Multivariable Control of a Boiler

An Application of Linear Quadratic Control Theory

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MULTIVARIABLE CONTROL OF A BOILER - AN APPLICATION OF LINEAR QUADRATIC CONTROL THEORY †

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ABSTRACT

An application of linear quadratic control theory to a multivariable system is presented. The process is a boiler and the object of the control is to keep the drum pressure and the drum level constant when the load changes. The load disturbances were modelled from measurements as a stationary stochastic process with rational spectral density function. The crucial difficulty when using optimal theory for design is to find the parameters of the loss functional. A method for choosing these parameters is outlined. A method to eliminate steady state errors is also presented. A Kalman filter for the estimation of the state vector as well as the load disturbance was included. The control situation was simulated on a hybrid computer. The results of these simulations as well as core memory requirements and execution time for the control algorithm are given.

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TABLE OF CONTENTS

1. INTRODUCTION

This report presents an application of linear quadratic control theory to a multivariable system. The process used is a simplified boiler which can be described with a linear constant coefficient dynamical system of the 5:th order. The process has three inputs and two outputs. There are considerable interactions between the inputs and outputs of the process. For such processes conventional synthesis methods are not very attractive. However, using linear quadratic control theory we can synthesize multivariate control laws in a systematic manner.

The boiler control problem is to keep the process output variables constant when the process is disturbed. This type of control problem is very common in process industries. Particular attention must be payed to formulate the control problem as an optimization problem.

A hybrid computer was used to simulate the control situation. This is as far as the control law implementation is concerned quite realistic. The process, however, will be the model equations simulated on the analog computer.

In section 2 we give a résumé of the linear quadratic control theory. The theory requires that we have models of the process and the disturbances available. In section 3 a short presentation of the model of the boiler is given and the models of the disturbances are derived and discussed in section 4. A method to eliminate steady state errors using the technique of feedforward is presented in section 5. It is also shown that the combination of a feedforward and a Kalman filter is equivalent to the introduction of an integrator. The crucial difficulty when using optimal control theory for design is to find the parameters of the loss functional. In section 6 we discuss this problem and outline a method for choosing these parameters. The sampling interval affects the quality of control which decrease with increasing length of the sampling interval. The choice of the sampling interval is discussed in section 7. The complete control law is given in section 8. In this section we also discuss the sensitivity of the Kalman filter to changes of the process parameters. In section 9 we give the core memory requirements and execution time for the control algorithm. The scaling problems which arise when we use fix point arithmetic are discussed. The results of analog and hybrid simulations are given in section 10.

2. RÉSUMÉ OF LINEAR QUADRATIC CONTROL THEORY

The theory can be developed both in the continuous and the discrete case. The résumé given here is restricted to the continuous case. In the discrete case the differential equations are replaced by difference equations but the structure of the solution is identical.

Consider the linear system

$$\frac{dx(t)}{dt} = A(t)\ x(t) + B(t)\ u(t) + w_1(t)$$

$$y(t) = C(t)\ x(t) + w_2(t) \tag{2.1}$$

for $t_o \leq t < \infty$. $x(t)$ is the state n-vector, $u(t)$ is the control m-vector and $y(t)$ is the output k-vector.

The formal expression (2.1) can be interpreted as a stochastic differential equation in the usual manner. Since we will not use (2.1) for any analysis we use this formal expression instead of the mathematically rigorous but more elaborate notions of stochastic differential equations.

The elements of the matrices $A(t)$, $B(t)$ and $C(t)$ are continuous and bounded functions of t. The variables $w_1(t)$ and $w_2(t)$ are white noise with zero mean and the covariance functions

$$E\ w_1(t)\ w_1^T(t+\tau) = R_1(t)\ \delta(\tau)$$

$$E\ w_2(t)\ w_2^T(t+\tau) = R_2(t)\ \delta(\tau) \tag{2.2}$$

where $\delta(\tau)$ is the Dirac measure. $R_1(t)$ is a symmetric nonnegative definite matrix and $R_2(t)$ is a symmetric positive definite matrix. The elements of $R_1(t)$ and $R_2(t)$ are continuous and bounded functions of t. The initial state is a random variable with

$$E\ x(t_o) = m$$

$$\text{cov}\ x(t_o)\ x^T(t_o) = R_o \tag{2.3}$$

The object of the control is to minimize the loss functional

$$V(x_o,t_o,t_1,u)=E\{x^T(t_1)Q_o x(t_1)+\int_{t_o}^{t_1}[x^T(s)Q_1(s)x(s)+u^T(s)Q_2(s)u(s)]ds\} \tag{2.4}$$

where Q_o and $Q_1(t)$ are symmetric nonnegative definite matrices and $Q_2(t)$ is a symmetric positive definite matrix. The parameter t_1 may be infinite. The elements of $Q_1(t)$ and $Q_2(t)$ are continuous and bounded functions of t.

The solution of this problem can be separated into two independent problems: 1) a deterministic control problem and 2) an estimation problem.

The solution of the deterministic control problem is given by

$$u(t) = - L(t)\,\hat{x}(t) \tag{2.5}$$

where $\hat{x}(t)$ is the estimated state vector and

$$L(t) = Q_2^{-1}(t)\,B^T(t)\,S(t;t_1) \tag{2.6}$$

$S(t;t_1)$ is the solution of a Riccati equation. This equation depends on $A(t)$, $B(t)$, Q_o, $Q_1(t)$, $Q_2(t)$ but does not depend on $C(t)$, R_o, $R_1(t)$, $R_2(t)$.

The minimum mean square estimate is given by

$$\frac{d\hat{x}(t)}{dt} = A(t)\hat{x}(t) + B(t)u(t) + K(t)\left[y(t) - C(t)\hat{x}(t)\right] \tag{2.7}$$

where

$$K(t) = P(t;t_o)\,C^T(t)\,R_2^{-1}(t) \tag{2.8}$$

The matrix $P(t;t_o)$ is the solution of a Riccati equation which depends on $A(t)$, $C(t)$, R_o, $R_1(t)$ and $R_2(t)$.

The deterministic control problem and the estimation problem are dual and the feedback matrix $L(t)$ and the filter gain matrix $K(t)$ can be computed using the same algorithm. If the time point t_1 is set equal to infinity we will obtain the stationary values of $L(t)$ and $K(t)$. There are no constraints on the state vector $x(t)$ and the control vector $u(t)$. In the time invariant case the closed system will be stable if (2.1) is controllable and observable and if the pair of matrices (Q_1,A) and (R_1,A^T) are observable. A detailed presentation of the theory is found in {1}, {5}.

3. BOILER MODEL

We consider a drum boiler with natural circulation. The configuration is given in Fig. 1.

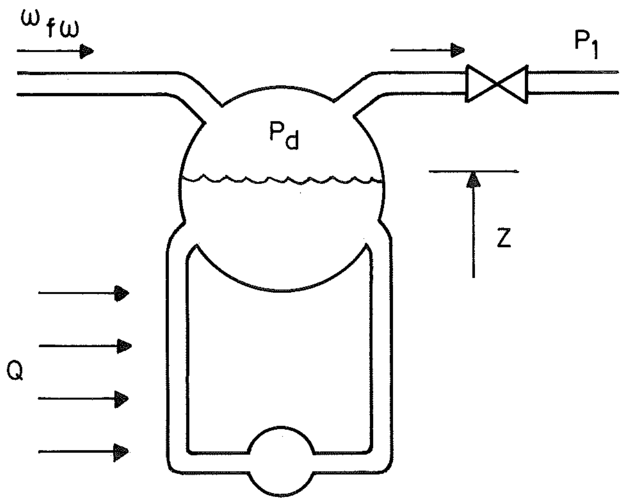

Fig. 1 - Simplified boiler configuration

We use a detailed model only for the drum-downcomer-riser loop of the boiler. The superheaters are simulated with a restriction only.

The linearized model on standard form is

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Fv_1(t)$$

$$y(t) = Cx(t) + v_2(t) \qquad (3.1)$$

where A,B,F and C are constant matrices. It is a fifth order model and the state variables are

$x_1(t)$ drum pressure p_d
$x_2(t)$ drum liquid level z
$x_3(t)$ drum liquid temperature
$x_4(t)$ riser wall temperature
$x_5(t)$ steam quality

The control variables are

$u_1(t)$ heat flow to the risers Q
$u_2(t)$ feedwater flow w_{fw}

and the output variables are

$y_1(t)$ the measured drum pressure
$y_2(t)$ the measured drum level.

The disturbances are

$v_1(t)$ load changes p_1
$v_2(t)$ measurement noise

The heat input variable is the heat flow to the risers and not the fuel flow. The feedwater enthalpy is taken as a constant and not as an input variable. In a power station boiler the pressure p_1 is the pressure before the throttle valve of the turbine. Changes in the demand for steam will instantaneously cause changes in this pressure. Thus we can use the pressure p_1 as a direct measure of the load changes. The controlled variables are the output variables and the object of control is to keep these variables constant when the load changes. A detailed discussion of the model is found in {2}.

Numerical values of the matrices A, B, C and F used in this report are found in Appendix A. The values apply to a power station boiler with a maximum steam flow of about 350 t/h. The drum pressure is 140 bar. The operating point is 90% of full load. The eigenvalues of the matrix A are

$-5.99 \cdot 10^{-2} \pm 1.72 \cdot 10^{-2}\, i$
$-1.81 \cdot 10^{-1}$
$-8.59 \cdot 10^{-2}$
0.00

It is not easy to give a simple physical interpretation of the eigenvalues because of the interaction in the system. Notice that the second column of A equals zero which gives a zero eigenvalue. This also means that the second state variable is not coupled to the other state variables.

The simulated responses of the state variables to step changes in the three input variables are given in Fig. 2. Notice the non-minimum phase characteristics of the drum level and steam quality responses. These two state variables are closely related. The step responses also show that we have a considerable interaction in the process.

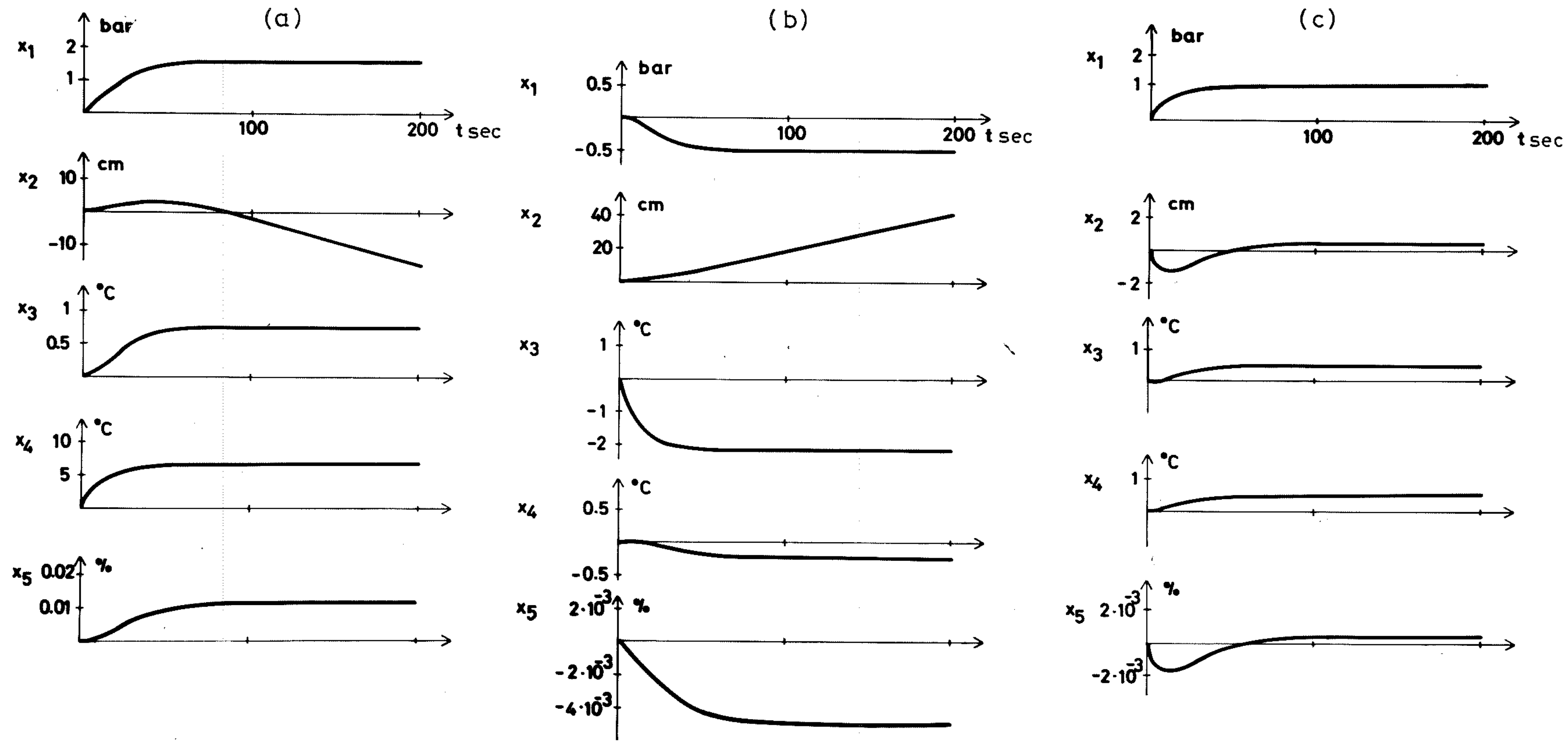

Fig. 2 - Responses of state variables to a step change in (a) heat flow to the risers (b) feedwater flow and (c) pressure p_1

4. CHARACTERISTICS OF DISTURBANCES

In section 2 it was stated that the solution of the formulated problem requires that we know the characteristics of the random processes involved. In the boiler application the input noise is the load disturbance $v_1(t)$ and the measurement noise is $v_2(t)$. (See equation (3.1)). In this section we will give the characteristics of the random processes which are used to describe these noises.

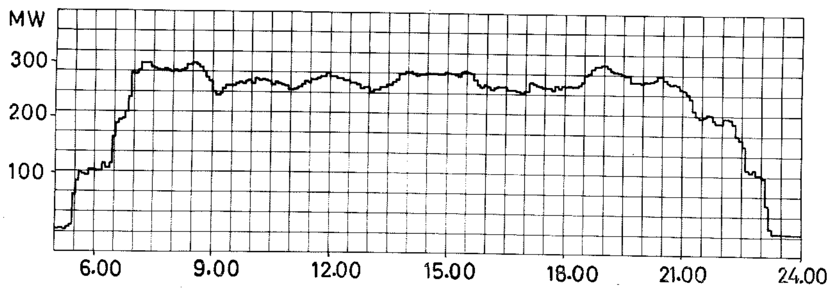

Fig. 3 - The power generated by four hydroelectric power stations in the time interval 5^{00}-24^{00} a weekday

Fig. 3 shows the power generated by four hydroelectric power stations in the north of Sweden a weekday. The sampling interval of the measurement is 300 sec. The load changes during 5^{00}-7^{00} and 21^{00}-24^{00} a'clock are for the main part ordered load changes. During the time interval 7^{00}-21^{00} the variations are mostly due to the control of the mains frequency. Since the dynamics of a hydroelectric power station are fast compared to the dynamics of a thermal power station we will use the recording in the interval 7^{00}-21^{00} as a measurement of the demand for power.

