathbf{L}}$, since for the same input signal $\{f(k)\}_{k \in \mathbf{Z}_+}$ in (9) and (11) and if $\bar{x}(0)' = \begin{bmatrix} \hat{x}(0)' & 0' \end{bmatrix}$, one has $\bar{x}(k)' = \begin{bmatrix} \hat{x}(k)' & 0' \end{bmatrix}$ and $\hat{y}(k) = \bar{y}(k)$, $\forall k \in \mathbf{Z}_+$. On these grounds, in the following $\hat{\mathbf{L}}$ will be used for both state space representations (9), (11). which one is meant will be clear from the context.

The idea of truncation presupposes that the states $x_2(k)$ are small in some appropriate sense. Mode dependent transformation matrices will be utilized to achieve this objective. Let $T[i] \in \mathbf{R}^{n[i] \times n[i]}$, $i \in \Theta$ be invertible matrices. Consider the change in coordinate system $x(k) = T[\theta(k)]\tilde{x}(k)$, the state space realization will transform to

$$\begin{aligned} \{A[\phi(k+1)], B[\phi(k+1)], C[\theta(k)]\} &\stackrel{T[\theta(k)]}{\rightarrow} \{\tilde{A}[\phi(k+1)], \tilde{B}[\phi(k+1)], \tilde{C}[\theta(k)]\} = \\ \{T[\theta(k+1)]^{-1}A[\phi(k+1)]T[\theta(k)], T[\theta(k+1)]^{-1}B[\phi(k+1)], C[\theta(k)]T[\theta(k)]\} &\end{aligned} \tag{12}$$

III. BALANCED TRUNCATION FOR MARKOV JUMP LINEAR SYSTEMS

A. Dissipation inequalities

A balanced truncation procedure for mean square stable MJLS's will be developed. The reduction procedure relies in the formulation of two sets of dissipation inequalities, in the form of linear matrix inequalities (LMI's). These two sets of LMI's, which will be referred to as input and output dissipation inequalities correspondingly, are

guaranteed to have solutions due to the mean square stability assumption. In fact it will be shown that mean square stability of the system is sufficient to guarantee solutions of a particular diagonal structure.

1) *Output dissipation inequalities:* Let $U \in \mathbf{H}_+^n$, the output dissipation inequalities are

$$|x|_{U[i]}^2 \geq \sum_{j \in \Theta} p_{ij} (|A[i, j]x|_{U[j]}^2) + |C[i]x|^2, \quad (13)$$

$$\forall x \in \mathbf{R}^{n[i]}, \forall i \in \Theta.$$

The above relations are LMI's and using the operator \mathcal{L} introduced in (5) they can be written more compactly as

$$\mathcal{L}[U] - U \leq -Q, \quad (14)$$

where $Q = [Q[1], \dots, Q[N]]$, with $Q[i] = C[i]'C[i] \geq 0, i \in \Theta$.

Lemma 3.1: Given a mean square stable MJLS \mathbf{L} , there exists $U \in \mathbf{H}_+^n$, such that (13) is satisfied.

Proof: The proof follows directly from theorem 2.1. Let $\tilde{Q}[i] = Q[i] + \epsilon I_{n[i]}, i \in \Theta$, where $\epsilon > 0$ is a small real number that guarantees that $\tilde{Q}[i] > 0$. Mean square stability is equivalent to the existence of a unique solution $U \in \mathbf{H}_+^n$ to

$$\mathcal{L}[U] - U = -\tilde{Q} \leq -Q.$$

Thus by construction the N-tuple U satisfies (14), which is equivalent to (13). ■

2) *Input dissipation inequalities:* Let $R \in \mathbf{H}_+^n$, the input dissipation inequalities are

$$|x|_{R[i]}^2 + |f|^2 \geq \sum_{j \in \Theta} p_{ij} (|A[i, j]x + B[i, j]f|_{R[j]}^2), \quad (15)$$

$$\forall x \in \mathbf{R}^{n[i]}, \forall f \in \mathbf{R}^m, \forall i \in \Theta.$$

The above set of LMI's can be written equivalently as

$$W[i] \leq \begin{bmatrix} R[i] & 0 \\ 0 & I_{n[i]} \end{bmatrix}, \quad \forall i \in \Theta, \quad (16)$$

where

$$W[i] = \begin{bmatrix} W_{11}[i] & W_{12}[i] \\ W_{12}'[i] & W_{22}[i] \end{bmatrix} = \sum_{j \in \Theta} p_{ij} \begin{bmatrix} A[i, j]' \\ B[i, j]' \end{bmatrix} R[j] \begin{bmatrix} A[i, j] & B[i, j] \end{bmatrix}, \quad i \in \Theta.$$

Lemma 3.2: Given a mean square stable MJLS \mathbf{L} , there exists $R \in \mathbf{H}_+^n$ such that (15) is satisfied.

