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1. INTRODUCTION.

The control theory for linear time invariant multivariable systems has developed very rapidly during the last two decades. This development has proceeded essentially along two lines which correspond to different ways to regard a linear system: from the point of view of internal system descriptions, the state space approach, or from the point of view of external system descriptions, the transfer function approach. There is a considerable conceptual and computational difference between the two approaches, which in the same problem can lead to quite different representations of the solution. Consider for instance the optimal filtering problem as formulated by Wiener [23] and Kalman [13]. In this thesis the emphasis is on the state space approach. The transfer function is, however, also used to gain insight into the problems and to interprete the results.

What are the desirable properties of a multivariable control system? This question cannot be answered in a straightforward way since the control objectives obviously must depend on the specific system. It is, however, still possible to give a basic set of properties which are common to a large class of systems. These objectives can be summarized in terms of the following demands on the controlled system: (a) satisfactory disturbance rejection, (b) satisfactory tracking ability and (c) satisfactory insensitivity and security. For a more detailed discussion on this topic the reader is referred to Mac Farlane [14]. The specifications (a) - (c) can be reduced to two basic control problems.

- <u>The regulator problem</u>: Find a control which makes the system insensitive to disturbance inputs and parameter changes.

- The servo problem: Find a control which makes the system react in a desired way to command inputs.

A large part of existing multivariable control theory treats various aspects of these problems. The regulator and servo problems can be formulated as optimal control problems, either in a deterministic frame-work, see e.g. [1, 6], or in a stochastic frame-work [2]. The regulator problem is the underlying problem in eigenvalue shifting techniques [5, 7, 8, 20, 22, 28]. Different aspects of the algebraic regulator problem are considered in [4, 9,12, 29, 30]. Model matching [18, 19, 25] and noninteraction control [11, 16, 17, 21, 26, 27] give different solutions to the servo problem. The main contribution of the transfer function approach is perhaps the extension of the well-known results of Bode and Nyquist to include also multivariable systems. A fairly complete description of these results can be found in [14, 20].

Design Philosophy.

Control systems can be designed analytically or heuristically. In the analytic approach, the control is obtained as a solution to a mathematical problem where the design objectives have been reformulated in precise mathematical terms. Optimal control theory and to some extent algebraic state space theory fall within this cathegory. In the heuristic approach, on the other hand, the control is designed stepwise starting with an initial guess which is successively improved by use of dynamical characteristics, physical insight and simulations. Design techniques based upon extended Nyquist criteria and some state space techniques use the heuristic approach. In the analytic approach it is often difficult to capture all requirements in precise mathematical criteria. It is also difficult to introduce constraints on the complexity of the control. In the heuristic approach, on the other hand, it is difficult to judge if a better control system could have been obtained by a different structure. Of course, there is no clear distinction between the two approaches and both may be needed at different stages of the design. No matter what approach is used, the computational burden in the multivariable case needs computer aided design techniques. Such techniques have been worked out both in the frequency domain, see e.g. [3, 15], and in the time domain, see e.g. [10, 24].

In this thesis a design philosophy which lies somewhere in between the heuristic and the analytic approach is adopted. The analytic approach is used to analyze the structural properties of the system and to arrive at a configuration which admits an ideal servo and an optimal regulator. The heuristic approach is used to reduce this ideal control system into a form which is in agreement with a set of practicality constraints, mainly on the complexity of the control system. With this approach it is clearly possible to evaluate how much is lost at each step of simplification and consequently to make appropriate design trade-offs.

In most actual designs the regulator and servo problems occur simultaneously. In this thesis a basic control configuration which considers the combined servo and regulator problem is presented. The inverse system is used to achieve the ideal servo as a static relation between the reference inputs and the controlled outputs. In this idealized situation the only task of the regulator is to recognize deviations due to disturbance inputs and parameter changes (i.e. not reference changes). This approach leads to a conceptual separation between the two problems which is valuable for the design. The ideal servo is only theoretically achievable and a trade-off must be accepted in the actual design. This trade-off can be done in two ways: (a) by approximating the inverse or (b) by specifying a model for the desired input-output behaviour. Case (b) is in fact a solution to the model matching problem, but in contrary to [18, 19, 25] the feedback is not used to match the model, but rather to obtain a regulation around the trajectory defined by the model.

The Servo Problem.

The inverse system plays an important role in the solution to the servo problem. A system can, however, have several inverses with different properties. It is then important to clearly understand the differences between various system inverses. For this purpose, the minimal system inverse is introduced as a left or right inverse of minimal dynamical order. Such inverses are not satisfactorily explained in existing state space theory, even if some upper bounds are obtained in [21]. In this thesis a minimal inverse is constructed by means of a sequence of basic operations on the original system, using some geometric concepts introduced by Wonham and Morse [27]. It turns out that a minimal system inverse has one particularly desirable property: its characteristic polynomial divides the characteristic polynomial of an arbitrary inverse. Another interesting feature is the following: the state of a minimal inverse is identical to a part of the state of the original system. Both these facts can be utilized in the combined servo and regulator problem.

Another notion which has not been clearly explained in state space theory is the concept of zeros. The poles and zeros have shown to be a valuable design aid in the single input-output case, and it could be expected that this is so also in the multivariable case.

The minimal inverse gives us, however, a tool for a proper state space definition of zeros. In fact, the zeros, here denoted the invariant zeros, have a simple interpretation in state space terminology in terms of the spectrum associated with a certain invariant subspace. Although the inverse system is used as starting point, the final definition of zeros will be valid also for non-invertible systems.

What does the invariant zeros tell us about the structural properties of the system? To answer this question a more detailed analysis is performed. It is shown that the invariant zeros have some properties which can be conjectured from the single input-output case, namely: (a) invariance under state feedback and duality, (b) under certain invertibility conditions uncontrollable and unobservable modes occur as "common factors" between the invariant zeros and the poles (i.e. the eigenvalues of A in $\dot{x}=Ax+Bu$).

The only way to change the invariant zeros is thus to change the input-output structure of the system. Therefore, we give criteria how to choose additional controls or measurements in the system in order to avoid a certain kind of zeros, e.g. all zeros in the right half-plane.

In the basic control configuration described above, reference values for the available outputs must be fed into the regulator. These reference values correspond to an idealized situation where the model is exact and no disturbances influence the system. The nominal track is generated in two ways: (a) by a reduced order state space model and (b) by properties from the inverse. The regulator is supposed to act whenever deviations from these nominal values are discovered.

A difficulty which can occur in the servo problem is the problem of unstable inverses. Systems with unstable inverses are called nonminimum phase in classical terminology. In this case the exact inverse, which generates unbounded inputs, must be replaced by a neighbouring stable system from which the design can be done. Two approaches to solve this problem are given, which both utilize stabilizing feedback on the inverse system: (a) a heuristic approach and (b) an optimal approach which uses minimum energy stabilization.

The Regulator Problem.

A regulator which has been frequently used in actual design is the PI regulator. Such regulators are designed to remove steady state deviations in the outputs due to disturbances which are constant or slowly varying. The notion of PI regulator is generalized to the multivariable case. The major results are (a) solvability conditions and (b) a design algorithm. In the design algorithm, a proportional and integral controller is designed stepwise allowing for the freedom to choose the "proportional" and "integral" parts independently. Moreover, the resulting controller contains a small number of integrators. If p denotes the number of controlled outputs and r the number of disturbance inputs, the number of integrators is always less than min(p,r).

We also face a problem which has a strong practical implication. In many actual systems there is a desire to restrict the complexity of the feedback structure, e.g. due to constraints in the measured signals or information exchange. This means that only output or restricted output feedback is allowed, where each output variable is permitted to be connected to a subset of the inputs.

In this thesis, the restricted feedback problem is attacked in two steps: (a) a state feedback regulator is designed under the hypothesis of a completely free feedback structure, (b) the state feedback regulator is fitted into a regulator with a constrained feedback structure using successive weightings on a dominant eigenspace. We thus utilize the fact that there are straightforward and rapid methods to find satisfactory state feedback regulators even for fairly large systems, e.g. using linear quadratic control theory [1] or modal control [22].

As a design principle, the procedure above is in line with the overall design philosophy of the thesis. By comparison with the performance of the initially designed state feedback regulator, it is possible to determine the trade-off which is caused by the constraints in the feedback structure and to change the structure if the decrease in the performance is too large.

Organization of the Material.

The thesis is organized in the following way:

In Chapter 2 a basic control configuration for the regulator and servo problems is introduced and analyzed. The concept of minimal system inverse, which plays a crucial role in this configuration, is the topic of Chapter 3. A state space theory for poles and zeros is developed in Chapter 4. In Chapter 5 we return to the servo problem and analyze two subproblems, reference values and nonminimum phase systems, in more detail. The multivariable version of the PI controller is resolved in Chapter 6. Finally, a heuristic method to consider practicality constraints is developed in Chapter 7.

Summing up, the following aspects of the multivariable control problem are thus treated in this thesis.

- A basic control configuration for regulators and servos: Chapter 2.
- Structural and algebraic concepts: Chapters 3 and 4.
- Dynamical characteristics: Chapter 4.
- The servo problem: Chapters 2 and 5.
- The regulator problem: Chapters 2, 6 and 7.
- Practicality constraints: Chapter 7.

All the chapters are supplied with local appendices and references which appear at the end of each chapter.

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2. A CONTROL SYSTEM CONFIGURATION.

A multivariable control system shall in many situations serve both as a regulator and a servo. The specifications can, however, be quite different or even contradictionary in the two cases which can rise difficulties in the design. The combined servo and regulator problem for multivariable systems does not seem to be exhaustively treated in the control literature. An accurate servo is often tied to high feedback gains in the feedback loops [2]. In model matching theory [3, 4, 5] the feedback is used rather to match the model than to achieve efficient regulation with respect to disturbances. Optimal model following in the asymptotic linear quadratic sense [1] requires a linear model of the command input.

In this chapter a basic control system configuration is suggested which makes a conceptual separation between the two problems. This configuration admits an ideal servo and an optimal regulator and is intended to serve as a starting point for the design. Different aspects on the regulator and servo are discussed in this context. The discussion starts with the single input-output case just to indicate some of the crucial points before the more general multivariable case is considered. A more detailed algebraic analysis is performed in Chapters 3 and 5.

2.1. The Single Input-Output Case.

Consider a conventional control scheme as in Fig. 2.1.

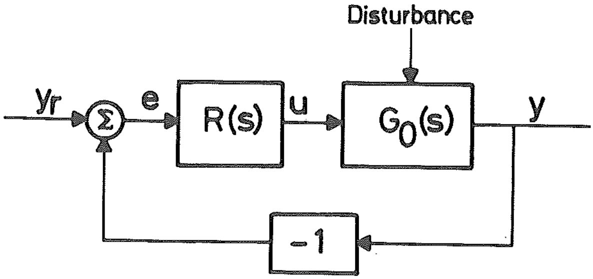

Fig. 2.1 - A conventional control loop.

The control system in Fig. 2.1 recognizes disturbances and reference changes in the same way, i.e. as a deviation in the control error e(t). This expresses a very simple control principle which only needs a fairly rough model of the process and its environment. However, conflicts between different design goals may arise in such a structure. Some examples are:

- An optimal setting of the regulator parameters with respect to disturbances may cause bad responses to reference inputs. Conversely, a regulator with satisfactory reference responses may have poor disturbance rejection.
- For security reasons large stability margins may be required. This may lead to slow responses for reference inputs.
- Additional measured signals are not readily included in the control loop, unless the system has some special structure (e.g. cascaded subsystems) and physical insight is exploited.

Some of these difficulties can be avoided if a somewhat more complex system configuration with a feedback and a feedforward structure is used, see Fig. 2.2.

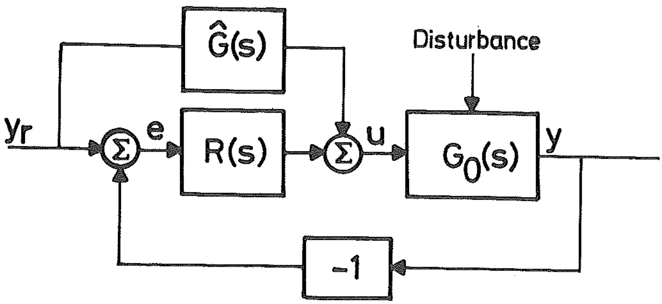

Fig. 2.2 - A conventional control loop with feedforward from the reference input.

The control systems in Fig. 2.1 and Fig. 2.2 react in the same way to disturbance inputs, but the responses for reference inputs are different. Consider the transfer function relating the reference and the output:

$$\frac{y(s)}{y_r(s)} = \left\{1 + \frac{G_0(s)\hat{G}(s) - 1}{1 + G_0(s)R(s)}\right\} \qquad (2.1)$$

If $\hat{G}(s)$ equals the inverse of the process, i.e.

$$\hat{G}(s) = G_0^{-1}(s) \qquad (2.2)$$

we have in a sense achieved the ideal servo since the transfer function between the reference and the output becomes unity irrespective of the choice of regulator R(s). In this case the regulator is not influenced by reference changes and does not become active until the system is driven away from its nominal values, e.g. due to disturbances or inaccuracies in system description. The behaviour is thus the opposite of the conventional control loop in Fig. 2.1, where the reference change is handled by the regulator alone.

The regulating ability can also be seen in the following way. Assume $\hat{G}(s)$ differs from the actual inverse by an amount $\Delta\hat{G}(s)$. The transfer function between the reference and the control error becomes then

$$\frac{e(s)}{y_r(s)} = \frac{G_0(s)\Delta\hat{G}(s)}{1 + R(s)G_0(s)} \tag{2.3}$$

If the gain in the regulator R(s) is high, we conclude that the servo system is insensitive to changes in $\hat{G}(s)$.

The appearance of the inverse $G_0^{-1}(s)$ in (2.2) creates some problems. In any practical application we must in some way avoid the pure differentiators appearing in the inverse. One way to do this is to "approximate" the inverse with some other transfer function $\hat{G}(s)$ such that the effect on the control error is small, i.e. such that the transfer function (2.3) is small over some relevant frequency interval. The conventional control loop in Fig. 2.1 can be considered as such an approximation with $\hat{G}(s) = 0$.

Another problem, which may be a more serious one, is the problem of unstable inverses. A more detailed analysis of such inverses are performed in Chapter 5. Let us just note here that systems with unstable inverses are called non-minimum phase in classical terminology.

The discussion above is illustrated by a simple example.

Example 2.1. Consider the system shown in Fig.2.3.

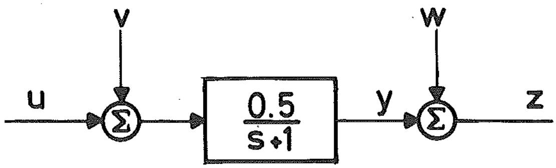

Fig. 2.3 - A first order system with process disturbance v and mesurement disturbance w.

The disturbances w and v are assumed to be independent white noise processes with covariances $r_w\delta(\tau)$ and $r_v\delta(\tau)$ respectively. Assume first that a control of the form

$$u = - K(z-y_r) \tag{2.4}$$

is used. The gain K is chosen to minimize the criterion

$$J = E(y-y_r)^2$$

in steady state operation. If $r_v/r_w = 32$, it can be shown that the minimum is obtained for

$$K = K_{opt} = 4$$

However, the response to reference inputs is rather slow for this value on K as is shown in Fig. 2.5. In order to get a better response we can take a control of the form

$$u = - K(z-y_r) + \hat{G}(s)y_r$$

If $\hat{G}(s)$ equals the inverse of the process, i.e. $\hat{G}(s) =$ $= 2(s+1)$, the transfer function between the reference and the output becomes unity irrespective of the value verse is "approximated" as is indicated in the asymptotic Bode diagram below.

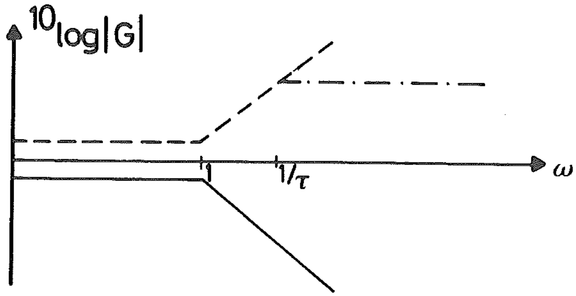

Fig. 2.4 - Asymptotic Bode diagram.

——— open loop system

----- exact inverse

-·-·- approximated inverse

The choice of breaking point $\omega_\tau = 1/\tau$ in Fig. 2.4 depends on how fast signals the system can track with realistic amplitudes of the control signal. The frequency ω_τ can be interpreted as the upper limit. The approximated inverse becomes

$$\hat{G}(s) = \frac{2(s+1)}{(1+\tau s)}$$

Using this approximation of the inverse, the control becomes

$$u = - K_{opt}(z-y_r) + \frac{2(s+1)}{(1+\tau s)} y_r \qquad (2.5)$$

The step responses for the control laws (2.4) and (2.5) are compared in Fig. 2.5. The breaking point in Fig. 2.4 is assumed to be $\tau = 1/5$.