A set of measurements for various weekdays have been used to determine the parameters of a model of the load disturbance. We assume that the disturbance is a stationary process with rational spectrum. Such a process can always be represented with a linear model

$$A(z^{-1})y(t) = \lambda\, C(z^{-1})e(t) \qquad (4.1)$$

where e(t) is a sequence of independent normal (0,1) random variables. z is the shift operator

$$z\, y(t) = y(t + T)$$

and T is the sampling interval. $A(z^{-1})$ and $C(z^{-1})$ are polynomials in the inverse shiftoperator z^{-1}. The identification method used is the maximum likelihood method. A presentation of the used method is found in {4}. The identification gives a first order system

$$y(t) = \lambda \frac{1 + c_1 z^{-1}}{1 + a_1 z^{-1}} e(t) \tag{4.2}$$

where the average values of the coefficients and standard deviations were

$a_1 = -0.92 \pm 0.036$
$c_1 = 0.10 \pm 0.071$
$\lambda = 5.6$

The coefficient c_1 is quite small and roughly zero within one standard deviation. We will therefore assume that c_1 equals zero. This is not a severe assumption and will simplify the computations. The variable $y(t)$ has the dimension MW and gives the deviation from the mean value of the generated power. The mean value is about 275 MW. To fit the boiler model we must find the equivalence between $y(t)$ and the pressure $v_1(t)$. If we also consider that the maximum power generated by the studied boiler is about 125 MW we get

$$v_1(t) = \lambda \cdot \frac{1}{1 + a\, z^{-1}} e(t) \tag{4.3}$$

where
$a = -0.92$
$\lambda = 0.225$

For the analog and hybrid simulations it is convenient to have a continuous approximation of the load disturbance model. Assume a first order continuous system

$$\frac{dx(t)}{dt} = \alpha\, x(t) + \mu\, \omega(t)$$
$$v_1(t) = x(t) \tag{4.4}$$

where $\omega(t)$ is white noise with zero mean and the covariance function

$$\operatorname{cov} \omega(t)\, \omega(t + \tau) = \delta(\tau)$$

The covariance functions for the discrete (4.3) and the continuous (4.4) representation of $v_1(t)$ are

$$r(n) = a^n \cdot \frac{\lambda^2}{1 - a^2} \tag{4.5}$$

$$r(\tau) = \frac{\mu^2}{2|\alpha|} \cdot e^{-|\alpha||\tau|} \tag{4.6}$$

Equating the covariance functions (4.5) and (4.6) for $n = 0$, $\tau = 0$ and $n = 1$, $\tau = 300$ sec we get

$\alpha = -2.78 \cdot 10^{-4}$

$\mu = 0.0130$

The continuous system (4.4) has a time constant T of 3600 sek. For our purpose we will use an observation time less than 2000 sec. During this time the system (4.4) will practically act as if α was equal to zero. With this approximation we get

$$\begin{aligned} \frac{dx(t)}{dt} &= \mu\ \omega(t) \\ v_1(t) &= x(t) \end{aligned} \tag{4.7}$$

Spectral density functions for the load disturbances generated by equations (4.4) and (4.7) are given in Fig. 4. The spectral density function of $\omega(t)$ equals a constant A.

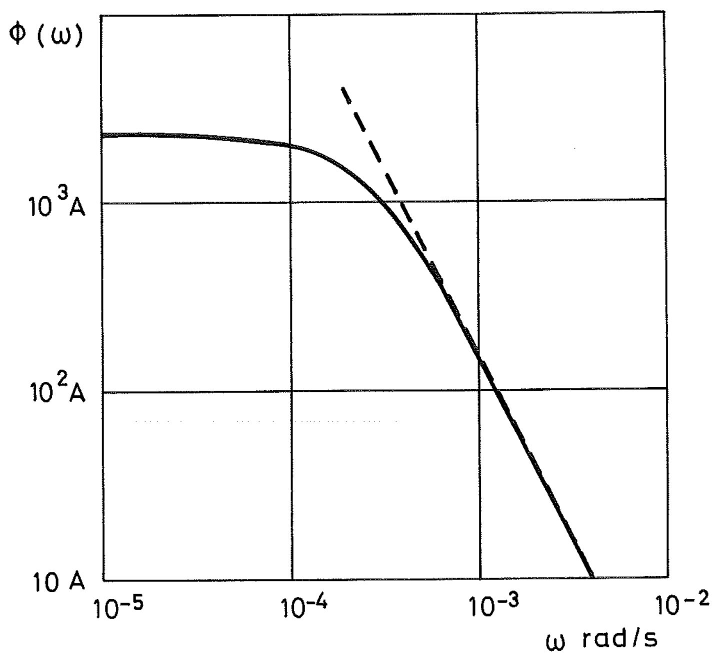

Fig. 4 - Spectral densities for the load disturbance generated by equation (4.4) (——) and equation (4.7) (----)

The frequency content above $\omega = 10^{-3}$ rad/s is very small. This is expected since the sampling interval in the original measurement was 300 sec. However, this is a reasonable noise considering the dynamics of the boiler.

No recordings of the measurement noise of the drum pressure and drum level signals were available. We will therefore assume that the measurement noises are pure random processes and that the amplitude distributions are normal with zero mean. The choice of the standard deviations are discussed in section 8.

5. ELIMINATION OF STEADY STATE ERRORS

When the control law (2.5) given in section 2 is used, there will be no steady state errors if the disturbance is an initial error in any state variable. But we also require that the steady state errors of the controlled variables are zero after e.g. a step change of the disturbance v(t), see equation (5.1). To achieve this, we will use the technique of feedforward.

We will first consider the system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Fv(t) \qquad (5.1)$$

where v(t) is a s-vector. We assume that the state vector x(t) and the disturbance vector v(t) can be measured directly. Using index o to indicate steady state values equation (5.1) gives

$$Ax_o + Bu_o + Fv_o = 0 \qquad (5.2)$$

We thus find that with a control law $u_o = - Lx_o$ there will in general be a steady state error. To eliminate this we add a feedforward term from the disturbance v(t) to the stationary control law. Hence

$$u_o = - Lx_o - Rv_o \qquad (5.3)$$

where R is a constant matrix which will be chosen in such a way that the steady state values of i components of the state vector are zero. We assume that equation (5.1) is arranged so that these components are the first i components. Introduce the notations

$$\bar{A}[nx(n-i)] = [a_{i+1} \; \dots \; a_n]$$

$$\bar{L}[mx(n-i)] = [\ell_{i+1} \; \dots \; \ell_n]$$

$$\bar{x} = [x_{i+1} \; \dots \; x_n]^T$$

where a_k and ℓ_k stand for the k:th column of A and L respectively. Introducing the zero error requirement in equation (5.2) and (5.3) we get

$$\bar{A}\bar{x}_o + Bu_o + Fv_o = 0 \qquad (5.4)$$

$$u_o = - \bar{L}\bar{x}_o - Rv_o \qquad (5.5)$$

or

$$[\bar{A}|B] \begin{bmatrix} \bar{x}_o \\ \hline v_o \end{bmatrix} = - Fv_o \tag{5.6}$$

$$Rv_o = - [\bar{L}|I] \begin{bmatrix} \bar{x}_o \\ \hline u_o \end{bmatrix} \tag{5.7}$$

The existence of a solution to equation (5.6) determines possible numbers i. For example if i equals the number of control variables a unique solution to equation (5.6) exists for all matrices F, if the inverse of $[\bar{A}|B]$ exists. If the inverse does not exist, we must require that the columns of F lie in the column space of $[\bar{A}|B]$. In this case the solution is obtained using the pseudo-inverse of $[\bar{A}|B]$. If equation (5.6) has a solution the feedforward matrix R is computed using equation (5.7).

In many physical systems the number i will equal the number of control variables. For this case it has been shown numerically for several specific problems that the feedforward matrix R obtained when using the technique described above can be obtained directly when the control law is computed. The details of this is given in Appendix B.

We will now consider the case when only the output vector y(t) can be measured. The system then is

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Fv(t)$$

$$y(t) = Cx(t) + \omega_2(t) \tag{5.8}$$

We assume that the disturbance v(t) is a Wiener process. Hence

$$\frac{dv(t)}{dt} = \omega_1(t) \tag{5.9}$$

$\omega_1(t)$ and $\omega_2(t)$ are white noise with zero mean and covariance functions given by equation (2.2). Especially the equations (5.8) and (5.9) hold for the boiler application. Combining equations (5.8) and (5.9) the system equations get the form of equation (2.1)

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \hline v(t) \end{bmatrix} = \left[\begin{array}{c|c} A & F \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} x(t) \\ \hline v(t) \end{bmatrix} + \begin{bmatrix} B \\ \hline 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \hline I \end{bmatrix} \omega_1(t)$$

$$y(t) = Cx(t) + \omega_2(t) \tag{5.10}$$

Equation (5.10) is used when the Kalman filter is computed. The filter equation gives the estimates of the state and disturbance vectors. The control law then is

$$u(t) = - L\hat{x}(t) - R\hat{v}(t) \tag{5.11}$$

The combination of a Kalman filter and a feedforward is equivalent to the introduction of an integrator, if the disturbance $v(t)$ is a Wienerprocess. To illustrate this we will consider an example.

Example

Consider a first order system with one control and one disturbance variable

$$\frac{dx_1(t)}{dt} = u(t) + v(t)$$

$$y(t) = x_1(t) + \omega_2(t) \tag{5.12}$$

The disturbance $v(t)$ is a Wienerprocess

$$\frac{dx_2(t)}{dt} = \omega_1(t)$$

$$v(t) = x_2(t) \tag{5.13}$$

The control law including the feedforward is

$$u(t) = - \ell\hat{x}_1(t) - \hat{v}(t)$$

Combining equations (5.12) and (5.13) we get

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega_1(t)$$

The filter equation then is

$$\frac{d\hat{x}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} k_1(t) \\ k_2(t) \end{bmatrix} \{y(t) - \hat{x}_1(t)\}$$

Using the stationary filter gains k_1 and k_2 we can compute the transfer function of the feedback loop. We get

$$U(s) = G(s)\, Y(s)$$

where

$$G(s) = - \frac{(\ell k_1 + k_2)s + \ell k_2}{s(s + \ell + k_1)}$$

Since G(s) contains an integrator the steady state error of y(t) equals zero.

Notice that if any of the conditions, a Wienerprocess disturbance or correct feedforward, are voilent there will be a steady state error. It is easy to verify that components of the state vector which have no steady state error are stationary processes. The feedforward does not influence the dynamics of the closed system and thus not the guaranteed stability associated with the optimal feedback. It should also be mentioned that this technique to compute the feedforward matrix R and the properties discussed above also apply in the discrete case.

6. CHOICE OF LOSS FUNCTIONAL

The feedback matrix L(t) given by equation (2.6) does not depend on the disturbances but only on the loss functional which determines the control law uniquely. Hence we will consider the system equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{6.1}$$

and the loss functional

$$V=\frac{1}{2} x^T(t_1)Q_o x(t_1) + \frac{1}{2}\int_{t_o}^{t_1} \{\alpha x^T(s)Q_1 x(s)+u^T(s)Q_2 u(s)\}ds \tag{6.2}$$

where the scalar α is used to vary the weight of the state variables in relation to the control variables.

The interpretation of Q_o, Q_1, Q_2 is apparent. Q_o represents the weight we put on the difference between the reached and the desired state at the terminal time t_1. Q_1 and Q_2 represents how we weight deviations from the desired state of the state vector x(t) and the use of the control vector u(t) in the control interval.

If we only use the diagonal elements of the loss functional matrices it is easy to qualitatively predict the effect of a parameter change on the closed loop dynamics. For example if we increase the ii-th element of Q_1 the deviations from the desired state of the state variable x_i will decrease. Since the relative weight of all other state variables then is decreased, the deviations in these variables will increase. If all the elements of Q_1 are increased the poles of the closed system will move to the left in the complex s-plane and the system becomes faster. At the same time the magnitude of the control variables will increase.

In the boiler application we will use the stationary value of the feedback matrix L(t). This is physically motivated since in the control problem defined for the boiler the terminal time t_1 can be regarded as plus infinity. This also means that Q_o can be set equal to zero.

In many cases there is no rational a priori choice of the parameters of the loss functional. Especially there is no rational way to match the relative magnitudes of Q_1 and Q_2. To find the parameters of the loss functional we will use the iteration procedure shown in Fig. 5.

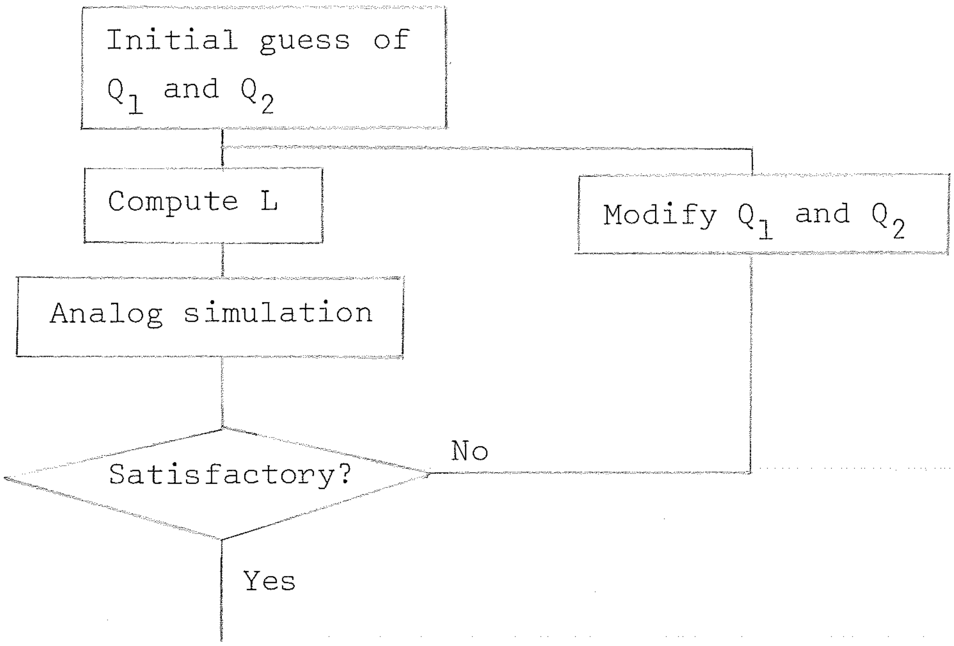

Fig. 5 - The iteration procedure for finding the loss functional matrices.

The idea behind the initial guess of Q_1 and Q_2 is to give all punished variables the same weight in the loss functional. This is achieved by normalizing the variables with an assumed maximum deviation. Notice that we have to punish all control variables but not all state variables since Q_2 must be positive definite and Q_1 only nonnegative definite. The control law is computed and evaluated by simulation. We can not take any constraints on the control vector explicitly into account. We thus have to balance a fast response of the closed system against the magnitude of the control variables for typical disturbances. It is important that the feedforward term is included in the simulations since this term alters the magnitude of the control variables. Notice, however, that we can not change the steady state value of the control vector by a change of a parameter of the loss functional.

The object of the control in the boiler application is to keep the drum pressure $x_1(t)$ and the drum level $x_2(t)$ constant with no steady state error. This error can be eliminated since the inverse of the matrix $[\bar{A}|B]$ exists in this case. The feedforward matrix R is computed using equations (5.6) and (5.7).

The initial guess of Q_1 and Q_2 is

$$Q_1 = \begin{bmatrix} \left(\frac{1}{x_{1_{max}}}\right)^2 & 0 & 0 & 0 & 0 \\ & \left(\frac{1}{x_{2_{max}}}\right)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$Q_2 = \begin{bmatrix} \left(\frac{1}{u_{1_{max}}}\right)^2 & 0 \\ 0 & \left(\frac{1}{u_{2_{max}}}\right)^2 \end{bmatrix}$$

where the assumed maximum deviations are

$x_{1_{max}} = 10$ bar

$x_{2_{max}} = 0.1$ m

$u_{1_{max}} = 10^4$ kJ/s

$u_{2_{max}} = 10$ kg/s

The disturbances used in the analog simulation are a 10% step change of the load and a disturbance in the initial value of $x_1(t)$ and $x_2(t)$.

Fig. 6 and 7 give the results of the simulation using the initially guessed Q_1 and Q_2 with $\alpha = 10$. The corresponding control law will be called control law I. The responses of the controlled variables $x_1(t)$ and $x_2(t)$ are not satisfactory. Especially the response of $x_2(t)$ is quite slow. Fig. 7 shows the responses to a load change with and without the feedforward matrix included in the control law.