Proof: By application of the Schur lemma to (16) it suffices to find $R \in \mathbf{H}_+^n$ such that

$$W_{11}[i] < R[i], \quad (17)$$

$$W_{22}[i] - W_{12}'[i](W_{11}[i] - R[i])^{-1}W_{12}[i] < I_{n[i]}, \quad (18)$$

$\forall i \in \Theta$ holds. Mean square stability is equivalent to the existence of $\tilde{R} \in \mathbf{H}_+^n$ such that

$$-\tilde{R}[i] + \sum_{j \in \Theta} p_{ij} A[i, j]' \tilde{R}[j] A[i, j] < 0, \quad i \in \Theta.$$

Set $R[i] = \alpha \tilde{R}[i]$, $i \in \Theta$, where $\alpha > 0$. Condition (17) is then automatically satisfied. Concerning condition (18) note that both terms in the left hand side scale linearly with α . Thus by taking α small enough one can satisfy (18) too. \blacksquare

The relations in (16) can be expressed in an equivalent form where the search variables are

$$Z[i] = R[i]^{-1}, \quad i \in \Theta. \quad (19)$$

This latter form is more convenient for computational purposes. In particular by using the Schur lemma and accounting for the fact that $Z[i] > 0$, $i \in \Theta$ relations (16) are equivalent to

$$V[i] = \begin{bmatrix} V_{11}[i] & V_{12}[i] \\ V_{12}'[i] & V_{22}[i] \end{bmatrix} \geq 0,$$

where

$$\begin{aligned} V_{11}[i] &= \begin{bmatrix} R[i] & 0 \\ 0 & I_{n[i]} \end{bmatrix}, \\ V_{12}[i] &= \begin{bmatrix} \sqrt{p_{i1}} A[i, 1]' & \dots & \sqrt{p_{iN}} A[i, N]' \\ \sqrt{p_{i1}} B[i, 1]' & \dots & \sqrt{p_{iN}} B[i, N]' \end{bmatrix}, \\ V_{22}[i] &= \text{diag}\{Z[1], \dots, Z[N]\}, \quad i \in \Theta. \end{aligned}$$

and the latter set of LMI's are equivalent to

$$\text{diag}\{Z[1], \dots, Z[N]\} \geq \bar{A}[i] Z[i] \bar{A}[i]' + \bar{B}[i] \bar{B}[i]', \quad (20)$$

where $\bar{A}[i]' = \begin{bmatrix} \sqrt{p_{i1}} A[i, 1]' & \dots & \sqrt{p_{iN}} A[i, N]' \end{bmatrix}$ and $\bar{B}[i]' = \begin{bmatrix} \sqrt{p_{i1}} B[i, 1]' & \dots & \sqrt{p_{iN}} B[i, N]' \end{bmatrix}$, $i \in \Theta$.

3) *Obtaining diagonal solutions to the dissipation inequalities:* Certain proofs in subsequent sections become more transparent if the solutions to the dissipation inequalities are simultaneously transformed to diagonal matrices.

Lemma 3.3: Let $U \in \mathbf{H}_+^n$, $R \in \mathbf{H}_+^n$ satisfy the dissipation inequalities (13) and (15) respectively. Consider the mode dependent coordinate transformation $x = T[i] \tilde{x}$, $i \in \Theta$, where $T \in \mathbf{H}^n$, $T[i]$ invertible, $i \in \Theta$. In the new coordinates, one has

$$\begin{aligned} |\tilde{x}|_{\tilde{U}[i]}^2 &\geq \sum_{j \in \Theta} p_{ij} (|\tilde{A}[i, j] \tilde{x}|_{\tilde{U}[j]}^2) + |\tilde{C}[i] \tilde{x}|^2, \\ |\tilde{x}|_{\tilde{R}[i]}^2 + |f|^2 &\geq \sum_{j \in \Theta} p_{ij} (|\tilde{A}[i, j] \tilde{x} + \tilde{B}[i, j] f|_{\tilde{R}[j]}^2), \\ \forall \tilde{x} \in \mathbf{R}^{n[i]}, \forall f \in \mathbf{R}^m, \forall i \in \Theta, \end{aligned}$$

where

$$\begin{aligned} \tilde{U}[i] &= T'[i] U[i] T[i] \\ \tilde{R}[i] &= T'[i] R[i] T[i], \quad i \in \Theta. \end{aligned} \quad (21)$$

Proof: Take into account the state space representation in the new coordinates as in (12). Replace $A[i, j]$ and $C[i]$ in (13) with $T[j]\tilde{A}[i, j]T[i]^{-1}$ and $\tilde{C}[i, j]T[i]^{-1}$ respectively. Use also the fact that $x = T[i]\tilde{x}$ and the result emerges. The proof for the input dissipation inequality is completely analogous and is obtained by replacing $A[i, j]$ and $B[i]$ in (15) with $T[j]\tilde{A}[i, j]T[i]^{-1}$ and $T[j]\tilde{B}[i, j]$ respectively. ■

Relations (21) allow also one to conclude that the eigenvalues of the product $Z[i]U[i]$, $i \in \Theta$ remain invariant under the aforementioned mode dependent coordinate transformation since

$$\tilde{Z}[i]\tilde{U}[i] = T[i]^{-1}Z[i]U[i]T[i], \quad i \in \Theta. \quad (22)$$

Lemma 3.4: Let $U \in \mathbf{H}_+^n$, $R \in \mathbf{H}_+^n$ satisfy the dissipation inequalities (13) and (15) respectively. There exists a mode dependent coordinate transformation $x = T[i]\tilde{x}$, $\theta(k) \in \Theta$, where $T \in \mathbf{H}^n$, $T[i]$ invertible, $i \in \Theta$, such that

$$\tilde{U}[i] = \tilde{Z}[i] = \text{diag}\{\beta_{1i}, \dots, \beta_{n[i]i}\}, \quad i \in \Theta.$$

