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System Analysis via Integral Quadratic Constraints Part I

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System Analysis via Integral Quadratic Constraints, Part I

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Abstract

This paper introduces a unified approach to robustness analysis with respect to nonlinearities, time-variations and uncertain parameters. From an original idea by Yakubovich, the approach has been developed under a combination of influences from the western and russian traditions of control theory. It is shown how a complex system can be described by using certain integral quadratic constraints (IQC's), derived for its elementary components. A stability theorem for systems described by IQC's is presented, that covers classical passivity/dissipativity arguments, but simplifies the use of multipliers and the treatment of causality.

The paper is divided into two parts. Part I presents the basic ideas for stability analysis, refering to a simple example. A systematic computational approach is described and relations to other methods of stability analysis are discussed. Last, but not least, it contains a summarizing list of IQC's for important types of system components, that exist in various forms in the literature.

1 Introduction

It is common engineering practice to work with simplest possible models for design of control systems. In particular, one often uses linear time-invariant plant models, for which there is a well established theory and commercially available computer tools that help in the design. To verify that the design also works well in practice one needs real experiments, often preceeded by simulations with more accurate models. However, there is also a strong need for more formal ways to analyse the systems. Such analysis can help to identify critical experimental circumstances or parameter combinations and estimate the power of the models.

In the 1960-70s, a large body of results was developed in this direction, often referred to as "absolute stability theory". The basic idea was to partition the system into a feedback interconnection of two positive operators. See [45, 78, 82, 75, 39, 17, 54] and the references therein. To improve the flexibility of the approach, so-called *multipliers* were used to select proper variables for the partitioning. The absolute stability theory is now considered as a fundamental component of the theory for nonlinear systems. However, the applicability of many of the results has been limited by computational problems and by restrictive causality conditions used in the multiplier theory.

For computation of multipliers, substantial progress has been made in the last decade, the most evident example being algorithms for computation of structured singular values (μ analysis) [19]. As a result, robustness analysis with respect to uncertain parameters and unmodeled dynamics, can be performed with great accuracy. A probably even more fundamental breakthrough in this direction is the development of polynomial time algorithms for convex optimization with constraints defined by linear matrix inequalities [40, 7]. Such problems appear not only in μ-analysis, but in almost any analysis method based on passivity-type concepts.

The purpose of this paper is to adress the second obstacle to efficeint analysis, by proving that multipliers can be introduced in a less restrictive manner, without causality restrictions. Not only does this make the theory more accessible by simplification of proofs, but also enhances the development of computer tools, that supports the transformation of assumptions on model structure into a numerically tractable optimization problem.

The term integral quadratic constraint (IQC) is used for several purposes:

- To exploit structural information about a complex or uncertain system component.
- To characterize properties of an external signal.
- To analyze combinations of several constraints on perturbations and signals in a system.

Implicitly, integral quadratic constaints have always been present in stability theory. For example, positivity of an operator F, can be expressed by the IQC

$$\int_{-\infty}^{\infty} \widehat{(Fv)}(j\omega)^* \hat{v}(j\omega) d\omega \geq 0 \quad \forall v \ .$$

In the 1960s, most of the stability theory was devoted to scalar feedback systems. This led to conveniently visualizable stability criteria based on the Nyquist diagram, which was particularly important in times when computers were less accessible.

In the 70-s, integral quadratic constaints were explicitly used (and named so) by Yakubovich to treat the stability problem for systems with advanced nonlinearities, including amplitude and frequency modulation systems. Some new IQC:s, unrelated to the passivity or small gain arguments, were introduced, and the so-called S-procedure was applied to the case of multiple constraints [79]. Willems also gave an energy related interpretation of the stability results, in terms of dissipativity, storage functions and supply rates [75]. Later on, Safonov interpreted the stability results geometrically, in terms of separation of the graphs of the two operators in the feedback loop.

An important step in the further development, was the introduction of analysis methods which essentially rely on the use of computers. One example is the theory for quadratic stabilization [30, 22, 15], another is the multiloop generalization of the circle criterion based on D-scaling, [55, 19]. Both the search for a Lyapunov function and the search for D-scales can be interpreted as optimization of parameters in an integral quadratic constaint. Another direction was the introduction of H^∞ optimization for synthesis of robust controllers [83, 61]. Again the results can be viewed in terms of integral quadratic constraints, since optimal design with respect to an IQC leads to H^∞ optimization.

During the last decade, a variety of methods has been developed within the area of robust control. As was pointed out in [35], many of

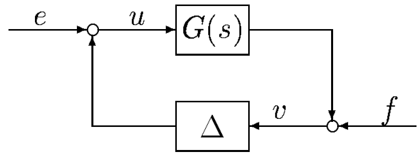

Figure 1: Perturbation in Feedback Form

them can be reformulated to fall within the framework of IQC's. This will be further demonstrated in the current paper, which is divided in two parts.

This first part presents some minimal framework for the stability analysis of feedback interconnections described in terms of IQC's. It is introduced by an extensive example, illustrating the main ideas on a feedback loop with saturation and an uncertain delay. In section 3, definitions and main theorem are stated in detail. After that follows sections with discussions and comparisons to well known results. Finally, we give a summarizing list of integral quadratic constraints for important types of system components.

The second part of the paper concerns analysis of robust performance, and generalizes the stability analysis to cases where the boundedness, causality and uniqueness assumptions of part one are violated.

2 Outline of the method

Consider a feedback configuration illustrated in Figure 1, consisting of a time-invariant linear operator with transfer matrix $G(s)$, interconnected with an operator Δ, that describes the "troublemaking" (nonlinear, time-varying or uncertain) components of the system. The notation G will in the sequel either denote a linear operator or a rational transfer matrix, depending on the context.

First, we describe Δ as accurately as possible by integral quadratic constraints (IQC's)

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix} d\omega \geq 0 \qquad (1)$$

which should hold for any square summable v with Fourier transform $\hat{v}$. The class Π_Δ of all rational hermitean matrix functions Π that define a valid IQC for a given Δ is convex, since the sum of two positive integrals is positive, and it is usually infinite-dimensional. For a large number of simple system components, a corresponding class Π_Δ is readily available in the litterature. In fact, IQC's are implicitly present in many results on robust/non-linear/time-varying stability. A list of such IQC's has been appended to this paper in section 7. When Δ consists of a combination of several simple blocks, IQC's can be generated by convex combinations of constraints for the simpler components.

Next, we search for a matrix function $\Pi \in \Pi_\Delta$, that satisfies the criterion

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \quad \forall\, \omega \in \mathbf{R} \cup \{\infty\}. \tag{2}$$

In combination with (1), this essentially proves stability of the interconnection. The search for a suitable Π can be carried out by numerical optimization, restricted to some finite-dimensional subset of Π_Δ. Roughly speaking, Π is expected to be of the form

$$\Pi(j\omega) = \sum_{q=1}^{q=q_0} x_q \Pi_q(j\omega),$$

where x_q are real parameters. Π and G are proper rational functions with no poles on the imaginary axis, so there exists $n > 0$, a Hurwitz matrix A of size $n \times n$, a matrix B of size $n \times m$, and a set of symmetric real matrices M_q of size $(n+m) \times (n+m)$, such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi_q(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} =$$

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* M_q \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

for all q. By application of the Kalman-Yakubovich-Popov Lemma, as stated by Willems [74], it follows that the inequality in (2) is equivalent to the existence of a symmetric $n \times n$ matrix $P = P^T$ such that

$$\begin{bmatrix} PA + A^TP & PB \\ B^TP & 0 \end{bmatrix} + \sum_{q=1}^{q=q_0} x_q M_q < 0. \tag{3}$$

Hence the search for x_q that produce a Π weight satisfying (2) (i.e. proving the stability) takes the form of a convex optimization problem defined by a linear matrix inequality (LMI) in the variables x_q, P. Such problems can be solved very efficiently using the recently developed numerical algorithms based on interior point methods [40, 7].