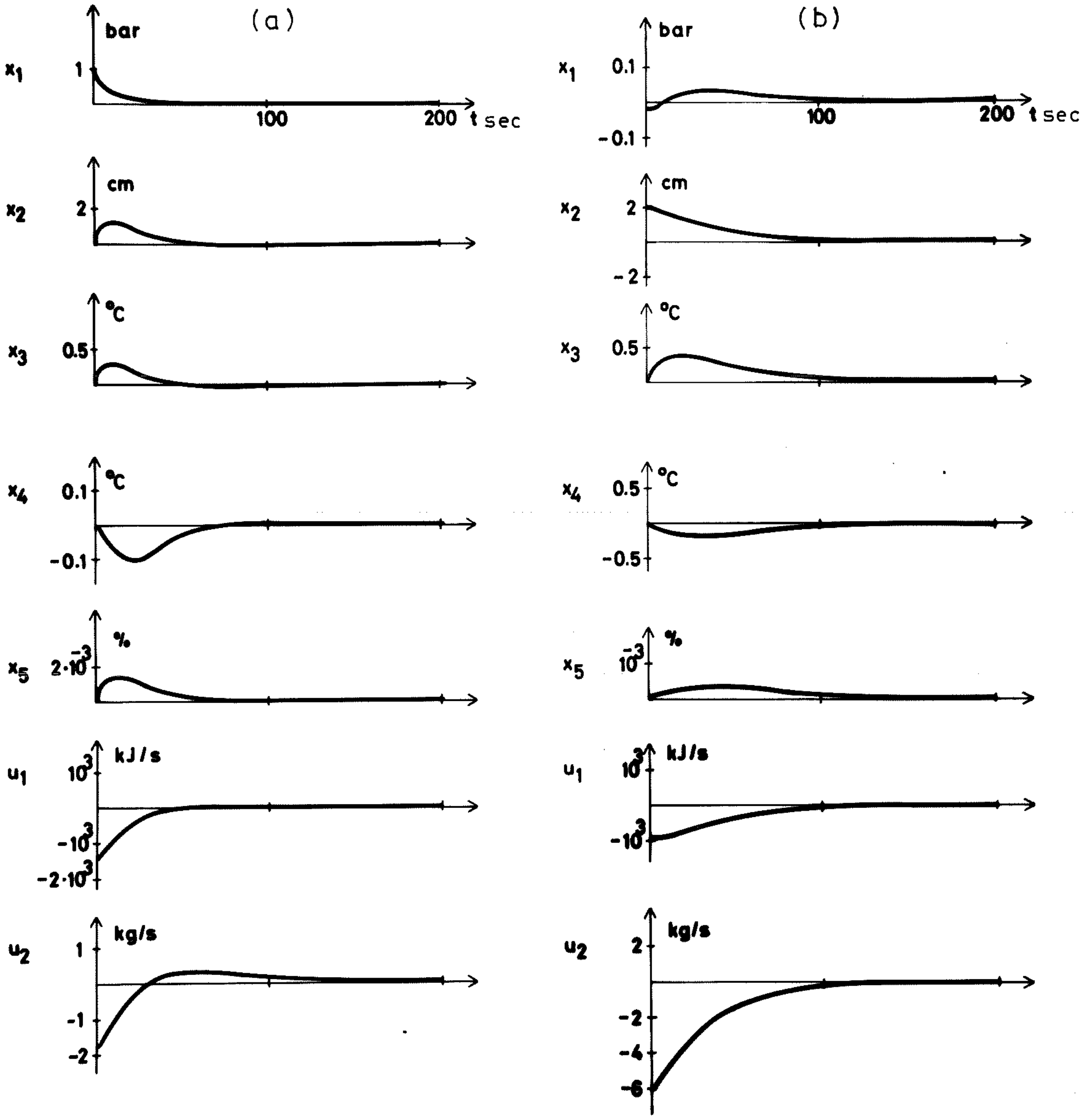

Fig. 6 - Responses of state variables to an initial value disturbance of (a) $x_1(t)$ and (b) $x_2(t)$. Control law I is used.

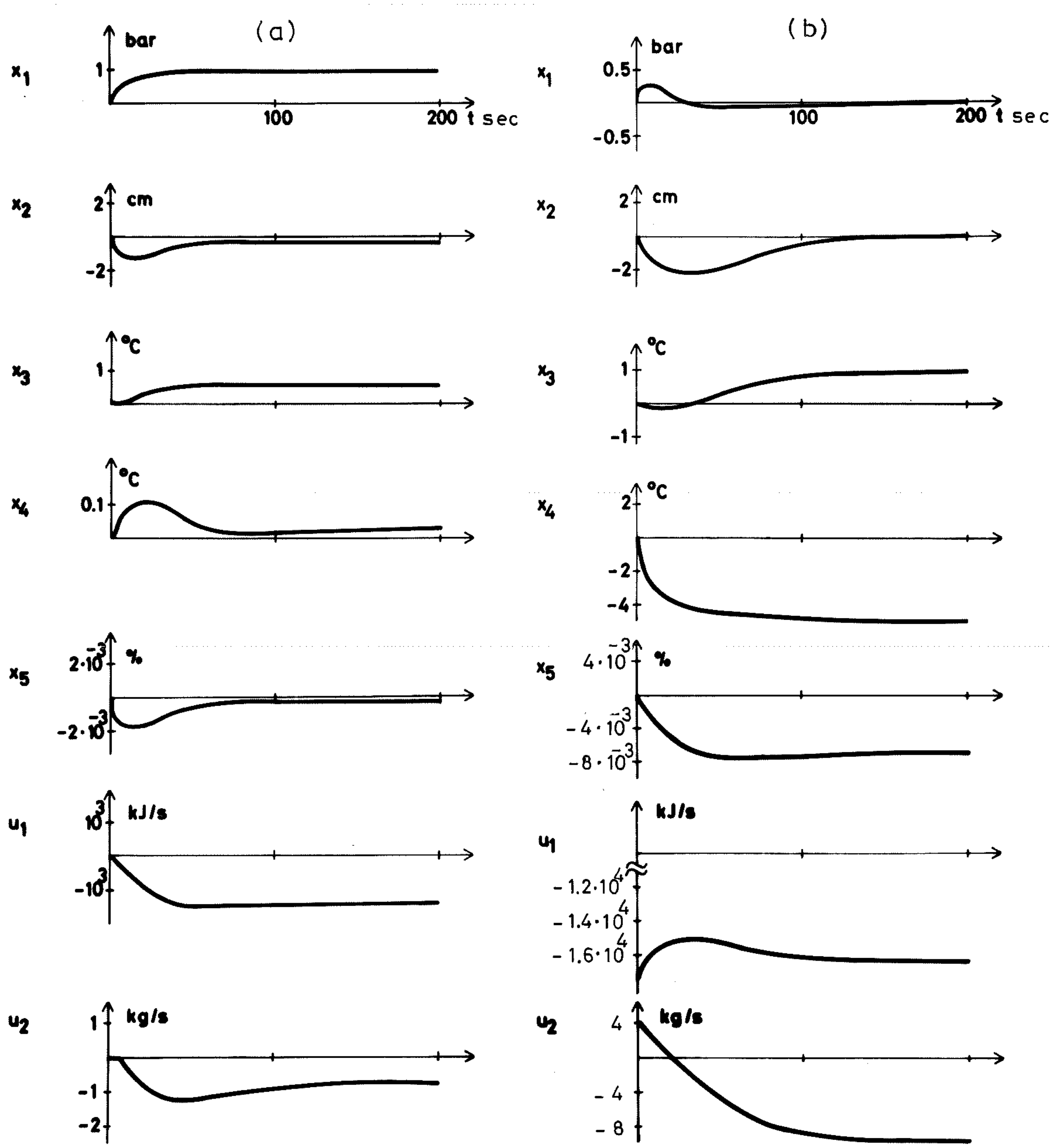

Fig. 7 - Responses of state variables to a 10% step change of the load. Control law I is used (a) without feedforward and (b) with feedforward.

To improve the response of $x_2(t)$ the 22-element of Q_1 was increased. After some iterations the final choice of the two non-zero elements of Q_1 were

$q_{11}' = 10^{-2}$

$q_{22}' = 10^{4}$

The matrix Q_2 was not altered. Fig. 8 and 9 show the responses to the disturbances using $\alpha = 1$. The corresponding control law will be called control law II. It is now appraised that the magnitude of $u_2(t)$ should not be further increased. The large positive value of $u_2(t)$ during the first moment after the load decrease is due to the increased drum pressure which causes a sudden decrease of the drum level, see Fig. 2c. The eigenvalues of the closed system matrix (A-BL) are in this case

$-7.55 \cdot 10^{-2} \pm 5.12 \cdot 10^{-2}\ i$

$-1.41 \cdot 10^{-1} \pm 1.70 \cdot 10^{-2}\ i$

$-4.90 \cdot 10^{-2}$

Numerical values of the used feedback and feedforward matrices both for the continuous and discrete cases are found in Appendix A. A detailed presentation of the programs used to compute the control law is given in {5}.

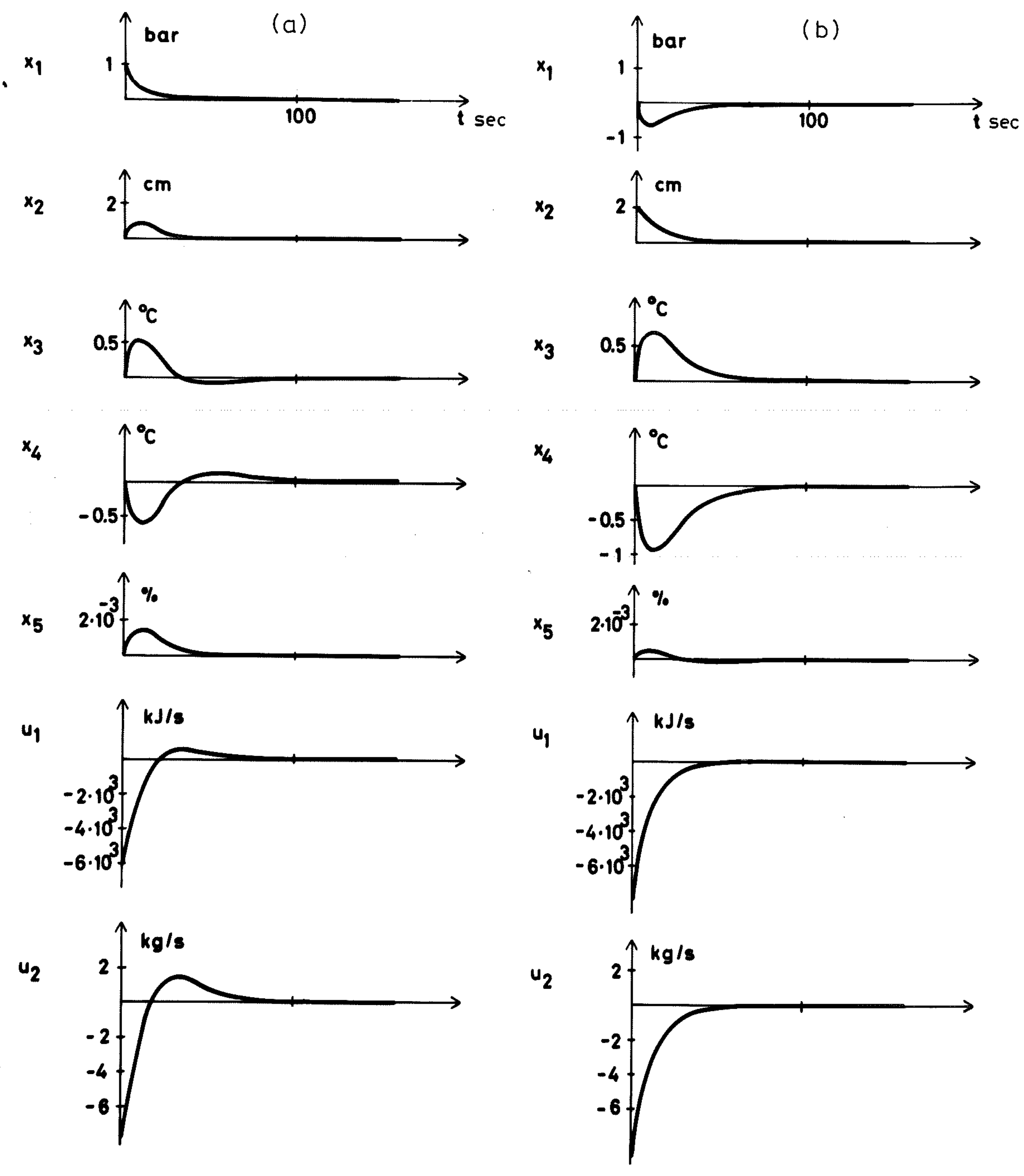

Fig. 8 - Responses of state variables to an initial value disturbance of (a) $x_1(t)$ and (b) $x_2(t)$. Control law II is used.

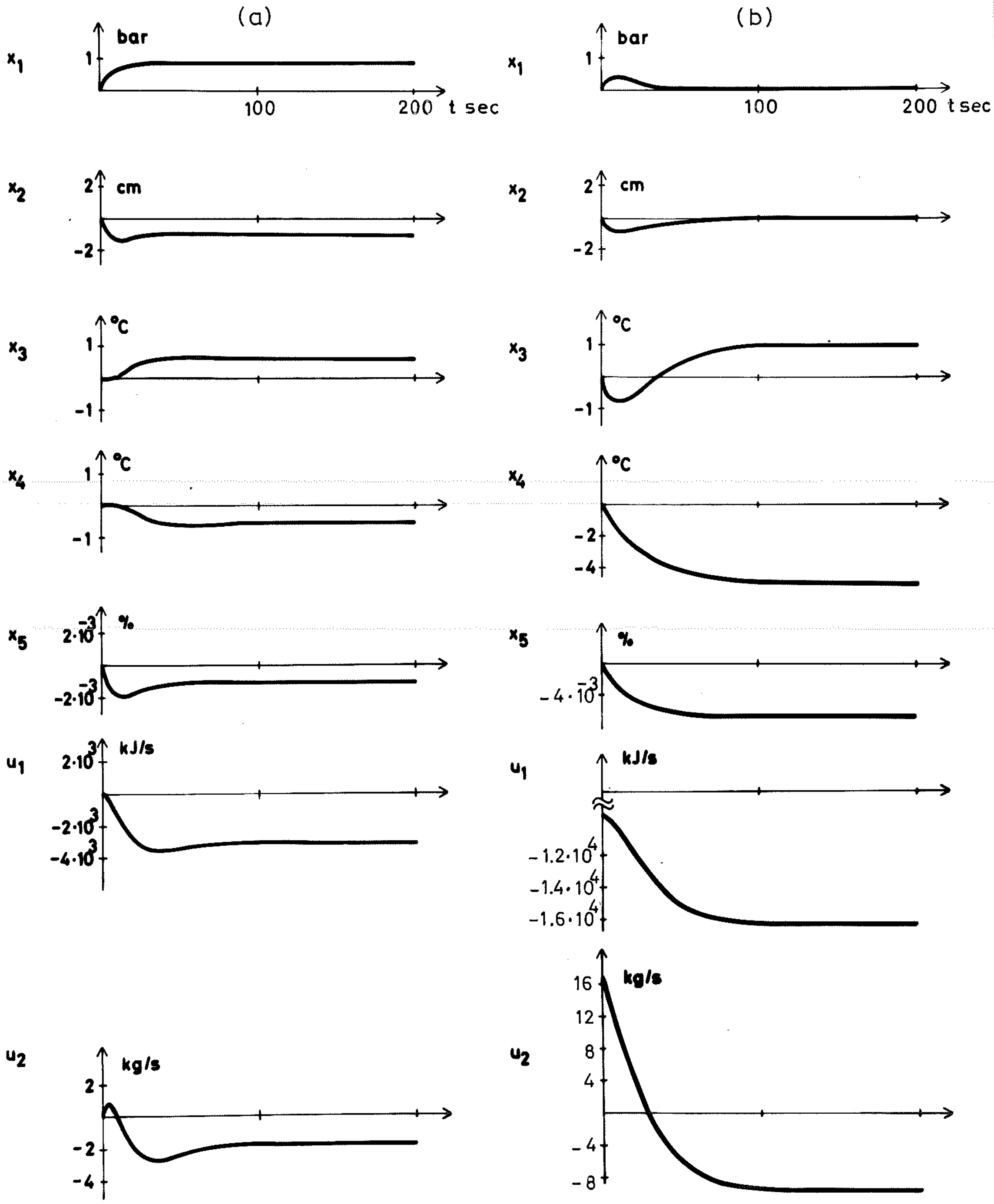

Fig. 9 - Reponses of state variables to a 10% step change of the load. Control law II is used (a) without feedforward and (b) with feedforward.