Proof: Compute a Cholesky factorization $Z[i] = F[i]F[i]'$ and subsequently an eigenvalue decomposition $F[i]U[i]F[i] = H[i]W[i]^2H[i]'$, where $H[i]H[i]' = H[i]'H[i] = I_{n[i]}$ and $W[i]$ is a positive definite diagonal matrix, i.e.,

$$W[i] = \text{diag}\{\beta_{1i}, \dots, \beta_{n[i]i}\}, \quad i \in \Theta.$$

The required transformation matrix is given by

$$T[i] = F[i]H[i]W[i]^{-\frac{1}{2}}, \quad i \in \Theta. \quad (23)$$

Substituting (23) into (21) gives

$$\begin{aligned} \tilde{U}[i] &= W[i]^{-\frac{1}{2}}H[i]'F[i]U[i]F[i]H[i]W[i]^{-\frac{1}{2}} \\ &= W[i]^{-\frac{1}{2}}H[i]'H[i]W[i]^2H[i]'H[i]W[i]^{-\frac{1}{2}} \\ &= W[i]. \end{aligned}$$

Similarly

$$\begin{aligned} \tilde{R}[i] &= W[i]^{-\frac{1}{2}}H[i]'F[i]'R[i]F[i]H[i]W[i]^{-\frac{1}{2}} \\ &= W[i]^{-\frac{1}{2}}H[i]'F[i]'(F[i]')^{-1}F[i]^{-1}F[i]H[i]W[i]^{-\frac{1}{2}} \\ &= W[i]^{-\frac{1}{2}}H[i]'H[i]W[i]^{-\frac{1}{2}} \\ &= W[i]^{-1} \end{aligned}$$

and therefore $\tilde{Z}[i] = W[i]$. ■

It has been established in this section that for a given mean square stable system MJLS \mathbf{L} there always exist solutions $U \in \mathbf{H}_+^n$, $R \in \mathbf{H}_+^n$ to the dissipation inequalities (13) and (15) respectively with $U[i] = Z[i] = R[i]^{-1}$, $i \in \Theta$ and diagonal.

B. Upper bound on the approximation error

This section is devoted in proving an upper bound to the approximation error with respect to the stochastic L_2 gain when the dimension of the continuous valued part of the state associated with a particular discrete mode is reduced by means of truncation.

Theorem 3.1: Consider a mean square stable system \mathbf{L} . Suppose that $U \in \mathbf{H}_+^n$, $R \in \mathbf{H}_+^n$ satisfy the dissipation inequalities (13), (15) respectively. Assume that for a particular mode $i^* \in \Theta$

$$U[i^*] = \begin{bmatrix} \Sigma_{U_{i^*}} & 0 \\ 0 & \beta I_{r[i^*]} \end{bmatrix}$$

and

$$R[i^*] = \begin{bmatrix} \Sigma_{R_{i^*}} & 0 \\ 0 & \frac{1}{\beta} I_{r[i^*]} \end{bmatrix}.$$

Let $\hat{\mathbf{L}}$ be the reduced order model obtained by truncating the last $r[i^*]$ continuous states corresponding to the mode i^* of \mathbf{L} . The stochastic L_2 gain of the error system $\mathbf{E}_{\mathbf{L}, \hat{\mathbf{L}}}$ is bounded from above by

$$\gamma_{\mathbf{E}_{\mathbf{L}, \hat{\mathbf{L}}}} \leq 2\beta. \quad (24)$$

Proof:

Introduce the matrix

$$E_{r[i]} = \begin{bmatrix} 0 & 0 \\ 0 & I_{r[i]} \end{bmatrix} \in \mathbf{R}^{n[i] \times n[i]}.$$

and note that $E_{r[i]} = 0$ unless $i = i^*$. Let $\hat{x}(k)' = (x_1(k)', 0')$ be the continuous part of the state variable of the reduced order model submerged in the original state space. The dynamics of the reduced order system are given by (11). The following variables are introduced to shorten subsequent notation,

$$\begin{aligned} z(k) &= x(k) + \hat{x}(k), \\ \delta(k) &= x(k) - \hat{x}(k) \\ h[\phi(k+1)] &= A[\phi(k+1)]\hat{x}(k) + B[\phi(k+1)]f(k), \\ e(k) &= y(k) - \hat{y}(k), \quad k \in \mathbf{N}. \end{aligned}$$

One obtains accordingly

$$\begin{aligned} z(k+1) &= A[\phi(k+1)]z(k) + 2B[\phi(k+1)]f(k) \\ &\quad - E_{r[\theta(k+1)]}h[\phi(k+1)], \\ \delta(k+1) &= A[\phi(k+1)]\delta(k) + E_{r[\theta(k+1)]}h[\phi(k+1)], \\ e(k) &= C[\theta(k)]\delta(k), \quad k \in \mathbf{N}. \end{aligned}$$

According to Lemma 2.1 it is sufficient to find a storage function such that :

$$|C[i]\delta|^2 + \Delta V_i \leq 4\beta^2|f|^2, \quad (25)$$

$$\forall x \in \mathbf{R}^{n[i]}, \forall \hat{x} \in V_{n[i]-r[i]}, \forall f \in \mathbf{R}^m, \forall i \in \Theta$$

$$\Delta V_i = \sum_{j \in \Theta} p_{ij} V[x(+), \hat{x}(+), j] - V[x, \hat{x}, i]$$

$$x(+) = A[i, j]x + B[i, j]f$$

$$\hat{x}(+) = (I_{n[j]} - E_{r[j]})(A[i, j]\hat{x} + B[i, j]f).$$

A quadratic storage function candidate is given by

$$V[x, \hat{x}, i] = \beta^2|x + \hat{x}|_{R[i]}^2 + |x - \hat{x}|_{U[i]}^2 = \beta^2|z|_{R[i]}^2 + |\delta|_{U[i]}^2.$$