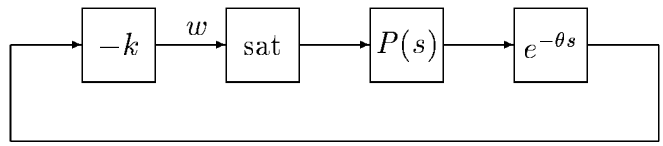

Figure 2: System with Saturation and Delay

2.1 Example with Saturation and Delay

Consider the following feedback system with control saturation and an uncertain delay.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\,\mathrm{sat}(w(t)) \\ w(t) &= -kCx(t-\theta) \end{aligned} \tag{4}$$

where r is the reference signal, $\theta \in [0, \theta_0]$ is an unknown constant,

$$P(s) = C(sI - A)^{-1}B = \frac{s^2}{s^3 + 2s^2 + 2s + 1}$$

is the transfer function of the controlled plant (see the Nyquist plot on Figure 3), and

$$\mathrm{sat}(w) = \begin{cases} w & , \ |w| \le 1, \\ w/|w| & , \ |w| \ge 1, \end{cases}$$

is the function that represents the saturation. The setup is illustrated in Figure 2.

Let us first consider stability analysis for the case of no delay. Then let Δ be the saturation, while $G(s) = -kP(s)$. Application of the circle criterion

$$-k^{-1} < \min_{\omega} \mathrm{Re}\, P(j\omega) \tag{5}$$

gives stability for

$$k < k_{\mathrm{circ}} \approx 8.12$$

(see dashed line in Figure 3). This corresponds to a Π_Δ containing only the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

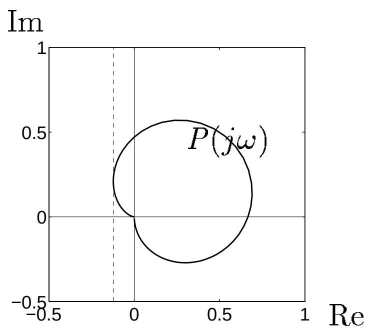

Figure 3: Nyquist plot for $P(j\omega)$ (solid line)

In the Popov criterion, Π_Δ consists of all linear combinations

$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} + \eta \begin{bmatrix} 0 & j\omega \\ -j\omega & 0 \end{bmatrix},$$

and the resulting inequality (2) gives the minor improvement

$$\begin{aligned} -k^{-1} &< \max_\eta \min_\omega \operatorname{Re}\left[(1 + j\omega\eta)P(j\omega)\right] \\ k &< k_{\text{Popov}} \approx 8.90 \end{aligned} \tag{6}$$

A Popov plot is shown in Figure 4.

Furthermore, because the saturation is monotone and odd, it is possible to apply a much stronger result, obtained by Zames and Falb, [84]. By their statement, a sufficient condition for stability is the existence of a function $H \in \mathbf{RL}_\infty$ such that

$$\begin{aligned} 0 &< \min_\omega \operatorname{Re}[(1 + H(j\omega)^*)(P(j\omega) + k^{-1})] \\ H(j\omega) &= \int_{-\infty}^{\infty} e^{-j\omega t} h(t)\,dt \\ 1 &\geq \int_{-\infty}^{\infty} |h(t)|\,dt \end{aligned}$$

This extends the class of valid IQC's further, by allowing all matrix functions of the form

$$\Pi(j\omega) = \begin{bmatrix} 0 & 1 + H(j\omega) \\ 1 + H(-j\omega) & -2(1 + \operatorname{Re} H(j\omega)) \end{bmatrix}$$

where H has an impulse response of L_1 norm no greater than one. For our problem, the choice $H(j\omega) = -(1 + j\omega)^{-1}$ gives for $\omega \in \mathbf{R}$ that

$$\operatorname{Re}[(1 + H(-j\omega))P(j\omega)] = |1 + H(-j\omega)|^2 \operatorname{Re}\left(\frac{j\omega}{-\omega^2 + j\omega + 1}\right) \geq 0$$

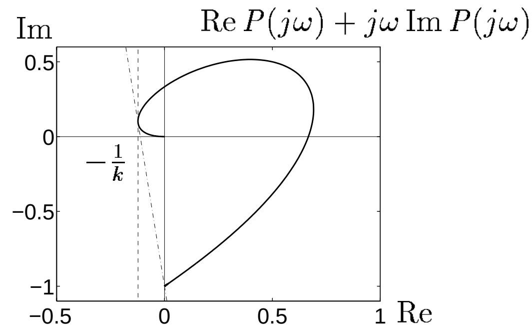

Figure 4: Popov plot for $P(j\omega)$ with stabilizing gain k

This shows that the feedback system is indeed stable for all $k > 0$ and concludes the stability analysis in the undelayed case.

Considering also the delay uncertainty, the problem is to find a bound on the maximal stabilizing feedback gain for a given delay bound. A crude bound can be received directly from the small gain theorem, stating that, because of the gain bound $\|e^{-\theta s}\,\mathrm{sat}(\cdot)\| < 1$, the feedback interconnection of $e^{-\theta s}\,\mathrm{sat}(\cdot)$ and $kP(s)$ is stable provided that

$$k \quad < \quad \|P\|_\infty{}^{-1} \approx 1.37.$$

Not surprisingly, this condition is conservative. For example, it does not utilize any bound on the delay. In order to do that, it is useful to generate more IQC's for the delay component. However, let us first step back and formulate the stability criterion more carefully. The example will be continued in section 6.

Notation

Let $\mathbf{RL}_\infty$ be the set of proper (bounded at infinity) rational functions with real coefficients. The subset consisting of functions without poles in the closed right half plane is denoted $\mathbf{RH}_\infty$. The set of $m \times n$ matrices with elements in $\mathbf{RL}_\infty$ ($\mathbf{RH}_\infty$) will be denoted $\mathbf{RL}_\infty^{m\times n}$ ($\mathbf{RH}_\infty^{m\times n}$).

$\mathbf{L}_2^l[0,\infty)$ can be thought of as the space of $\mathbf{R}^l$-valued signals (i.e. functions $f:\ [0,\infty) \to \mathbf{R}^l$) of finite energy

$$\|f(\cdot)\| = \int_0^\infty |f(t)|^2 dt.$$

This is a subset of the space $\mathbf{L}_{2e}^l[0,\infty)$, whose members only need to be square integrable on finite intervals. By an *operator* we mean a function $F : \mathbf{L}_{2e}^a[0,\infty) \to \mathbf{L}_{2e}^b[0,\infty)$ from one $\mathbf{L}_{2e}[0,\infty)$ space to another. The *gain* of an operator $F : \mathbf{L}_{2e}^a[0,\infty) \to \mathbf{L}_{2e}^b[0,\infty)$ is given by

$$\|F\| = \sup\{\|F(f)\|/\|f\| : \quad f \in \mathbf{L}_2^a[0,\infty),\ f \neq 0\}$$

(same notation for the gain as for the energy). An important example of an operator is given by the *past projection* (truncation) P_T, which leaves a function unchanged on the interval $[0,T]$ and gives the value zero on $(T,\infty]$. *Causality* of an operator F means that $P_T F = P_T F P_T$ for any $T > 0$.

3 A Basic Stability Theorem

The following feedback configuration, illustrated in Figure 1, is the basic object of the theoretical study in this paper.

$$\begin{cases} v &= Gu + f \\ u &= \Delta(v) + e, \end{cases} \tag{7}$$

Here $f \in \mathbf{L}_{2e}^l[0,\infty)$, $e \in \mathbf{L}_{2e}^m[0,\infty)$ represent the "interconnection noise", G and Δ are the two causal operators on $\mathbf{L}_{2e}^m[0,\infty)$ and $\mathbf{L}_{2e}^l[0,\infty)$ respectively. It is assumed that G is a linear time-invariant operator with the transfer function $G(s)$ in $\mathbf{RH}_\infty^{l\times m}$, and Δ is bounded (but not necessarily linear or time-invariant).

An important assumption about system (7) will be its well-posedness.

Definition The feedback system (7) is said to be *well-posed*, if the operator

$$I - G\Delta : \ \mathbf{L}_{2e}^l[0,\infty) \to \mathbf{L}_{2e}^l[0,\infty),$$

which maps v to $v - G\Delta(v)$, is causally invertible, i.e. if there exists a causal operator Γ such that $I = (I - G\Delta)\circ\Gamma = \Gamma\circ(I - G\Delta)$.

In most applications, this definition of well-posedness (a more general definition will be introduced in the second part of the paper) is equivalent to the existence, uniqueness and continuability of solutions of the underlying differential equations, and is therefore easy to verify.