7. CHOICE OF SAMPLING INTERVAL

The choice of the sampling interval is usually a difficult problem. A common method to find a suitable sampling interval is to determine the value of the loss functional for different sampling intervals. The value of the loss functional, which is a measure of the quality of control, will increase quadratically with increasing length of the sampling interval. There are methods available which give good estimates of the influence of the sampling interval on the loss functional, see e.g. {6}. However, in this study a very rough estimate is used.

The increase of the loss functional due to increasing sampling interval will be judged from the first two diagonal elements of the stationary S matrix. These elements correspond to the controlled variables $x_1(t)$ and $x_2(t)$. In Table 1 numerical values for the case of control law II are given.

Sampling interval	$S_{11} \cdot 10^{-1}$	$S_{22} \cdot 10^{-4}$
0 sec	1.734897	1.450922
1	1.735418	1.451257
2	1.736984	1.452263
5	1.748047	1.459299
10	1.789885	1.484359

Table 1. The first two diagonal elements of the stationary S matrix for different sampling interval

In Fig. 10 the percentile increase Δs of the elements are plotted against the sampling interval.

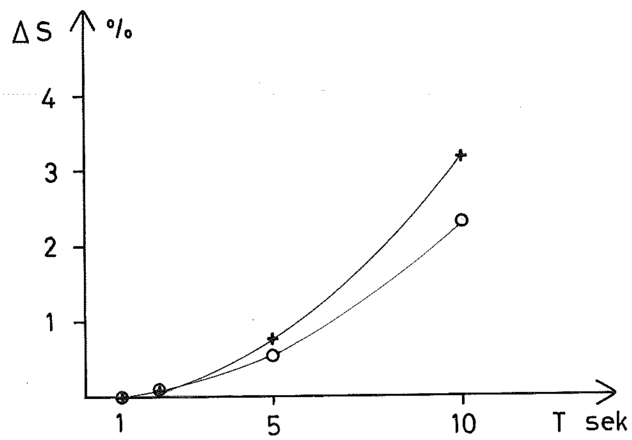

Fig. 10 - The increase Δs of the 11-element (x) and the 22-element (o) for different sampling intervals.

The two diagonal elements increase with approximately 3% for a sampling interval of 10 sec. The increase of the loss functional can roughly be estimated to the same amount. An increase less than 5% is acceptable and we choose a sampling interval of 10 sec.

8. COMPLETE CONTROL LAW

Having obtained the feedback and feedforward matrices the complete control law is obtained by adding the Kalman filter equation for reconstruction of the state vector. Using the model of the process (3.1) and the model of the load disturbance (4.7) we get

$$x(t + 1) = \phi_1 x(t) + \Gamma_1 u(t) + \Gamma_F v_1(t) \tag{8.1 a}$$

$$y(t) = \theta_1 x(t) + e_2(t) \tag{8.1 b}$$

and

$$v_1(t + 1) = v_1(t) + e_1(t) \tag{8.2}$$

where

$$E\, e_1(t)\, e_1(t) = r_1 \tag{8.3}$$
$$E\, e_2(t)\, e_2^T(t) = R_2$$

The conversion to discrete form is done in the usual manner. The sampling interval has been taken as the time unit. Introducing

$$x_6(t) = v_1(t)$$

and combining equations (8.1) and (8.2) we get

$$x(t+1) = \begin{bmatrix} \phi_1 & \Gamma_F \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} \Gamma_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_1(t)$$

$$y(t) = \begin{bmatrix} \theta_1 0 \end{bmatrix} x(t) + e_2(t)$$

or

$$x(t + 1) = \phi\, x(t) + \Gamma\, u(t) + \Gamma_e\, e_1(t) \tag{8.4 a}$$

$$y(t) = \theta\, x(t) + e_2(t) \tag{8.4 b}$$

where the covariance matrix of the noise term in equation (8.4 a) is

$$R_1 = \Gamma_e r_1 \Gamma_e^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.69\cdot 10^{-3} \end{bmatrix}$$

The Kalman filter equation is given by

$$\hat{x}(t + 1) = \phi\ \hat{x}(t) + \Gamma\ u(t) + K[y(t) - \theta\ \hat{x}(t)] \qquad (8.5)$$

If we assume that the covariance matrix R_2 of the measurement noise equals zero the steady state covariance matrix of the reconstruction error and the filter gains can be computed in the following manner {7}. The control variables are omitted since as usual they only represent an additional term in the equations and do not influence the solution. Then equation (8.4) gives

$$x(t + 1) = \phi\ x(t) + \Gamma_e\ e_1(t) \qquad (8.6\ a)$$

$$y(t) = \theta\ x(t) \qquad (8.6\ b)$$

The input-output relation is given by

$$y(t) = \frac{B_2 z^{-2} + \dots + B_n z^{-n}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}}\ e_1(t) \qquad (8.7)$$

where $a_1,\dots,a_n$ are the coefficients of the characteristic polynomial and $B_2,\dots,B_n$ are coefficient matrices of order 2x1. The coefficient matrix B_1 is zero since B_1 equals $\theta \cdot \Gamma_e$.

Assume that the system is initialized at t = 0. Equation (8.6 a) then gives

$$x(t) = \phi^t\ x(0) + \sum_{s=0}^{t-1} \phi^{t-1-s}\ \Gamma_e\ e_1(s) \qquad (8.8)$$

Given $y(0),\dots,y(t)$, $t>n$ the stochastic variables $e(0),\dots,e(t-2)$ can be computed exactly using equation (8.7). Assume that the initial value $x(0)$ is known then we can compute $x(t-1)$ exactly from equation (8.8). Given $x(t-1)$ the minimum mean square estimate of $x(t+1)$ is

$$\hat{x}(t + 1) = \phi^2\ x(t - 1) \qquad (8.9)$$

The true value of state vector at t+1 is

$$x(t + 1) = \phi^2\ x(t - 1) + \phi\Gamma_e\ e_1(t - 1) + \Gamma_e\ e_1(t)$$

Then the steady state value of the covariance matrix P of the reconstruction error is

$$P = E\{x(t) - \hat{x}(t)\}\{x(t) - \hat{x}(t)\}^T = \phi\Gamma_e\ r_1\ \Gamma_e^T\phi^T + \Gamma_e r_1 \Gamma_e^T \qquad (8.10)$$

or

$$P = r_1 (\varphi_6 \varphi_6^T + \Gamma_e \Gamma_e^T) \tag{8.11}$$

where φ_6 is the sixth column of ϕ. To compute the filter gains we use equation (8.6) and the fact that $\theta\Gamma_e$ is zero. We get

$$y(t) = \theta\phi^2\ x(t-2) + \theta\phi\Gamma_e\ e_1(t-2) \tag{8.12}$$

$\theta\phi\Gamma_e$ is nonzero and we can solve equation (8.12) using the pseudo-inverse of $\theta\phi\Gamma_e$. Hence

$$e_1(t-2) = (\theta\phi\Gamma_e)^\dagger\ [y(t) - \theta\phi^2\ x(t-2)] \tag{8.13}$$

Combining this equation and equation (8.6 a) we get the following recursive equation for the state vector

$$x(t-1) = \phi\ x(t-2) + (\theta\phi\Gamma_e)^\dagger\ [y(t) - \theta\phi^2\ x(t-2)] \tag{8.14}$$

Equations (8.9) and (8.14) now give the filter equation (8.5) and the filter gains are

$$K = \phi^2\ \Gamma_e(\theta\phi\Gamma_e)^\dagger \tag{8.15}$$

Notice that the filter gains are not uniquely determined. This fact can be exploited to adjust the weight given to the different measured variables in the Kalman filter. Equation (8.13) gives us two equations which both can be used to compute the scalar $e_1(t-2)$. The use of these equations can be weighted as

$$e_1(t-2) = \beta\ e_1'(t-2) + (1-\beta)\ e_1''(t-2)$$

where $e_1'(t-2)$ and $e_1''(t-2)$ are computed from the first and second equation of (8.13) respectively. β is the weighting factor. The eigenvalues of matrix $(\phi-K\theta)$, which give the dynamics of the reconstruction error will be independent of the factor β, only if these two equations are identical. In the boiler application there is a small difference between the two equations given by (8.13) which will slightly alter the eigenvalues when β is changed. Choosing β so that k_{11} roughly equals k_{22} we get

$$K = \begin{bmatrix} 0.69 & -28.7 \\ -0.01 & 0.44 \\ 0.21 & -8.72 \\ 0.21 & -8.94 \\ -0.001 & 0.06 \\ 0.88 & -36.9 \end{bmatrix} \tag{8.16}$$

The eigenvalues of ϕ-Kθ are

0.00
0.00
0.41
0.32
0.99
0.83

There is one eigenvalue very near the unit circle in the complex plane. This means that the transient response of the reconstruction error will contain a very slow mode. But this also means that the filter equation is very sensitive to changes of the process parameters. In Appendix D the following expression is derived for the steady state reconstruction error

$$\tilde{x}_o = (I - \phi + K\theta)^{-1}(\phi^{*} - \phi)x_o$$

where ϕ^{*} is the disturbed process matrix.

We have

$$(I-\phi+K\theta)^{-1} = \begin{bmatrix} -5.5\cdot10^{-1} & 5.8\cdot10^{3} & -2.2\cdot10^{1} & -1.7\cdot10^{1} & -4.8\cdot10^{4} & 5.7\cdot10^{-1} \\ -1.3\cdot10^{-2} & 1.4\cdot10^{2} & -5.2\cdot10^{-1} & -4.0\cdot10^{-1} & -1.1\cdot10^{3} & -1.4\cdot10^{-2} \\ -5.7\cdot10^{-1} & 2.8\cdot10^{3} & -8.8 & -8.0 & -2.3\cdot10^{4} & 5.8\cdot10^{-2} \\ -6.1\cdot10^{-1} & 2.9\cdot10^{3} & -1.1\cdot10^{1} & -6.5 & -2.4\cdot10^{4} & 1.1\cdot10^{-1} \\ 4.6\cdot10^{-3} & 2.0 & -4.8\cdot10^{-3} & -3.4\cdot10^{-3} & -1.4\cdot10^{1} & -1.7\cdot10^{-3} \\ -2.2 & 5.7\cdot10^{3} & -2.2\cdot10^{1} & -1.7\cdot10^{1} & -4.7\cdot10^{4} & 2.0 \end{bmatrix}$$

The large elements of the second and fifth column of $(I-\phi+K\theta)^{-1}$ indicate that the Kalman filter is very sensitive to changes of the parameters of the second and fifth row of ϕ. If we change the 25:th element of ϕ 1% and let the steady state value of the state vector x_o correspond to a step change of v_1 of 1 bar (a 10% load change) the steady state reconstruction error is

$$\tilde{x}_o = \begin{bmatrix} 1.75 \\ 4.18\cdot10^{-2} \\ 8.37\cdot10^{-1} \\ 8.72\cdot10^{-1} \\ 5.86\cdot10^{-4} \\ 1.71 \end{bmatrix}$$

These reconstruction errors are not acceptable. Especially the two controlled variables $x_1(t)$ and $x_2(t)$ will deviate considerably from their steady state value. The stationary P matrix given by equation (8.11) equals

$$P=\begin{bmatrix} 5.4\cdot10^{-4} & -1.3\cdot10^{-5} & 5.5\cdot10^{-5} & 1.0\cdot10^{-4} & -1.7\cdot10^{-6} & 9.6\cdot10^{-4} \\ -1.3\cdot10^{-5} & 3.1\cdot10^{-7} & -1.3\cdot10^{-6} & -2.4\cdot10^{-6} & 4.0\cdot10^{-8} & -2.3\cdot10^{-5} \\ 5.5\cdot10^{-5} & -1.3\cdot10^{-6} & 5.7\cdot10^{-6} & 1.0\cdot10^{-5} & -1.7\cdot10^{-7} & 9.8\cdot10^{-5} \\ 1.0\cdot10^{-4} & -2.4\cdot10^{-6} & 1.0\cdot10^{-5} & 1.9\cdot10^{-5} & -3.1\cdot10^{-7} & 1.8\cdot10^{-4} \\ -1.7\cdot10^{-6} & 4.0\cdot10^{-8} & -1.7\cdot10^{-7} & -3.1\cdot10^{-7} & 5.1\cdot10^{-9} & -2.9\cdot10^{-6} \\ 9.6\cdot10^{-4} & -2.3\cdot10^{-5} & 9.8\cdot10^{-5} & 1.8\cdot10^{-4} & -2.9\cdot10^{-6} & 3.4\cdot10^{-3} \end{bmatrix}$$

The standard deviations $\sigma_{\tilde{x}_i}$ of the reconstruction errors then are

$$\begin{aligned} \sigma_{\tilde{x}_1} &= 2.3\cdot10^{-2} \text{ bar} \\ \sigma_{\tilde{x}_2} &= 5.6\cdot10^{-4} \text{ m} \\ \sigma_{\tilde{x}_3} &= 2.4\cdot10^{-3}\ {}^{o}C \\ \sigma_{\tilde{x}_4} &= 4.3\cdot10^{-3}\ {}^{o}C \\ \sigma_{\tilde{x}_5} &= 7.1\cdot10^{-5}\ \% \\ \sigma_{\tilde{x}_6} &= 5.8\cdot10^{-2} \text{ bar} \end{aligned} \qquad (8.17)$$

The values for the 3:rd, 4:th and 5:th components of the state vector are unrealistically small since obviously the model is not that accurate.

One way to introduce uncertainties in the model is to add white noise with a given variance to each component of the state vector. Choosing standard deviations of this noise as roughly 1% of the maximum deviations of the state variables when the load is changed 10% we get

$$R_1 = \begin{bmatrix} 10^{-4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\cdot10^{-8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^{-4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5\cdot10^{-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.69\cdot10^{-3} \end{bmatrix} \qquad (8.18)$$

The covariance matrix of the measurement noise is chosen to

$$R_2 = \begin{bmatrix} 10^{-4} & 0 \\ 0 & 10^{-7} \end{bmatrix} \tag{8.19}$$

Notice that the nonzero elements of R_2 have roughly the same magnitude as the variance of the reconstruction errors of $x_1(t)$ and $x_2(t)$ respectively.