One needs to verify (25). Let $x \in \mathbf{R}^{n[i]}$, $\hat{x} \in VC_{n[i]-r[i]}$, $f \in \mathbf{R}^m$, $i \in \Theta$, one has

$$\begin{aligned} \Delta V_i &= \sum_{j \in \Theta} p_{ij} |A[i, j]\delta + E_{r[j]}h[i, j]|_{U[j]}^2 + \\ &\quad \beta^2 \sum_{j \in \Theta} p_{ij} |A[i, j]z + 2B[i, j]f - E_{r[j]}h[i, j]|_{R[j]}^2 + \\ &\quad -\beta^2|z|_{R[i]}^2 - |\delta|_{U[i]}^2. \end{aligned}$$

Expanding the individual terms in the above expressions, one obtains

$$\begin{aligned} \Delta V_i &= \sum_{j \in \Theta} p_{ij} |A[i, j]\delta|_{U[j]}^2 - |\delta|_{U[i]}^2 + \\ &\quad \beta^2 \sum_{j \in \Theta} p_{ij} |A[i, j]z + 2B[i, j]f|_{R[j]}^2 - \beta^2|z|_{R[i]}^2 + \\ &\quad 2\beta \sum_{j \in \Theta} p_{ij} |E_{r[j]}h(i, j)|^2 - \\ &\quad 2\beta \sum_{j \in \Theta} p_{ij} (E_{r[j]}h(i, j))'(A[i, j]z + 2B[i, j]f - A[i, j]\delta). \end{aligned} \quad (26)$$

Applying the dissipation inequality (13) to the first two terms of (26) gives

$$\sum_{j \in \Theta} p_{ij} |A[i, j]\delta|_{U[j]}^2 - |\delta|_{U[i]}^2 \leq -|C[i]\delta|^2.$$

Applying the dissipation inequality (15) to the second line in (26) gives

$$\beta^2 \sum_{j \in \Theta} p_{ij} |A[i, j]z + 2B[i, j]f|_{R[j]}^2 - \beta^2|z|_{R[i]}^2 \leq 4\beta^2|f|^2.$$

For the last term of (26) note that $A[i, j]z + 2B[i, j]f - A[i, j]\delta = 2h[i, j]$, and that $E_{r[j]}^2 = E_{r[j]}$. Using the above relations we obtain

$$\Delta V_i + |C[i]\delta|^2 \leq 4\beta^2|f|^2 - 2\beta \sum_{j \in \Theta} p_{ij} |E_{r[j]}h[i, j]|^2.$$

Since $2\beta \sum_{j \in \Theta} p_{ij} |E_{r[j]}h[i, j]|^2 \geq 0$ relation (25) is satisfied, completing the proof. ■

The above result can be generalized to the case where truncation is applied recursively in order to achieve further reduction. The recursive truncation is enabled by the following lemma.

Lemma 3.5: Consider the same setting as in theorem 3.1. For the reduced order model $\hat{\mathbf{L}}$ one has

$$\begin{aligned} |\hat{x}|_{\hat{U}[i]}^2 &\geq \sum_{j \in \Theta} p_{ij} (|\hat{A}[i, j] \hat{x}|_{\hat{U}[j]}^2) + |\hat{C}[i] \hat{x}|^2, \\ |\hat{x}|_{\hat{R}[i]}^2 + |f|^2 &\geq \sum_{j \in \Theta} p_{ij} (|\hat{A}[i, j] \hat{x} + \hat{B}[i, j] f|_{\hat{R}[j]}^2), \\ \forall \hat{x} \in \mathbf{R}^{\hat{n}[i]}, \forall f \in \mathbf{R}^m, \forall i \in \Theta, \end{aligned}$$

where

$$\begin{aligned} \hat{U}[i] &= U[i], \hat{R}[i] = R[i], \text{ when } i \in \Theta, i \neq i^*, \\ \hat{U}[i^*] &= \Sigma_{U_{i^*}}, \hat{R}[i^*] = \Sigma_{R_{i^*}}. \end{aligned}$$

Proof: Note that $\hat{n}[i] = n[i]$ when $i \in \Theta, i \neq i^*$ and $\hat{n}[i^*] < n[i^*]$. Let $i = i^*$ and consider (13) evaluated at $x' = [\hat{x}', 0']$, $\hat{x} \in \mathbf{R}^{\hat{n}[i^*]}$. This gives

$$\begin{aligned} \hat{x}' \Sigma_{U_{i^*}} \hat{x} &\geq p_{i^* i^*} \hat{x}' A_{11}[i^*, i^*]' \Sigma_{U_{i^*}} A_{11}[i^*, i^*] \hat{x} + \\ &\quad p_{i^* i^*} \beta \hat{x}' A_{21}[i^*, i^*]' A_{21}[i^*, i^*] \hat{x} + \\ &\quad \sum_{j \in \Theta, j \neq i^*} p_{i^* j} (|\hat{A}[i^*, j] \hat{x}|_{\hat{U}[j]}^2) + |\hat{C}[i^*] \hat{x}|^2 \\ &\geq p_{i^* i^*} \hat{x}' A_{11}[i^*, i^*]' \Sigma_{U_{i^*}} A_{11}[i^*, i^*] \hat{x} + \\ &\quad \sum_{j \in \Theta, j \neq i^*} p_{i^* j} (|\hat{A}[i^*, j] \hat{x}|_{\hat{U}[j]}^2) + |\hat{C}[i^*] \hat{x}|^2 \\ &= \sum_{j \in \Theta} p_{i^* j} (|\hat{A}[i^*, j] \hat{x}|_{\hat{U}[j]}^2) + |\hat{C}[i^*] \hat{x}|^2. \end{aligned}$$