The following kind of input/output stability in the L_2-setting, will be convenient.

Definition The feedback system (7) is said to be *stable* if there exists a $C > 0$ such that

$$\int_0^T (|v|^2 + |u|^2)dt \ \ \leq \ \ C\int_0^T (|f|^2 + |e|^2)dt \tag{8}$$

for any $T \geq 0$ and for any solution of (7).

Indeed, a well-posed system (7) is stable if and only if $(I - G\Delta)^{-1}$ is a bounded causal operator. In many cases, it is also desirable to verify some kind of *exponential stability*. One might expect that this requires separate analysis. However, for general classes of ordinary differential equations, exponential stability turns out to be equivalent to the input/output stability introduced above (compare [67], section 6.3).

Proposition 1 *Let ϕ be such that*

$$\sup_{x,t} |\phi(x,t)|/|x(t)| < \infty.$$

Assume that for any $g \in \mathbf{L}_2^n[0,\infty)$, $x_0 \in \mathbf{R}^n$, $t_0 \geq 0$ the system

$$\dot{x}(t) \quad = \quad \phi(x(t),t) + g(t), \quad t \geq t_0 \tag{9}$$

has a solution $x(\cdot)$. Then the following two conditions are equivalent.

(i) *There exists a $c > 0$ such that*

$$\int_0^T |x(t)|^2 dt \quad \leq \quad c\int_0^T |g(t)|^2 dt \quad \forall\ T > 0 \tag{10}$$

for any solution of (9) with $x(0) = 0$.

(ii) *There exist $\epsilon, d > 0$ such that*

$$|x(t_1)|^2 \quad \leq \quad de^{\epsilon(t_0 - t_1)}|x(t_0)|^2 + d\int_{t_0}^{t_1} |g(t)|^2 dt \tag{11}$$

for any solution x of (9).

Proof. Parts, if not all, of this result can be found in standard references on nonlinear systems. However, for easy reference, a complete proof is given in section 8. □

Next, we need a formal definition of the term IQC.

Definition Suppose $\Pi : j\mathbf{R} \to \mathbf{C}^{(l+m)\times(l+m)}$ is a bounded measurable function taking Hermitean values. Let σ be the quadratic form defined on $\mathbf{L}_2^l[0,\infty) \times \mathbf{L}_2^m[0,\infty)$ by

$$\sigma(v,u) \quad = \quad \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{u}(j\omega) \end{bmatrix} d\omega$$

A bounded operator $\Delta : \mathbf{L}_{2e}^l[0,\infty) \to \mathbf{L}_{2e}^m[0,\infty)$ is said to *satisfy the IQC defined by* Π if

$$\sigma(v, \Delta v) \quad \geq \quad 0 \qquad \forall\ v \in \mathbf{L}_2^l[0,\infty). \tag{12}$$

Theorem 2 *Assume that*

(i) *for any $\tau \in [0,1]$, system (7) with Δ replaced by $\tau\Delta$ is well-posed.*

(ii) *for any $\tau \in [0,1]$, the IQC defined by Π is satisfied by $\tau\Delta$.*

(iii) *there exists $\epsilon > 0$ such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall\ \omega \in \mathbf{R}. \tag{13}$$

Then the feedback system (7) is stable.

Remark 1 Note that $\Pi(j\omega) = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ gives a version of the small gain theorem, while $\Pi(j\omega) = \begin{bmatrix} \epsilon & I \\ I & -\epsilon/\|\Delta\|^2 \end{bmatrix}$ gives a passivity theorem.

Remark 2 In many applications, (see, for example, Remark 1), the upper left corner of $\Pi(j\omega)$ is positive semi-definite and the lower right corner is negative semi-definite, so $\tau\Delta$ satisfies the IQC defined by Π for $\tau \in [0,1]$ if and only if Δ does so. This simplifies assumption (ii).

Remark 3 It is important to note that if $\tau\Delta$ with $\tau \in [0,1]$ satisfies several IQC:s, defined by $\Pi_1, \ldots, \Pi_n$, then a sufficient condition for stability is existence of $x_1, \ldots, x_n \geq 0$ such that (13) holds for $\Pi = x_1\Pi_1 + \cdots + x_n\Pi_n$. Hence, the more IQC:s that can be verified for Δ, the better. Moreover, it can be proved along the lines of [60, 36] that if no such $x_1, \ldots, x_n \geq 0$ exist, then there is a bounded operator that destabilizes (7), but satisfies all the IQC:s. In this sense, the stability condition of Theorem 2 is non-conservative.

Proof of Theorem 2.
STEP 1. *Show that there exists $c_0 > 0$ such that*

$$\|v\| \leq c_0\|v - \tau G\Delta(v)\| \quad \forall\ v \in \mathbf{L}_2^l[0,\infty). \tag{14}$$

Introduce m_{11}, m_{12}, m_{22} as the norms $m_{ij} = \sup_\omega \|\Pi_{ij}(j\omega)\|$ for the matrix blocks of

$$\Pi(j\omega) = \begin{bmatrix} \Pi_{11}(j\omega) & \Pi_{12}(j\omega) \\ \Pi_{12}(j\omega)^* & \Pi_{22}(j\omega) \end{bmatrix}.$$

For $\epsilon > 0$, let $c(\epsilon) = m_{11} + m_{11}^2/\epsilon + m_{12}^2/\epsilon$. Then

$$\begin{aligned} |\sigma(v,u) - \sigma(v+\delta,u)| &\leq m_{11}\|\delta\|^2 + 2\|\delta\|(m_{11}\|v\| + m_{12}\|u\|) \\ &\leq c(\epsilon)\|\delta\|^2 + \epsilon(\|u\|^2 + \|v\|^2) \end{aligned}$$

for all $v, \delta \in \mathbf{L}_2^l[0,\infty)$, $u \in \mathbf{L}_2^m[0,\infty)$. Note that (13) implies that

$$\sigma(Gu, u) \leq -\epsilon\|u\|^2 \quad \forall u \in \mathbf{L}_2^m[0,\infty)$$

Let $\tau \in [0,1]$, $u = \tau\Delta(v)$, $v \in \mathbf{L}_2^l[0,\infty)$, $\epsilon_1 = \epsilon/(2+2\|\Delta\|^2)$. Since $\tau\Delta$ satisfies the IQC defined by Π, we have

$$\begin{aligned} 0 &\leq \sigma(v,u) = \sigma(Gu,u) + \sigma(v,u) - \sigma(Gu,u) \\ &\leq -\epsilon\|u\|^2 + c(\epsilon_1)\|v - Gu\|^2 + \epsilon_1(\|u\|^2 + \|v\|^2) \\ &\leq -\frac{\epsilon}{2}\|u\|^2 + c(\epsilon_1)\|v - Gu\|^2 \end{aligned}$$

Hence $\|u\| \leq \sqrt{2c/\epsilon}\|v - Gu\|$ and

$$\begin{aligned} \|v\| &\leq \|Gu\| + \|v - Gu\| \\ &\leq (1 + \|G\|\sqrt{2c/\epsilon})\|v - \tau G\Delta(v)\|. \end{aligned}$$

Step 2. *Show that if $(I - \tau G\Delta)^{-1}$ is bounded for some $\tau \in [0,1]$ then $(I - \nu G\Delta)^{-1}$ is bounded for any $\nu \in [0,1]$ with $|\tau - \nu| < (c_0\|G\|\cdot\|\Delta\|)^{-1}$.* By the well-posedness assumption, the inverse $(I - \tau G\Delta)^{-1}$ is well defined on $\mathbf{L}_{2e}^l[0,\infty)$. Boundedness of the inverse means that

$$\|P_T v\| \leq \text{const}\|P_T(v - \tau G\Delta(v))\| \quad \forall v \in \mathbf{L}_{2e}^l[0,\infty)$$

Furthermore, when this inequality holds for some constant, it follows from (14) that it holds with the constant c_0. Then

$$\begin{aligned} \|P_T v\| &\leq c_0\|P_T(v - \tau G\Delta(v))\| \\ &\leq c_0\|P_T(v - \nu G\Delta(v)) + (\nu - \tau)P_T G\Delta(v)\| \\ &\leq c_0\|P_T(v - \nu G\Delta(v))\| + c_0\|G\|\cdot\|\Delta\|\cdot|\tau - \nu|\cdot\|P_T v\|. \end{aligned}$$

Boundedness of $(I - \nu G\Delta)^{-1}$ follows, since $c_0\|G\|\cdot\|\Delta\|\cdot|\tau - \nu| < 1$.