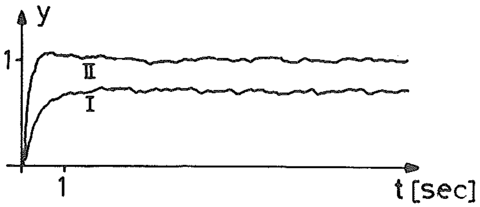

Fig. 2.5 - Responses to a unit step in y_r.
I. Control law (2.4).
II. Control law (2.5).

2.2. The Multivariable Case.

The basic control principles for single input-output systems discussed above are also applicable to multivariable systems. Consider the system shown in Fig. 2.6.

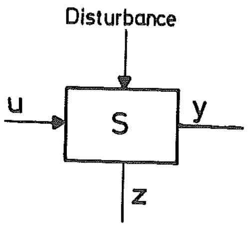

Fig. 2.6 - Open loop multivariable system.

u is the m-vector of control inputs, y the p-vector of controlled outputs and z the q-vector of measured outputs. The controlled outputs may coincide with the measured outputs, although it is not necessary in the sequel. Moreover, let S denote the input-output operator of this system relating u and y and assume for simplicity that the initial state is zero.

Both the servo and the regulator problems can be approached using the same principles as were applied in the single input-output case. Let S^{-1} be a right inverse of the operator S, i.e.

$$SS^{-1} = I$$

A solution to the servo problem is given by

$$u_r = S^{-1} y_r \tag{2.6}$$

This input can be regarded as the ideal servo since the operator relating the reference and the output becomes the identity. If (2.6) is applied to the system in Fig. 2.6, nominal values z_r on the available outputs are obtained. A disturbance input or a parameter change in the system then causes a deviation in z from its nominal track. In order to make the servo system insensitive to disturbances and parameter changes, a regulating part Δu is added to (2.4) which recognizes such deviations. Hence

$$u = u_r + \Delta u$$

$$= u_r + R(z_r - z)$$

where R is an arbitrary dynamic operator. The combined regulator and servo system is shown in Fig. 2.7.

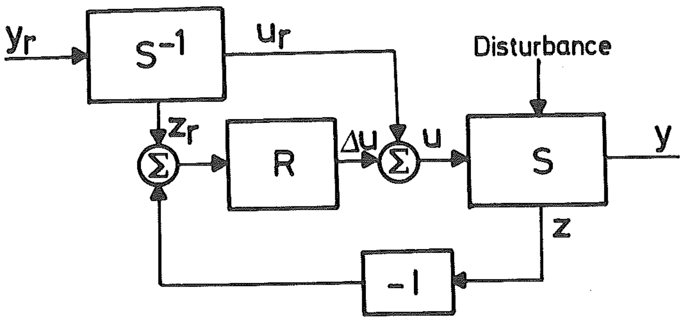

Fig. 2.7 - A basic control system configuration for regulators and servos for multivariable systems.

At this stage it is not obvious that z_r is given by y_r as is indicated in Fig. 2.7. This point is clearified in Chapter 5. For a linear system with state x

$$\dot{x} = Ax + Bu$$

$$z = Hx$$

controlled by state feedback and a state estimator, of conventional type, the control system becomes

$$u = u_r + \Delta u$$

$$= u_r + L\Delta\hat{x}$$

$$\Delta\dot{\hat{x}} = A\Delta\hat{x} + B\Delta u + K(z_r - z + C\Delta\hat{x})$$

where u_r is given by the inverse S^{-1}.

The inverse S^{-1} will often contain pure differentiators. In this sense, the basic control configuration in Fig. 2.7 is only theoretically achievable. This problem can be overcome in two different ways:

- The inverse is "approximated" by another system $\hat{S}$, which contains no pure differentiators. Such an approximation can either be valid for a certain frequency interval (in a frequency domain description) or for a certain class of inputs (e.g. stationary responses for step and/or ramp inputs). Compare with the arguments of Example 2.1.

- The desired output y_r is described by a model S_m as is shown in Fig. 2.8.

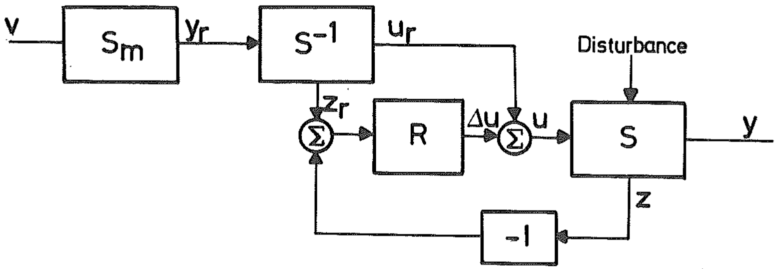

Fig. 2.8 - The basic control system configuration with reference model S_m and command input v.

The model should be chosen so that the composite system $S^{-1}S_m$ contains no pure differentiators. This will be the case if the model is compatible with the process dynamics in the sense that no more is required from the model than can be achieved by the process with say bounded inputs.

The preceding discussion suggests a two-step design for regulator and servo systems. In the first step a regulator is designed such that the closed loop system has satisfactory disturbance rejection and insensitivity. In the second step reference signals are introduced into the system according to the principles presented above, in such a way that the closed loop system including the regulator responds satisfactorily to command inputs. This conceptual "separation" between the regulator and servo problems is valuable at the design stage.

Finally we notice that some problems of importance remain to be solved:

- The problem of nonunique inverses. A given system may have several right inverses, some of which are stable while others are not, cf. a discussion in [3].
- A given system may have <u>no</u> stable inverse. It is then impossible to use the structure in Fig. 2.7 since unbounded inputs u_r will occur. How shall the basic configuration be changed to cover this case?
- The nominal values z_r on the measured outputs should be available to the regulator. A simple (but important) special case is $z = y$ where z_r equal the desired output y_r. How is z_r obtained if $z \neq y$?

<u>Remark 1</u>. For discrete time systems the pure differentiators in the inverse S^{-1} are replaced by forward shift operators. This means that u_r needs future values on y_r to be realized. In actual process design the implication is that a change of operation point must be preplanned some time-steps ahead, which is often a realistic assumption at least for large operation changes. If such preplanned action is not permitted, some of the schemes given above for continuous time systems must be used.

Remark 2. The control scheme in Fig. 2.8 is a solution to the model matching problem. In contrary to [3, 4, 5], where the feedback is used to match the model, the feedback is present to achieve regulation around the trajectory defined by the model.

Remark 3. Observe that the model S_m in Fig. 2.8 may be nonlinear. In fact, nonlinear models for the output behaviour during transfer between different operating points are desirable in many cases, e.g. in order to restrict the amplitudes in the controlled variables and its derivatives.

Let us illustrate the discussion by some simple examples:

Example 2.2. Consider the first order system described in Example 2.1. Instead of approximating the inverse, the reference is generated from a model.

$$y_r = G_m(s)v = \frac{q_m(s)}{p_m(s)} v$$

where v is the command input. The control becomes thus

$$u = - K_{opt}(y-y_r) + G_0(s)^{-1} y_r$$

$$= - K_{opt}\left(y-G_m(s)\, v\right) + 2(s+1)G_m(s)v$$

The pure differentiators are avoided if $\deg\left(q_m(s)\right) \leq \deg\left(p_m(s)\right)-1$. With this control, the system behaves as the model under reference changes, while maintaining its optimal disturbance rejection properties. ■

Example 2.3. The problem of nonunique inverses is illustrated by the following system

$$\dot{x} = \begin{pmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 2 \end{pmatrix} u$$

$$y = [1 \quad 0 \quad 0] x$$

The transfer function for this system is

$$G(s) = \begin{pmatrix} \dfrac{1 - s}{s^3 + 2s^2 + s + 1} & \dfrac{2 - s}{s^3 + 2s^2 + s + 1} \end{pmatrix}$$

Two different right inverses of G(s) are given by

$$\hat{G}_1(s) = \begin{pmatrix} \dfrac{s^3 + 2s^2 + s + 1}{1 - s} \\ 0 \end{pmatrix} \qquad \hat{G}_2(s) = \begin{pmatrix} 0 \\ \dfrac{s^3 + 2s^2 + s + 1}{2 - s} \end{pmatrix}$$

which are both unstable. An inverse with nice properties exists, however:

$$\hat{G}_3(s) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} (s^3+2s^2+s+1)$$

The implication is that a multivariable system can be easy to control even if each individual loop has nonminimum phase properties. ■

2.3. References.

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3. MINIMAL SYSTEM INVERSES.

The inverse system plays an important role in linear system theory. One reason being that the properties of the inverse tell much about the original system such as tracking ability and stabilizability. Many control and estimation problems are consequently closely related to system inversion. A few examples are decoupling [3, 9, 10, 14, 18], model matching [7, 8, 19] and feedforward control. It has also been shown that systems with unstable inverses can be very difficult to control [11]. In this thesis, the inverse system is of importance for the servo problem as was described in Chapter 2.

The inversion problem consists in fact of two different subproblems, left and right inversion. Left invertibility is sometimes called input functional observability and right invertibility output functional controllability, see e.g. [12].

For linear time invariant systems with zero initial state, the inversion problem can be treated completely algebraically using the transfer function of the system. The problem then becomes a problem of inverting a matrix of rational functions. This can be done, Forney [4], for instance using the invariant factor theorem. An algorithmic approach has recently been described by Wang and Davison [17].

Silverman [13] and Silverman and Payne [14] have developed a quite different inversion theory using state space terminology. The inverse system is constructed by means of an algorithm, the structure algorithm, from which some properties can be extracted. Related work has been done by Sain and Massey [12]. Quite recently, Wonham and Morse [10] have given necessary and sufficient conditions for left invertibility in terms of a certain invariant subspace.

We introduce the concept of minimal system inverse as an inverse system having the least dynamical order. Such inverses are in the frequency domain obtained as a consequence of the invariant factor theorem [4, 17], where the order is defined by the McMillan degree. Here, the problem is treated in a state space terminology, where upper bounds of order previously have been given in [13, 14].

Unlike Silverman and Payne [13, 14], we do not hinge our results on the properties of a specific algorithm, but rather on a set of geometric concepts introduced by Wonham and Morse [10]. We utilize a sequence of basic operations on the original system which leads to a structure from which the inverse system is obtained almost directly.

Minimal left inverses are constructed for systems with unknown and known (zero) initial states and minimal right inverses for zero initial states. The dynamical order of a minimal inverse equals the order of a certain invariant subspace. The same invariant subspace also determines its characteristic polynomial.

The characteristic polynomial for the minimal system inverse is shown to be unique in the sense that it divides the characteristic polynomial for an arbitrary inverse. Thus, the minimal system inverse extracts the stability properties from all other inverses and provides a solution to the problem of nonunique inverses mentioned in Section 2.2 in connection with the servo problem. It also leads to a proper definition of the concept of zeros in the multivariable case. This will be further discussed in Chapter 4.

3.1. Preliminaries.

Notations.

The Euclidian n-dimensional vector space over the real numbers is written R^n, and over the complex numbers $\mathbb{C}^n$. Script letters $\mathcal{V}$, $\mathcal{W}$, $\mathcal{X}$ denote linear subspaces and capital letters A, B, C denote linear maps (matrices). For summation of linear subspaces we will use the symbol $+$, and for the direct sum $\oplus$. The range space of a linear map B is denoted $\{B\}$, or sometimes by the corresponding script letter $\mathcal{B}$. The null space of B is written ker(B). A matrix V is said to be a basis matrix for $\mathcal{V}$ if the columns of V are linearly independent and span $\mathcal{V}$.

Assume A: $R^n \to R^n$ is a linear map and $\mathcal{V} \subset R^n$ a subspace. The subspaces $A\mathcal{V} = \{x \in R^n \mid Az = x,\ z \in \mathcal{V}\}$ and $A^{-1}\mathcal{V} = \{x \in R^n \mid Ax \in \mathcal{V}\}$ denote the image and inverse image respectively of $\mathcal{V}$ under A. A subspace $\mathcal{V}$ satisfying $A\mathcal{V} \subset \mathcal{V}$ is said to be A-invariant. If $\mathcal{V}$ is A-invariant and V is a basis matrix for $\mathcal{V}$, the restriction of A to $\mathcal{V}$, $A|\mathcal{V}$, is defined by $AV = V\bar{A}$ and $\bar{A} = A|\mathcal{V}$, cf. [19].

If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two independent subspaces, i.e. $\mathcal{V}_1 \cap \mathcal{V}_2 = 0$, we define the projection onto $\mathcal{V}_1$ along $\mathcal{V}_2$ as a map P such that $Px_1 = x_1$ for all $x_1 \in \mathcal{V}_1$ and $Px_2 = 0$ for all $x_2 \in \mathcal{V}_2$.

Let A: $R^n \to R^n$ and B: $R^m \to R^n$ be a pair of linear maps. The controllable subspace for the pair (A,B) is written $\{A|\mathcal{B}\}$ and is defined by

$$\{A|\mathcal{B}\} = \mathcal{B} + A\mathcal{B} + \dots + A^{n-1}\mathcal{B}$$

A subspace $\mathcal{V}$ is said to be (A,B)-invariant if $(A+BL)\mathcal{V} \subset \mathcal{V}$ for some linear map L. A controllability subspace $\mathcal{R}$

is any subspace satisfying $\mathcal{R} = \{A + BL | \mathcal{B} \cap \mathcal{R}\}$ for some linear map L. A more detailed discussion on these concepts can be found in [10, 18] and below where a geometric interpretation is given.

System description.

The control system of interest is described by the differential equation

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0 \tag{3.1a}$$

$$y = Cx \tag{3.1b}$$

where $x \in R^n$ is the vector of states, $u \in R^m$ is the vector of inputs and $y \in R^p$ is the vector of outputs. Here, A, B and C are linear time invariant maps (matrices). For convenience, a system of the form (3.1) is denoted S(A,B,C).

Some basic geometric concepts.

The geometric concepts (A,B)-invariant and controllability subspaces were introduced in control theory essentially to provide a well-posed formulation and solution to the decoupling problem, Wonham and Morse [10, 18]. These concepts have shown to give valuable insight to different problems dealing with structure of linear systems, e.g. the algebraic regulator problem as formulated by Wonham, Bhattacharyya and Pearson [2] and the model matching problem discussed by Morse [8]. Here, the same concepts will be used to explain properties of inverse systems.

The concepts A-invariant, (A,B)-invariant and controllability subspaces, which are defined algebraically above, can be given a simple interpretation in terms of a basic control problem. It is felt that the algebraic treatment of the inversion problem later can be more easily understood with this interpretation in mind. Consider a subspace $\mathcal{V} \subset R^n$ and a linear time-invariant system S(A, B,C).

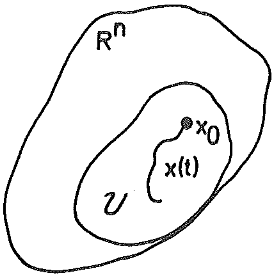

Fig. 3.1 - A geometric illustration of a basic control problem.

As is indicated in Fig. 3.1, the task is to keep the state trajectory x(t) within the given subspace $\mathcal{V}$. Three different versions of this problem are considered below.

Consider first the uncontrolled version of (3.1), i.e.

$$\dot{x} = Ax \qquad\qquad x(t_0) = x_0$$

The state trajectory remains within the given subspace $\mathcal{V}$ if and only if the velocity vector $\dot{x}$ does not point out of $\mathcal{V}$ for any point of time. Since x_0 is an arbitrary vector in $\mathcal{V}$, x(t) belongs to $\mathcal{V}$ if and only if

$$A\mathcal{V} \subset \mathcal{V} \tag{3.2}$$

i.e. if and only if the subspace is A-invariant.

Let us now proceed to the controlled version. The system is then described by

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

In this case we have the additional freedom to choose the control vector u. As above, we must force the velocity vector to remain within $\mathcal{V}$, i.e. there must exist vectors $v_1 \in \mathcal{V}$ and $u \in R^m$ such that

$$v_1 = Av + Bu$$

for all $v \in \mathcal{V}$. We can express this condition in an alternative way. The subspace $\mathcal{V}$ must satisfy

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B} \qquad (3.3)$$

Moreover, (3.3) is satisfied if and only if [1, 10, 18]

$$(A+BL)\mathcal{V} \subset \mathcal{V}$$

for some linear map L, i.e. if and only if the subspace $\mathcal{V}$ is (A,B)-invariant. Assume now that $\mathcal{D}$ is a given subspace. It can be shown [1, 18] that there exists a unique maximal (A,B)-invariant subspace $\mathcal{V}^M$ contained in $\mathcal{D}$, i.e. $\mathcal{D} \supset \mathcal{V}^M \supset \mathcal{V}$ where $\mathcal{V}$ is any (A,B)-invariant subspaces contained in $\mathcal{D}$. This maximal subspace can be obtained by the following sequence [18]

$$\mathcal{V}_0 = \mathcal{D} \qquad (3.4)$$

$$\mathcal{V}_i = \mathcal{D} \cap A^{-1}(\mathcal{V}_{i-1}+\mathcal{B}) \qquad i = 1, 2, \ldots, n$$

$$= \mathcal{V}_{i-1} \cap A^{-1}(\mathcal{V}_{i-1}+\mathcal{B})$$

Let k be the first integer such that $\mathcal{V}_{k+1} = \mathcal{V}_k$, then $\mathcal{V}^M = \mathcal{V}_k$. This sequence converges after at most ν steps where $\nu = \dim(\mathcal{D})$.