Now let $i \neq i^*$ and recall that in this case

$$A[i, i^*] = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} [i, i^*].$$

Evaluating (13) at $x = \hat{x} \in \mathbf{R}^{\hat{n}[i]}$ gives

$$\begin{aligned} |\hat{x}|_{\hat{U}[i]}^2 &\geq p_{ii^*} \hat{x}' A_{11}[i, i^*]' \Sigma_{U_{i^*}} A_{11}[i, i^*] \hat{x} + \\ &\quad p_{ii^*} \beta \hat{x}' A_{21}[i, i^*]' A_{21}[i, i^*] \hat{x} + \\ &\quad \sum_{j \in \Theta, j \neq i^*} p_{ij} (|\hat{A}[i, j] \hat{x}|_{\hat{U}[j]}^2) + |\hat{C}[i] \hat{x}|^2 \\ &\geq p_{ii^*} \hat{x}' A_{11}[i, i^*]' \Sigma_{U_{i^*}} A_{11}[i, i^*] \hat{x} + \\ &\quad \sum_{j \in \Theta, j \neq i^*} p_{ij} (|\hat{A}[i, j] \hat{x}|_{\hat{U}[j]}^2) + |\hat{C}[i] \hat{x}|^2 \\ &= \sum_{j \in \Theta} p_{ij} (|\hat{A}[i, j] \hat{x}|_{\hat{U}[j]}^2) + |\hat{C}[i] \hat{x}|^2. \end{aligned}$$

This establishes the result for the output dissipation inequalities. The proof for the input dissipation inequalities is completely analogous and is included here for reasons of completeness. Let $i = i^*$ and consider (15) evaluated at $x' = [\hat{x}', 0']$, $\hat{x} \in \mathbf{R}^{\hat{n}[i^*]}$, $f \in \mathbf{R}^m$.

$$\begin{aligned}
\hat{x}' \Sigma_{R_{i^*}} \hat{x} + |f|^2 &\geq p_{i^* i^*} (\hat{x}' A_{11}[i^*, i^*]' + f' B_1[i^*, i^*]) \Sigma_{R_{i^*}} (A_{11}[i^*, i^*] \hat{x} + B_1[i^*, i^*] f) + \\
&\quad p_{i^* i^*} \frac{1}{\beta} |A_{21}[i^*, i^*] \hat{x} + B_2[i^*, i^*] f|^2 + \\
&\quad \sum_{j \in \Theta, j \neq i^*} p_{i^* j} (|\hat{A}[i^*, j] \hat{x} + \hat{B}[i^*, j] f|_{\hat{R}[j]}^2) \\
&\geq p_{i^* i^*} |A_{11}[i^*, i^*] \hat{x} + B_1[i^*, i^*] f|_{\Sigma_{R_{i^*}}}^2 + \\
&\quad \sum_{j \in \Theta, j \neq i^*} p_{i^* j} (|\hat{A}[i^*, j] \hat{x} + \hat{B}[i^*, j] f|_{\hat{R}[j]}^2) \\
&= \sum_{j \in \Theta} p_{i^* j} (|\hat{A}[i^*, j] \hat{x} + \hat{B}[i^*, j] f|_{\hat{R}[j]}^2).
\end{aligned}$$

Now let $i \neq i^*$ and note that in this case

$$A[i, i^*] = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} [i, i^*], \quad B[i, i^*] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [i, i^*]$$

Evaluating (15) at $x = \hat{x} \in \mathbf{R}^{\hat{n}[i]}$ gives

$$\begin{aligned}
\hat{x}' \hat{R}_i \hat{x} + |f|^2 &\geq p_{ii^*} (\hat{x}' A_{11}[i^*, i^*]' + f' B_1[i^*, i^*]) \Sigma_{R_{i^*}} (A_{11}[i^*, i^*] \hat{x} + B_1[i^*, i^*] f) + \\
&\quad p_{ii^*} \frac{1}{\beta} |A_{21}[i^*, i^*] \hat{x} + B_2[i^*, i^*] f|^2 + \\
&\quad \sum_{j \in \Theta, j \neq i^*} p_{ij} (|\hat{A}[i^*, j] \hat{x} + \hat{B}[i^*, j] f|_{\hat{R}[j]}^2) \\
&\geq p_{ii^*} |A_{11}[i^*, i^*] \hat{x} + B_1[i^*, i^*] f|_{\Sigma_{R_{i^*}}}^2 + \\
&\quad \sum_{j \in \Theta, j \neq i^*} p_{ij} (|\hat{A}[i^*, j] \hat{x} + \hat{B}[i^*, j] f|_{\hat{R}[j]}^2) \\
&= \sum_{j \in \Theta} p_{ij} (|\hat{A}[i^*, j] \hat{x} + \hat{B}[i^*, j] f|_{\hat{R}[j]}^2).
\end{aligned}$$

■

The next theorem is obtained readily given the developed machinery.