Step 3. Now, since $(I - \tau G\Delta)^{-1}$ is bounded for $\tau = 0$, step 2 shows that $(I - \tau G\Delta)^{-1}$ is bounded for $\tau < (c_0\|G\|\cdot\|\Delta\|)^{-1}$, then for $\tau < 2(c_0\|G\|\cdot\|\Delta\|)^{-1}$, etc. By induction, $(I - G\Delta)^{-1}$ is bounded as well. $\square$

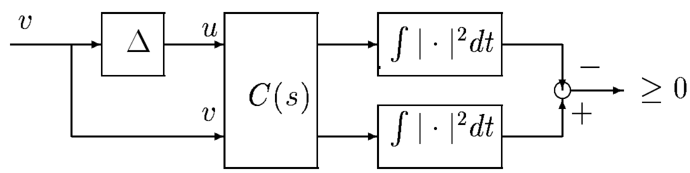

Figure 5: Testing a block Δ for an IQC.

4 Hard and soft IQC's

As a rule, an integral quadratic constraint is an inequality describing correlation between the input and output signals of a causal block Δ. Verifying an IQC can be viewed as a virtual experiment with the setup shown on Fig. 5, where Δ is the block tested for an IQC, f is the test signal of finite energy and $C(s)$ is a stable linear transfer matrix with two vector inputs, two vector outputs and zero initial data. The blocks with $\int |\cdot|^2 dt$ indicate calculation of the energy integral of the signal. We say that Δ satisfies the IQC described by the test setup, if the energy of the second output of C is always at least as large as the energy of the first output. Then the IQC can be represented in the form (1), where

$$\Pi(j\omega) = C(j\omega)^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} C(j\omega). \tag{15}$$

The most commonly used IQC is the one that expresses a gain bound on the operator Δ. For example $C(s) = I$ corresponds to the bound $\|\Delta\| \leq 1$. The energy bounds have the particular property that the energy difference until time T will be non-negative at any moment T, not just $T = \infty$. Such IQC's are called *hard* IQC's, in contrast to the more general *soft* IQC's, which need not hold for finite time intervals. Some of the most simple IQC's are hard, but the "generic" ones are not. In the theory of absolute stability, the use of soft IQC's was often referred to as allowing "non-causal multipliers". While for scalar systems this was usually not a serious problem, the known conditions for applicability of non-causal multipliers were far too restrictive for multivariable systems. The formulation of Theorem 2 makes it possible (and easy) to use soft IQC's in a very general situation. For example, consider the following corollary.

Corollary 3 (Non-causal multipliers) *Assume that condition (i) of Theorem 2 is satisfied. If there exist some* $M \in \mathbf{RL}_\infty^{l\times m}$ *and* $\epsilon > 0$

such that

$$\begin{array}{rcll} \int_{-\infty}^{\infty} \mathrm{Re}\,(\hat{v}^* M \widehat{\Delta v}) d\omega & \geq & 0 & \textit{for } v \in \mathbf{L}_2^l[0,\infty) \\ M^*G + G^*M & \leq & -\epsilon G^*G & \textit{on } j\mathbf{R} \end{array}$$

then the system is input/output stable.

Proof. This is Theorem 2 with

$$\Pi(j\omega) \quad = \quad \begin{bmatrix} \epsilon & M(j\omega) \\ M(j\omega)^* & -\epsilon/\|\Delta\|^2 \end{bmatrix}$$

□

For multivariable systems, the above conditions on M are much weaker than factorizability as $M = M_- M_+$, with $M_+, {M_+}^{-1}, {M_-}^*, ({M_-}^*)^{-1}$ all being stable, which is required for example in [84] and [17]. The price paid for this in Theorem 2 is the very mild assumption that the feedback loop is well-posed not only for $\tau = 1$, but for all $\tau \in [0,1]$.

Another example is provided by the classical Popov criterion.

Corollary 4 (Popov criterion) *Assume that $\phi : \mathbf{R} \to \mathbf{R}$ is such that $0 \leq \sigma\phi(\sigma) \leq \mathrm{const} \cdot \sigma^2$ for $\sigma \in \mathbf{R}$. Let $H(s) = C(sI - A)^{-1}B$, where A is Hurwitz. Assume that the system*

$$\dot{x}(t) = Ax(t) + B\tau\phi(Cx(t)) + f(t) \tag{16}$$

has unique solution on $[0,\infty)$ for any $\tau \in [0,1]$ and for any square summable f. If for some $q \in \mathbf{R}$

$$\inf_{\omega > 0} \mathrm{Re}\,[(1 + j\omega q)H(j\omega)] \quad > \quad 0 \tag{17}$$

then the system (16) with $\tau = 1$ is exponentially stable.

Remark 5 In fact, the condition of existence and uniqueness, used to define Δ as an operator, is not really important in the stability analysis. In the second part of this paper, a stronger version of Theorem 2 is given, which allows us to drop the uniqueness assumption.

Proof. For $q \in \mathbf{R}$ and a differentiable $w \in \mathbf{L}_2^l[0,\infty)$, we have the soft IQC

$$\int_0^\infty (w + q\dot{w})\phi(w) dt \geq q \left[\int_0^{w(t)} \phi(\sigma) d\sigma \right]_0^\infty = 0$$

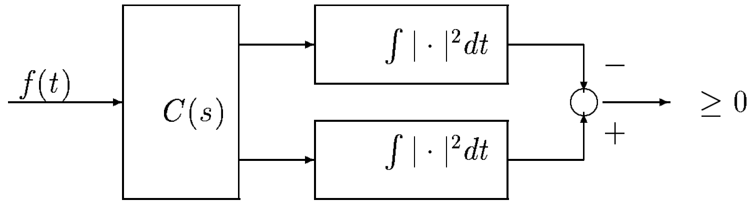

Figure 6: Testing a signal f for an IQC.

Application of Corollary 3 with

$$\begin{aligned} G(s) &= (s+1)H(s) \\ (\Delta v)(t) &= \phi\left(\int_0^t e^{-\tau} v(\tau)d\tau\right) \\ M(s) &= (1+qs)/(s+1) \end{aligned}$$

shows that the conditions of Proposition 1 hold, which ensures the exponential stability. □

Integral quadratic constraints can be used to describe an external signal (noise or a reference) entering the system. The "virtual experiment" setup for a signal f is shown on Fig. 6. The setup clearly shows the "spectral analysis" nature of IQC's describing the signals. Mathematically, the resulting IQC has the form

$$\int_{-\infty t}^{\infty} \hat{f}(j\omega)^* \Pi(j\omega) \hat{f}(j\omega) d\omega \geq 0,$$

where Π is given by (15). In the second part of this paper, performance analysis of systems with both interior blocks and external signals described in terms of IQC's is considered.

5 IQC's and Quadratic Stability

There is a close relationship between quadratic stability and stability analysis based on IQC's. As a rule, if a system is quadratically stable then its stability can also be proved by using a simple IQC. Conversely, in some generalized sense, a system that can be proved to be stable via IQC's always has a quadratic Lyapunov function. However, to actually present this Lyapunov function, one has to extend the state space of the system (by adding the states of $C(s)$ from Figure 5). Even then, in the case of soft IQC's, the Lyapunov function does not need to be sign-definite, and may not decrease monotonically along the system trajectories. In any case, use of IQC's replaces the "blind" search for a quadratic Lyapunov function, which is typical for the quadratic stability, by a more intelligent search. In general, for example in the case of so-called "parameter-dependent" Lyapunov functions, the relationship with the IQC type analysis is yet to be clarified.