The third version of the control problem in Fig. 3.1 is the following. Consider the system

$$\dot{x} = Ax + Bu \qquad\qquad x(t_0) = 0$$

with the control

$$u = Lx + Gv$$

Under what conditions do there exist linear maps L and G such that the subspace controllable from the input v is exactly $\mathcal{V}$? It can be shown, [10, 18], that this problem can be solved if and only if the subspace $\mathcal{V}$ is a controllability subspace, i.e.

$$\{A+BL \mid \mathcal{B} \cap \mathcal{V}\} = \mathcal{V} \tag{3.5}$$

for some linear map L. It is fairly easy to show that a controllability subspace is also an (A,B)-invariant subspace [10, 18]. As was the case for (A,B)-invariant subspaces, there is a unique maximal controllability subspace $\mathcal{R}^M$ contained in a given subspace $\mathcal{D}$. The subspace $\mathcal{R}^M$ can be constructed as [10]

$$\mathcal{R}^M = \{A+BL \mid \mathcal{B} \cap \mathcal{V}^M\} \tag{3.6}$$

where $\mathcal{V}^M$ is the maximal (A,B)-invariant subspace contained in $\mathcal{D}$ and L is a map such that $(A+BL)\mathcal{V}^M \subset \mathcal{V}^M$. Alternatively, $\mathcal{R}^M$ is produced by the sequence [10]

$$\mathcal{S}_0 = 0$$

$$S_i = V^M \cap (AS_{i-1}+B) \qquad i = 1, 2, \ldots, n \qquad (3.7)$$

$$S_n = R^M$$

A computational algorithm for the sequence (3.4) can be found in Appendix 3B.

Example 3.1. Consider a linear time invariant system

$$\dot{x} = Ax + Bu \qquad x(t_0) \in V$$

$$y = Cx$$

where V is a subspace satisfying

$$V \subset \ker(C)$$

Case 1. V is A-invariant. According to the discussion above the state trajectory will remain within V for $u = 0$. Since $V \subset \ker(C)$ we have thus

$$y(t) = Cx(t) = 0 \qquad t \geq t_0$$

V is thus an unobservable subspace to the pair (C,A) and the unobservable modes equal the eigenvalues of the matrix representing the mapping $A|V$.

Case 2. V is (A,B)-invariant. In this case there is a linear map L such that V is (A+BL)-invariant. According to Case 1 this means that V is an unobservable subspace to the pair (C,A+BL), i.e. unobservability is introduced into the system with a control of the form $u = Lx$.

Case 3. $\mathcal{V}$ is a controllability subspace. According to the discussion above there exist linear maps L and G such that the subspace controllable from v in u = Lx + + Gv is exactly $\mathcal{V}$. Since $\mathcal{V} \subset \ker(C)$ this means that

$$y(t) = Cx(t) = 0$$

for all possible choices of the input v. ■

3.2. The Inversion Problem.

Definitions.

The solution of (3.1) is

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-s)}Bu(s)ds$$

which is written formally as

$$y = \bar{S}(x_0,u) = S_0x_0 + S_1u \qquad (3.8)$$

The expression (3.8) can thus be regarded as an input-output map for the system S(A,B,C) parametrized by the initial state x_0. For our purpose, the initial state will be of minor importance and is therefore set equal to zero. The system is then simply described by

$$y = S_1u$$

where $u \in U$, $y \in Y$. The input space U and the output space Y consist of all real-valued piecewise continuous m-vector and p-vector functions respectively, defined on $[t_0,\infty)$. S_1 is a linear operator which maps U onto $Y_1 \subset Y$.

An inverse to the operator S_1 can either be applied on the left hand side or the right hand side. An illustration of the difference between left and right inverses is given in Fig. 3.2.

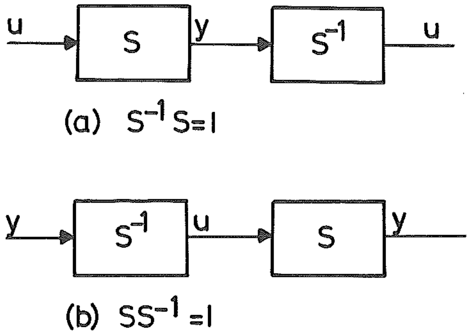

Fig. 3.2 - An illustration of (a) left inverses and (b) right inverses.

Note the change in consequtive order between the block diagrams and the algebraic expressions in Fig. 3.2.

In the left inversion problem, the intention is to reconstruct the inputs from the available outputs, or more precisely.

Definition 3.1 (Left inverse). A left inverse to the system S(A,B,C) is any linear operator $\hat{S}$ such that

$$\hat{S}y = \hat{S}S_1u = u \tag{3.9}$$

for all input-output pairs $(u,y) \in U \times Y$. ■

Remark 1. In Section 3.3 we will also consider the case of unknown initial states. In this case the inverse $\hat{S}$ shall satisfy $\hat{S}S_0 = 0$ and $\hat{S}S_1 = I$.

Remark 2. The case of known initial states can be handled in the same way as the case of zero initial states, cf. Remark 10.

Without loosing generality, it can be assumed that a left inverse is of the form

$$\dot{w} = \hat{A}w + N_1(p)y \qquad w(t_0) = 0 \tag{3.10}$$

$$u = \hat{C}w + N_2(p)y$$

where $N_1(p)$ and $N_2(p)$ are polynomial matrices and $p = d/dt$. It can be shown [15] that any system with a rational transfer function can be transformed into the form (3.10) by a sequence of equivalence operations. This representation of the left inverse is assumed in the sequel. The concept of minimal left inverse can now be concisely defined.

Definition 3.2 (Minimal left inverse). A minimal left inverse of S(A,B,C) is any operator $\hat{S}$ of the form (3.10) such that condition (3.9) is satisfied and w in (3.10) is of minimal dimension. ∎

The corresponding right inversion problem, i.e. to find an input u to the system which produces an arbitrary output y_r is of more direct interest in many control problems. Such an input is produced by the right inverse operating on y_r. In order to avoid differentiability problems it is assumed that the desired output y_r belongs to the space

$$Y_r = \{\text{all functions } y_r\colon R \to R^p \text{ such that } y_r(t) = 0 \text{ for } t < t_0 \text{ and } y_r \in C^n(-\infty,\infty)\}$$

where $C^n(-\infty,\infty)$ denotes the space of n times differentiable functions on $(-\infty,\infty)$.

Definition 3.3 (Right inverse). A right inverse to the system S(A,B,C) is any operator $\hat{S}$ such that

$$y_r = S_1\hat{S}(y_r) \tag{3.11}$$

for all $y_r \in Y_r$. ■

Without loosing generality it can be assumed that the right inverse is of the form

$$\dot{w} = Aw + By_r \qquad w(t_0) = 0 \tag{3.12}$$

$$u_r = N_1(p)w + N_2(p)y_r$$

where $N_1(p)$ and $N_2(p)$ are polynomial matrices and p = = d/dt. A minimal right inverse is then defined by

Definition 3.4 (Minimal right inverse). A minimal right inverse of S(A,B,C) is any operator $\hat{S}$ of the form (3.12) such that condition (3.11) is satisfied and w in (3.12) is of minimal dimension. ■

The benefit of using different representations (3.10) and (3.12) for left and right inverses will be evident from the discussion on duality below.

Duality.

The left and right inversion problems are dual problems [12]. This is easily verified using transform theory. Assume that a left inverse $\hat{S}$ to the system $S(A^T, C^T, B^T)$ is given by

$$\dot{w} = \hat{A}w + N_1(p)y$$

$$u = \hat{C}w + N_2(p)y$$

Since $\hat{S}$ is a left inverse it follows that

$$\left(\hat{C}(sI-\hat{A})^{-1}N_1(s) + N_2(s)\right)\left(B^T(sI-A^T)^{-1}C^T\right) = I$$

Taking transposes we obtain

$$\left(C(sI-A)^{-1}B\right)\left(N_1(s)^T(sI-\hat{A}^T)^{-1}\hat{C}^T + N_2(s)^T\right) = I$$

It thus follows that

$$\dot{w} = \hat{A}^T w + \hat{C}^T y_0$$

$$u_0 = N_1(p)^T w + N_2(p)^T y_0$$

is a right inverse to the system $S(A,B,C)$. For zero initial state, a right inverse for $S(A,B,C)$ can be constructed from a left inverse of the system $S(A^T, C^T, B^T)$ and vice versa. For notational convenience the system $S(A^T, C^T, B^T)$ will be denoted the dual system in the sequel. Note, however, that in order to have duality in the strict mathematical sense, a minus sign should be inserted before A^T.

Geometric interpretation.

Before dealing with the inversion problem in more detail, let us first give an interpretation of the problem in terms of the geometric concepts discussed in Section 3.2. Consider the system S(A,B,C)

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

$$y = Cx$$

and its left inverse

$$\dot{w} = \hat{A}w + N_1(p)y \qquad w(t_0) = w_0$$

$$u = \hat{C}w + N_2(p)y$$

Assume there exists an input $u = u_0$ and a subspace $\mathcal{V}$ such that the output y is kept zero for $x_0 \in \mathcal{V}$. Since $y(t) \equiv 0$ and the left inverse is supposed to reconstruct the input, we have

$$\dot{w} = \hat{A}w \qquad w(t_0) = w_0$$

$$u_0 = \hat{C}w$$

i.e.

$$u_0(t) = \hat{C}e^{\hat{A}(t-t_0)}w_0$$

which can be written as

$$u_0(t) = \sum_{i=1}^{q} \Pi_i(t)e^{\alpha_i t}$$

where α_i, i = 1, 2, ..., q, are the eigenvalues of $\hat{A}$. We find that the input u_0 given above contains information of the dynamical properties of the inverse system. In particular we find that u_0 is bounded if all the eigenvalues α_i have negative real parts, i.e. if the left inverse is stable. The problem of finding a left inverse to a given system is thus closely related to the problem of finding an input to the system such that the output is identically zero over an interval of time. Problems of this type were discussed in Example 3.1 and is illustrated in Fig. 3.3.

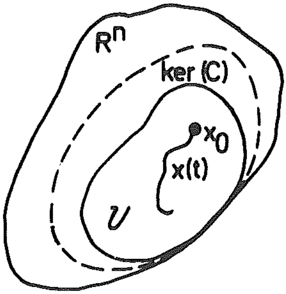

Fig. 3.3 - A geometric illustration of the problem to keep the output zero.

The maximal (A,B)-invariant subspace $\mathcal{V}^M$ contained in ker(C) will be of particular importance in this context.

3.3. Minimal Left Inverses.

We will first consider systems with arbitrary unknown initial state. Left inversion is in this case possible only under rather restrictive conditions, and the results are therefore of fairly limited practical interest. Naturally, the class of invertible systems is much larger when the initial state is known to be zero. The results obtained in the case of unknown initial state are, however, needed when the more interesting case of zero initial state is considered. A fairly detailed analysis of the minimal left inverse, particularly its stability properties, is performed in this case.

It is assumed that

i) the system S(A,B,C) is completely observable

ii) ker(B) = 0

The first assumption is introduced merely for convenience. If it is not satisfied, the results below are applied to the observable subsystem. The second assumption is a necessary condition for the system to be left invertible.

Systems with unknown initial states.

Consider the system S(A,B,C) and assume that the initial state is arbitrary and unknown. The left inverse shall reproduce the input irrespective of the initial state. We can express this in terms of the following conditions on the inverse $\hat{S}$, cf. Remark 1 and (3.8).

$$\hat{S}S_0 = 0 \qquad \hat{S}S_1 = I \tag{3.13}$$

Introduce V^M as the maximal (A,B)-invariant subspace contained in ker(C). The following lemma will be needed in the proof of the theorem below.

Lemma 3.1. If $V^M = 0$ there are maps N_i: $R^p \to R^n$, i = 0, 1, ..., n, such that

$$\sum_{i=0}^{n} N_i C A^i = I_n$$

$$\sum_{i=k}^{n} N_i C A^{i-k} B = 0 \qquad k = 1, 2, \ldots, n \quad \blacksquare$$

A proof of the lemma is given in Appendix 3A.

Remark 3. The matrices N_i, i = 0, 1, ..., n, can be constructed in the following way. Introduce block matrices Q and R such that

$$Q = \begin{pmatrix} C \\ CA \\ \ldots \\ CA^n \end{pmatrix} \qquad R = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ CB & 0 & \ldots & 0 \\ CAB & CB & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ CA^{n-1}B & \ldots & \ldots & CB \end{pmatrix} \tag{3.14}$$

The condition $V^M = 0$ guarantees that there exists a matrix N such that NQ = I and NR = 0. Then $N = [N_0, N_1, \ldots, N_n]$. $\blacksquare$

The minimal left inverse in the case of arbitrary unknown initial states is characterized in the following theorem.

Theorem 3.1. Assume the system S(A,B,C) is completely observable. There exists a left inverse with the property (3.13) if and only if $\mathcal{V}^M = 0$. Moreover, if $\mathcal{V}^M = 0$ there is a polynomial matrix N(p) such that for all $(u,y) \in U \times Y$ and all x_0

$$
\begin{aligned}
x &= N(p)y \qquad\qquad t \in (t_0,\infty) \\
u &= \hat{B}(pI-A)N(p)y
\end{aligned}
\tag{3.15}
$$

where $p = d/dt$ and $\hat{B}$ is a left inverse of B.

Proof. First assume that there is an operator $\hat{S}$ with the property (3.13) and $\mathcal{V}^M \neq 0$. Let L_M be a map such that $(A+BL_M)\mathcal{V}^M \subset \mathcal{V}^M$ and consider the input $u_1 = L_M x$ with $x_0 \in \mathcal{V}^M$. Then

$$
\begin{aligned}
\dot{x} &= (A+BL_M)x \qquad\qquad x(t_0) = x_0 \\
y &= Cx \\
u_1 &= L_M x
\end{aligned}
\tag{3.16}
$$

Since $x_0 \in \mathcal{V}^M$ and $(A+BL_M)\mathcal{V}^M \subset \mathcal{V}^M$ it follows that $x(t) \in \mathcal{V}^M$ and thus $y(t) = 0$ for $t \geq t_0$. The input u_1 is not identically zero for all $x_0 \in \mathcal{V}^M$, since this would imply that $L_M\mathcal{V}^M = 0$ and $\ker(C) \supset \mathcal{V}^M \supset (A+BL)\mathcal{V}^M = A\mathcal{V}^M$ and the observability assumption is contradicted. The same output is, however, produced by $x_0 = 0$ and $u_2 = 0$, and it will be impossible to distinguish between the inputs u_1 and u_2 by observing the output and left invertibility in the sense of (3.13) fails.

Conversely, assume $\mathcal{V}^M = 0$. By successive differentiation of y in (3.1) we have using Lemma 3.1 and the substitution

$$x(t) = A\int^{t} x(s)ds + B\int^{t} u(s)ds + x_0$$

$$N_n y = N_n Cx = N_n CA \int x(s)ds + N_n CB \int u(s)ds + N_n Cx_0 =$$

$$= N_n CA \int x(s)ds + N_n Cx_0$$

$$N_n y^{(1)} = N_n CAx$$

$$N_n y^{(1)} + N_{n-1} y = N_n CAx + N_{n-1} Cx = (N_n CA^2 + N_{n-1} CA) \int x(s)ds +$$

$$+ (N_n CAB + N_{n-1} CB) \int u(s)ds + (N_n CA + N_{n-1} C)x_0 =$$

$$= (N_n CA^2 + N_{n-1} CA) \int x(s)ds + (N_n CA + N_{n-1} C)x_0$$

$$N_n y^{(2)} + N_{n-1} y^{(1)} = (N_n CA^2 + N_{n-1} CA)x$$

Proceeding recursively in this way we have after n steps

$$\sum_{i=1}^{n} N_i y^{(i)} = \sum_{i=1}^{n} N_i CA^i x$$

and adding $N_0 y = N_0 Cx$ to either side

$$\sum_{i=0}^{n} N_i y^{(i)} = \sum_{i=0}^{n} N_i CA^i x = x \qquad (3.17)$$

where Lemma 3.1 has been used once more. The last expression can be written in operator form as $N(p)y = x$ with $N(p) = N_0 + N_1 p + \ldots + N_n p^n$. From (3.1) we have $(pI-A)x = Bu$. Substitute $x = N(p)y$ and multiply from left by $\hat{B}$, where $\hat{B}$ is a left inverse of B. We obtain

$$\hat{B}(pI-A)N(p)y = u \tag{3.18}$$

The last relation holds for all x_0 and a left inverse in the sense of (3.13) exists. The second statement in the theorem is also proven by (3.17) and (3.18). ■

Remark 4. The inverse operator $\hat{S} = \hat{B}(pI-A)N(p)$ is obviously in the required minimal form since w in the representation (3.10) has zero dimension. The construction of the operator N(p) can be done as is outlined in Remark 3.