Theorem 3.2: Given a mean square stable system \mathbf{L} and matrices $U \in \mathbf{H}_+^n$, $R \in \mathbf{H}_+^n$ such that the dissipation inequalities (13), (15) are satisfied, and suppose for mode $i^* \in \Theta$

$$U[i^*] = \text{diag}\{\Sigma_{1i^*}, \beta_1 I_{r_1[i^*]}, \dots, \beta_s I_{r_s[i^*]}\}$$

and

$$R[i^*] = \text{diag}\{\bar{\Sigma}_{1i^*}, \frac{1}{\beta_1} I_{r_1[i^*]}, \dots, \frac{1}{\beta_s} I_{r_s[i^*]}\}$$

Let $\hat{\mathbf{L}}$ be the reduced order model obtained by truncating the last $r_1[i^*] + \dots + r_s[i^*]$ continuous states corresponding to the mode i^* of \mathbf{L} . Then, the stochastic L_2 gain of the error system $\mathbf{E}_{\mathbf{L}, \hat{\mathbf{L}}}$ is bounded from above by

$$\gamma_{\mathbf{E}_{\mathbf{L}, \hat{\mathbf{L}}}} \leq 2(\beta_1 + \dots + \beta_s). \quad (27)$$

Proof: First remove the last $r_s[i^*]$ and call the truncated system \mathbf{L}_s . By theorem 3.1 one has

$$\gamma_{\mathbf{E}_{\mathbf{L}}, \mathbf{L}_s} \leq 2\beta_s.$$

Notice due to lemma 3.5 the truncated system \mathbf{L}_s still satisfies the corresponding dissipation inequalities (13), (15), thus one can proceed iteratively and repeat the truncation process until $\mathbf{L}_1 = \hat{\mathbf{L}}$ is reached. Then by invoking the triangle inequality one has

$$\gamma_{\mathbf{E}_{\mathbf{L}}, \hat{\mathbf{L}}} \leq \gamma_{\mathbf{E}_{\mathbf{L}}, \mathbf{L}_s} + \dots + \gamma_{\mathbf{E}_{\mathbf{L}_2}, \mathbf{L}_1} = 2(\beta_1 + \dots + \beta_s).$$

■

The derived error bound readily generalizes to the case where continuous states associated with different modes are truncated. Each mode can be treated successively by virtue of lemma 3.5.

C. Computational considerations

In this section it will be discussed how to obtain solutions to the dissipation inequalities which are suitable for truncating the continuous valued part of the state of a particular discrete mode, call it i . Suppose that $U \in \mathbf{H}_+^n$, $R \in \mathbf{H}_+^n$ satisfy the dissipation inequalities (13), (15). In lemma 3.4 it was established that due to the simultaneous diagonalization argument one can assume that

$$U[i] = Z[i] = W[i] = \text{diag}\{\beta_{1i}, \dots, \beta_{n[i]i}\}, \quad \forall i \in \Theta. \quad (28)$$

Furthermore relation (22) implies that

$$\text{Tr}[U[i]Z[i]] = \sum_{j=1}^{n[i]} \beta_{ji}^2, \quad \forall i \in \Theta..$$

Denote the subset of \mathbf{H}_+^n , whose elements satisfy (14) with \mathbf{H}_U^n . Similarly let \mathbf{H}_Z^n denote the subset of \mathbf{H}_+^n , whose elements satisfy (20). Given that the error bound (27) is controlled by the sum of the nonrepeated eigenvalues corresponding to the truncated states a reasonable objective is

$$\min_{U \in \mathbf{H}_U^n, Z \in \mathbf{H}_Z^n} \text{Tr}[U[i]Z[i]] \quad (29)$$

This is a nonconvex optimization problem, which needs to be relaxed for the sake of computation tractability. Note for fixed $Z[i]$, the objective function in (29) is monotonic in $U[i]$. Thus from an error bound point of view it is desirable to find a minimal solution $U_- \in \mathbf{H}_U^n$, in the sense that

$$U_-[j] \leq U[j], \quad \forall j \in \Theta, \quad \forall U \in \mathbf{H}_U^n.$$

Lemma 3.6: The output dissipation inequalities possess a minimal solution.

Proof: Let $Q[i] = C[i]'C[i] \geq 0$, $i \in \Theta$. Relation (14) is repeated for the sake of clarity.

$$\mathcal{L}[U] - U \leq -Q. \quad (30)$$

Consider also the corresponding Lyapunov like equation

$$\mathcal{L}[U_-] - U_- = -Q. \quad (31)$$

Subtracting (31) from (30) and by letting $\Delta = U - U_-$, one gets

$$\mathcal{L}[\Delta] - \Delta = -Q_\Delta \leq 0.$$

Mean square stability implies $r_\sigma[\mathcal{L}] < 1$ and $\Delta = \sum_{i=0}^{\infty} \mathcal{L}^i[Q_\Delta]$ solves the above Lyapunov like equation. By construction $\Delta \geq 0$ proving the minimality of U_- among all solutions of (13). ■

The N-tuple of matrices U_- can be computed as the limit of the nondecreasing sequence $\{U(k)\}$, where

$$U(k+1) = Q + \mathcal{L}[U(k)], \quad U(0) = Q, \quad k \in \mathbf{N}. \quad (32)$$

The convergence to the fixed point U_- is exponential. The situation concerning the computation of U_- is completely analogous to the balanced truncation algorithm for the LTI case. For $N = 2$ one can compute U_- for systems up to about 1000 states per discrete mode on a standard PC.

Having obtained U_- and in particular $U_-[i]$ one can revisit the objective function in (29). The matrix $Z[i]$ can now be obtained as the result of the optimization problem

$$\min_{Z \in \mathbf{H}_2^z} \text{Tr}[U_-[i]Z[i]]. \quad (33)$$

The optimization problem in (33) is a semidefinite program, which is convex and can be solved efficiently using interior point methods [17]. However this step of the reduction algorithm is the limiting factor since the computational cost for obtaining Z is higher than the the matrix product iterations (32) required for computing U . On a standard PC using SeDuMi [18] together with YALMIP [19] one can compute solutions to (33), when $N = 2$, for systems up to about 100 states per discrete mode.