Numerical values of the obtained filter gains are given in Appendix A. The eigenvalues of ϕ-$K\theta$ are

$0.10 \pm 0.10\, i$

$0.26 \pm 0.14\, i$

0.51

0.34

and the sensitivity matrix is

$$(I-\phi+K\theta)^{-1} = \begin{bmatrix} -7.1\cdot10^{-4} & 7.6 & -2.8\cdot10^{-2} & -2.2\cdot10^{-2} & -6.2\cdot10^{1} & 7.4\cdot10^{-1} \\ -8.0\cdot10^{-5} & 8.5\cdot10^{-1} & -3.1\cdot10^{-3} & -2.4\cdot10^{-3} & -6.9 & -1.3\cdot10^{-2} \\ -3.1\cdot10^{-1} & -1.2 & 1.6 & 5.9\cdot10^{-3} & -3.9\cdot10^{1} & 1.1\cdot10^{-1} \\ -3.3\cdot10^{-1} & -1.2 & 1.9\cdot10^{-2} & 1.8 & 9.8 & 1.1\cdot10^{-1} \\ 4.8\cdot10^{-3} & -1.3\cdot10^{-2} & 2.5\cdot10^{-3} & 2.3\cdot10^{-3} & 1.9 & -2.5\cdot10^{-3} \\ -1.6 & 1.4\cdot10^{1} & -4.2\cdot10^{-1} & -3.3\cdot10^{-1} & -9.5\cdot10^{1} & 2.3 \end{bmatrix}$$

Notice that no eigenvalue is close to the unit circle in the complex plane and that the elements of the sensitivity matrix have been reduced with about a factor 100. The standard deviations $\sigma_{\tilde{x}_i}$ of the reconstruction errors in this case are

$\sigma_{\tilde{x}_1} = 3.0\cdot10^{-2}$ bar

$\sigma_{\tilde{x}_2} = 8.5\cdot10^{-4}$ m

$\sigma_{\tilde{x}_3} = 1.2\cdot10^{-2}$ $^\circ$C

$\sigma_{\tilde{x}_4} = 5.6\cdot10^{-2}$ $^\circ$C

$\sigma_{\tilde{x}_5} = 1.5\cdot10^{-4}$ %

$\sigma_{\tilde{x}_6} = 6.2\cdot10^{-2}$ bar

Compared to (8.17) the standard deviations of the 3:rd, 4:th and 5:th components of the state vector have increased about ten times. Thus the filter gains obtained using the disturbed model give reasonable properties of the filter equation and these gains will be used.

9. IMPLEMENTATION OF CONTROL LAW ON A PROCESS CONTROL COMPUTER

The whole problem was simulated on the hybrid computer at the Research Institute of National Defense in Stockholm, Sweden. The process was patched on the analog computer EAI 8800 and the control law was implemented on the digital computer EAI 640. The details of the simulation are found in {3}.

A simplified flow diagram of the control algorithm is shown in Fig. 11.

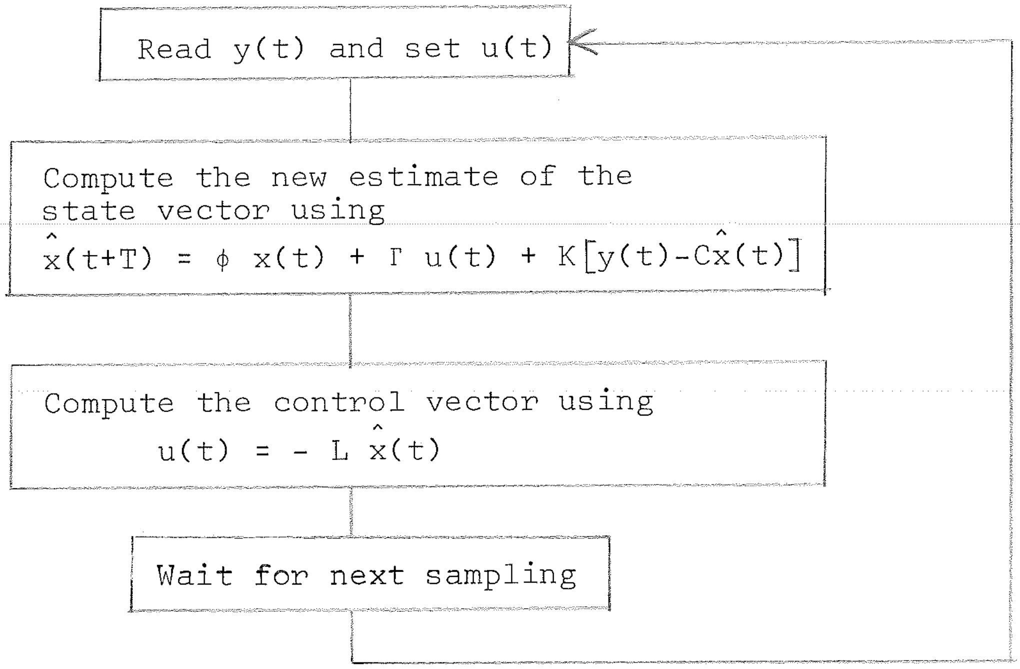

Fig. 11 - Simplified flow diagram of control algorithm

The matrices ϕ and Γ are the sampled A and B matrices. Notice that the filter equation is of the 6:th order since one state has been added for the load disturbance. The filter equation gives an estimate of the state vector and the load disturbance one sampling interval ahead and the control vector can be computed using this estimate. When the next sampling interrupt occurs the control variables are set and a new measurement of the output variables is made. Numerical figures of sampled matrices are given in Appendix A.

The control law was implemented using fix point arithmetic and single precision. The word length of the computer is 16 bits including the sign bit which gives an accuracy of about 4 decimal digits. The numbers in the computer are regarded as fractionals. Then we must make sure that no constants or sums become larger than one. Otherwise overflow will occur. Each estimated state variable is scaled according to the largest matrix element on the right hand side in the filter equation. These scale factors are then introduced in the coefficients of the L matrix. The control variables are then also scaled according to the largest element. Before the storing and setting of $\hat{x}(t)$ and u(t) they are rescaled. It is obvious that some caution must be exercised so that the accuracy not unnecessarily is decreased and that the scaling requires a considerable knowledge about intermediate results during the calculations.

The control algorithm (CALG) is programmed in assembler language. The program listings are given in Appendix C. The matrix calculations are performed using subroutines for vector addition (VADD) vector subtraction (VSUB) and matrix-vector multiplication (MVMULT) The rescaling subroutine is called RESCA. A detailed presentation is found in {8}.
The core memory requirements for the control algorithm and subroutines are shown in Table 3. Figures are given for a 6:th and a 15:th order system both with 2 inputs and 2 outputs.

n	6	15
CALG	121 words	157 words
VADD	49	49
VSUB	5	5
MVMULT	81	81
RESCA	45	45
Matrix storage array	84	345
SUM	385	682

Table 3. The storage requirements for the control algorithm and subroutines for a 6:th and a 15:th order system both with 2 inputs and 2 outputs.

The program list for CALG apply to a 15:th order system with 10 inputs and 10 outputs. There is some unnecessary storage arrays in CALG since some intermediate results are saved.

The execution time for CALG is 6.7 ms.

10. SIMULATION

In the analog simulations of the boiler control it was assumed that all state variables and the disturbance v_1 could be measured directly. Hence the Kalman filter was not included.

Fig. 12 gives the open loop responses of the state variables when the disturbance v_1 is a stochastic process given by equation (4.7). Fig. 13 and 14 give the responses of the state variables when control law II, without and with feedforward respectively, is used. Notice that the realizations of $v_1(t)$ are different in the figures referred to above. A measurement of the variance of $x_1(t)$ and $x_2(t)$ on the analog computer gave

$$E\, x_1^2(t) = 1.1 \cdot 10^{-3}\ \text{bar}^2$$
$$E\, x_2^2(t) = 6.0 \cdot 10^{-4}\ \text{cm}^2 \qquad (10.1)$$

The results of the hybrid simulations are presented in Fig. 15, 16, 17, 18.

Fig. 15a gives the responses of the state variables to a step change of the load disturbance v_1. The corresponding estimated state and disturbance variables are given in Fig. 15b. Control law II with feedforward is used and the filter gains correspond to the covariance matrices R_1 and R_2 given by equations (8.18) and (8.19). Notice that the control variables are zero during the first two sampling intervals. This follows from that the control algorithm CALG is given starting values for $\hat{x}(0)$ and $u(0)$ which equal zero.

Fig. 16 illustrates the sensitivity of the Kalman filter using the filter gains given by equation (8.16). The model disturbance is a change of the 25:th element of the matrix A with 0.25%. The estimation errors of especially $x_1(t)$, $x_2(t)$ and $v_1(t)$ are considerable.

Fig. 17 which should be compared with Fig. 14 gives the responses of the state variables to the stochastic process $v_1(t)$ defined by equation (4.7). The estimates are presented in Fig. 18. The measured variances of the two controlled variables are

$$E\, x_1^2(t) = 2.8 \cdot 10^{-3}\ \text{bar}^2$$
$$E\, x_2^2(t) = 6.6 \cdot 10^{-3}\ \text{cm}^2 \qquad (10.2)$$

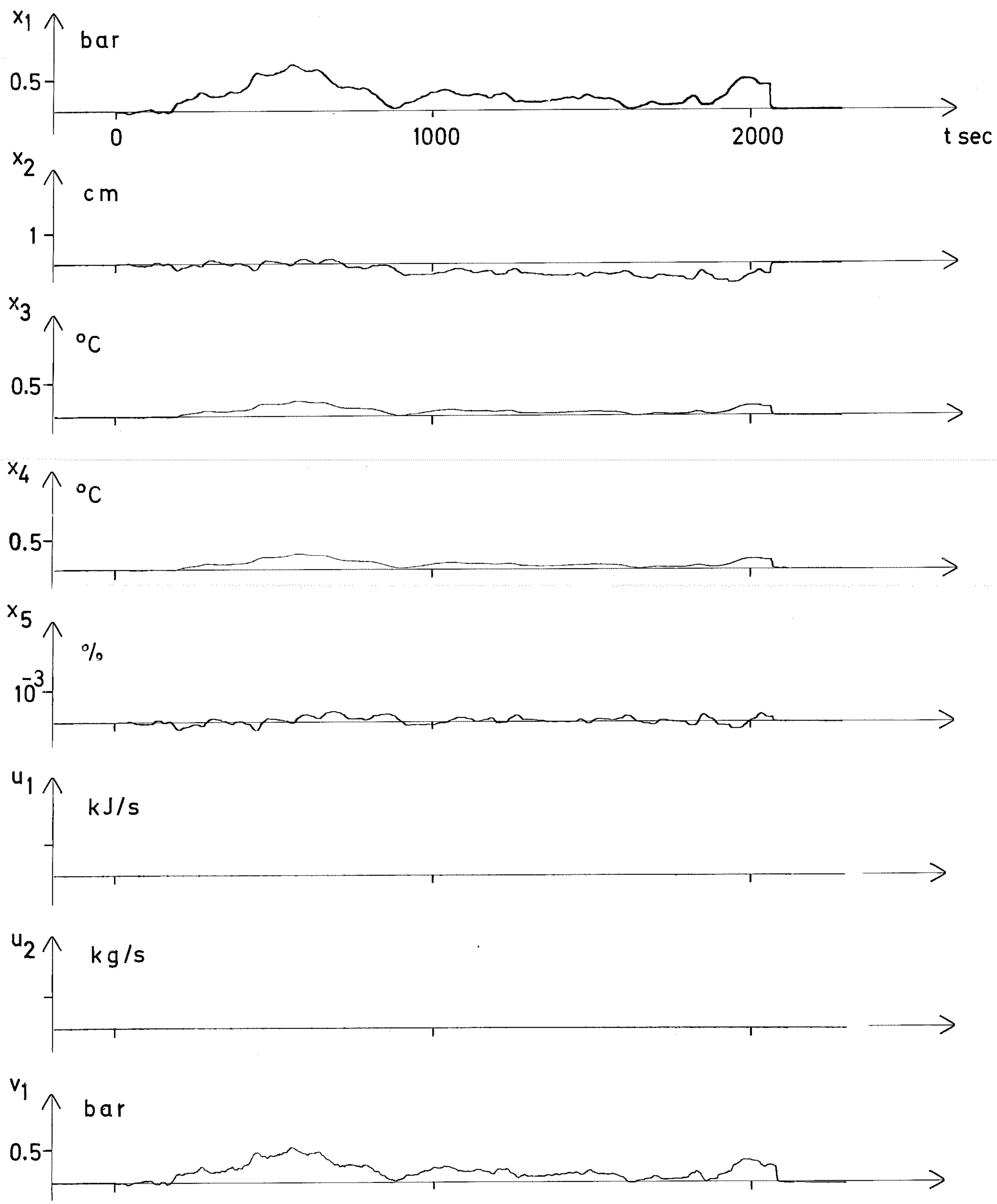

Fig. 12 - Open loop responses of the state variables. The disturbance $v_1(t)$ is given by equation (4.7).

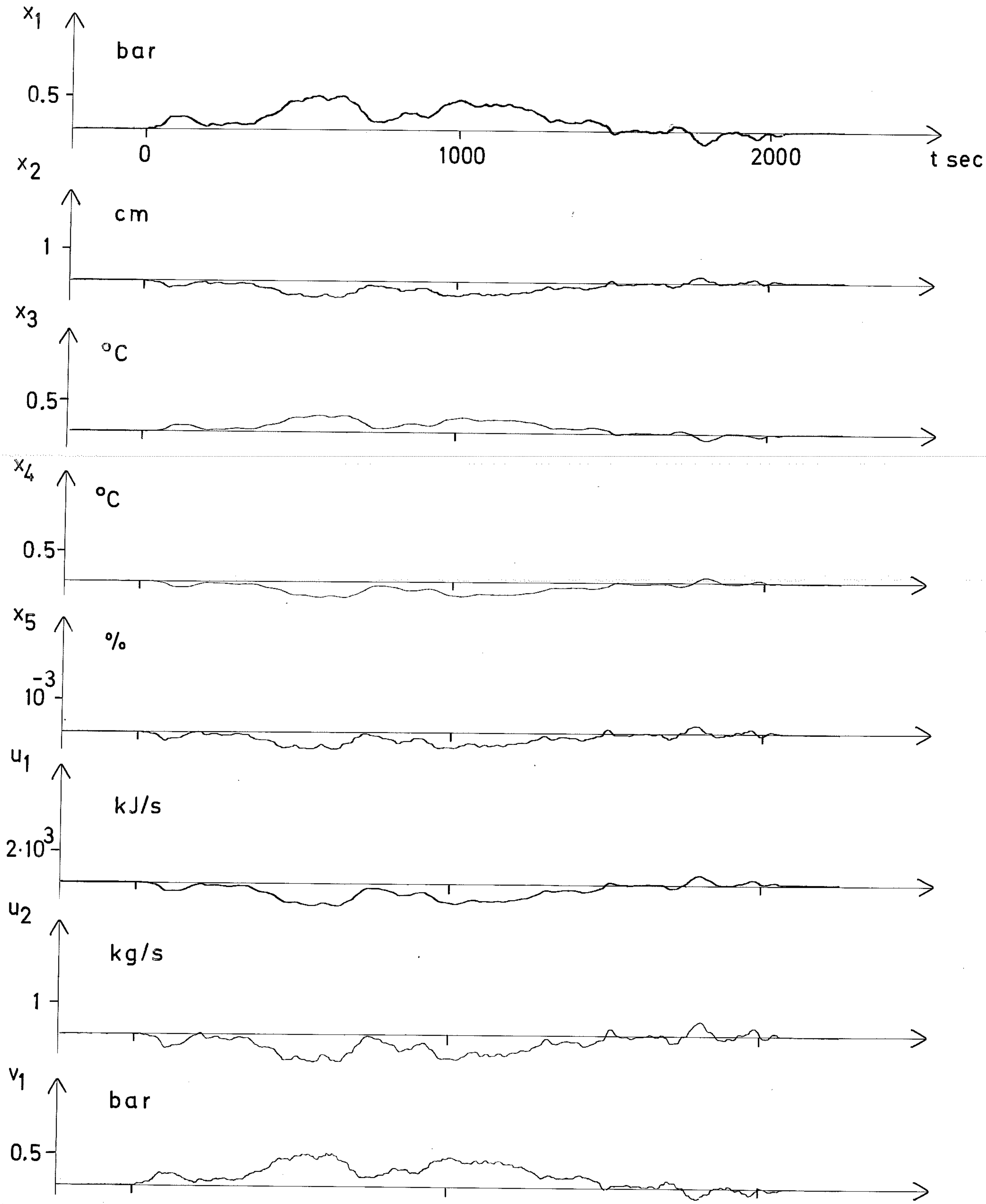

Fig. 13 - Responses of state variables to load disturbance $v_1(t)$ defined by equation (4.7). Control law II without feedforward is used.