Below we formulate and prove a result on the relationship between a simple version of quadratic stability and IQC's. Let $\mathcal{D}$ be a polytope of $m \times l$ matrices Δ, containing the zero matrix $\Delta = 0$. Let $\Delta_1, ..., \Delta_N$ be the extremal points of $\mathcal{D}$. Consider the system of differential equations

$$\dot{x}(t) = (A + B\Delta(t)C)x(t), \ \Delta(t) \in \mathcal{D}, \tag{18}$$

where A, B, C are given matrices of appropriate size, A is a Hurwitz $n \times n$ matrix. (The most often considered case of system (18) is obtained when $m = l$ and $\mathcal{D}$ is the set of all diagonal matrices with the norm not exceeding 1. Then $N = 2^m$, and Δ_i are the diagonal matrices with ± 1 on the diagonal). The system is called stable if $x(t) \to 0$ for any solution of (18) where $\Delta(\cdot)$ is a measureable function and $\Delta(t) \in \mathcal{D}$ for all t. There are no efficient general conditions, that are both *necessary and sufficient* for stability of system (18). Instead, we will be concerned with stability conditions that are only *sufficient.*

The system (18) is called *quadratically stable* if there exists a matrix $P = P^T$ such that

$$P(A + B\Delta_i C) + (A + B\Delta_i C)^T P < 0 \quad \forall \ i. \tag{19}$$

Note that, since $0 \in \mathcal{D}$ and A is a Hurwitz matrix, this condition implies that $P > 0$. It follows that $V(x) = x^T P x$ is a Lyapunov function for the system (18), in the sense that V is positive definite, and $dV(x(t))/dt$ is negative definite on the trajectories. Quadratic stability is a *sufficient* condition for stability of the system and (19) can be solved efficiently with respect to $P = P^T$ as a system of linear matrix ineqalities.

An IQC-based approach to stability analysis of system (18) can be formulated as follows. Note that stability of (18) is equivalent to stability of the feedback interconnection (7), where G is the linear time invariant operator with transfer function $G(s) = C(sI - A)^{-1}B$, and Δ is the operator of multiplication by $\Delta(t) \in \mathcal{D}$. One can apply Theorem 2, using the fact that Δ satisfies the IQC's given by the constant multiplier matrix

$$\Pi(j\omega) = \left[\begin{array}{cc} Q & S \\ S^T & R \end{array} \right],$$

where $Q = Q^T, R = R^T, S$ are real matrices such that

$$Q + S\Delta + \Delta^T S^T + \Delta^T R \Delta > 0 \quad \forall\ \Delta \in \mathcal{D}. \tag{20}$$

For a fixed matrix Π satisfying (20), a sufficient condition of stability given by Theorem 2 is

$$\left[\begin{array}{c} G(j\omega) \\ I \end{array} \right]^* \Pi \left[\begin{array}{c} G(j\omega) \\ I \end{array} \right] < 0 \quad \forall\ \omega \in \mathbf{R} \cup \{\infty\},$$

which is equivalent (by the Kalman-Yakubovich-Popov Lemma) to the existence of a matrix $P = P^T$ such that

$$\left[\begin{array}{cc} PA + A^T P + C^T QC & PB + C^T S \\ B^T P + S^T C & R \end{array} \right] < 0. \tag{21}$$

For an indefinite matrix R, condition (20) may be difficult to verify. However (21) yields $R < 0$. In that case, it is sufficient to check (20) at the vertices $\Delta = \Delta_i$ of $\mathcal{D}$ only, i.e. (20) can be replaced by

$$Q + S\Delta_i + \Delta_i^T S^T + \Delta_i^T R \Delta_i > 0 \quad \forall\ i. \tag{22}$$

It is easy to see that the existence of the matrices $P = P^T$, $Q = Q^T$, S, $R = R^T$, such that (21),(22) hold, is a sufficient condition of stability of system (18).

Now we have the two seemingly different conditions of stability of system (18), both expressed in terms of systems of LMI's: quadratic stability (19), and IQC-stability (21),(22). Condition (19) has $n(n+1)/2$ free variables (the components of the matrix $P = P^T$), while conditions (21),(22) have $n(n+1)/2 + (n+m)(n+m+1)/2$ free variables. However, the advantage of using (21),(22) is that the overall "size" of the corresponding LMI is $n + m + Nl$ while the "size" of (19) is Nn. If N is a large number and n is significantly larger than l and m, modest (about 2 times) increase of the number of free variables in (21),(22) results in a significant (about n/l times) decrease in the size of the corresponding LMI. The following result shows that the two sufficient conditions of stability (21),(22) and (19) are equivalent from the theoretical point of view.

Theorem 5 *Assume that A is a Hurwitz matrix, and that zero belongs to the convex hull of matrices $\Delta_1, ..., \Delta_N$. Then a given symmetric matrix P solves the system of LMI's (19), if and only if P together with the matrices $Q = Q^T$, $R = R^T$, S solves (21) and (22).*

A proof is given in section 8.

6 Example Revisited

In order to apply the results to system (4), we rewrite it as a feedback interconnection on Fig. 1, with

$$G(s) \quad = \quad \begin{bmatrix} -kP(s) & -k \\ P(s) & 0 \end{bmatrix},$$

and

$$\Delta(v)(t) \quad = \quad \begin{bmatrix} \mathrm{sat}(v_1(t)) \\ v_2(t-\theta) - v_2(t) \end{bmatrix}$$

The equations for the interconnection are then

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + Bu_1(t) \\ u_1(t) &=& \mathrm{sat}[v_1(t)] + e_1(t) \\ u_2(t) &=& v_2(t-\theta) - v_2(t) + e_2(t) \\ v_1(t) &=& -kCx(t) - ku_2(t) + f_1(t) \\ v_2(t) &=& Cx(t) + f_2(t) \end{array} \tag{23}$$

where $x(0) = 0$ and $v_2(t-\theta) = 0$ for $t < \theta$. One can see that (23) is equivalent to the equations from (4), disturbed by the "interconnection noise" e, f.

For the uncertain time delay, several types of IQC's are given in the list. Here we shall use a simple (and not complete) set of IQC's for the uncertain delay

$$\hat{u}_2(j\omega) = (e^{-j\theta\omega} - 1)\hat{v}_2(j\omega), \quad \theta \in [0, \theta_0],$$

based on the bounds

$$\begin{array}{rcl} |\hat{v}_2(j\omega)|^2 - |\hat{v}_2(j\omega) - \hat{u}_2(j\omega)|^2 &\geq& 0 \\ \psi_0(\theta_0\omega)|\hat{v}_2(j\omega)|^2 - |\hat{u}_2(j\omega)|^2 &\geq& 0 \end{array} \tag{24}$$

where

$$\psi_0(\omega) = \frac{\omega^2 + 0.08\omega^4}{1 + 0.13\omega^2 + 0.02\omega^4}$$

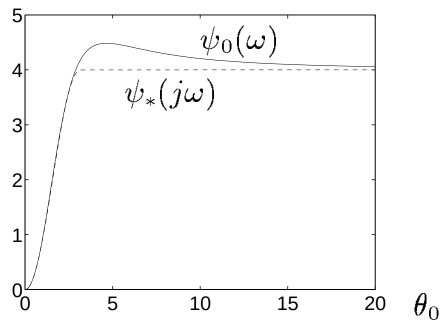

Figure 7: Comparison of $\psi_0(\omega)$ and $\psi_*(j\omega)$

is chosen as a rational upper bound (see Figure 7) of

$$\psi_*(j\omega) = \max_{\theta\in[0,\theta_0]} |e^{-j\omega\theta/\theta_0} - 1|^2 = \begin{cases} 4\sin^2(\omega/2), & \omega < \pi \\ 4 & \omega \geq \pi \end{cases}$$

By integrating the pointwise inequalities (24) with some nonnegative rational functions, one can obtain a huge set of IQC's valid for the uncertain delay. Using these in combination with some set of IQC's for the saturation nonlinearity, one can estimate the region of stability for the system given in (4). In Figure 8, we have plotted the resulting stability bound for the case when only one IQC

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}_1 \\ \hat{u}_1 \end{bmatrix}^* \begin{bmatrix} 0 & 1+H \\ 1+H^* & -2(1+\mathrm{Re}\ H) \end{bmatrix} \begin{bmatrix} \hat{v}_1 \\ \hat{u}_1 \end{bmatrix} d\omega \geq 0,$$

with $H(s) = -(s+1)^{-1}$, describes the saturation, while (24) utilizes the information about the delay. The guaranteed instability region was obtained analytically by considering the behavior of the system in the linear "unsaturated" region around the origin.