D. Remarks

Markov jump linear systems contain as special cases LTI systems as well linear time varying periodic systems. For the latter two classes of systems balanced truncation algorithms have already been developed in the literature. The two sets of output and input dissipation inequalities proposed in this work reduce for these special cases to the observability and reachability Lyapunov inequality respectively, see for instance [10] for the case of periodic systems.

The system class in this work is not the standard MJLS model considered in the literature. In this paper the matrices in the state space recursion are allowed to depend on the mode transition rather than the mode alone as is the case with standard MJLS's. This was done to accommodate mode varying dimension of the continuous valued part of the state. Applying the balanced truncation algorithm to a standard MJLS with state space representation

$$\begin{aligned} x(k+1) &= A[\theta(k)]x(k) + B[\theta(k)]f(k), \\ y(k) &= C[\theta(k)]x(k), \quad k \in \mathbf{N}, \end{aligned} \quad (34)$$

will lead to a reduced order model where again the matrices in the state space recursion will depend on the mode transition even in the case where equally many continuous states have been truncated at each mode. The only way of getting a reduced order model in the standard form is by finding mode independent solutions to the corresponding dissipation inequalities.

IV. A NUMERICAL EXAMPLE

To illustrate the model reduction algorithm developed in this paper, consider a network control example based on [20], [21]. A one dimensional platoon consists of $m + 1$ vehicles. Let x_0 denote the position of the lead car and x_i , $i \in \{1, \dots, m\}$ denote the position of the i 'th follower in the platoon. The spacing error is given by $e_i(t) = x_{i-1}(t) - x_i(t) - \delta$, $i \in \{1, \dots, m\}$, where δ is the desired vehicle spacing, which is constant. It is assumed that $x_0(0) = 0$ and that there is no initial spacing error, $e_i(0) = 0$, $i \in \{1, \dots, m\}$.

Two control schemes have been designed, whose goal is to achieve disturbance attenuation between the leader motion, which is considered as a reference signal and the spacing error among any two successive followers in the platoon. The assumptions are that every vehicle has the same simple model of a double integrator with first order actuator dynamics, $X_i(s) = H(s)U_i(s) - i\frac{\delta}{s}$, $i \in \{1, \dots, m\}$, where

$$H(s) = \frac{1}{Ms^2(\tau s + 1)},$$

and $M = 1$, $\tau = 0.1$. The control loop runs at a sampling time of $T_s = 20$ ms and a zero-order hold is used on the control input of each vehicle.

The first scheme is decentralized, based on local measurements from on-board sensors. Its performance cannot be satisfactory due to fundamental limitations, which have been elaborated in [20]. The second control scheme utilizes information about the lead car and exhibits better performance. However it requires communication between the lead car and the followers $\{2, \dots, m\}$, which occurs through a wireless network idealized as a two state Markov chain. Note that follower 1 has always information about the motion of the leader based on the measurement of its on-board sensors. The interpretation of the two states of the Markov chain used to model the network is the following. State one corresponds to low load and state two to high load in the network. If there is a transition from high load to high load the leader motion is not transmitted to the followers $\{2, \dots, m\}$ and the first control scheme based on the on-board measurements is implemented for these vehicles in that particular sample. If there is a transition from low load to low load the leader motion is transmitted to the followers $\{2, \dots, m\}$ and the second control scheme is utilized for these vehicles. If there is a transition from high load to low load or vice versa then only the followers 2 and 3 get information about the leader motion, they implement the second control scheme, whereas followers $\{4, \dots, m\}$ receive no information about the leader motion and utilize the first control scheme. The transition probability matrix of the two state Markov chain is denoted by P . The first control scheme, is called predecessor following and the control law is of the form

$$U_i(s) = K(s)E_i(s), \quad i \in \{1, \dots, m\}.$$

Accordingly one has

$$\begin{aligned} E_1(s) &= \frac{1}{1 + H(s)K(s)} X_0(s) \\ E_i(s) &= \frac{H(s)K(s)}{1 + H(s)K(s)} E_{i-1}(s), i \in \{2, \dots, m\}. \end{aligned}$$

The second control scheme, which requires communication, is called predecessor and leader following and is of the form

$$\begin{aligned} U_i(s) &= K_p(s)E_i(s) + K_l(s)(X_0(s) - X_i(s) - \frac{i\delta}{s}), \\ i &\in \{1, \dots, m\}. \end{aligned}$$

Accordingly one has

$$\begin{aligned} E_1(s) &= \frac{1}{1 + H(s)(K_p(s) + K_l(s))} X_0(s) \\ E_i(s) &= \frac{H(s)K_p(s)}{1 + H(s)(K_p(s) + K_l(s))} E_{i-1}(s), \end{aligned}$$

where $i \in \{2, \dots, m\}$. The control parameters are

$$K(s) = \frac{2s + 1}{0.05s + 1}$$

and $K_p(s) = K_l(s) = 0.5K(s)$, note that with this choice of parameters the first follower uses the same control law regardless of the state of the network.

For the exposition of this paper, what is important is not the actual control design, but the fact that the closed loop system is a MJLS, which can serve the purposes of demonstrating the reduction algorithm. An example where $m = 8$ is considered, so there are 8 followers. The input to the system is the reference signal $x_o(t)$ and the output is taken to be the spacing error between the last two followers, $e_8(t)$. The transition probability matrix is chosen to be

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$$

The original model has 32 states per discrete mode. The semidefinite programs involved in the calculation of the stochastic L_2 gain and the reduction process are solved using standard tools in MATLAB such as SeDuMi [18] together with YALMIP [19]. Transformation matrices as in (23) enable the calculation of $U[1] = Z[1] = W[1]$ and $U[2] = Z[2] = W[2]$ where $W[1], W[2]$ are diagonal matrices with positive entries. The diagonal entries of $W[1], W[2]$ control the error bound in terms of the stochastic L_2 gain and are depicted in the following figure.