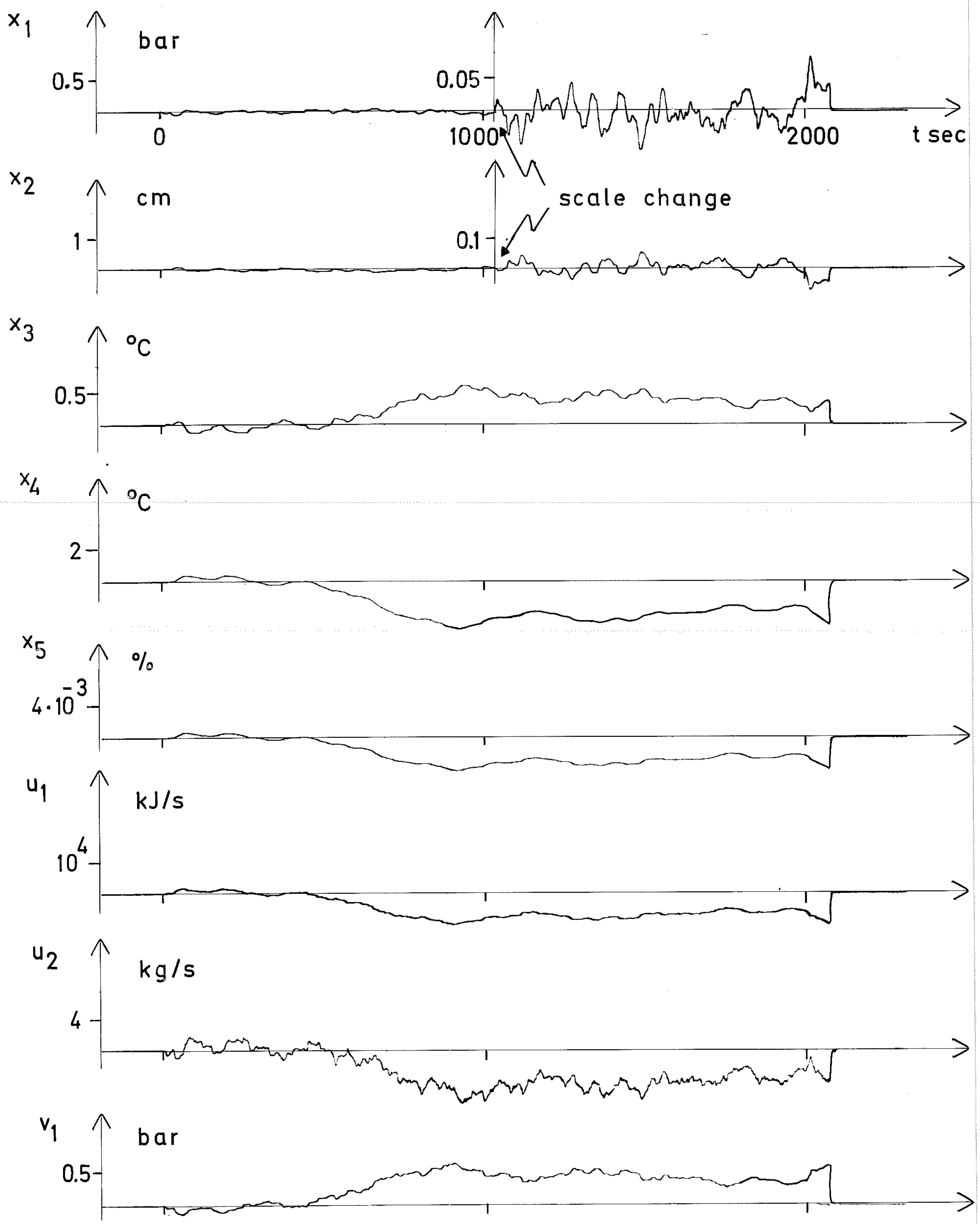

Fig. 14 - Responses of state variables to load disturbance $v_1(t)$ defined by equation (4.7). Control law II with feedforward is used.

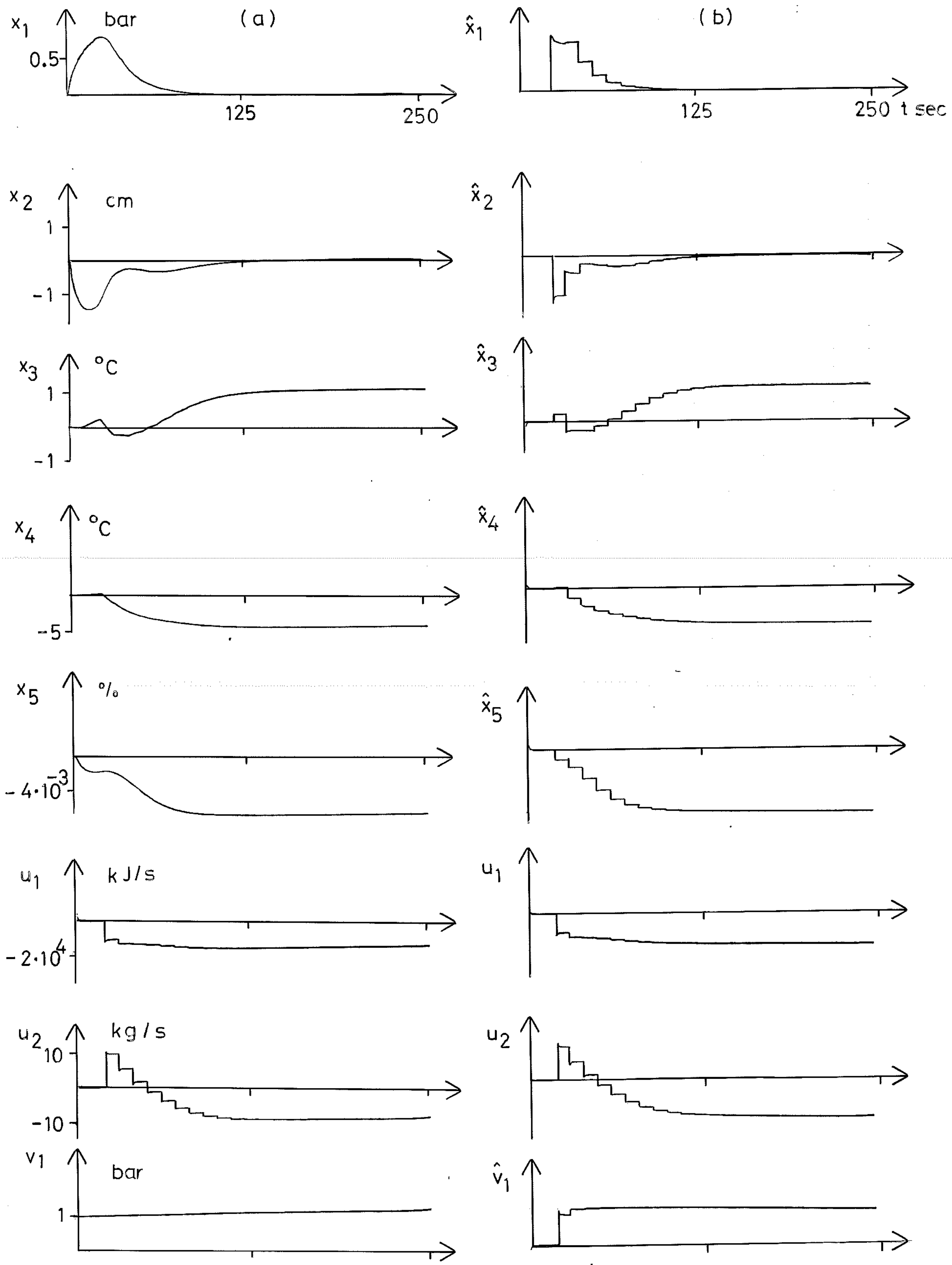

Fig. 15 - Responses of state variables (a) and estimated state and disturbance variables (b) to a step change of 1 bar of load disturbance $v_1(t)$. Control law II with feedforward is used. Covariance matrices R_1 and R_2 given by equations (8.18) and (8.19) define the filter gains used.

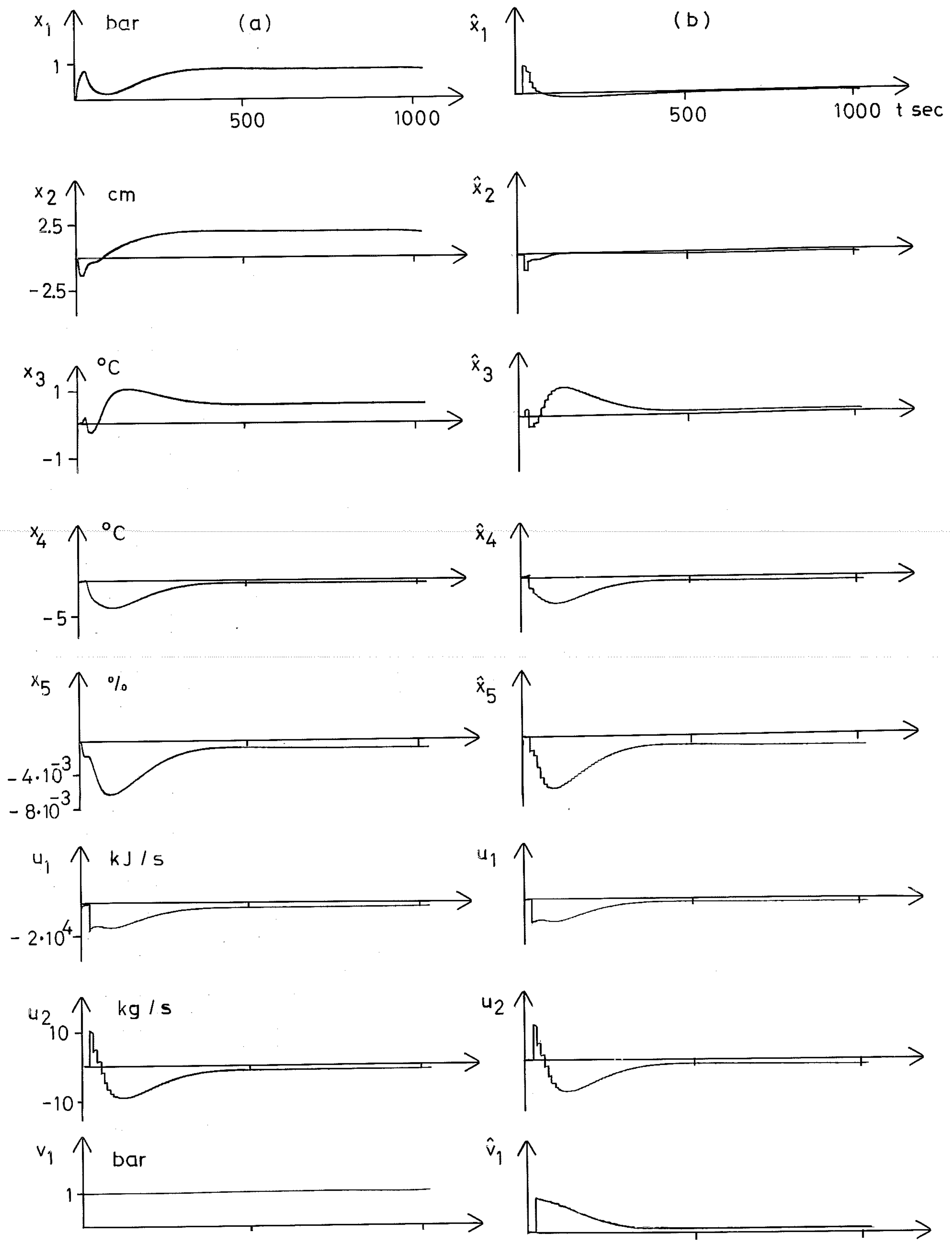

Fig. 16 - Responses of state variables (a) and estimated state and disturbance variables (b) to a step change of 1 bar of load disturbance $v_1(t)$. Control law II with feed-forward is used. The used filter gains are given by equation (8.16).

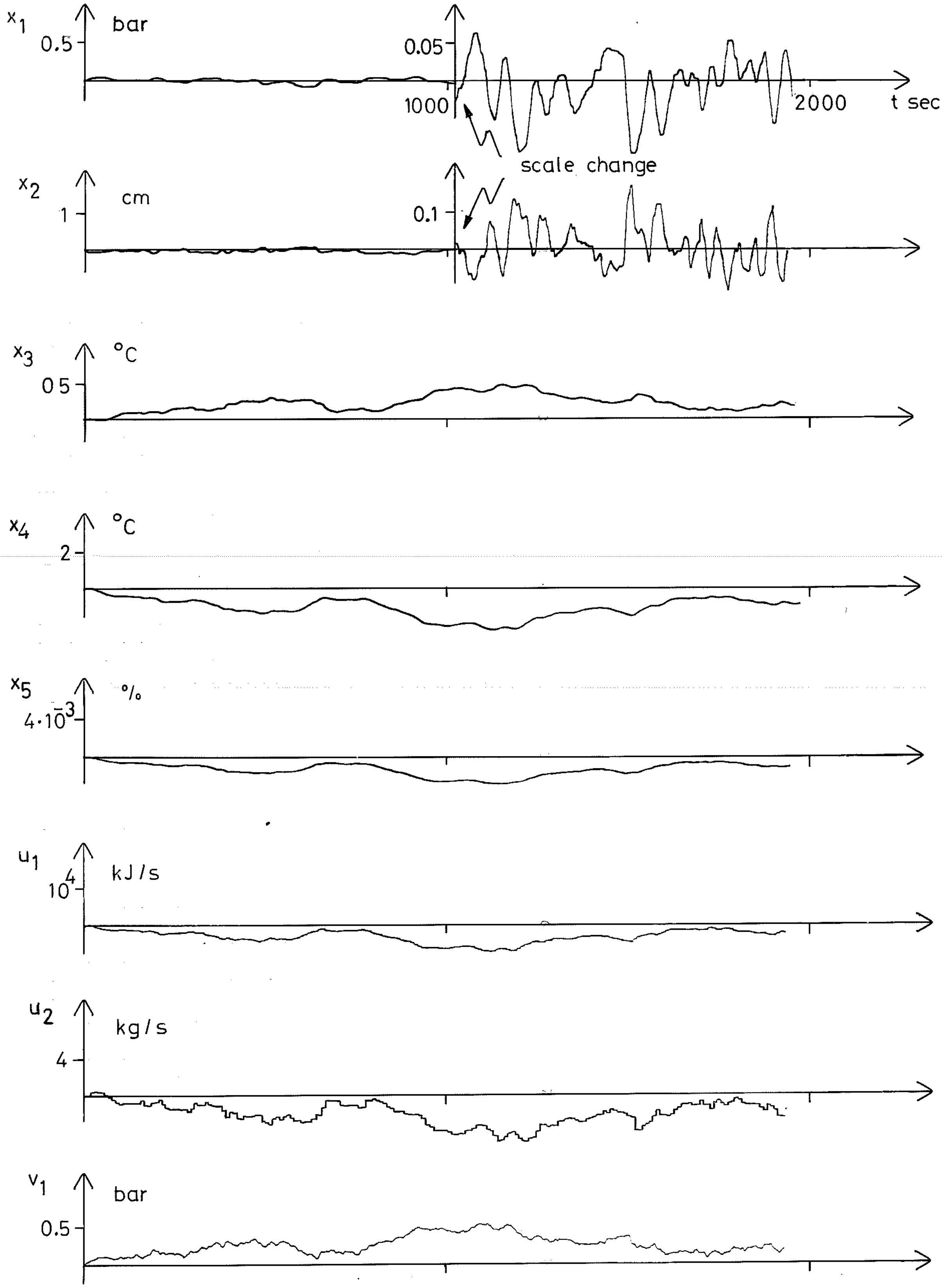

Fig. 17 - Responses of state variables to load disturbance $v_1(t)$ given by equation (4.7). Control law II with feedforward is used. Covariance matrices R_1 and R_2 given by equations (8.18) and (8.19) define the filter gains used.