7 A List of IQC's

The collection of IQC:s presented in this section is far from being complete. However, the authors hope it will support the idea that many important properties of basic system interconnections used in stability analysis can be characterized by IQC:s.

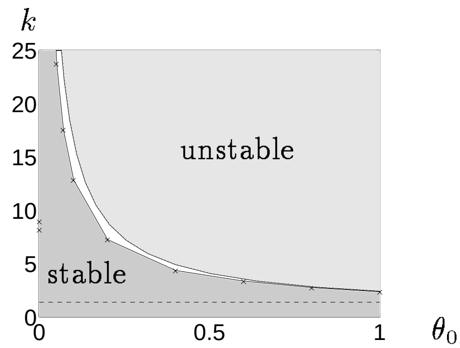

Figure 8: Bound on stabilizing gain k versus delay uncertainty θ_0

7.1 Uncertain LTI Dynamics

Let Δ be any linear time-invariant operator with gain (H_∞ norm) less than one. Then Δ satisfies all IQC's of the form

$$\begin{bmatrix} x(j\omega)I & 0 \\ 0 & -x(j\omega)I \end{bmatrix}$$

where $x(j\omega) \geq 0$ is a bounded measurable function.

7.2 Constant Real Scalar

If Δ is defined by multiplication with a real number of absolute value ≤ 1, then it satisfies all IQC:s defined by matrix functions of the form

$$\begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega) \end{bmatrix} \tag{25}$$

where $X(j\omega) = X(j\omega)^* \geq 0$ and $Y(j\omega) = -Y(j\omega)^*$ are bounded and measurable matrix functions.

This IQC and the previous one are the basis for standard upper bounds for structured singular values [20, 80].

7.3 Time-varying Real Scalar

Let Δ be defined by multiplication in the time-domain with a scalar function $\delta \in \mathbf{L}_\infty$ with $\|\delta\|_\infty \leq 1$. Then Δ satisfies IQC:s defined by a matrix of the form

$$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$$

where $X = X^T \geq 0$ and $Y = -Y^T$ are real matrices.

7.4 Coefficients From a Polytope

Let Δ be defined by multiplication in the time-domain with a measurable matrix $\Delta(\cdot)$, such that $\Delta(t) \in \mathcal{D}$ for any t, where $\mathcal{D}$ is a polytope of matrices with the extremal points (vertices) $\Delta_1, ..., \Delta_N$. Δ satisfies the IQC's given by the constant weight matrices

$$\Pi(j\omega) = \begin{bmatrix} Q & F \\ F^T & R \end{bmatrix},$$

where $Q = Q^T, F, R = R^T$ are real matrices such that $R \leq 0$, and

$$Q + F\Delta_i + \Delta_i^T F^T + \Delta_i^T R \Delta_i > 0 \quad \forall\ i.$$

This IQC corresponds to quadratic stability and was studied in section 5.

7.5 Periodic Real Scalar

Let Δ be defined by multiplication in the time-domain with a periodic scalar function $\delta \in \mathbf{L}_\infty$ with $||\delta||_\infty \leq 1$ and period T. Then Δ satisfies IQC:s defined by (25) where X and Y are bounded, measurable matrix functions satisfying

$$\begin{aligned} X(j\omega) &= X(j(\omega + 2\pi/T)) = X(j\omega)^* \geq 0 \\ Y(j\omega) &= Y(j(\omega + 2\pi/T)) = -Y(j\omega)^*. \end{aligned}$$

This set of IQC:s gives the result by Willems on stability of systems with uncertain periodic gains [74].

7.6 Multiplication by a Harmonic Oscillation

If $(\Delta v)(t) = v(t)\cos(\omega_0 t)$ then Δ satisfies the IQC's defined by

$$\Pi(j\omega) = \begin{bmatrix} X(j\omega - j\omega_0) + X(j\omega + j\omega_0) & 0 \\ 0 & -2X(j\omega) \end{bmatrix},$$

where $X(j\omega) = X(j\omega)^* \geq 0$ is any bounded matrix-valued rational function. Multiplication by a more complicated (almost periodic) function can be represented as a sum of several multiplications by a harmonic oscillation, with the IQC's derived for each of them separately. For example,

$$v(t)\{a_1\cos(\omega_1 t) + a_2\cos(\omega_2 t)\} = a_1(\Delta_1 v)(t) + a_2(\Delta_2 v)(t),$$

where $(\Delta_1 v)(t) = v(t)\cos(\omega_1 t)$, and $(\Delta_2 v)(t) = v(t)\cos(\omega_2 t)$.

7.7 Slowly Time-varying Real Scalar

Here Δ is the operator of multiplication by a slowly time-varying scalar, $\Delta v = \delta(t)v(t)$, where $|\delta(t)| \leq 1$, $|\dot{\delta}(t)| \leq d$. Since the 60-s, various IQC:s have been discovered that hold for such time-variations. See, for example, [21, 34, 24].

Here we describe a simple but representative family of IQC:s describing the redistribution of energy among frequencies, caused by the multiplication by a slowly time-varying coefficient. For any transfer matrix

$$H(s) = H_0 + \int_{-\infty}^{+\infty} e^{-ts} h(t) dt,$$

where $h(\cdot) \in L_1^{n\times m}(-\infty, +\infty)$ and H_0 is a constant, let $\phi(H, d)$ be an upper bound of the norm of the commutator $\Delta \circ H - H \circ \Delta$, for example

$$\phi(H, d) = \int_{-\infty}^{+\infty} \|h(t)\| \min\{2, d|t|\} dt.$$

The following weighting matrices then define valid IQC:s:

$$\Pi = \begin{bmatrix} (1+\rho)\{H^*H + \frac{\phi(H,d)^2}{\rho} I_m\} & 0 \\ 0 & -H^*H \end{bmatrix} \tag{26}$$

where $\rho > 0$ is a parameter, and H is a causal transfer function ($h(t) = 0$ for $t < 0$). Another set of IQC:s is given by

$$\Pi = \begin{bmatrix} \phi(H, d)I & H \\ H^* & 0 \end{bmatrix} \tag{27}$$

where H is skew-Hermitean along the imaginary axis (i.e. $H(j\omega) = -H(j\omega)^*$), but not necessarily causal. Since

$$\phi(H, d) = O(d) \quad \text{as} \quad d \to 0$$

whenever $\|h(t)\| = O(t^{-2-\epsilon})$, the constraints used in the "μ" case (multiplication by a constant gain $\delta \in [-1, 1]$) can be recovered from (26) and (27) as $d \to 0$. Similarly, the "time-varying real scalar" IQC's will be recovered as $d \to \infty$ by using constant transfer matrices $H(s) = H_0$.

In [49, 50], IQC's are instead derived for uncertain time-varying parameters with bounds on the support of the Fourier transform $\hat{\delta}$. Slow variation then means that $\hat{\delta}(j\omega)$ is zero except for ω in some small interval $[-a, a]$.

7.8 Delay

The uncertain bounded delay operator $(\Delta v)(t) = v(t-\theta) = u(t)$, where $\theta \in [0, \theta_0]$, satisfies the "pointwise" quadratic constraints in the frequency domain:

$$|\hat{u}(j\omega)|^2 = |\hat{v}(j\omega)|^2, \tag{28}$$

$$\psi_1(\omega_*)(|j\omega_*\hat{u}(j\omega) + \hat{v}(j\omega)|^2 - (1+\omega_*^2)|\hat{v}(j\omega)|^2) \geq \psi_2(j\omega_*)|\hat{v}(j\omega) - \hat{u}(j\omega)|^2, \tag{29}$$

where $\omega_* = \omega\theta_0/2$, and $\psi_{1,2}$ are the functions defined by

$$\psi_1(\omega) = \begin{cases} \frac{\sin\omega}{\omega} & , |\omega| \leq \pi \\ 0 & , |\omega| > \pi. \end{cases},$$

$$\psi_2(\omega) = \begin{cases} \cos\omega & , |\omega| \leq \pi \\ 0 & , |\omega| > \pi. \end{cases}.$$

Note that (29) is just a sector inequality for the relation between $\hat{v}(j\omega) - \hat{u}(j\omega)$ and $j(\hat{v}(j\omega) + \hat{u}(j\omega))$:

$$j(\hat{v}(j\omega) + \hat{u}(j\omega)) = \frac{\cos(\omega\theta/2)}{\sin(\omega\theta/2)}(\hat{v}(j\omega) - \hat{u}(j\omega)).$$