The approximation error and the upper bounds to the approximation error are depicted for various truncation levels, showing that for this particular example the bound is rather conservative.

REFERENCES

- [1] O. Costa, M. Fragoso, and R. Marques, *Discrete-time Markov Jump Linear Systems*. Springer, 2005.

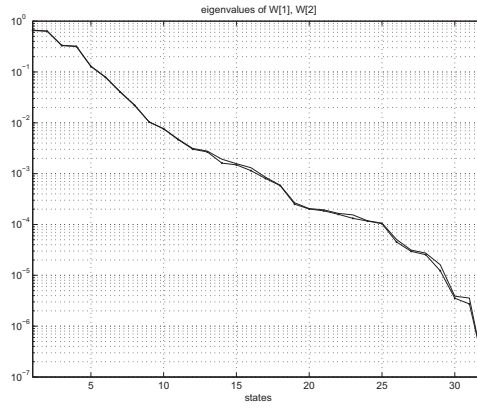


Fig. 2. Entries in the diagonal of $W[1], W[2]$ in logarithmic scale.

TABLE I

ACTUAL APPROXIMATION $\delta_{\mathbf{L}, \hat{\mathbf{L}}}$ AND BOUND $\gamma_{\mathbf{L}, \hat{\mathbf{L}}}$

$\hat{n}[1] = \hat{n}[2]$	$\delta_{\mathbf{L}, \hat{\mathbf{L}}}$	$\gamma_{\mathbf{L}, \hat{\mathbf{L}}}$
7	0.0257	0.2319
6	0.0497	0.3951
5	0.1809	0.7122
4	0.1889	1.2274
3	0.5616	2.5027

- [2] L. Q. Zhang, B. Huang, and J. Lam., "Model reduction of Markovian jump linear systems," *Systems and Control Letters*, vol. 50, no. 2, pp. 103–118, 2003.
- [3] B. Moore, "Principal component analysis in linear systems, controllability, observability, and model reduction," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 1, pp. 17–32, Feb. 1981.
- [4] D. Enns, "Model reduction with balanced realizations: An error bound and a frequency weighted generalization," in *Proc. IEEE Conf. Decision Control*, Las Vegas, USA, Dec. 1984, pp. 127 – 132.
- [5] U. M. Al-Saggaf and G. F. Franklin, "An error bound for a discrete reduced order model of a linear multivariable system," *IEEE Trans. Automat. Contr.*, vol. AC-32, no. 9, pp. 815–819, Sept. 1987.
- [6] D. Hinrichsen and A. J. Pritchard, "An improved error estimate for reduced-order models of discrete-time systems," *IEEE Trans. Automat. Contr.*, vol. 35, no. 3, pp. 317–320, Mar. 1990.
- [7] C. L. Beck, J. Doyle, and K. Glover, "Model reduction of multidimensional and uncertain systems," *IEEE Trans. Automat. Contr.*, vol. 41, no. 10, pp. 1466–1477, Oct. 1996.
- [8] G. Wood, P. Goodard, and K. Glover, "Approximation of linear parameter-varying systems," in *Proc. IEEE Conf. Decision Control*, Dec. 1996, pp. 406 – 411.
- [9] S. Lall and C. L. Beck, "Error-bounds for balanced model-reduction of linear time-varying systems," *IEEE Trans. Automat. Contr.*, vol. 48, no. 6, pp. 946–956, June 2003.
- [10] H. Sandberg and A. Rantzer, "Balanced truncation of linear time-varying systems," *IEEE Trans. Automat. Contr.*, vol. 49, no. 2, pp. 217 – 229, Feb. 2004.
- [11] G. Kotsalis, A. Megretski, and M. Dahleh, "Balanced truncation for stochastic jump linear systems and a model reduction algorithm for hidden markov models," *IEEE Trans. Automat. Contr.*, vol. 53, no. 11, pp. 2543–2557, 2008.

- [12] A. A. Agung and G. J. Pappas, "Approximations of stochastic hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 54, no. 6, pp. 1193–1203, 2009.
- [13] H. A. P. Blom, "Overlooked potential of systems with markovian coefficients," in *Proc. IEEE Conf. Decision Control*, Athens, Greece, Dec. 1986, pp. 1758 – 1764.
- [14] M. Petreczky and R. Vidal, "Realization theory of stochastic jump-markov linear systems," in *Proc. IEEE Conf. Decision Control*, New Orleans, USA, Dec. 2007, pp. 4668 – 4674.
- [15] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–162, Jan. 2007.
- [16] J. Chizeck, X.Feng, and K. Loparo, "Stability and control of discrete time jump linear systems," *Control Theory Adv. Tech.*, vol. 7, no. 2, pp. 247–270, 1991.
- [17] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan., *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.
- [18] J. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11–12, pp. 625–653, 1999. [Online]. Available: citeseer.ist.psu.edu/sturm99using.html
- [19] J. Löfberg, "Yalmip : A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. [Online]. Available: <http://control.ee.ethz.ch/~joloef/yalmip.php>
- [20] P. Seiler, A. Pant, and K. Hedrick, "Disturbance propagation in vehicle strings," *IEEE Trans. Automat. Contr.*, vol. 49, no. 10, pp. 1835 – 1841, Oct. 2004.
- [21] P. Seiler and R. Sengupta, "An H_∞ approach to networked control," *IEEE Trans. Automat. Contr.*, vol. 50, no. 3, pp. 356 – 361, Mar. 2005.