Remark 4'. Since the input u is permitted to be piecewise continous, x is differentiable almost everywhere. In those points where x is not differentiable, we may use the convention that $\dot{x}$ equals its left limit.

Remark 5. For controllable and observable systems with one input and output, the condition $\mathcal{V}^M = 0$ is equivalent to the condition that the transfer function has no zeros. The necessary and sufficient conditions for left invertibility in the case of unknown initial state is thus rather restrictive.

Systems with zero initial state.

For zero initial states, the input-output operators of S(A,B,C) and its controllable and observable subsystem are the same. Therefore it is no restriction to assume the system is completely controllable and observable. This property is assumed in the sequel. In this case the inverse shall satisfy $\hat{S}S_1 = I$, which can be compared with (3.13).

If $\mathcal{V}^M$ is the maximal (A,B)-invariant subspace contained in ker(C), a necessary and sufficient condition for the system to be left invertible in the case of zero initial state is given by [10]

i) $\mathcal{V}^M \cap \mathcal{B} = 0$

$$\text{ii)} \quad \ker(B) = 0 \qquad (3.19)$$

where $\mathcal{B}$ denotes the range space of B. The second condition is here satisfied by assumption.

To construct the minimal inverse it will be convenient to first make the transformation

$$S(A,B,C) \overset{(T,L)}{\rightarrow} S\left(T^{-1}(A+BL)T,\ T^{-1}B,\ CT\right) = S(\bar{A},\bar{B},\bar{C}) \qquad (3.20)$$

with suitable T and L. This transformation is achieved by a state feedback $u = Lx + u_0$ followed by a coordinate transformation $v = T^{-1}x$.

Let L_M be a map such that $(A+BL_M)\mathcal{V}^M \subset \mathcal{V}^M$. From the invertibility condition (3.19) it can be seen that the whole space can be factorized into independent subspaces as $R^n = \hat{\mathcal{X}} \oplus \mathcal{B} \oplus \mathcal{V}^M$, where $\hat{\mathcal{X}}$ is any extension space. Introduce

$$T_M = [\hat{X} \quad B \quad V_M] \qquad (3.21)$$

where $\hat{X}$, B and V_M are basis matrices for $\hat{\mathcal{X}}, \mathcal{B}$ and $\mathcal{V}^M$ respectively. Consider now the transformation (3.20) with (T_M, L_M). Since $\mathcal{V}^M$ is $(A+BL_M)$-invariant and contained in ker(C) the transformed system is of the form

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} \bar{B}_1 \\ 0 \end{pmatrix} u_0 \qquad v(t_0) = 0 \qquad (3.22a)$$

$$y = [\bar{C}_1 \quad 0] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$(3.22b)$$

$$L_M x = L_M T_M v = \bar{L}_{M1} v_1 + \bar{L}_{M2} v_2$$

where v_1 and v_2 are given by $x = [\hat{X} \quad B]v_1 + V_M v_2$. Some properties of the system (3.22) are given below.

Lemma 3.2. Consider the system (3.22). There exists a polynomial matrix N(p) such that

$$\begin{aligned} v_1 &= N(p)y \\ u_0 &= \hat{B}_1(pI-\bar{A}_{11})N(p)y \end{aligned} \tag{3.23}$$

where $\hat{B}_1$ is a left inverse of $\bar{B}_1$ and $p = d/dt$. ∎

Lemma 3.3. The pair $(\bar{L}_{M2},\bar{A}_{22})$ is completely observable. ∎

Proofs of these lemmas can be found in Appendix 3A.

Remark 6. The coefficient matrices in the operator $N(p) = N_0 + N_1 p + \dots + N_q p^q$ are constructed as in Remark 3 applied to the subsystem $S(\bar{A}_{11},\bar{B}_1,\bar{C}_1)$.

With these notations the following theorems may now be stated characterizing the minimal inverse for systems with zero initial states.

Theorem 3.2. Denote the characteristic polynomial of $\bar{A}_{22}$ by $\alpha_M(s)$. Let $\hat{S}$ be an arbitrary left inverse of S(A,B,C) with $x_0 = 0$ and let $\hat{\alpha}(s)$ be the characteristic polynomial of $\hat{A}$ in the representation (3.10). Then $\alpha_M(s)$ divides $\hat{\alpha}(s)$.

Proof. Let $x_1 \in \mathcal{V}^M$. Since the system S(A,B,C) is completely controllable, there exists an input $u_1 \in U$ such that $x(t_1) = x_1$ for some fixed point of time $t_1 > t_0$. Consider then the input

$$u(t) = \begin{cases} u_1(t) & t_0 \leq t < t_1 \\ L_M x(t) & t \geq t_1 \end{cases}$$

with L_M as in (3.22). Obviously $u \in U$. For $t \geq t_1$ the solution of S(A,B,C) becomes

$$\dot{x} = (A+BL_M)x \qquad x(t_1) = x_1$$

$$y = Cx$$

$$u = L_M x$$

Consider now the transformation $v = T_M^{-1}x$ with T_M as in (3.21). The transformed system is described by (3.22) with $u_0 = 0$ and subject to the initial condition $v(t_1)^T = [0;\ v_2(t_1)^T]$ since $x(t_1) \in \mathcal{V}^M$. Thus for $t \geq t_1$

$$u(t) = \bar{L}_{M2} e^{\bar{A}_{22}(t-t_1)} v_2(t_1); \quad y(t) \equiv 0$$

However, u is also produced as the output of any left inverse $\hat{S}$ with y as input.

Since $y(t) \equiv 0$ for $t \geq t_1$ we have from (3.10)

$$u(t) = \hat{C} e^{\hat{A}(t-t_1)} w(t_1) = \tilde{C} e^{\tilde{A}(t-t_1)} \tilde{w}(t_1)$$

where $(\tilde{C},\tilde{A})$ denotes the observable subsystem of $(\hat{C},\hat{A})$.

It is easy to show that the characteristic polynomial $\tilde{\alpha}(s)$ of $\tilde{A}$ divides $\hat{\alpha}(s)$. Since $v_2(t_1)$ is an arbitrary ν-vector where $\nu = \dim(V^M)$ ($x(t_1)$ is arbitrary in V^M) we have derived the following relation between $\bar{A}_{22}$ and $\tilde{A}$

$$\bar{L}_{M2} e^{\bar{A}_{22}(t-t_1)} = \tilde{C} e^{\tilde{A}(t-t_1)} \tilde{W} \tag{3.24}$$

for some matrix $\tilde{W}$. Introduce the observability matrices

$$Q_1 = \begin{pmatrix} \bar{L}_{M2} \\ \bar{L}_{M2}\bar{A}_{22} \\ \vdots \\ \bar{L}_{M2}\bar{A}_{22}^{n-1} \end{pmatrix} \qquad Q_2 = \begin{pmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{pmatrix}$$

where $n \geq \max(\dim(\bar{A}_{22}), \dim(\tilde{A}))$. A successive differentiation of (3.24) gives

$$Q_1 e^{\bar{A}_{22}(t-t_1)} = Q_2 e^{\tilde{A}(t-t_1)} \tilde{W} \tag{3.25}$$

According to Lemma 3.3, the pair $(\bar{L}_{M2}, \bar{A}_{22})$ is completely observable, i.e. $\text{rank}(Q_1) = \nu$. Setting $t = t_1$ in (3.25) we have $Q_1 = Q_2\tilde{W}$ and it follows that $\text{rank}(\tilde{W}) = \nu$. Since the pair $(\tilde{C},\tilde{A})$ is completely observable Q_2 has a left inverse $\hat{Q}_2$ and $\hat{Q}_2 Q_1 = \tilde{W}$. Another differentiation of (3.25) gives with $t = t_1$

$$Q_1\bar{A}_{22} = Q_2\tilde{A}\tilde{W}$$

Multiply from left by $\hat{Q}_2$

$$\tilde{W}\bar{A}_{22} = \tilde{A}\tilde{W}$$

From the last expression we conclude that $\mathcal{W} = \{\tilde{W}\}$ is $\tilde{A}$-invariant and $\bar{A}_{22} = \tilde{A}|\mathcal{W}$. Thus $\alpha_M(s)$ divides $\tilde{\alpha}(s)$. Since $\tilde{\alpha}(s)$ divides $\hat{\alpha}(s)$, the theorem follows trivially. ■

<u>Remark 7</u>. The theorem above gives a lower bound on the dynamical order of any inverse $\hat{S}$ of the form (3.10). This bound equals $\deg\left(\alpha_M(s)\right) = \dim(\mathcal{V}^M)$.

<u>Theorem 3.3</u>. Assume the system S(A,B,C) is left invertible. With notations as above a minimal inverse of dynamical order $\nu = \dim(\mathcal{V}^M)$ is given by

$$\begin{aligned} \dot{w} &= \bar{A}_{22}w + N_1(p)y \qquad w(t_0) = 0 \\ u &= \bar{L}_{M2}w + N_2(p)y \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} N_1(p) &= \bar{A}_{21}N(p) \\ N_2(p) &= \left(\bar{L}_{M1} + \hat{B}_1(pI-\bar{A}_{11})\right)N(p) \end{aligned} \tag{3.27}$$

and N(p) is given by Lemma 3.2 and $\hat{B}_1$ is a left inverse of $\bar{B}_1$.

<u>Proof</u>. Notice first that it follows from Theorem 3.2 that the dynamical order ν_0 of any inverse must satisfy $\nu_0 \geq \dim(\mathcal{V}^M)$. Let $u \in U$ be an arbitrary input and define u_0 by $u_0 = u - L_Mx$. Make the transformation (3.20) with (T_M,L_M). From (3.22)

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} \bar{B}_1 \\ 0 \end{pmatrix} u_0 \qquad v(t_0) = 0 \tag{3.28a}$$

$$y = \bar{C}_1 v_1$$

$$u = \bar{L}_{M1} v_1 + \bar{L}_{M2} v_2 + u_0 \qquad (3.28b)$$

The input-output operator for this system equals the input-output operator for the system $S(\bar{A}_{11}, \bar{B}_1, \bar{C}_1)$ by neglecting the unobservable part. From Lemma 3.2

$$N(p)y = v_1$$

$$\hat{B}_1(pI-\bar{A}_{11})N(p)y = u_0$$

where $N(p)$ is a polynomial matrix and $p = d/dt$. Using (3.28)

$$u = \bar{L}_{M1} v_1 + \bar{L}_{M2} v_2 + u_0$$

$$= \bar{L}_{M2} v_2 + \left(\bar{L}_{M1} + \hat{B}_1(pI-\bar{A}_{11})\right)N(p)y \qquad (3.29)$$

where v_2 satisfies

$$\dot{v}_2 = \bar{A}_{22} v_2 + \bar{A}_{21} v_1 = \bar{A}_{22} v_2 + \bar{A}_{21} N(p)y \qquad (3.30)$$

$$v_2(t_0) = 0$$

Then (3.29) and (3.30) obviously constitutes a left inverse for $S(A,B,C)$. ■

The characteristic polynomial of the minimal left inverse satisfies a uniqueness condition.

Corollary 3.1. The characteristic polynomial of a minimal left inverse is unique and divides the characteristic polynomial of any other left inverse.

Proof. Follows directly from Theorem 3.2 and Theorem 3.3 and the uniqueness of the subspace $\mathcal{V}^M$. ■

Remark 8. The minimal left inverse is constructed by a sequence of operations on the original system S(A,B,C):

- Apply feedback of the form $u = L_M x + u_0$.

- Change coordinates in the state space $v = T_M^{-1} x$ where T_M is given by (3.21). These two steps coincide with the system transformation (3.20) with (T_M, L_M). The structure (3.22) is obtained.

- Find a polynomial operator N(p) such that $v_1 = N(p)y$ in the system (3.22).

Once these steps have been performed, a minimal left inverse is obtained almost immediately by substitution into (3.26) and (3.27).

Remark 9. Note that v in (3.28) is directly related to the state x of the original system S(A,B,C) by the transformation $x = T_M v$. This means that the state of the inverse (3.26) coincides with a part of the state of the original system.

Remark 10. From the preceding remark it also follows that the case of known initial state x_0 is handled in the same way as zero initial state. The only difference is that $v(t_0) = 0$ in (3.28) is replaced by $v(t_0) = T_M^{-1} x(t_0)$. The minimal left inverse is then given an initial value $w(t_0)$ which equals the last part of the partition $v(t_0)^T =$ $= [v_1(t_0)^T \quad v_2(t_0)^T]$.

Example 3.2. Consider a system S(A,B.C) with

$$A = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

The transfer function of the system is

$$G(s) = \begin{pmatrix} \dfrac{s^2 - s - 2}{s^3 + 2s - 1} \\ \\ \dfrac{s^2 - 1}{s^3 + 2s - 1} \end{pmatrix}$$

To construct a minimal left inverse we first find the maximal (A,B)-invariant subspace $\mathcal{V}^M$ contained in ker(C). Using the sequence (3.4) we have

$$\mathcal{V}_0 = \ker(C) = \left\{ \begin{matrix} 1 \\ -1 \\ 1 \end{matrix} \right\}$$

$$\mathcal{V}_1 = \ker(C) \cap A^{-1}(\mathcal{V}_0 + \mathcal{B}) = \left\{ \begin{matrix} 1 \\ -1 \\ 1 \end{matrix} \right\} \cap \left\{ \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{bmatrix} 1 & \vdots & 1 \\ -1 & \vdots & 0 \\ 1 & \vdots & 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{matrix} 1 \\ -1 \\ 1 \end{matrix} \right\} \cap \left\{ \begin{matrix} -1 & 0 \\ 1 & 0 \\ 3 & 1 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ -1 \\ 1 \end{matrix} \right\}$$

Since $\mathcal{V}_1 = \mathcal{V}_0$ the sequence has converged and

$$\mathcal{V}^M = \left\{ \begin{matrix} 1 \\ -1 \\ 1 \end{matrix} \right\}$$

The system is left invertible since $\mathcal{V}^M \cap \mathcal{B} = 0$ and the dynamical order of a minimal inverse becomes $\dim(\mathcal{V}^M) = 1$ according to Theorem 3.3. A map L_M such that $(A+BL_M)\mathcal{V}^M \subset \subset \mathcal{V}^M$ is found as described in Appendix 3B, i.e.

$$[V_M \quad B]^\dagger = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}^\dagger = \begin{bmatrix} 0 & -0.5 & 0.5 \\ \cdots & \cdots & \cdots \\ 1 & 0.5 & -0.5 \end{bmatrix}$$

where $(\cdot)^\dagger$ denotes the pseudo inverse [6, 16]. Then

$$L_M = -[1 \quad 0.5 \quad -0.5]A = [-0.5 \quad 2.5 \quad -1]$$

The matrix T_M in (3.21) is chosen as

$$T_M = [\hat{X} \quad B \quad V_M] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Next the system transformation (3.20) with (T_M, L_M) is performed

$$T_M^{-1}(A+BL_M)T_M = \left(\begin{array}{cc:c} 1 & 1 & 0 \\ -0.5 & -0.5 & 0 \\ \hdashline 1 & 0 & -1 \end{array}\right) \qquad T_M^{-1}B = \begin{pmatrix} 0 \\ 1 \\ \cdots \\ 0 \end{pmatrix}$$

$$CT_M = \left(\begin{array}{cc:c} -1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \qquad L_M T_M = [2.5 \quad -0.5 \;\vdots\; -4]$$

The blocks $\bar{A}_{11}$, $\bar{A}_{21}$, $\bar{A}_{22}$, $\bar{B}_1$, $\bar{C}_1$, $\bar{L}_{M1}$ and $\bar{L}_{M2}$ can now be identified by comparison with (3.22). The operator N(p) in Theorem 3.3 is constructed according to Remark 6, i.e.

$$Q = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_1\bar{A}_{11} \\ \bar{C}_1\bar{A}_{11}^2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ \cdots & \cdots \\ -1.5 & -1.5 \\ -0.5 & -0.5 \\ \cdots & \cdots \\ -0.75 & -0.75 \\ -0.25 & -0.25 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 0 \\ \bar{C}_1\bar{B}_1 & 0 & 0 \\ \bar{C}_1\bar{A}_{11}\bar{B}_1 & \bar{C}_1\bar{B}_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -0.5 & 1 & 0 \end{bmatrix}$$

A matrix N such that NQ = I and NR = 0 is given by

$$N = \begin{bmatrix} \overbrace{\begin{matrix} -1 & 1 \\ 0 & 1 \end{matrix}}^{N_0} & \vdots & \overbrace{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}}^{N_1} & \vdots & \overbrace{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}}^{N_2} \end{bmatrix}$$

The operator N(p) thus becomes

$$N(p) = \sum N_i p^i = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

All the steps are now performed, and a substitution into (3.26) gives a minimal left inverse of the system:

$$\dot{w} = -1w + [-1 \quad 1]y$$

$$u = -4w + [-3 \quad p+3]y$$

where p = d/dt. The transfer function of the minimal inverse is

$$\hat{G}(s) = \left(\frac{1 - 3s}{1 + s} \quad \frac{s^2 + 4s - 1}{1 + s} \right)$$

It is easily verified that $\hat{G}(s)G(s) = 1$. We observe that the computational steps above include only operations of linear algebra, which can be performed by standard computer programs. ■

3.4. Minimal Right Inverses.

The duality between the left and right inversion problems discussed in Section 3.2 implies that the results for left inverses in Section 3.3 are applicable to right inverses via duality. A minimal right inverse for $S(A,B,C)$ is thus directly obtained from a minimal left inverse for $S(A^T,C^T,B^T)$.