Multiplying (28) by any rational function and integrating over the imaginary axis yields a set of IQC's for the delay. Unfortunately, these IQC's do not utilize the bound on the delay. To improve the IQC-description, one can multiply (29) by any non-negative weight function and integrate over the imaginary axis. The resulting IQC's, however, will have non-rational weight matrices $\Pi(\cdot)$. To fix the problem, one should use a rational upper bound ψ_{1+} of ψ_1 and rational lower bounds ψ_{1-} and ψ_{2-} of ψ_1 and ψ_2 respectively. For example, a reasonably good approximation is given by

$$\psi_{1+} = \frac{(1-0.0646\omega^2)^2}{1+0.038\omega^2+0.0001\omega^4+0.00085\omega^6},$$

$$\psi_{1-} = \frac{1-\omega^2/\pi^2}{1+(1/6-1/\pi^2)\omega^2+(2/\pi^4-1/6\pi^2)\omega^4},$$

$$\psi_{2-} = \frac{1-0.4073\omega^2}{1+0.0927\omega^2+0.0085\omega^4}.$$

Then the pointwise inequality (29) holds with ψ_2 replaced by ψ_{2-}, and with ψ_1 replaced by $\psi_{1\pm}$ (the upper bound for the $|j\omega_*\hat{u}+\hat{w}|^2$ multiplier, the lower bound for the $|\hat{v}|^2$ multiplier) respectively, and can be integrated with a non-negative rational weight function to get rational IQC's utilizing the upper bound on the delay.

A simpler, but less informative, set of IQC:s is defined for $(\Delta v)(t) = v(t-\theta) - v(t)$, $\theta \le \theta_0$, by

$$\begin{bmatrix} \tau(j\omega)\psi_0(\omega\theta_0/2) & 0 \\ 0 & -\tau(j\omega) \end{bmatrix},$$

where $\tau(\cdot)$ is any non-negative rational weighting function, and $\psi_0(\omega)$ is any rational upper bound of

$$\psi_*(j\omega) = \max_{\theta\in[0,\theta_0]} |e^{-j\omega\theta/\theta_0} - 1|^2 = \begin{cases} 4\sin^2(\omega/2), & \omega < \pi \\ 4 & \omega \ge \pi \end{cases},$$

for example,

$$\psi_0(\omega) = \frac{\omega^2 + 0.08\omega^4}{1 + 0.13\omega^2 + 0.02\omega^4}.$$

7.9 Memoryless nonlinearity in a sector

If $(\Delta v)(t) = \phi(v(t), t)$, where $\phi : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ is a function, such that

$$\alpha v^2 \le \phi(v,t)v \le \beta v^2 \quad \forall\ v \in \mathbf{R}, t \ge 0,$$

then obviously the IQC with

$$\Pi(j\omega) = \begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix}$$

holds.

7.10 The "Popov" IQC

If $u(t) = (\Delta v)(t) = \phi(v(t))$, where $\phi : \mathbf{R} \to \mathbf{R}$ is a continuous function, $v(0) = 0$, and both $u(\cdot)$ and $\dot{v}(\cdot)$ are square summable, then

$$\int_0^\infty \dot{v}(t)u(t)dt = 0.$$

In the frequency domain, this looks like an IQC with

$$\Pi(j\omega) = \pm\begin{bmatrix} 0 & j\omega \\ -j\omega & 0 \end{bmatrix}.$$

However, this is not a "proper" IQC, because $\Pi(\cdot)$ is not bounded on the imaginary axis. To fix the problem, consider $\Delta_1 = \Delta \circ \frac{1}{s+1}$ instead of Δ, i.e. $u(t) = (\Delta_1 f)(t) = \phi(v(t))$, where $\dot{v}(t) = -v(t) + f(t)$, $v(0) = 0$. Now, Δ_1 satisfies the IQC with

$$\Pi(j\omega) = \pm \begin{bmatrix} 0 & \frac{j\omega}{1+j\omega} \\ -\frac{j\omega}{1-j\omega} & 0 \end{bmatrix}.$$

Together with the IQC for a memoryless nonlinearity in a sector, this IQC yields the well-known Popov criterion.

7.11 Monotonic Odd Nonlinearity

Suppose Δ operates on scalar signals according to the nonlinear map $(\Delta v)(t) = \delta(v(t))$, where δ is an odd function on $\mathbf{R}$ such that $\dot{\delta}(x) \in [0, k]$ for some constant k. Then Δ satisfies the IQC:s defined by

$$\begin{bmatrix} 0 & 1 + H(j\omega) \\ 1 + H(-j\omega) & -(2 + 2\,\mathrm{Re}\, H(j\omega))/k \end{bmatrix},$$

where $H \in \mathbf{RL}_\infty$ is arbitrary except that the L_1-norm of its impulse response is no larger than one [84].

7.12 IQC:s for Signals

Performance of a linear control system is often measured in terms of disturbance attenuation. An important issue is then the definition of the set of expected external signals. Here again, integral quadratic constraints can be used as a flexible tool, for example to specify bounds on auto correlation, frequency distribution, or even to characterize a given finite set of signals. Then, the information given by the IQC:s can be used in the performance analysis, along the lines discussed in [33, 44] and further in the second part of this paper.

7.13 IQC's from robust performance

One of the most appealing features of IQC's is their ability to widen the field of application of already existing results. This means that almost any robustness result derived by some method (possibly unrelated to the IQC techniques) for a special class of systems can be translated into an integral quadratic constraint.

As an example of such "translation", consider the feedback interconnection of a particular linear time-invariant system $G_0 = G_0(s)$ with an "uncertain" block Δ:

$$v = G_0 u + f, \quad u = \Delta(v), \tag{30}$$

where f is the external disturbance. Assume that stability of this interconnection (i.e. the invertibility of the operator $I - G_0\Delta$) is already proved, and, moreover, an upper bound on the induced L_2 gain "from f to v ("robust performance") is known: $\|v\|^2 \leq d\|f\|^2$ for any square summable f, v satisfying (30). Then, since for any square summable v there exists a square summable $f = v - G_0\Delta(v)$ satisfying (30), the block Δ satisfies the IQC given by

$$\Pi(j\omega) = \begin{bmatrix} d-1 & -dG_0(j\omega) \\ -dG_0(-j\omega) & d|G_0(j\omega)|^2 \end{bmatrix}. \tag{31}$$

This IQC implies stability of system (30) via Theorem 2, but can also be used in the analysis of systems with additional feedback blocks, as well as with different nominal transfer functions.

For example, consider the uncertain block Δ which represents multiplication of a scalar input by a scalar time-varying coefficient $k = k(t)$, such that $k(t) \in [-1, 1]$. There is one obvious IQC for this block, stating that the L_2-induced norm of Δ is not greater than 1. Let us show how additional non-trivial IQC's can be derived based on a particular robust performance result. Consider the feedback interconnection of Δ with a given LTI block with a stable transfer function $G_0(s) = C(sI - A)^{-1}B$. This is the case of a system with *one* uncertain fast time-varying parameter $k = k(t)$, $k(t) \in [-1, 1]$:

$$\dot{x}(t) = Ax(t) + Bk(t)(Cx(t) + f(t)), \tag{32}$$

where A, B, C are given constant matrices, A is a Hurwitz matrix, $f(\cdot)$ is the external disturbance. It is known that, for this system, the norm bound $\|v\|^2 \leq \|\Delta(v)\|^2$, yields the circle stability criterion $|G_0(j\omega)| < 1$, which gives only *sufficient* contitions of stability. Nevertheless, for a large class of trasnsfer functions $G_0(s)$, not satisfying the circle criterion, system (32) is robustly stable. A proof of such stability usually involves using a non-quadratic Lyapunov function $V = V(x)$, and provides an upper bound d of the worst-case L_2-induced gain "from v to $y = Cx + v$". This upper bound, in turn, yields the IQC given by (31), describing the uncertain block Δ. The fact that stability of system (32) can be proved from this new IQC, but not from the simple norm bound $\|v\|^2 \le \|\Delta(v)\|^2$, shows that the new IQC indeed carries additional information about Δ.