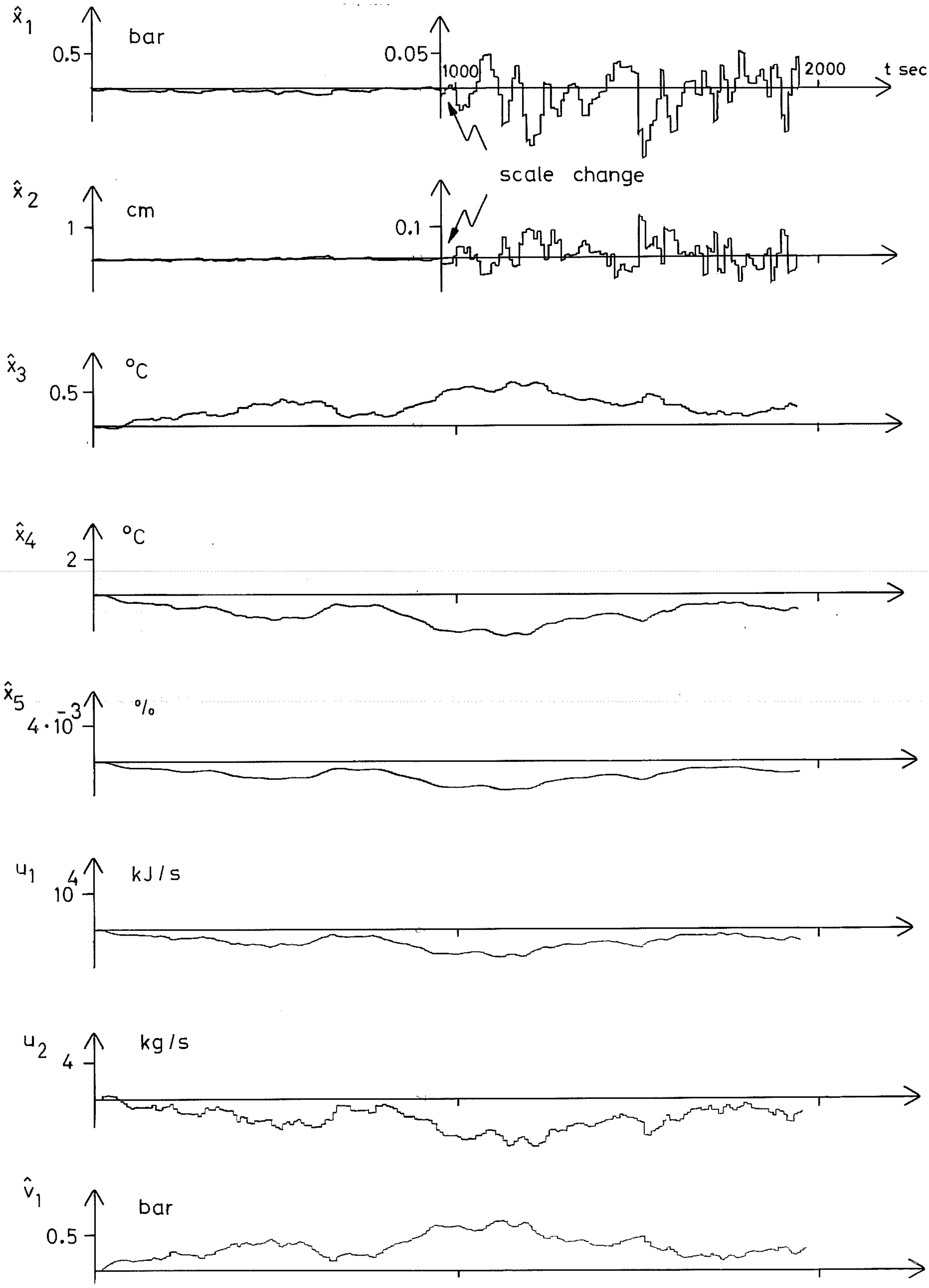

Fig. 18 - Responses of estimated state and disturbance variables to load disturbance $v_1(t)$ given by equation (4.7). Control law II with feedforward is used. Covariance matrices R_1 and R_2 given by equations (8.18) and (8.19) define the filter gains used.

Compared to (10.1) the variances have increased roughly by a factor 2 and 10 respectively. The larger increase of the variance of $x_2(t)$ was expected considering the frequency content of the control signals $u_1(t)$ and $u_2(t)$ in the continuous case, see Fig. 14.

11. ACKNOWLEDGEMENTS

Civ.ing. I. Gustavsson and Civ.ing. K. Mårtensson have written the programs used for the identification of time series and for the computation of optimal control laws. The recordings of the load disturbance were supplied by Sydsvenska Kraftaktiebolaget. The author also wish to record his appreciation to Miss L. Jönsson who typed the manuscript and to Mrs. B. Tell who drew the figures.

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APPENDIX A

Numerical values of the matrices A, B, F and C of the boiler model used in this report are given. The values apply to a power station boiler with a maximum steam flow of about 350 t/h. The operating point is 90% of full load. Also the discrete boiler model matrices as well as the feedback, feedforward and filter matrices are given.

The continuous system matrices are:

MATRIX A

-1.2917081500-001	-0.0000000000+000	3.9622910300-002	2.5023257900-002	1.9119354000-002
3.2907472400-003	-0.0000000000+000	-7.7913590600-005	1.2162226000-004	-6.2135344500-001
7.1772161300-002	-0.0000000000+000	-1.0041304100-001	8.8726400000-004	-3.8507722100+000
4.1127161600-002	-0.0000000000+000	-0.0000000000+000	-8.2254323200-002	-0.0000000000+000
3.6122785500-004	-0.0000000000+000	3.5023976300-005	4.2559545699-005	-7.4327812999-002

MATRIX B

-0.0000000000+000	1.3935922200-003
-0.0000000000+000	3.5944889200-005
-0.0000000000+000	-9.8919025499-003
2.4884535800-005	-0.0000000000+000
-0.0000000000+000	-5.3430336699-006

MATRIX F

9.9473651899-002
-3.1814050400-003
-2.3209124400-002
-0.0000000000+000
-3.8138205800-004

MATRIX C

1	0	0	0	0	0
0	1	0	0	0	0

The discrete system matrices for T = 10 sec:

MATRIX ϕ

3.3340778199-001	0.0000000000+000	1.3393072381-001	9.3340709882-002	-2.7947667816+000
1.3097281272-002	1.0000000000+000	1.3841302424-003	1.7094978276-003	-4.4004041692+000
2.1665335495-001	0.0000000000+000	4.0911537056-001	3.0770091572-002	-1.6702744064+001
1.5301957496-001	0.0000000000+000	2.9803090464-002	4.5939395088-001	-3.9128795615-001
1.4803653130-003	0.0000000000+000	4.2575896441-004	3.8294082174-004	4.6900439390-001

MATRIX Γ

1.6417868651-005	-1.4696276466-003
2.1800773794-007	5.6284718163-004
3.6464543804-006	-6.1419331881-002
1.7175937227-004	2.3135682378-004
5.1649590092-008	-4.5215376170-005

SAMPLED MATRIX F

5.6700616003-001
-1.3561195318-002
5.7902031018-002
1.0504507399-001
-1.7297831589-003

CONTINUOUS FEEDBACK MATRIX L OF CONTROL LAW I

1.4851558112+003	4.7087735868+004	4.9025667321+002	4.4297695742+002	-1.0435741927+005
1.8410253907+000	3.1270232798+002	-1.8653363861-001	-2.1287153831-002	-2.1008260647+003

CONTINUOUS FEEDFORWARD MATRIX R OF CONTROL LAW I

1.7367719797+004
-4.1942240773+000

CONTINUOUS FEEDBACK MATRIX L OF CONTROL LAW II

6.6780720481+003	4.1803405923+005	1.3552544432+003	1.3718685270+003	-1.7532529445+006
8.0344481077+000	9.0843133789+002	4.8581215785-001	8.1554038939-001	-4.3102102217+003

CONTINUOUS FEEDFORWARD MATRIX R OF CONTROL LAW II

9.6400244546+003
-1.6681729147+001

APPENDIX A
(continued)

SAMPLED FEEDBACK MATRIX L OF CONTROL LAW I

1.2585005228+003	4.5273256515+004	4.4963971804+002	4.0898954185+002	-1.9188271739+005
1.5363789856+000	2.7413239833+002	-2.0105521722-001	-5.0266359095-002	-1.9753315105+003

SAMPLED FEEDFORWARD MATRIX R OF CONTROL LAW I

1.6647427054+004
-3.4646600566+000

SAMPLED FEEDBACK MATRIX L OF CONTROL LAW II

4.7553637319+003	2.9068339261+005	1.1620722849+003	1.1433438861+003	-1.6021789753+006
6.0422080146+000	6.4567754380+002	3.9813452707-001	6.4496091534-001	-3.8272260016+003

SAMPLED FEEDFORWARD MATRIX R OF CONTROL LAW II

1.0016191581+004
-1.3177689952+001

Using R_1 and R_2 according to (8.18) and (8.19) the filter gain matrix K is

1.0073571793+000	-5.7569950558+000
6.3120600146-003	1.0757017094+000
3.7910573573-001	5.3174449364-001
4.0181264417-001	2.6182033238-001
-1.7632880588-003	7.1557604587-003
1.1591905537+000	-1.0347879562+001

APPENDIX B

Numerically it has been shown for several specific problems that the feedforward matrix R discussed in section 5 also can be computed in the following way. Consider the system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Fv(t) \tag{1}$$

Set

$$v_k = x_{n+k} \qquad k = 1,\dots,s$$

$$\dot{x}_{n+k} = 0 \qquad k = 1,\dots,s$$

and add these new state variables to the system equation (1). We get

$$\frac{dx(t)}{dt} = \begin{bmatrix} A & F \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) = A_1x(t) + B_1u(t) \tag{2}$$

Introduce the notation

$$\bar{A}_1 = [0\ 0\ \dots\ 0\ a^1_{i+1}\ \dots\ a^1_n\ ]$$

where a^1_k is the k:th column of A. We require that the steady state error of the first i components of the state vector equals zero, and that i equals the number of control variables. The loss functional

$$V=\frac{1}{2}\ x^T(t_1)Q_ox(t_1)+\frac{1}{2}\int_{t_o}^{t_1}\{x^T(s)Q_1x(s)+(\bar{A}_1x(s)+B_1u(s))^TQ_2(\bar{A}_1x(s)+ +B_1u(s))\}ds$$

then gives the control law

$$u(t) = -L_1(t)\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} - L_2(t)\begin{bmatrix} x_{n+1} \\ \vdots \\ x_{n+s} \end{bmatrix}$$