However, to be able to identify the set of operations which lead up to a minimal right inverse, we proceed through the steps of the preceding section applied to the system $S(A^T,C^T,B^T)$, but the intermediate results are interpreted as operations on the original system $S(A,B,C)$. It is assumed that the system $S(A,B,C)$ is completely controllable and observable and $\ker(C^T) = 0$.

Systems with Zero Initial State.

Consider the system S(A,B,C) with zero initial state:

$$\dot{x} = Ax + Bu \qquad x(t_0) = 0$$

$$y = Cx$$

The intention is to find an operator $\hat{S}$ with representation (3.12) such that if the input $u = \hat{S}y_r$ is applied to the system, the output equals the specified function $y_r \in Y_r$. In addition $\hat{S}$ shall be a minimal inverse according to Definition 3.4.

Let us first establish a necessary and sufficient condition for the existence of a right inverse $\hat{S}$. Using the fact that S(A,B,C) is right invertible if and only if $S(A^T,C^T,B^T)$ is left invertible, the condition (3.19) can be applied. Let V_*^M be the maximal (A^T,C^T)- invariant subspace contained in $\ker(B^T)$ and let L_M^* be a map such that $(A+L_M^*C)^T V_*^M \subset V_*^M$. The necessary and sufficient conditions for right invertibility in the case of zero initial state are:

$$\text{i)}\quad V_*^M \cap \{C^T\} = 0$$
$$\text{ii)}\quad \ker(C^T) = 0 \qquad (3.31)$$

These conditions imply that the whole space can be factorized into independent subspaces $\hat{X}$, $\{C^T\}$ and V_*^M as

$$R^n = \hat{X} \oplus \{C^T\} \oplus V_*^M \qquad (3.32)$$

where $\hat{X}$ is any extension space. Introduce $\hat{X}^T$, C^T and $(V_M^*)^T$ as basis matrices for $\hat{X}$, $\{C^T\}$ and V_*^M respectively. From the independence in (3.32) it follows that the matrix

$$T_M^* = \begin{pmatrix} \hat{X} \\ C \\ V_M^* \end{pmatrix} \tag{3.33}$$

is nonsingular. Observe that T_M^* and the factorization of R^n correspond to (3.21) in the case of left inverses.

A right inverse of minimal dynamical order can be constructed as follows. Rewrite the system (3.1) in the following way:

$$\begin{aligned} \dot{x} &= Ax + Bu + L_M^*(Cx-y) \\ &= (A+L_M^*C)x + Bu - L_M^*y \\ y &= Cx \end{aligned} \tag{3.34}$$

A change of coordinates in the state space $v = T_M^*x$ gives

$$\begin{aligned} \dot{v} &= T_M^*(A+L_M^*C)T_M^{*-1}v + T_M^*Bu - T_M^*L_M^*y \\ y &= CT_M^{*-1}v \end{aligned} \tag{3.35}$$

The transformed system will have a convenient block structure, which can be seen directly from duality by considering the dual of (3.35) and the block structure in (3.22). Thus

$$\begin{aligned} \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} &= \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} \bar{B}_1 \\ 0 \end{pmatrix} u - \begin{pmatrix} \bar{L}_{M1}^* \\ \bar{L}_{M2}^* \end{pmatrix} y \\ y &= (\bar{C}_1 \quad 0) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{aligned} \tag{3.36}$$

where the subvectors v_1 and v_2 are related to the state x as

$$v_1 = \begin{pmatrix} \hat{X} \\ \cdots \\ C \end{pmatrix} x$$

$$v_2 = V_M^* x$$

The system (3.36) can be written

$$\dot{v}_1 = \bar{A}_{11}v_1 + \bar{A}_{12}v_2 + \bar{B}_1 u - \bar{L}_{M1}^* y \tag{3.37a}$$

$$\dot{v}_2 = \bar{A}_{22}v_2 - \bar{L}_{M2}^* y \tag{3.37b}$$

$$y = \bar{C}_1 v_1 \tag{3.37c}$$

The following lemmas follow by duality from Lemma 3.2 and Lemma 3.3.

Lemma 3.4. The pair $(\bar{A}_{22}, \bar{L}_{M2}^*)$ is controllable. ■

Lemma 3.5. There exists a polynomial matrix N(p) such that the systems

$$\dot{v}_1 = \bar{A}_{11}v_1 + \bar{B}_1 N(p) u_1 \qquad p = \frac{d}{dt}$$

$$y = \bar{C}_1 v_1$$

and

$$\dot{w}_1 = \bar{A}_{11}w_1 + u_1$$

$$y = \bar{C}_1 w_1$$

are input-output equivalent for zero initial state. ■

Remark 11. To construct the operator

$$N(p) = \sum_i N_i^* p^i$$

apply Remark 3 to the system $S(\bar{A}_{11}^T, \bar{C}_1^T, \bar{B}_1^T)$. The coefficient matrices are given by $N_i^* = N_i^T$, cf. Remark 6. ■

Let

$$u = N(p)(-\bar{A}_{12}v_2 + \bar{L}_{M1}^* y + u_0) \quad (3.38)$$

in (3.37) where N(p) is given by Lemma 3.5. According to Lemma 3.5 the system (3.37) then becomes input-output equivalent to the following system

$$\dot{w}_1 = \bar{A}_{11}w_1 + (-\bar{A}_{12}v_2 + \bar{L}_{M1}^* y + u_0) + \bar{A}_{12}v_2 - \bar{L}_{M1}^* y$$

$$= \bar{A}_{11}w_1 + u_0 \quad (3.39a)$$

$$y = \bar{C}_1 w_1$$

Let $y_r \in Y_r$ be a specified function and choose

$$u_0 = (pI - \bar{A}_{11})\hat{C}_1 y_r \quad (3.39b)$$

where $\hat{C}_1$ is a right inverse of $\bar{C}_1$. By substitution into (3.39) it follows by simple calculations that $y = y_r$. Thus, (3.37b), (3.38) and (3.39b) constitute a right inverse. This inverse will also be of minimal dynamical order.

Summarizing we have:

Corollary 3.2. Assume the system S(A,B,C) is right invertible. With notations as above a minimal right inverse of dynamical order $\nu = \dim(V_*^M)$ is given by

$$\dot{w} = \bar{A}_{22}w - \bar{L}_{M2}^* y_r \qquad (3.40a)$$

$$u_r = N_1(p)w + N_2(p)y_r$$

where

$$N_1(p) = -N(p)\bar{A}_{12} \qquad (3.40b)$$

$$N_2(p) = N(p)\left(\bar{L}_{M1}^* + (pI-\bar{A}_{11})\hat{C}_1\right)$$

where N(p) is given by Lemma 3.5 and $\hat{C}_1$ is a right inverse of $\bar{C}_1$. ■

The uniqueness of the inverse spectrum, i.e. the spectrum of the matrix $\bar{A}_{22}$ in (3.40a), follows by duality from Corollary 3.1.

Corollary 3.3. The characteristic polynomial of a minimal right inverse is unique and divides the characteristic polynomial for all other right inverses. ■

Remark 12. Corollaries 3.1 and 3.3 express an important property of minimal system inverses: the uniqueness of its spectrum. We are thus guaranteed that the minimal inverse is stable if there exists any stable inverse. The minimal inverse thus provides a solution to the problem of nonunique inverses mentioned in Section 2.3.

Remark 13. The uniqueness property does not include the polynomial operators $N_1(p)$ and $N_2(p)$ in Theorem 3.3 and Corollary 3.2. It can be shown that two minimal inverses can have different input-output relations.

Remark 14. In analogy with left inversion, a minimal right inverse is constructed by a sequence of basic operations on the original system:

- Add and subtract $\bar{L}_M^* y$ from the right-hand side of (3.1a)
- Transform the system by $v = T_M^* x$ where T_M^* is given by (3.33). The structure (3.37) is obtained.
- Find an operator $N(p)$ with the property described in Lemma 3.5.

Remark 15. Note that the state of the minimal right inverse is related to the state of the original system as $w = V_M^* x$.

Example 3.3. Consider the same system as in Example 2.3, i.e. the system $S(A,B,C)$ with

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 2 \end{pmatrix} \qquad C = [1 \quad 0 \quad 0]$$

The computational details have already been demonstrated in Example 3.2 and are therefore not repeated here. The maximal (A^T, C^T) invariant subspace $\mathcal{V}_*^M$ contained in $\ker(B^T)$ is given by

$$\mathcal{V}_*^M = 0$$

Thus $\mathcal{V}_*^M \cap \{C^T\} = 0$ and the system is right invertible. Moreover, a minimal right inverse has zero dynamical order according to Corollary 3.2. Since $\mathcal{V}_*^M = 0$, the transformation $v = T_M^* x$ needs not to be performed. The polynomial matrix $N(p)$ in Corollary 3.2 is found as described in Remark 10 with $\bar{A}_{11} = A$, $\bar{B}_1 = B$ and $\bar{C}_1 = C$. Thus

$$N(p) = \begin{pmatrix} -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} p$$

A substitution into (3.40) gives a minimal right inverse of the system:

$$u_r = N(p)(pI-A)\hat{C}y_r$$

$$= \begin{pmatrix} -p^2-4p-7 \\ p+4 \end{pmatrix} y_r$$

The minimal right inverse consists in this case of pure differentiators since the dimension of $\mathcal{V}_*^M$ is zero. ■

<u>An alternative representation of minimal right inverses.</u>

In some applications it is preferable to use other representations of the minimal right inverse than (3.40). It is shown below that the model

$$\dot{w} = \hat{A}w + \hat{B}y$$

$$u = N_1(p)w + N_2(p)y \tag{3.41}$$

which is used in Corollary 3.2, easily can be expressed in the alternative form

$$\dot{w} = \hat{A}w + \hat{B}y$$
$$u = \hat{C}w + \hat{N}(p)y \tag{3.42}$$

The latter form is more convenient to handle for instance in the basic control configuration shown in Fig. 2.8, since the pure differentiators appear in a single term.

Let $N_1(p)$ be described by

$$N_1(p) = \sum_{i=0}^{q} N_{1i}p^i$$

By successive differentiations of w in (3.41) it is easily shown that

$$N_1(p)w = \left(\sum_{i=0}^{q} N_{1i}\hat{A}^i\right)w + \sum_{i=0}^{q-1}\left(\sum_{k=i+1}^{q} N_{1k}\hat{A}^{k-i-1}\hat{B}\right)y^{(i)}$$

The intermediate calculations are tedious but straightforward. Thus the model (3.41) is replaced by the model (3.42) if

$$\hat{C} = \sum_{i=0}^{q} N_{1i}A^i \tag{3.43}$$

$$\hat{N}(p) = N_2(p) + N_1'(p)$$

where

$$N_1'(p) = \sum_{i=0}^{q-1}\left(\sum_{k=i+1}^{q} N_{1k}\hat{A}^{k-i-1}\hat{B}\right)p^i$$

3.5. Characterization of an Arbitrary Minimal Inverse.

Besides from the properties of its spectrum, cf. Corolary 3.1 and 3.3, a minimal inverse is generally not unique. There is one exception, namely systems with the same number of inputs and outputs. In this case the input-output relation of the minimal inverse is unique since the system is described by a square transfer function matrix, which by necessity must have a unique inverse. This is, however, not true for systems with different numbers of inputs and outputs. Consider for instance the simple example

$$G(s) = \begin{pmatrix} \frac{1}{s+1} \\ \frac{2}{s+2} \end{pmatrix}$$

Two left inverses of this system are given by

$$G_1^{-1}(s) = [0.5(s+1) \quad 0.25(s+2)]$$

$$G_2^{-1}(s) = [s+1 \quad 0]$$

These inverses are both minimal, but they have different input-output relations. In many cases, e.g. in the servo problem in Chapter 2, it is desirable to have a description of all minimal inverses to a given system in order to select a suitable inverse. Such a description is given below.

Let us first introduce some notations. With $\bar{A}_{11}$, $\bar{B}_1$ and

$\bar{C}_1$ defined as in (3.22), introduce the following block matrices

$$Q_\sigma = \begin{pmatrix} \bar{C}_1 \\ \bar{C}_1\bar{A}_{11} \\ \vdots \\ \bar{C}_1\bar{A}_{11}^{\sigma} \end{pmatrix} \qquad R_\sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \bar{C}_1\bar{B}_1 & 0 & \cdots & 0 \\ \bar{C}_1\bar{A}_{11}\bar{B}_1 & \bar{C}_1\bar{B}_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \bar{C}_1\bar{A}_{11}^{\sigma-1}\bar{B}_1 & \cdots & \cdots & \bar{C}_1\bar{B}_1 \end{pmatrix} \tag{3.45}$$

Moreover, to any polynomial matrix $L(p) = L_0+L_1p+\ldots+L_\ell p^\ell$ we define a corresponding matrix $\tilde{L}$ as

$$\tilde{L} = [L_0 \quad L_1 \quad \ldots \quad L_\ell] \tag{3.46}$$

Consider the system S(A,B,C) and let two arbitrary minimal left inverses to this system be described by

$$\begin{aligned} \dot{w}_1 &= \hat{A}_1w_1 + N_1(p)y \qquad w_1(t_0) = 0 \\ u &= \hat{C}_1w_1 + M_1(p)y \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \dot{w}_2 &= \hat{A}_2w_2 + N_2(p)y \qquad w_2(t_0) = 0 \\ u &= \hat{C}_2w_2 + M_2(p)y \end{aligned} \tag{3.48}$$

where $N_1(p)$, $N_2(p)$, $M_1(p)$ and $M_2(p)$ are polynomial matrices and $p = d/dt$. The following theorem gives a general relation between two minimal left inverses to the same system.

Theorem 3.4. Assume that the systems (3.47) and (3.48) are two minimal left inverses to the system $S(A,B,C)$. There exist a nonsingular matrix T and a polynomial matrix $M(p)$ such that

$$w_1 = Tw_2 + M(p)y \qquad p = d/dt \tag{3.49}$$

Moreover, by this transformation the system (3.48) becomes

$$\begin{aligned} \dot{w}_1 &= \hat{A}_1 w_1 + N_1(p)y \\ u &= \hat{C}_1 w_1 + (M_1(p) + L(p))y \end{aligned} \tag{3.50}$$

where

$$L(p) = M_2(p) - M_1(p) - \hat{C}_1 M(p) \tag{3.51}$$

and

$$\tilde{L}Q_\ell = Q \qquad \tilde{L}R_\ell = 0 \tag{3.52}$$

Conversely, any system of the form (3.50) with $\tilde{L}$ satisfying (3.52) is a minimal left inverse to the system $S(A,B,C)$.

In order to prove the theorem some constructional steps are needed. These steps are given in the form of lemmas below. Proofs of these lemmas can be found in Appendix 3B.

Lemma 3.6. There exists a nonsingular matrix T such that

$$T\hat{A}_2 T^{-1} = \hat{A}_1$$

$$\hat{C}_2 T^{-1} = \hat{C}_1 \qquad \blacksquare$$

Lemma 3.7. There exist a polynomial matrix $M(p)$ and a matrix K such that

$$(pI - \hat{A}_1)M(p) + TN_2(p) = N_1(p) + K \quad \blacksquare$$

Proof of Theorem 3.4. The proof will be performed for the corresponding discrete time case, i.e. $S(A,B,C)$ denotes

$$x(t+1) = Ax(t) + Bu(t) \qquad\qquad x(t_0) = 0$$

$$y(t) = Cx(t)$$

Cf. the arguments of Remark 17. In the discrete time case, all the time derivatives above are replaced by the shift operator q.