8 Proofs

Proof of Proposition 1

(i)⇒(ii): For $t_0 \ge 1$ and $c_0 > c$, define the Lyapunov function

$$V(x_0, t_0) = \sup_{g \in L_2, x(t_0)=x_0} \int_{t_0}^{\infty} \{|x(t)|^2 - c_0 |g(t)|^2\} dt$$

where x, g satisfy (9).

Our first objectives are to show convergence of the integral for any $g \in \mathbf{L}_2[t_0, \infty)$ and existence of $\alpha, \beta > 0$ such that

$$\alpha |x_0|^2 \le V(x_0, t_0) \le \beta |x_0|^2.$$

Any solution x, g of (9) on $[t_0, \infty)$ with $x(t_0) = x_0$ can be extended to $[0, \infty)$ with $x(0) = 0$, by setting

$$\begin{cases} g(t) &= x_0/t_0 - \phi(x_0 t/t_0, t) \\ x(t) &= x_0 t/t_0 \end{cases} \qquad 0 \le t \le t_0.$$

Let $\|\phi\| = \sup_{x,t} |\phi(x,t)|/|x(t)|$ and note that

$$\begin{aligned} \int_0^{t_0} |g|^2 dt &\le 2|x_0/t_0|^2 + 2\int_0^{t_0} \phi(x_0 t/t_0, t)^2 dt \\ &\le 2|x_0/t_0|^2 (1 + \|\phi\|^2 {t_0}^3/3) \\ &= c_2 |x_0|^2 \end{aligned}$$

for some $c_2 > 0$. The inequality (10) implies that

$$\int_{t_0}^{\infty} |x(t)|^2 dt \le \|x\|^2 \le c\|g\|^2 \le c\int_{t_0}^{\infty} |g|^2 dt + c c_2 |x_0|^2$$

This proves convergence of the integral in the definition of V and with $\beta = c_2$, it shows that $V(x_0, t_0) \le \beta |x_0|^2$. To prove the existence of α, let $g \equiv 0$ and note that

$$\begin{aligned} |\dot{x}| &\le \|\phi\| \cdot |x| \\ \left| \frac{d}{dt} \ln |x| \right| &\le \|\phi\| \\ |x(t)| &\ge |x_0| e^{(t_0 - t)\|\phi\|}, \quad t \ge t_0 \\ V(x_0, t_0) &\ge \alpha |x_0|^2. \end{aligned}$$

for some $\alpha > 0$.

Now consider a fixed solution x, g of (9). By definition of V

$$V(x(t_0), t_0) \;\; \geq \;\; V(x(t_1), t_1) + \int_{t_0}^{t_1} \{|x(t)|^2 - c_0|g(t)|^2\} dt$$

for any $t_1 \geq t_0 \geq 1$. Hence, with $k(t) = V(x(t), t) \geq 0$, the measure $dk(t)$ is absolutely continuous, and satisfies the inequalities

$$\begin{aligned}
dk(t) &\leq [c_0|g(t)|^2 - |x(t)|^2]dt \leq [c_0|g(t)|^2 - k(t)/\beta]dt \\
d[e^{t/\beta}k(t)] &\leq c_0 e^{t/\beta}|g(t)|^2 dt \\
k(t_1) &\leq e^{(t_0-t_1)/\beta}k(t_0) + c_0 \int_{t_0}^{t_1} |g(t)|^2 dt \\
&\leq \beta e^{(t_0-t_1)/\beta}|x(t_0)|^2 + c_0 \int_{t_0}^{t_1} |g(t)|^2 dt
\end{aligned}$$

This implies (11) for $t_1 \geq t_0 \geq 1$. The result follows for arbitrary $t_1 \geq t_0 \geq 0$, since

$$\begin{aligned}
|\dot{x}| &\leq \|\phi\| \cdot |x| + |g| \\
|x(1)|^2 &\leq c_3|x(t_0)|^2 + c_3 \int_{t_0}^{1} |g(t)|^2 dt,
\end{aligned}$$

for some $c_3 > 0$.

(ii)⇒(i): Let $T > 0$ be such that $d_1 := de^{\epsilon T} < 1$ where d is the constant from (11). Then, by (11)

$$|x(kT+T)|^2 \leq d_1|x(kT)|^2 + d \int_{kT}^{kT+T} |g(t)|^2 dt$$

for $k = 0, 1, 2, \ldots$ Hence

$$\sum_{k=0}^{\infty} |x(kT)|^2 \;\; \leq \;\; d_2 \sum_{k=0}^{\infty} \int_{kT}^{kT+T} |g(t)|^2 dt = d_2\|g\|^2$$

for some $d_2 > 0$ if $x(0) = 0$. Also, the inequality (11) applied for $t_0 = kT$, $t_1 \in [kT, kT+T]$ yields

$$\begin{aligned}
|x(t)|^2 &\leq d\left(|x(kT)|^2 + \int_{kT}^{kT+T} |g(t)|^2 dt\right), \quad t \in [kT, kT+T] \\
\int_{kT}^{kT+T} |x(t)|^2 dt &\leq dT\left(|x(kT)|^2 + \int_{kT}^{kT+T} |g(t)|^2 dt\right) \\
\|x\|^2 &\leq (d_2+1)dT\|g\|^2,
\end{aligned}$$

which completes the proof. $\square$

Proof of Theorem 5 The *sufficiency* is straightforward: multiplying (21) by $[I \;\; C^T\Delta_i^T]$ from the left, and by $[I \;\; C^T\Delta_i^T]^T$ from the right yields

$$P(A+B\Delta_iC)+(A+B\Delta_iC)^TP+C^T(Q+S\Delta_i+\Delta_i^TS^T+\Delta_i^TR\Delta_i)C<0,$$

which implies (19) because of the inequality in (22).

To prove the *necessity*, let $P=P^T$ satisfy (19). Let $\sigma_0:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}$ be the quadratic form

$$\sigma_0(x,\xi)=-\epsilon(|x|^2+|\xi|^2)-2x^TP(Ax+B\xi),$$

where $\epsilon>0$ is a small parameter. Define $\sigma:\mathbf{R}^l\times\mathbf{R}^m\to\mathbf{R}$ by

$$\sigma(y,\xi)=\inf\{\sigma_0(x,\xi):\;Cx=y\}, \tag{33}$$

where the infimum is taken over all $x\in\mathbf{R}^n$ such that $Cx=y$. Since the zero matrix belongs to the convex hull of $\mathcal{D}$, (19) implies that $PA+A^TP<0$. Hence, for a sufficiently small $\epsilon>0$, σ is strictly convex in the first argument, and a finite minimum in (33) exists. Moreover, since σ_0 is a quadratic form, the same is true for σ and the matrices Q,R,S can be introduced by

$$\sigma(y,\xi)=y^TQy+2y^TS\xi+\xi^TR\xi.$$

Let us show that the inequalities (21), (22) are satisfied. First, by (19), for any y we have

$$\begin{aligned}
&y^T(Q+S\Delta_i+\Delta_i^TS^T+\Delta_i^TR\Delta_i)y\\
&\quad=\sigma(y,\Delta_iy)\\
&\quad=\inf\{\sigma_0(x,\Delta_iCx):\;Cx=y\}\\
&\quad=\inf\{-x^T(P(A+B\Delta_iC)+(A+B\Delta_iC)^TP)x\\
&\qquad\quad-\epsilon(|x|^2+|\Delta_iCx|^2):\;Cx=y\}\\
&\quad\geq\epsilon_1|y|^2,
\end{aligned}$$

(provided that ϵ and ϵ_1 are sufficiently small). Hence (22) holds. Similarly, for any x,ξ we have

$$\begin{aligned}
&x^TP(Ax+B\xi)+\sigma(Cx,\xi)\\
&\quad=x^TP(Ax+B\xi)+\inf\{\sigma_0(x_1,\xi):\;Cx_1=Cx\}\\
&\quad\leq x^TP(Ax+B\xi)-\epsilon(|x|^2+|\xi|^2)-x^TP(Ax+B\xi)\\
&\quad\leq-\epsilon(|x|^2+|\xi|^2),
\end{aligned}$$

and hence (21) holds, since the matrix in (21) is the matrix of the quadratic form $x^T P(Ax + B\xi) + \sigma(Cx, \xi)$. $\square$

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