The stationary value of $L_2(t)$ obtained when $t_1\to\infty$ is given by

$$L_2 = R$$

where R satisfy equation (5.7) using the stationary value of $L_1(t)$. Notice that $\bar{A}_1x_o+B_1u_o$ equals the left hand side of equation (5.4).

```
00001:                         * SUBROUTINE   CALG
00002:                         * COMMON AREA MATR
00003:                                  COMMON    MATR
00004:                   00001    N     BSS       1
00005:                   00001    M     BSS       1
00006:                   00001    P     BSS       1
00007:                   00341    FI    BSS       225
00008:                   00226    GAMMA BSS       150
00009:                   00226    C     BSS       150
00010:                   00226    K     BSS       150
00011:                   00226    L     BSS       150
00012:                         * COMMON AREA VECT
00013:                                  COMMON    VECT
00014:                   00017    XHAT  BSS       15
00015:                   00012    U     BSS       10
00016:                   00012    Y     BSS       10
00017:                         * COMMON AREA ERROR
00018:                                  COMMON    ERROR
00019:                   00010    EFDC  BSS       8
00020:                   00012    E     BSS       10
00021:                   00014    UTFY2 BSS       12          FILL UP TO 30
00022:                         * EXT ASSEMBLER CALLING SEQUENCE
00023:                         *            CALL CALG
00024:                         * SUBROUTINES CALLED
00025:                         *            VADD
00026:                         *            VSUB
00027:                         *            MVMULT
00028:  01000            00000          REL       0
00029:                                  NAME      CALG
00030:  00000    00000   00000    CALG  ADR       0
00031:  00001    60XXX                  CALL      MVMULT,N,N,FI,XHAT,FIX,E,0
00031:  00002    XXXXX   00000 ←--
00031:  00003    XXXXX   00000 ←--
00031:  00004    XXXXX   00003 ←--
00031:  00005    XXXXX   00000 ←--
00031:  00006    XXXXX   00122 ←--
00031:  00007    XXXXX   00010 ←--
00031:  00010    00000   00000 ←--
00032:  00011    60XXX          F1      CALL      MVMULT,N,M,GAMMA,U,GU,E+1,
00032:  00012    XXXXX   00000 ←--
00032:  00013    XXXXX   00001 ←--
00032:  00014    XXXXX   00344 ←--
00032:  00015    XXXXX   00017 ←--
00032:  00016    XXXXX   00141 ←--
00032:  00017    XXXXX   00011 ←--
00032:  00020    00000   00000 ←--
00033:  00021    60XXX          F2      CALL      VADD,N,FIX,GU,FIXPGU,E+2,0
00033:  00022    XXXXX   00000 ←--
00033:  00023    XXXXX   00122 ←--
00033:  00024    XXXXX   00141 ←--
00033:  00025    XXXXX   00160 ←--
00033:  00026    XXXXX   00012 ←--
00033:  00027    00000   00000 ←--
00034:  00030    60XXX          F3      CALL      MVMULT,P,N,C,XHAT,CX,E+3,0
00034:  00031    XXXXX   00002 ←--
00034:  00032    XXXXX   00000 ←--
00034:  00033    XXXXX   00572 ←--
00034:  00034    XXXXX   00000 ←--
00034:  00035    XXXXX   00177 ←--
00034:  00036    XXXXX   00013 ←--
00034:  00037    00000   00000 ←--
00035:  00040    60XXX          F4      CALL      VSUB,P,Y,CX,YMCX,E+4,0
00035:  00041    XXXXX   00002 ←--
```

```
00035: 00042  XXXXX 00031 ←←←
00035: 00043  XXXXX 00177 ←←←
00035: 00044  XXXXX 00211 ←←←
00035: 00045  XXXXX 00014 ←←←
00035: 00046  00000 00000 ←←←
00036: 00047  60XXX             F5      CALL       MVMULT,N,P,K,YMCX,KYCX,E+5
00036: 00050  XXXXX 00000 ←←←
00036: 00051  XXXXX 00002 ←←←
00036: 00052  XXXXX 01020 ←←←
00036: 00053  XXXXX 00211 ←←←
00036: 00054  XXXXX 00223 ←←←
00036: 00055  XXXXX 00015 ←←←
00036: 00056  00000 00000 ←←←
00037: 00057  60XXX             F6      CALL       VADD,N,FIXPGU,KYCX,XHAT,E+
00037: 00060  XXXXX 00000 ←←←
00037: 00061  XXXXX 00160 ←←←
00037: 00062  XXXXX 00223 ←←←
00037: 00063  XXXXX 00000 ←←←
00037: 00064  XXXXX 00016 ←←←
00037: 00065  00000 00000 ←←←
00038: 00066  60XXX             F7      CALL       MVMULT,M,N,L,XHAT,LXHAT,E+
00038: 00067  XXXXX 00001 ←←←
00038: 00070  XXXXX 00000 ←←←
00038: 00071  XXXXX 01246 ←←←
00038: 00072  XXXXX 00000 ←←←
00038: 00073  XXXXX 00242 ←←←
00038: 00074  XXXXX 00017 ←←←
00038: 00075  00000 00000 ←←←
00039: 00076  60XXX             F8      CALL       VSUB,M,ZEROV,LXHAT,U,E+8,0
00039: 00077  XXXXX 00001 ←←←
00039: 00100  XXXXX 00254 ←←←
00039: 00101  XXXXX 00242 ←←←
00039: 00102  XXXXX 00017 ←←←
00039: 00103  XXXXX 00020 ←←←
00039: 00104  00000 00000 ←←←
00040:                  * RESCALING OF XHAT AND U.
00041: 00105  60XXX                     CALL       RESCA,N,FXHAT,XHAT,E+10,0
00041: 00106  XXXXX 00000 ←←←
00041: 00107  XXXXX 00266 ←←←
00041: 00110  XXXXX 00000 ←←←
00041: 00111  XXXXX 00022 ←←←
00041: 00112  00000 00000 ←←←
00042: 00113  60XXX                     CALL       RESCA,M,FU,U,E+11,0
00042: 00114  XXXXX 00001 ←←←
00042: 00115  XXXXX 00274 ←←←
00042: 00116  XXXXX 00017 ←←←
00042: 00117  XXXXX 00023 ←←←
00042: 00120  00000 00000 ←←←
00043: 00121  45657 00000               J,I        CALG
00044:                  * TAG TABLE
00045: 00122        00017       FIX     BSS        15
00046: 00141        00017       GU      BSS        15
00047: 00160        00017       FIXPGU  BSS        15
00048: 00177        00012       CX      BSS        10
00049: 00211        00012       YMCX    BSS        10
00050: 00223        00017       KYCX    BSS        15
00051: 00242        00012       LXHAT   BSS        10
00052: 00254  00000 00000       ZEROV   BSS        10,0
00053:                                  NAME       FXHAT
00054:                                  NAME       FU
00055: 00266  00012             FXHAT   DEC         2,2,1,1,1,1
00055: 00267  00004       ←←←
00055: 00270  00002       ←←←
```

```
00055: 00271  00001        ←←←
00055: 00272  00001        ←←←
00055: 00273  00001        ←←←
00056: 00274  00002           FU    DEC    1, 4,
00056: 00275  00010        ←←←
00057:              00000         END    0
```

```

                          *SUBROUTINE MVMULT, PERFORMS
                          *MATRIX-VECTOR MULTIPLICATION.
                          *PROGRAMMER: JONAS AGERBERG.
                          *DATE        9.11.67
                          *REVISION 17.SEPT.68
                          *A MATRIX, A(N,M), (N ROWS, M COLUMNS)
                          *IS POSTMULTIPLIED BY A VECTOR, X(M).
                          *THE PRODUCT IS A VECTOR, Y(N).
                          *ALL ELEMENTS ARE IN SINGLE PRECISION
                          *ON OVERFLOW ON ANY ELEMENT OPERATION
                          *THE ELEMENT WILL BE SET TO MAX OR MIN
                          *FRACTIONAL VALUE (OCT 77777 OR 100000)
                          *AND CELL 'ERROR' WILL BE INCREMENTED BY
                          *ONE. THIS CELL CAN BE TESTED BY MAIN
                          *PROGRAM
                          *
                          *CALLING SEQUENCE:
                          **  L    MVMULT
                          **  ADR N
                          **  ADR M
                          **  ADR A
                          **  ADR X
                          **  ADR Y
                          **      ADR      ERROR
                          **      ADR      0
                          *
                          *DATA FIELD SHOULD BE DEFINED BY
                          **   A    BSS M.N
                          **   X    BSS M
                          **   Y    BSS N
                          *
                          *
                          *
                          *PROGRAM STARTS HERE
                                     NAME     MVMULT
01000             00000              REL      0
                  00000     SHIFT    EQU      0           OR SET AT ASSEMB
00000 100000 00000          MVMULT   ADR,I    0
00001  51076 00077                   STX      SAVX
00002 145776 00000                   LA,I     MVMULT      FETCH N
00003 161062 00065                   STA      N
00004  20100                         TCA
00005  26400                         SSP                  SET MINUS N
00006 161061 00067                   STA      MNX         FOR INDEX
00007  71771 00000                   AOM      MVMULT
00010 145770 00000                   LA,I     MVMULT      FETCH M
00011 161055 00066                   STA      M
00012  20100                         TCA
00013  26400                         SSP                  SET MINUS M
00014 161054 00070                   STA      MMX         FOR INDEX
00015 141763 00000                   LA       MVMULT      FETCH STRING
00016  26400                         SSP                  ADDRESSES AND
00017  26500                         EX                   STORE THEM
00020 142001 00001                   LA,X     1
00021 151045 00066                   A        M
00022 161047 00071                   STA      ADRA
00023 142002 00002                   LA,X     2
00024 151042 00066                   A        M
00025 161045 00072                   STA      ADRX
00026 142003 00003                   LA,X     3
00027 151036 00065                   A        N
00030 161043 00073                   STA      ADRY
```

```
00031 142004 00004          LA,X     4
00032 161044 00076          STA      ERRA
00033  22006 00006          ICX      6
00034  51041 00075          STX      EXIT
00035  53032 00067          LX       MNX
00036  51031 00067  LOOP2   STX      MNX
00037  26740                CLR                  ZERO TEMP
00040 161034 00074          STA      YMS         Y-CELLS
00041  53027 00070          LX       MMX
00042 147027 00071  LOOP1   LA,IX    ADRA
00043  37027 00072          M,IX     ADRX        MPLY.PROD IN DP
00044  27416                SNO
00045  61047 00114          L        MOV         SET +1 IFOVFL
00046 151026 00074          A        YMS         ADD MS PART OF Y
00047  27416                SNO                  OVERFL:(NO,YES-
00050  61030 00100          L        OVFL        GOTO SET MIN/MAX
00051 161023 00074          STA      YMS
00052  22001 00001          ICX      1           VECTOR PROD OK:
00053  41767 00042          J        LOOP1       (YES,NO-LOOP1)
00054  53013 00067          LX       MNX
00055 167016 00073          STA,IX   ADRY
00056 141013 00071          LA       ADRA        MOVE A-PTR
00057 151007 00066          A        M
00060 161011 00071          STA      ADRA
00061  22001 00001          ICX      1           N Y-VALUES DONE?
00062  41754 00036          J        LOOP2       (YES,NO-LOOP2)
00063  53014 00077          LX       SAVX
00064  45011 00075          J,I      EXIT
00065        00001  N       BSS      1
00066        00001  M       BSS      1
00067        00001  MNX     BSS      1           MINUS N FOR INDX
00070        00001  MMX     BSS      1             '    M    '     '
00071        00001  ADRA    BSS      1           PTR TO A
00072        00001  ADRX    BSS      1           PTR TO X
00073        00001  ADRY    BSS      1           PTR TO Y
00074        00001  YMS     BSS      1           TEMP FOR DP
00075  00000 00000  EXIT    BSS      1,0
00076  00000 00000  ERRA    BSS      1,0
00077  00000 00000  SAVX    BSS      1,0
                    *HERE IF OVERFLOW ON ADD
00100  00000 00000  OVFL    ADR      0
00101  24040                SKP                  WAS OVFL POS
00102  41004 00106          J        POS
00103  26740                CLR
00104  26440                SSN
00105  41004 00111          J        OUT
00106  26740        POS     CLR
00107  20200                OCA
00110  26400                SSP
00111  75765 00076  OUT     AOM,I    ERRA
00112  45766 00100          J,I      OVFL
00113  45765 00100          J,I      OVFL
                    *
                    *HERE IF OVERFLOW ON MULT
00114  00000 00000  MOV     ADR      0
00115  26740                CLR
00116  20200                OCA                  SET (AR) AND (QR
00117  26400                SSP                  TO '77777
00120  45774 00114          J,I      MOV
             00000          END      0
```

```
                                 *
         01000         00000              REL         0
                                          NAME        VSUB
                                          NAME        VADD
                                 * PERFORMS VECTOR ADD/SUBTRACT, C=A+B OR
                                 * C=A-B IN SINGLE PRECISION
                                 * DIMENSION OF VECTORS = DIM'
                                 * IF AN ELEMENT OF C OVERFLOWS IT IS SET
                                 * TO MAX (OR MIN) FRACTIONAL VALUE
                                 * ON EACH OVERFLOW (ERR) IS INCREMENTED BY ONE
                                 * NO ERROR EXITS
                                 * VSUB USES VADD AFTER FIXING
                                 * PROGRAMMER     :J AGERBERG
                                 * DATE           12 SEPT 68
                                 * REVISED
                                 *
                                 * FORTRAN CALL :
                                 **  CALL VADD (DIM,A,B,C,   ERR)
                                 * OR  CALL VSUB (DIM,A,B,C,   ERR)
                                 *
                                 * ASSEMBLER CALL :
                                 *       L          VADD               (OR VSUB)
                                 *       ADR        DIM
                                 *       ADR        A
                                 *       ADR        B
                                 *       ADR        C
                                 *       ADR        ERR
                                 *       ADR        0
                                 *
                                 *
                                 *
00000 100000 00000      VSUB     ADR,I      0           FIX RETURN
00001 141777 00000               LA         *-1         ADDRESS
00002 161004 00006               STA        VADD        FIX SUBTRACT
00003 141033 00036               LA         OP          INSTRUCTION
00004 101057 00063               OR         SUBMSK
00005  41004 00011               J          ADD
00006 100000 00000      VADD     ADR,I      0
00007 141027 00036               LA         OP          FIX ADD
00010 131052 00062               AND        ADDMSK      INSTR.
00011 161025 00036      ADD      STA        OP
00012  51054 00066               STX        SAVX
00013 141773 00006               LA         VADD
00014  26400                     SSP                    REMUVE INDBIT
00015  26500                     EX                     PT TO L+1
00016 142001 00001               LA,X       1           FETCH ARG
00017 161036 00055               STA        ABASE       ADDRESSES
00020 142002 00002               LA,X       2
00021 161035 00056               STA        BBASE
00022 142003 00003               LA,X       3
00023 161034 00057               STA        CBASE
00024 142004 00004               LA,X       4
00025 161033 00060               STA        ADRERR
00026  22006 00006               ICX        6
00027  51032 00061               STX        EXIT
00030 145756 00006               LA,I       VADD        FETCH DIM
00031  26500                     EX                     (XR) ← DIM
00032  22777 00001      MORE     DCX        1
00033  27417                     SKU
00034  41030 00064               J          OUT
00035 147020 00055               LA,IX      ABASE
00036 157020 00056      OP       A,IX       BBASE       ADD/SUBTR
00037 167020 00057               STA,IX     CBASE
```

```
00064: 00040  27401                    SO                 OVFL?
00065: 00041  41771 00032              J         MORE     NO
00066: 00042  24040                    SKP                YES, OVFL NEG?
00067: 00043  41004 00047              J         POS      NO, GOTO POS
00068: 00044  26740                    CLR                YES, SET
00069: 00045  26440                    SSN                (AR) =-1
00070: 00046  41004 00052              J         STOVF
00071: 00047  26740            POS     CLR                SET (AR)
00072: 00050  20200                    OCA                MAX POS
00073: 00051  26400                    SSP
00074: 00052 167005 00057      STOVF   STA,IX    CBASE    STORE MAX/MIN
00075: 00053  75005 00060              AOM,I     ADRERR   INCR. ERRCOUNT
00076: 00054  41756 00032              J         MORE
00077: 00055  00000 00000      ABASE   BSS       1,0
00078: 00056  00000 00000      BBASE   BSS       1,0
00079: 00057  00000 00000      CBASE   BSS       1,0
00080: 00060  00000 00000      ADRERR  BSS       1,0
00081: 00061  00000 00000      EXIT    BSS       1,0
00082: 00062 157777            ADDMSK  OCT       157777
00083: 00063 170000            SUBMSK  OCT       170000
00084: 00064  53002 00066      OUT     LX        SAVX
00085: 00065  45774 00061              J,I       EXIT
00086: 00066  00000 00000      SAVX    BSS       1,0
00087:              00000              END       0
```

```
                        *  SUBROUTINE RESCA.
                        * PERFORMS
                        * VECTOR A IN FRACTIONAL MULT WITH
                        * VECTOR SCALE IN INTEGER.
                        * RESULT IN FRACTIONAL  PLACED IN
                        * VECTOR A.
                        * IF OVERFLOW OCCUR ,AOM E,MAX VALUE
                        * WITH SIGN STORED.
                        * CALLING SEQ.
                        *    CALL   RESCA,N,SCALE,A,E,0
                        * SCALE N-DIM VECTOR IN INTEGER.
                        * A N-DIM VECTOR IN FRACTIONAL
                        * E NAME OF OF ERROR CELL.
                        *
                        *
01000         00000                   REL       0
                                      NAME      RESCA
00000 100000 00000            RESCA   ADR,I     0
00001  53777 00000                    LX        *-1
00002 142000 00000                    LA,X      0           LOAD FIRST ARG
00003  20100                          TCA
00004 161046 00052                    STA       COUNT
00005  26500                          EX                    ADR TO SCALE
00006  26400                          SSP                   STORED IN TAS
00007  26500                          EX
00010 142001 00001                    LA,X      1
00011 161040 00051                    STA       TAS
00012 142002 00002                    LA,X      2           ADR TO A
00013 161035 00050                    STA       TAA         STORED IN TAA
                        * MULTIPLICATION
00014  53037 00053                    LX        ZERO
00015 147033 00050            LP      LA,IX     TAA
00016  37033 00051                    M,IX      TAS
00017 161030 00047                    STA       TEMP
00020  26117 00017                    ALD       15
00021  27416                          SNO
00022  41011 00033                    J         ERR
00023 167025 00050            AE      STA,IX    TAA
00024  22001 00001                    ICX       1
00025  71025 00052                    AOM       COUNT
00026  41767 00015                    J         LP
00027 141751 00000                    LA        RESCA       EXIT
00030  26400                          SSP
00031  26500                          EX
00032  42005 00005                    J,X       5
00033  51021 00054            ERR     STX       TMP2
00034  53744 00000                    LX        RESCA
00035  72003 00003                    AOM,X     3           4TH ARG
00036 141011 00047                    LA        TEMP
00037 151014 00053                    A         ZERO
00040 141006 00046                    LA        MAX
00041  53013 00054                    LX        TMP2
00042  27402                          SM
00043  41760 00023                    J         AE
00044  20100                          TCA
00045  41756 00023                    J         AE
00046  77777                  MAX     OCT       77777
00047  00000 00000            TEMP    BSS       1,0
00050  00000 00000            TAA     BSS       1,0         ADR TO A
00051  00000 00000            TAS     BSS       1,0         ADT TO SCALE
00052  00000 00000            COUNT   BSS       1,0         NO OF ELEMENTS
00053  00000                  ZERO    DEC       0
00054  00000 00000            TMP2    BSS       1,0
```

```
00064:          00000          END     0
```

APPENDIX D

Consider the system

$$x(t+1) = \phi^{*} x(t) + \Gamma u(t) + e_1(t)$$

$$y(t) = \theta x(t) + e_2(t) \qquad (1)$$

where ϕ^{*} is the disturbed matrix ϕ. The Kalman filter is

$$\hat{x}(t+1) = \phi \hat{x}(t) + \Gamma u(t) + K[y(t) - \theta \hat{x}(t)] \qquad (2)$$

Equations (1) and (2) give

$$x(t+1) - \hat{x}(t+1) = (\phi - K\theta)(x(t) - \hat{x}(t)) + (\phi^{*} - \phi)x(t) + e_1(t) - Ke_2(t)$$

The deterministic part of the reconstruction error $\tilde{x}(t) = x(t) - \hat{x}(t)$ then is

$$\tilde{x}(t+1) = (\phi - K\theta)\tilde{x}(t) + (\phi^{*} - \phi)x(t)$$

In steady state we get

$$\tilde{x}_o = (I - \phi + K\theta)^{-1} (\phi^{*} - \phi) x_o \qquad (3)$$

Using the steady state value of the state vector corresponding to the undisturbed system equation (3) will give an estimate of the true reconstruction error.