Consider the matrix T in Lemma 3.6. It follows immediately that by $w_3 = Tw_2$, (3.48) becomes

$$qw_3 = \hat{A}_1 w_3 + TN_2(q)y \qquad\qquad w_3(t_0) = 0$$

$$u = \hat{C}_1 w_3 + M_2(q)y$$

It is assumed in the sequel that the initial state is zero. Moreover if

$$w_4 = w_3 + M(q)y = Tw_2 + M(q)y \tag{3.53}$$

with $M(q)$ as in Lemma 3.7, we have

$$qw_4 = qw_3 + qM(q)y$$

$$= \hat{A}_1 w_3 + TN_2(q)y + qM(q)y$$

$$\doteq \hat{A}_1 w_4 + ((qI-\hat{A}_1)M(q) + TN_2(q))y$$

$$= \hat{A}_1 w_4 + (N_1(q) + K)y \qquad (3.54a)$$

$$u = \hat{C}_1 w_4 + (M_2(q) - \hat{C}_1 M(q))y \qquad (3.54b)$$

where the second last equality follows from Lemma 3.7. Subtract (3.47) from (3.54) and let $\Delta w = w_4 - w_1$

$$\begin{aligned} q\Delta w &= \hat{A}_1 \Delta w + Ky \\ \Delta u &= \hat{C}_1 \Delta w + L(q)y = 0 \end{aligned} \qquad (3.55)$$

where

$$L(q) = M_2(q) - \hat{C}_1 M(q) - M_1(q) \qquad (3.56)$$

The input y to (3.55) is the output of S(A,B,C). Redefine the input of S(A,B,C) as

$$u_0 = u - L_M x$$

with L_M as in (3.22). Moreover, change coordinates in the state space as $v = T_M^{-1} x$ with T_M given by (3.21). Then S(A,B,C) becomes as in (3.22), i.e.

$$\begin{aligned} qv_1 &= \bar{A}_{11} v_1 + \bar{B}_1 u_0 \\ qv_2 &= \bar{A}_{22} v_2 + \bar{A}_{21} v_1 \\ y &= \bar{C}_1 v_1 \end{aligned} \qquad (3.57)$$

Introduce vectors $\tilde{y}_\sigma$ and $\tilde{u}_\sigma$ as

$$\tilde{y}_\sigma(t) = \begin{pmatrix} y(t) \\ y(t+1) \\ \vdots \\ y(t+\sigma) \end{pmatrix} \qquad \tilde{u}_\sigma(t) = \begin{pmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ u_0(t+\sigma) \end{pmatrix}$$

Then from (3.57)

$$\tilde{y}_\sigma = Q_\sigma v_1 + R_\sigma \tilde{u}_{\sigma-1}$$

where Q_σ and R_σ are defined as in (3.45). Thus with $\tilde{L}$ given by (3.46)

$$L(q)y = \tilde{L}\tilde{y}_\ell = \tilde{L}O_\ell v_1 + \tilde{L}R_\ell \tilde{u}_{\ell-1} \qquad \ell = \deg(L(p)) \qquad (3.58)$$

Then from (3.55), (3.57) and (3.58)

$$\begin{pmatrix} v_1 \\ \Delta w \end{pmatrix} = \begin{pmatrix} \bar{A}_{11} & 0 \\ K\bar{C}_1 & \hat{A}_1 \end{pmatrix} \begin{pmatrix} v_1 \\ \Delta w \end{pmatrix} + \begin{pmatrix} \bar{B}_1 \\ 0 \end{pmatrix} u_0$$

$$0 = \Delta u = \begin{pmatrix} \tilde{L}O_\ell & \hat{C}_1 \end{pmatrix} \begin{pmatrix} v_1 \\ \Delta w \end{pmatrix} + \tilde{L}R_\ell \tilde{u}_{\ell-1} \qquad (3.59)$$

Assume first that $K\bar{C}_1 \neq 0$ in (3.59). Since $(\bar{A}_{11}, \bar{B}_1)$ is a controllable pair (this follows from the fact that the original system S(A,B,C) is controllable), there is a sequence $u_0(t_0)$, $u_0(t_0+1)$, ..., $u_0(t_1-1)$ followed by all zeros such that $v_1(t_1) = 0$ and $\Delta w(t_1) \neq 0$, where $t_1 \geq t_0 + n$. Since the sequence is followed by all zeros, i.e. $u_0(t) = 0$ for $t \geq t_1$, we also have $\tilde{u}_{\ell-1}(t) = 0$ for $t \geq t_1$. Thus from (3.59)

$$\Delta w(t+1) = \hat{A}_1 \Delta w \qquad \Delta w(t_1) = w_0$$

$$0 = \hat{C}_1 \Delta w$$

Since $(\hat{C}_1, \hat{A}_1)$ is an observable pair, cf. Lemma 3.2 and Lemma 3.6, it follows that $\Delta w(t_1) = 0$ which is a contradiction. Thus $K\bar{C}_1 = 0$, and since the rows of $\bar{C}_1$ are linearly independent, $K = 0$. Then from (3.55)

$$L(q)y = 0$$

and using (3.58)

$$\tilde{L}Q_\ell v_1 + \tilde{L}R_\ell \tilde{u}_{\ell-1} = 0$$

Now, $u_0(t_0)$, $u_0(t_0+1)$, ..., $u_0(t_1-1)$ can be chosen so that $v_1(t_1)$ is an arbitrary vector. Moreover, $\tilde{u}_{\ell-1}(t_1)$ can be chosen arbitrarily since it consists of the sequence $u_0(t_1)$, $u_0(t_1+1)$, ..., $u_0(t_1+\ell-1)$. We must thus have

$$\tilde{L}Q_\ell = 0 \qquad \tilde{L}R_\ell = 0 \tag{3.60}$$

Then (3.49) and (3.50) follow from (3.53) and (3.54) since $K = 0$. (3.52) follows from (3.60). This completes the first part of the proof. The converse statement follows from the fact that (3.52) implies that $L(q)y = 0$. Since (3.47) is a minimal left inverse to the system $S(A,B,C)$, then (3.50) is also a minimal inverse. ■

Remark 16. Note that the transfer function of the minimal left inverse changes if $L(p)$ in (3.50) changes.

Remark 17. Since we are only concerned with triple of linear maps, it is immaterial if the continous or the discrete time case is considered. See [12] for more arguments on this point. The main reason why the discrete time system is used here, is that the arguments of the proof become more illuminant.

The following characterization of all possible minimal left inverses to a given system follows directly from Theorem 3.4.

Corollary 3.4. Consider the specific minimal left inverse of S(A,B,C) derived in Theorem 3.3, i.e.

$$\dot{w} = \bar{A}_{22}w + N_1(p)y$$
$$u = \bar{L}_{M2}w + N_2(p)y \qquad (3.61)$$

Any other minimal left inverse to the same system is input-output equivalent to a system of the form

$$\dot{w} = \bar{A}_{22}w + N_1(p)y$$
$$u = L_{M2}w + (N_2(p) + L(p))y \qquad (3.62)$$

where L(p) is a polynomial matrix such that

$$\tilde{L}Q_\ell = 0 \qquad \tilde{L}R_\ell = 0 \qquad \ell = \deg(L(p)) \qquad (3.63)$$

Conversely, any system of the form (3.62) with L satisfying (3.63) is a minimal left inverse to the system S(A,B,C). ∎

Remark 18. Note that we may add an arbitrary polynomial K(p) with the property corresponding to (3.63) to the polynomial $N_1(p)$ in (3.62)

Remark 19. The corresponding results for right invertible systems are easily obtained using the duality. Since this extension is straightforward, it is omitted here.

3.6. A Generalization.

It is sometimes desirable to invert systems of more general types than (3.1), e.g. systems described by the following differential equation

$$\dot{x} = Ax + Bu \qquad x(t_0) = 0$$
$$y = Cx + D(p)u \qquad p = \frac{d}{dt} \tag{3.64}$$

where A, B and C are linear time invariant maps as before and $D(p) = D_0 + D_1 p + \dots + D_q p^q$ is a polynomial operator with constant coefficient.

Due to the term D(p), the transfer function G(s) of (3.64) contains elements where the degree of the numerator is greater or equal to the degree of the denominator. In order to obtain a model of the original form (3.1), we instead consider the problem of inverting a system whose transfer function is

$$G_E(s) = \frac{1}{s^{q+1}} G(s) \tag{3.65}$$

The inverse is

$$G_E^{-1}(s) = s^{q+1} G^{-1}(s) \tag{3.66}$$

which determines the inverse to our original system.

The integrators in (3.65) can be introduced in two ways in a state space representation. If they are introduced at the input side, i.e.

$$u_E = u^{(q+1)}$$

$$y_E = y$$

the following system is obtained

$$A_E = \begin{pmatrix} A & B & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & I \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix} \qquad B_E = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ I \end{pmatrix} \tag{3.67}$$

$$C_E = [C \quad D_0 \quad D_1 \quad \ldots \quad D_q]$$

On the other hand, if we introduce the integrators at the output side, i.e.

$$y_E^{(q+1)} = y$$

$$u_E = u$$

we have

$$A_E = \begin{pmatrix} A & 0 & \ldots & \ldots & 0 \\ C & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I & 0 \end{pmatrix} \qquad B_E = \begin{pmatrix} B \\ D_0 \\ \vdots \\ D_q \end{pmatrix} \tag{3.68}$$

$$C_E = [0 \ \ldots\ldots \ 0 \quad I]$$

It is easily verified that (3.67) is completely controllable and (3.68) completely observable. A minimal inverse is constructed in the usual way for the extended system $S(A_E, B_E, C_E)$. An inverse for the original system is obtained by the relation (3.66).

Conjecture. The inverse which is constructed according to rules above is a minimal inverse.

The procedure above is formal and not quite satisfactorily from a theoretical viewpoint. Minimal inverses for systems of the form (3.64) can probably be constructed similar to minimal inverses for the system (3.1). To do this it is necessary to consider a generalized version of the "zeroing-the-output" problem mentioned in Example 3.1. Since we are primarily interested in systems of the form (3.1), this problem is not pursued further in this thesis.

3.7. References.

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APPENDIX 3A - Proof of Lemmas.

Proof of Lemma 3.1.

Show first that $\mathcal{V}^M = 0$ implies that $\{Q\} \cap \{R\} = 0$ where $\{\cdot\}$ denotes the range space and R and Q are the following block matrices

$$Q = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^n \end{pmatrix} \qquad R = \begin{pmatrix} 0 & 0 & \dots\dots\dots & 0 \\ CB & 0 & \dots\dots\dots & 0 \\ CAB & CB & \dots\dots\dots & 0 \\ \dots & \dots & \dots & \dots \\ CA^{n-1}B & \dots\dots & CAB & CB \end{pmatrix} \tag{A.1}$$

Assume that $\{Q\} \cap \{R\} \neq 0$. There exist vectors x and $r^T = [r_1^T;\ r_2^T;\ \dots;\ r_n^T]$ such that $Qx = Rr$, i.e. using (A.1)

$$\begin{aligned} &Cx = 0 \\ &CAx - CBr_1 = 0 \\ &CA^2x - CABr_1 - CBr_2 = 0 \\ &\vdots \end{aligned} \tag{A.2}$$

Introduce $v_1 = x$ and $v_{i+1} = Av_i - Br_i$, $i = 1,2,\dots,n$. The sequence (A.2) then becomes $Cv_i = 0$, $i = 1,2,\dots,n+1$. Since $Av_i = v_{i+1} + Br_i$ we can write in subspace notations

$$A\{v_i\} \subset \{v_{i+1}\} + \mathcal{B};\ \{v_i\} \subset \ker(C) \tag{A.3}$$

where $\mathcal{B}$ denotes the range space of B. Now define a sequence of subspaces $\mathcal{V}_k$, $k = 0,1,2,\dots,n$, by

$$\mathcal{V}_k = \sum_{i=1}^{n-k+1} \{v_i\}$$

Using (A.3):

$$AV_k = \sum_{i=1}^{n-k+1} A\{v_i\} \subset \sum_{i=1}^{n-k+1} \{v_{i+1}\} + B \subset V_{k+1} + B \qquad (A.4)$$

$$V_k \subset \ker(C)$$

We then have a sequence of subspaces satisfying $0 \neq V_n \subset \subset V_{n-1} \ldots \subset V_0 \subset \ker(C)$. Since V_n is nonzero and ker(C) has dimension at most n-1 it follows that $V_j = V_{j+1}$ for some j. Then from (A.4)

$$AV_{j+1} \subset V_{j+1} + B;\ V_{j+1} \subset \ker(C)$$

Thus $V^M \neq 0$ and $\{Q\} \cap \{R\} = 0$ by contradiction. However, by the observability assumption, $\dim(\{Q\}) = n$ and the columns $q_1, q_2, \ldots, q_n$ of Q is a basis for {Q}. Moreover, let $w_1, w_2, \ldots, w_s$ be a basis for {R}. Since $\{Q\} \cap \{R\} = 0$ from above, the vectors $q_1, q_2, \ldots, q_n$, $w_1, w_2, \ldots, w_s$ are linearly independent. This implies that there is a map $N: R^{p(n+1)} \rightarrow R^n$ such that $Nq_i = e_i$, where e_i is the i:th unit vector, and $Nw_j = 0$. For this map we have $NQ = I_n$ and $NR = 0$. Partion $N = [N_0\ N_1\ \ldots\ N_n]$ compatibly with the blocks in Q and R. An evaluation of the matrix products $NQ = I_n$ and $NR = 0$ give the sums in the lemma. ∎

Proof of Lemma 3.2.

Let $\mathcal{V}_1^M$ be the maximal $(\bar{A}_{11},\bar{B}_1)$-invariant contained in $\ker(\bar{C}_1)$. By the maximal property of $\mathcal{V}^M$ it follows that $\mathcal{V}_1^M = 0$.

Consider then the system (3.22). Since the initial state $v(t_0) = 0$, the input-output operator of (3.22) becomes equal to the input-output operator for the subsystem $S(\bar{A}_{11},\bar{B}_1,\bar{C}_1)$ by neglecting the unobservable part, i.e.

$$\dot{v}_1 = \bar{A}_{11}v_1 + \bar{B}_1u_0$$

$$y = \bar{C}_1v_1$$

The lemma then follows directly from Theorem 3.1. ∎

Proof of Lemma 3.3.

If the pair $(\bar{L}_{M2},\bar{A}_{22})$ is not completely observable there is an $\bar{A}_{22}$-invariant subspace $\mathcal{W}$ contained in $\ker(\bar{L}_{M2})$. If W is a basis matrix for $\mathcal{W}$ this implies that $\bar{A}_{22}W = WQ$ for some matrix Q and $\bar{L}_{M2}W = 0$. Introduce $\bar{A} = T_M^{-1}(A+BL_M)T_M$ and $\bar{W}^T = [0;\ W^T]$. From the special block form of $\bar{A}$ shown in (3.22) it immediately follows that $\bar{A}\bar{W} = \bar{W}Q$. Consider then $V = T_M\bar{W}$. By some simple manipulations

$$(A+BL_M)V = AV + BL_MT_M\bar{W} = AV + B\bar{L}_{M2}W = AV$$

$$(A+BL_M)V = T_M\bar{A}\bar{W} = T_M\bar{W}Q = VQ$$

Thus $AV = VQ$ and $\mathcal{V} = \{V\}$ is A-invariant. Moreover by the form of T_M (3.21) and $\bar{W}$, $V = T_M\bar{W} = V_MW$ and thus $\mathcal{V} \subset \mathcal{V}^M$. Since $\mathcal{V}^M \subset \ker(C)$ this implies that $\mathcal{V}$ is an A-invariant contained in $\ker(C)$ and the observability assumption is contradicted. ∎

Proof of Lemma 3.4.

Follows directly from Lemma 3.2 using the duality. ■

Proof of Lemma 3.5.

This lemma follows from Lemma 3.3 by the duality. Since a direct proof is of interest in a later context, one is given here.

Apply Lemma 3.1 to the system $S(\bar{A}_{11}^T, \bar{C}_1^T, \bar{B}_1^T)$. Since $V_*^M = 0$ there are matrices N_i, $i=1,2,\ldots,n_1$, such that

$$\sum_{i=0}^{n_1} \bar{A}_{11}^i \bar{B}_1 N_i = I \qquad (A.5)$$

and

$$\sum_{i=k}^{n_1} \bar{C}_1 \bar{A}_{11}^{i-k} \bar{B}_1 N_i = 0 \qquad k = 1, 2, \ldots, n_1 \qquad (A.6)$$

where n_1 is the dimension of $\bar{A}_{11}$. Let

$$N(p) = N_0 + N_1 p + \ldots. + N_{n_1} p^{n_1}$$

Consider the system

$$\dot{v}_1 = \bar{A}_{11} v_1 + \bar{B}_1 N(p) u_1$$

$$y = \bar{C}_1 v_1$$

Apply a sequence of transformations. Consider first

$$q_1 = v_1 - \bar{B}_1 N_{n_1} u_1^{(n_1-1)}$$

Then

$$\dot{q}_1 = \bar{A}_{11} q_1 + (\bar{B}_1 N_0 + \bar{B}_1 N_1 p + \dots + \bar{B}_1 N_{n_1-2} p^{n_1-2} +$$

$$+ (\bar{B}_1 N_{n_1-1} + \bar{A}_{11} \bar{B}_1 N_{n_1}) p^{n_1-1}) u_1$$

$$y = \bar{C}_1 q_1 + \bar{C}_1 \bar{B}_1 N_{n_1} u_1^{(n_1-1)}$$

$$= \bar{C}_1 q_1$$

where the last equality follows from (A.6). Next set

$$q_2 = q_1 - (\bar{B}_1 N_{n_1-1} + \bar{A}_{11} \bar{B}_1 N_{n_1}) u_1^{(n_1-2)}$$

In the same way as above we have

$$\dot{q}_2 = \bar{A}_{11} q_2 + (\bar{B}_1 N_0 + \bar{B}_1 N_1 p + \dots + \bar{B}_1 N_{n_1-3} p^{n_1-3} +$$

$$+ (\bar{B}_1 N_{n_1-2} + \bar{A}_{11} \bar{B}_1 N_{n_1-1} + \bar{A}_{11}^2 \bar{B}_1 N_{n_1-1}) p^{n_1-2}) u_1$$

$$y = \bar{C}_1 q_2$$

After n_1 steps we have

$$\dot{q}_{n_1} = \bar{A}_{11} q_{n_1} + (\bar{B}_1 N_0 + \bar{A}_{11} \bar{B}_1 N_1 + \dots + \bar{A}_{11}^{n_1} \bar{B}_1 N_{n_1}) u_1$$

$$= \bar{A}_{11} q_{n_1} + u_1$$

$$y = \bar{C}_1 q_{n_1}$$

where the second last equality follows from (A.5). Moreover, since $q_{n_1} = w_1$

$$w_1 = v_1 - M(p) u_1 \tag{A.7}$$

where the coefficient matrices in M(p) are

$$M_{n_1-1} = \bar{B}_1 N_{n_1} \tag{A.8}$$

Proof of Lemma 3.6.

For convenience set $A = \hat{A}_1$ and $C = \hat{C}_1$. Using the same approach as in the proof of Theorem 3.2, cf. (3.24), we have

$$\bar{L}_{M2} e^{\bar{A}_{22}(t-t_1)} = \hat{C} e^{\hat{A}(t-t_1)} W \tag{A.9}$$

Let ν be the order of a minimal left inverse and introduce the following block matrices

$$Q_1 = \begin{pmatrix} \bar{L}_{M2} \\ \bar{L}_{M2}\bar{A}_{22} \\ \vdots \\ \bar{L}_{M2}\bar{A}_{22}^{\nu-1} \end{pmatrix} \qquad Q_2 = \begin{pmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \vdots \\ \hat{C}\hat{A}^{\nu-1} \end{pmatrix}$$

By successive differentiation of (A.9) we have

$$Q_1 e^{\bar{A}_{22}(t-t_1)} = Q_2 e^{\hat{A}(t-t_1)} W$$

Since minimal left inverses are observable, cf. Lemma 3.2, Q_1 has a left inverse $\hat{Q}_1$. Hence

$$e^{\bar{A}_{22}(t-t_1)} = \hat{Q}_1 Q_2 e^{\hat{A}(t-t_1)} W \tag{A.10}$$

Setting $t = t_1$, we thus have $I = \hat{Q}_1 Q_2 W$. Let $T = \hat{Q}_1 Q_2$ and $T^{-1} = W$. By another differentiation of (A.10), it follows that

$$\bar{A}_{22} = T\hat{A}T^{-1} \tag{A.11}$$

if $t = t_1$. Finally from (A.9) with $t = t_1$

$$\bar{L}_{M2} = \hat{C}T^{-1} \tag{A.12}$$

There are thus two nonsingular matrices T_1 and T_2 such that (A.11) and (A.12) are satisfied for (C_1, A_1) and (C_2, A_2) respectively. From this fact the lemma follows directly. ■

Proof of Lemma 3.7.

Set

$$R(p) = TN_2(p) - N_1(p)$$

we shall thus choose M(p) such that

$$(pI - \hat{A}_1)M(p) + R(p) = K$$

where K is a constant matrix. Let $\ell = \deg(R(p))$ and set

$$M(p) = M_0 + M_1 p + \dots\dots + M_{\ell-1}p^{\ell-1}$$

$$R(p) = R_0 + R_1 p + \dots\dots + R_\ell p^\ell$$

Expanding the lefthand side

$$(R_\ell + M_{\ell-1})p^\ell + (R_{\ell-1} + M_{\ell-2} - \hat{A}_1 M_{\ell-1})p^{\ell-1} + \dots\dots\dots +$$

$$+ (R_1 + M_0 - \hat{A}_1 M_1)p + (R_0 - \hat{A}_1 M_0) = K$$

Now select

$$M_{\ell-1} = - R_\ell$$

$$M_i = \hat{A}_1 M_{i+1} - R_i \qquad i = 0,1, \dots,\ell-2$$

Then $K = R_0 - \hat{A}_1 M_0$ and the Lemmma is proven. ■

APPENDIX 3B - Algorithms.

The purpose of this Appendix is to translate some of the geometric concepts introduced in Chapter 3 into algorithms suitable for computational purposes. It is not claimed that these algorithms are the most efficient, but they are simple and straightforward to apply. Most of the algorithms hinge on a basic algorithm which is simply an orthogonalization procedure which can be performed in many ways. The effectiveness of the remaining algorithms depends on the effectiveness of this basic algorithm.

B.1. A Basic Algorithm.

Let

$$R = [r_1 \; r_2 \; \dots \; r_q]$$

be a basis matrix for a linear subspace $\mathcal{R}$ and let P_0 be an orthogonal projection onto $\mathcal{R}^{\perp}$. The purpose is to extend the basis R by some vectors from

$$V_0 = [v_1 \; v_2 \; \dots \; v_s]$$

to form a basis for the subspace spanned by the joint columns of V_0 and R, i.e. for the subspace $\mathcal{V}_0 + \mathcal{R}$. Denote this extension basis by V_1 and let n_1 be its dimension. Moreover, the orthogonal projection onto $(\mathcal{V}_0+\mathcal{R})^{\perp}$ is denoted by P_1.

Consider now an arbitrary algorithm which produces P_1, V_1, n_1 given P_0, V_0. The outcome of such an algorithm is written formally as

$$(P_1, V_1, n_1) = T(P_0, V_0) \qquad \text{(B.1)}$$

where T denotes the algorithm.

There are many possible ways to construct such an algorithm. One is described below.

Project all the vectors V onto $R^{\perp}$, i.e. calculate

$$s_i^0 = P_0 v_i \tag{B.2}$$

calculate the quantity,

$$\sigma_i = \frac{\| s_i^0 \|}{\| v_i \|}$$

If $\sigma_i < \varepsilon$, where ε is a small number, the vector v_i is deleted as a candidate to the extension basis. Let i_0 be the integer which maximizes the quantity σ_i. The first member of the extension basis is given by v_{i_0}. Calculate the projection onto $(R \oplus \{v_{i_0}\})^{\perp}$ by the updating formula

$$P^{(1)} = P_0 - \frac{s_{i_0} s_{i_0}^T}{s_{i_0}^T s_{i_0}} \tag{B.3}$$

Next, the remaining vectors are projected onto $(R \oplus \{v_{i_0}\})^{\perp}$ by

$$s_i^{(1)} = P^{(1)} s_i^0$$

The procedure is then repeated from (B.2) and gives the second vector v_{i_1} of the extension basis. The procedure is stopped when either $\sigma_i < \varepsilon$ for all i or no vectors remain to be tested.

Remark 1. Note that the vector which has the largest "angle" to the previously selected vectors is chosen in each step. To obtain an accurate projection, it has turned out to be favourable to replace $P^{(k)}$ by $(P^{(k)} \cdot P^{(k-1)})^T(\cdot)$ in each iteration.

B.2. (A,B)-Invariant Subspaces.

The feedback matrix.

Assume that $\mathcal{V}$ is an (A,B)-invariant subspace. Find a matrix L such that

$$(A+BL)\mathcal{V} \subset \mathcal{V} \tag{B.4}$$

The subspace $\mathcal{V}$ satisfies also (3.3)

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$$

i.e.

$$AV = [V \quad B]S \tag{B.5}$$

has a solution S. This solution can always be written as

$$S = [V \quad B]^{\dagger}AV \tag{B.6}$$

where $(\cdot)^{\dagger}$ denotes the pseudoinverse (6, 16). Partition

$$[V \quad B]^{\dagger} = \begin{pmatrix} Q_1 \\ \cdots \\ Q_2 \end{pmatrix} \tag{B.7}$$

compatibly with the matrices V and B. Evaluating the matrix products in (B.5) we have

$$AV = [V \quad B][V \quad B]^{+}AV$$

$$= [V \quad B]\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} AV$$

$$= VQ_1AV + BQ_2AV$$

which can be written as

$$(A-BQ_2A)V = V(Q_1AV)$$

i.e.

$$L = - Q_2A \tag{B.8}$$

$$(A+BL)|\mathcal{V} = Q_1AV \tag{B.9}$$

Maximal subspaces.

The maximal (A,B)-invariant subspace $\mathcal{V}^M$ contained in a given subspace $\mathcal{D}$ is produced by the sequence (3.4). Taking orthogonal complements we have

$$\mathcal{V}_0^{\perp} = \mathcal{D}^{\perp}$$

$$\mathcal{V}_i^{\perp} = \mathcal{V}_{i-1}^{\perp} + A^T(\mathcal{V}_{i-1}+\mathcal{B})^{\perp}$$

A translation of this sequence into an algorithm is given below using (B.1). Let D be a basis matrix for $\mathcal{D}$ and I the unit matrix.

1°. Perform $(Q_0,\cdot,n_0) = T(I,D)$ and calculate $P_0 = I - Q_0$.

2°. Perform $(S_i,\cdot,\sigma_i) = T(Q_i,B)$. If $\sigma_i = 0$ go to 4°, otherwise go to 3°.

3^o. Perform $(P_{i+1},\cdot,n_{i+1}) = T(P_i, A^T S_i)$ and calculate $Q_{i+1} = I - P_{i+1}$. If $n_{i+1} = 0$ or $i+1 = n_0$ go to 4^o, otherwise to 2^o.

4^o. Perform $(P_M, V_M, n_m) = T(I, Q_i)$. We have

V_M = basis matrix for $\mathcal{V}^M$

P_M = orthogonal projection onto $\mathcal{V}^{M\perp}$

n_m = dimension of $\mathcal{V}^M$

Remark 2. To verify that the algorithm above produces a basis matrix for the maximal (A,B)-invariant subspace, use the following facts:

- P_i is the orthogonal projection onto $\mathcal{V}_i$.
- Q_i is the orthogonal projection onto $\mathcal{V}_i^\perp$.
- S_i is the orthogonal projection onto $(\mathcal{V}_i + \mathcal{B})^\perp$.

Remark 3. Note that $x \in \mathcal{V}^M$ if and only if $P_M x = 0$. For computational purposes, use the quantity $\| P_M x \| / \| x \|$ for testing of zero.

Remark 4. In this chapter the maximal (A,B)-invariant subspace contained in ker(C) is of particular importance. A basis matrix for ker(C) is obtained by the following two steps

$(P_0,\cdot,\cdot) = T(I, C^T)$

$(\cdot, N, \cdot) = T(I, P_0)$

Alternatively, the first step above can be replaced by

1°. $(P_0, \cdot, r_0) = T(I, C^T)$

$$Q_0 = I - P_0$$

$$n_0 = n - r_0$$

4. POLES AND ZEROS.

The poles and zeros of the transfer function play an important role in classical systems theory. They constitute a small set of dynamical characteristics, which determine the behaviour of the system under control. For single input single output systems, the transfer function is a rational function and the poles and zeros are easily defined as the zeros of the numerator and denominator polynomials. Several attempts have been made to extend these concepts to multivariable systems. The most successful approach seems to be that of [10, 11, 12, 13], which use the Smith-McMillan form of the transfer function matrix. Alternative ways to define zeros for multivariable systems have also been reported [1, 6].

The lack of a proper equivalence of the concept of zeros has been regarded as a considerable drawback of state space synthesis [7]. It is obvious that the inverse system must be the key concept behind such an equivalence, but unfortunately a given system may have several (right or left) inverses, some of which are stable while others are not. Compare with a discussion in [8] on this point. Therefore, earlier inversion theory [14, 15, 16] cannot be used since the redundant dynamics is not clearly explained.

In this chapter we present a state space definition of zeros based upon the notion of minimal system inverse introduced in Chapter 3 and the uniqueness of its spectrum. It turns out that the zeros can be expressed in simple terms from a set of basic geometric concepts associated with the state space description of the system. A certain polynomial, the inverse characteristic polynomial (ch.p), is introduced, and the zeros of this polynomial are called the invariant zeros of the system. The invariant zeros equal the spectrum of the minimal system inverse for (left or right) invertible systems. However, since the inverse ch.p. is properly defined also for noninvertible systems, the definition can be used for an arbitrary linear time invariant multivariable system.

It is shown that the invariant zeros have some properties which could be conjectured from the single input-output case, namely (a) the invariant zeros are invariant under state feedback, and (b) the invariant zeros for $S(A,B,C)$ and $S(A^T,C^T,B^T)$ are the same. New results are obtained in the area of controllability and observability of multivariable systems using a state space analogy of "pole-zero cancellations" in the transfer function approach.

How do the invariant zeros change if the input and output vectors are changed? This problem may be considered as a "zero assignment" problem where the goal is to avoid a certain kind of zeros, e.g. all zeros in the right half plane. It is shown how this problem can be solved by selecting new controls and measurements in the systems. This problem has, of course, no analogy for single input single output systems.

It is felt that the poles and zeros as defined here could be a valuable aid at an initial stage of the design, for instance to select a suitable feedback structure or to choose an appropriate number of controls and measurements in a multivariable system. The results of this chapter will probably also provide an interesting link between the geometric state space theory and certain frequency domain results due to Rosenbrock [13].

4.1. Definition of the Invariant Zeros.

In Chapter 3, the spectrum of minimal system inverses was characterized in different ways for left and right invertible system. As is shown below, it is possible to give a unified description in terms of a certain invariant polynomial, the inverse ch.p. This polynomial is defined using the geometric concepts due to Wonham and Morse [9, 18], which were introduced in Section 3.2.

Preliminary discussion.

Before considering the general case let us briefly return to the single input single output case. The transfer function of S(A,B,C) becomes for $m = p = 1$

$$G(s) = C(sI-A)^{-1}B = \frac{q(s)}{p(s)} \tag{4.1}$$

where q(s) and p(s) are relatively prime polynomials with $\deg(p(s)) > \deg(q(s))$. According to classical control theory the zeros of the system are the zeros of the numerator polynomial q(s), or equivalently the spectrum of the inverse system

$$G(s)^{-1} = \frac{p(s)}{q(s)} \tag{4.2}$$

A similar construction can be made for multivariable systems using the Smith-McMillan form of G(s). See e.g. [13].

Introduce the following concepts associated with the state space description S(A,B,C):

Definition 4.1. The subspaces $\mathcal{V}^M$ and $\mathcal{R}^M$ are the (unique) maximal (A,B)-invariant and controllability subspace respectively contained in ker(C). Associated with $\mathcal{V}^M$ is the feedback class $\underline{L}^M = \{L | (A+BL)\mathcal{V}^M \subset \mathcal{V}^M\}$. ■

The corresponding concepts for the dual system $S(A^T, C^T, B^T)$ are also introduced.

Definition 4.2. The subspaces $\mathcal{V}_*^M$ and $\mathcal{R}_*^M$ are the maximal (A^T, C^T)-invariant and controllability subspace respectively contained in $\ker(B^T)$. Associated with $\mathcal{V}_*^M$ is the feedback class $\underline{L}_*^M = \{L_* | (A+L_*C)^T \mathcal{V}_*^M \subset \mathcal{V}_*^M\}$. ■

Note that the controllability subspace in Def. 4.2 is taken with respect to the pair (A^T, C^T). The existence and uniqueness of such maximal subspaces with respect to a given subspace are shown in [18]. Using the concepts above, the spectrum of the minimal system inverse is described by, cf. Corollaries 3.1 and 3.3,

the spectrum of $(A+BL)|\mathcal{V}^M$ where $L \in \underline{L}^M$ for left invertible systems, (4.3a)

the spectrum of $(A+L_*C)^T|\mathcal{V}_*^M$ where $L_* \in \underline{L}_*^M$ for right invertible systems, (4.3b)

where complete controllability and observability of the system is assumed. The spectra (4.3a) and (4.3b) are unique in the sense that they are subsets of the spectrum of an arbitrary left and right inverse respectively. The intention is to describe the spectra (4.3) by the zeros of a single polynomial. For this purpose we need some connections between dual systems.

Remark 1. The subspace notion $\mathcal{R}^M$ in Definitions 4.1 and 4.2 should not be mixed up with R^m which denotes the ordinary m-dimensional Euclidian space over the real numbers. The use of the script letter $\mathcal{R}$ to denote controllability subspaces is more or less standard, cf. [9].

Relations between dual systems.

Consider the systems $S(A,B,C)$ and $S(A^T,C^T,B^T)$ and the associated subspaces introduced in Definitions 4.1 and 4.2. The following formulas for $\mathcal{R}^M$ and $\mathcal{R}_*^M$ give an alternative to (3.6) and (3.7).

Theorem 4.1. The subspaces $\mathcal{R}^M$ and $\mathcal{R}_*^M$ are given by

$$\mathcal{R}^M = \mathcal{V}^M \cap (\mathcal{V}_*^M)^\perp$$

$$\mathcal{R}_*^M = \mathcal{V}_*^M \cap (\mathcal{V}^M)^\perp$$

The following lemma is needed in the proof of this theorem.

Lemma 4.1. Let $\mathcal{D} \subset R^n$ be an arbitrary subspace. The following identity is valid

$$\left[\mathcal{B} + A(\mathcal{D} \cap \ker(C))\right] \cap \mathcal{V}^M \equiv (\mathcal{B} + A(\mathcal{D} \cap \mathcal{V}^M)) \cap \mathcal{V}^M \quad ■$$

(Proof in Appendix 4A)

Proof of Theorem 4.1. The subspace $\mathcal{V}_*^M$ is produced by applying the following sequence, cf. (3.4)

$$\mathcal{V}_0 = R^n$$

$$\mathcal{V}_i = \ker(B^T) \cap (A^T)^{-1}(\mathcal{V}_{i-1} + \{C^T\}) \; ; \; i = 1,2,\ldots,n \qquad (4.4)$$

$$\mathcal{V}_n = \mathcal{V}_*^M$$

For notational convenience the sequence is started with $\mathcal{V}_0 = R^n$, but since $\mathcal{V}_1 = \ker(B^T)$ the sequence (4.4) is identical to (3.4). Taking orthogonal complements of (4.4):

$$\mathcal{V}_0^\perp = 0$$

$$\mathcal{V}_i^\perp = \mathcal{B} + A\left(\mathcal{V}_{i-1}^\perp \cap \ker(C)\right)$$

$$\mathcal{V}_n^\perp = (\mathcal{V}_*^M)^\perp$$

and taking intersections with $\mathcal{V}^M$:

$$\mathcal{V}_0^\perp \cap \mathcal{V}^M = 0$$

$$\begin{aligned} \mathcal{V}_i^\perp \cap \mathcal{V}^M &= \left[\mathcal{B} + A\left(\mathcal{V}_{i-1}^\perp \cap \ker(C)\right)\right] \cap \mathcal{V}^M \\ &= \left[\mathcal{B} + A\left(\mathcal{V}_{i-1}^\perp \cap \mathcal{V}^M\right)\right] \cap \mathcal{V}^M \end{aligned} \qquad (4.5)$$

$$\mathcal{V}_n^\perp \cap \mathcal{V}^M = \mathcal{V}^M \cap (\mathcal{V}_*^M)^\perp$$

where the second last equality follows from Lemma 4.1 with $\mathcal{D} = \mathcal{V}_{i-1}^\perp$. Let $\mathcal{S}_i \triangleq \mathcal{V}_i^\perp \cap \mathcal{V}^M$. Substituting $\mathcal{S}_i$ into (4.5), the sequence becomes identical to a sequence converging toward $\mathcal{R}^M$, (3.7), and thus $\mathcal{S}_n = \mathcal{R}^M = \mathcal{V}^M \cap (\mathcal{V}_*^M)^\perp$. The formula for $\mathcal{R}_*^M$ is obtained by symmetry. ∎

A second relation concerns a certain invariant polynomial. To describe this polynomial some properties of invariant subspaces are used:

Lemma 4.2. Assume $A : R^n \to R^n$ is a linear map with ch.p. $d(s)$. Let V_i, $i = 1,2,3,4$, be A-invariant subspaces and $d_i(s)$ the ch.p. of $A|V_i$.

(i) $d_1(s)$ divides $d_2(s)$ if $V_1 \subset V_2$.

(ii) $V_1^\perp$ is A^T-invariant and $d(s) = d_1(s)\hat{d}_1(s)$ with $\hat{d}_1(s)$ being the ch.p. for $A^T|V_1^\perp$.

(iii) if $V_1 = V_2 + V_3$ and $V_4 = V_2 \cap V_3$, $d_1(s) = \hat{d}_2(s) \cdot$ $\cdot\, d_3(s)$ where $\hat{d}_2(s)$ divides $d_2(s)$ and is defined by the factorization $d_2(s) = \hat{d}_2(s)d_4(s)$. ■

(Proof in Appendix 4A)

First make the observation that $\underline{L}^M$ is also a feedback class for R^M in the sense that $(A+BL)R^M \subset R^M$ for all $L \in \underline{L}^M$. Then let $L \in \underline{L}^M$ and introduce

$$d_v(s) = \text{ch.p. for } (A+BL)|V^M \tag{4.6a}$$

$$d_r(s) = \text{ch.p. for } (A+BL)|R^M \tag{4.6b}$$

and analogously for the dual system with $L_* \in \underline{L}_*^M$

$$d_v^*(s) = \text{ch.p. for } (A+L_*C)^T|V_*^M \tag{4.7a}$$

$$d_r^*(s) = \text{ch.p. for } (A+L_*C)^T|R_*^M \tag{4.7b}$$

Since $R^M \subset V^M$ by the maximality of V^M, it follows from Lemma 4.2(i) that $d_r(s)$ divides $d_v(s)$. By the same arguments, $d_r^*(s)$ divides $d_v^*(s)$, and we have for some unique polynomials $d_z(s)$ and $d_z^*(s)$.

$$d_v(s) = d_r(s)d_z(s) \tag{4.8a}$$

$$d_v^*(s) = d_r^*(s)d_z^*(s) \tag{4.8b}$$

The polynomials $d_z(s)$ and $d_z^*(s)$ are related in the following way.

<u>Theorem 4.2</u>. The polynomial $d_z(s)$ is invariant for all $L \in \underline{L}^M$ and $d_z(s) = d_z^*(s)$.

<u>Proof</u>. Let $L \in \underline{L}^M$ and $L_* \in \underline{L}_*^M$ and introduce $A_0 = A + BL + L_*C$. Since $(A+BL)V^M \subset V^M \subset \ker(C)$ and $(A+L_*C)^T V_*^M \subset V_*^M \subset \ker(B^T)$ by construction, we have for arbitrary $x \in V^M$ and $v \in V_*^M$

$$A_0 x = (A+BL+L_*C)x = (A+BL)x \in V^M$$

$$A_0^T v = (A+BL+L_*C)^T v = (A+L_*C)^T v \in V_*^M$$

Thus

$$A_0 | V^M = (A+BL)|V^M \ ; \quad A_0^T | V_*^M = (A+L_*C)^T | V_*^M \tag{4.9}$$

and $d_v(s)$ and $d_v^*(s)$ both divide the ch.p. $d_0(s)$ of A_0. Hence for some polynomials $d(s)$ and $d^*(s)$

$$\begin{aligned} d_0(s) &= d_v(s)d(s) = d_z(s)d_r(s)d(s) \\ &= d_v^*(s)d^*(s) = d_z^*(s)d_r^*(s)d^*(s) \end{aligned} \tag{4.10}$$

where (4.8) has been used. Since $d_r^*(s)$ and $d_v^*(s)$ are ch.p.

for $A_0^T | R_*^M$ and $A_0^T | V_*^M$ respectively by (4.9), we have using Lemma 4.2(ii) in connection with (4.10)

$$d_z^*(s) d^*(s) = \text{ch.p. for } A_0 | (R_*^M)^\perp \qquad (4.11)$$

$$d^*(s) = \text{ch.p. for } A_0 | (V_*^M)^\perp \qquad (4.12)$$

From Theorem 4.1

$$(R_*^M)^\perp = V^M + (V_*^M)^\perp \; ; \quad R^M = V^M \cap (V_*^M)^\perp \qquad (4.13)$$

by taking orthogonal complements of the formula for R_*^M. Applying Lemma 4.2(iii) to (4.13) in connection with (4.8) and (4.12) with $V_1 = (R_*^M)^\perp$, $V_2 = V^M$, $V_3 = (V_*^M)^\perp$ and $V_4 = R^M$

$$d_z(s) d^*(s) = \text{ch.p. for } A_0 | (R_*^M)^\perp \qquad (4.14)$$

A comparison with (4.11) shows that $d_z(s) = d_z^*(s)$. Since this identity holds for arbitrary $L \in \underline{L}^M$ and $L_* \in \underline{L}_*^M$, the polynomial $d_z(s)$ must be invariant for all $L \in \underline{L}^M$. ■

The inverse characteristic polynomial.

It is now possible to describe the spectra (4.3a) and (4.3b), characterizing the spectrum of a minimal left and right system inverse respectively by the zeros of a single polynomial. Assume the system S(A,B,C) is completely observable and controllable.

Theorem 4.3. The spectra (4.3) coincide with the zeros of the polynomial $d_z(s)$ defined by the factorization (4.8) for (left or right) invertible systems.

Proof. For left invertible systems, the theorem is an immediate consequence of the invertibility assumption $\mathcal{V}^M \cap \mathcal{B} = 0$, cf. (3.19). From (3.6) it follows that $\mathcal{R}^M = 0$ and $d_r(s)$ defined by (4.6b) equals unity. Thus $d_v(s) = d_z(s)$ from (4.8) and the theorem is proven for left invertible systems. The dual system $S(A^T,C^T,B^T)$ is left invertible if $S(A,B,C)$ is right invertible [12]. By the same arguments as above we obtain $d_v^*(s) = d_z^*(s)$, and using Theorem 4.2, $d_v^*(s) = d_z(s)$. ■

The theorem above also holds for systems which are not completely controllable and observable, if $d_z(s)$ is replaced by the corresponding polynomial $\bar{d}_z(s)$ for the controllable and observable subsystem $S(\bar{A},\bar{B},\bar{C})$ in a canonical representation of $S(A,B,C)$. Moreover, the following relation can be shown

$$d_z(s) = \bar{d}_z(s)q(s)$$

where the zeros of $q(s)$ coincide with uncontrollable or unobservable modes. Using terminology from transform theory, $q(s)$ can be interpreted as a "common factor" which cancel out in forming the transfer function of the system.

From Theorem 4.3 and the preceding discussion, we see that the zeros of $d_z(s)$ can be introduced as the zeros of the system if it is left or right invertible. However, since the polynomial $d_z(s)$ is well defined for arbitrary systems $S(A,B,C)$, the following definition is taken to be valid in general.

Definition 4.3. Consider an arbitrary linear time invariant system S(A,B,C) and let $d_v(s)$ and $d_r(s)$ be the ch.p. of $(A+BL)|\mathcal{V}^M$ and $(A+BL)|\mathcal{R}^M$ respectively where $L \in \underline{L}^M$. The inverse ch.p. $d_z(s)$ is the unique polynomial defined by the factorization $d_z(s) \triangleq d_v(s)/d_r(s)$. The zeros $Z = \{z_1, z_2, \ldots, z_q\}$ of this polynomial are introduced as the invariant zeros of the system. ■

Note that the inverse ch.p. is well defined for arbitrary linear time invariant multivariable systems, although the interpretations are restricted to (left or right) invertible systems in this thesis.

Remark 2. Algorithms for the calculation of the invariant zeros are given in Appendix 3B.

A simple interpretation.

If we consider the special case of left invertible systems we see from Theorem 4.3 that the inverse ch.p. is

$$d_z(s) = d_v(s) = \text{ch.p. for } (A+BL)|\mathcal{V}^M$$

where $\mathcal{V}^M$ is the maximal (A,B)-invariant subspace contained in ker(C) and $L \in \underline{L}^M$. Using Example 3.1 we can thus interpret $\mathcal{V}^M$ as an unobservable subspace to the system

$$\dot{x} = (A+BL)x + Bu_0$$

$$y = Cx$$

In fact, by its maximality, $\mathcal{V}^M$ is the maximal unobservable subspace that can be achieved by state feedback. Moreover, the invariant zeros are the unobservable modes associated with $\mathcal{V}^M$, i.e. to the pair (C,A+BL).

This fact leads us to a simple connection with the transfer function for single input single output systems. The transfer function of S(A,B,C) is

$$G(s) = C(sI-A)^{-1}B \triangleq \frac{q(s)}{p(s)}$$

It is also well known that q(s) is invariant under state feedback

$$G_*(s) = C(sI-A-BL)^{-1}B = \frac{q(s)}{p^*(s)}$$

where p*(s) can be arbitrarily specified by a proper choice of L. In order to achieve maximal unobservability, we choose L such that q(s) is completely cancelled out, i.e. such that $p^*(s) = q(s)p_1(s)$. For this choice of L:

$$G_*(s) = C(sI-A-BL)^{-1}B = \frac{q(s)}{q(s)p_1(s)} = \frac{1}{p_1(s)}$$

The zeros of q(s) are then unobservable modes to the pair (C,A+BL), i.e. q(s) equals the inverse ch.p. according to the discussion above.

Example 4.1. Consider a system S(A,B,C) with

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The transfer function of this system is

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

and has been used by Rosenbrock [10] to illustrate non-minimum phase behaviour of multivariable systems. For this system we have using the sequence (3.4)

$$\mathcal{V}_0 = \ker(C) = \left\{ \begin{matrix} 1 \\ -1 \\ -1 \end{matrix} \right\}$$

$$\mathcal{V}_1 = \mathcal{V}_0 \cap A^{-1}(\mathcal{V}_0 + \mathcal{B}) = \left\{ \begin{matrix} 1 \\ -1 \\ -1 \end{matrix} \right\} \cap \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \right\}$$

$$= \left\{ \begin{matrix} 1 \\ -1 \\ -1 \end{matrix} \right\}$$

Since $\mathcal{V}_0 = \mathcal{V}_1$ we have

$$\mathcal{V}^M = \left\{ \begin{matrix} 1 \\ -1 \\ -1 \end{matrix} \right\}$$

Moreover, since $\mathcal{V}^M \cap \mathcal{B} = 0$, we have from (3.5)

$$\mathcal{R}^M = 0$$

It then follows from Definition 4.3

$$d_z(s) = d_v(s) = \text{ch.p. } (A+BL)|\mathcal{V}^M$$

where $L \in \underline{L}^M$. An appropriate feedback matrix L and a matrix representation for $(A+BL)|\mathcal{V}^M$ can be calculated as is shown in Appendix 3B, cf. (B.8) and (B.9). Thus

$$[V^M \quad B]^{\dagger} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ \hdashline 1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

and

$$L = -Q_2A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 3 \end{pmatrix}$$

$$(A+BL)|\mathcal{V}^M = Q_1AV_M = 1$$

Thus $d_z(s) = s + 1$ and the invariant zeros become

$$z = \{1.0\}$$

which is in agreement with [10]. Note that the calculations only include standard operations of linear algebra. ■

4.2. Invariance Properties.

Returning to the classical single input-output case for a moment, we recall that the numerator polynomial q(s) in (4.1) is invariant under state feedback. From (4.1) it is also evident that the system S(A,B,C) and its dual $S(A^T,C^T,B^T)$ have the same set of zeros by taking a formal transpose of G(s). These fundamental properties are also true in the multivariable case using Definition 4.3.

Theorem 4.4. The invariant zeros are invariant under state feedback in the sense that S(A,B,C) and S(A+BL,B,C) have the same set of zeros for all linear maps L.

Proof. This is an immediate consequence of the construction of the polynomial $d_z(s)$. If $L_0 \in \underline{L}^M$ has been used for the system S(A,B,C) to construct $d_z(s)$, take $L_1 = L_0 - L$ for the system S(A+BL,B,C). ■

Theorem 4.5. The system S(A,B,C) and its dual $S(A^T,C^T,B^T)$ have the same set of invariant zeros.

Proof. Follows directly from the definition and Theorem 4.2. ■

Remark 3. Theorem 4.4 can be interpreted in the following way: an unsatisfactory input-output behaviour due to non-minimum phase cannot be improved by applying feedback.

4.3. Controllability and Observability.

In single input single output theory it is well known that cancellations between the numerator and denominator polynomials of the transfer function G(s) correspond to unobservable and uncontrollable modes of the system S(A,B,C). A state space interpretation of the same fact is given below in terms of the invariant zeros and the eigenvalues of A.

The concepts of mode controllability and mode observability as an alternative to the ordinary controllability and observability concepts are discussed in [2, 17]. Consider a canonical decomposition of the system due to Kalman [5]. By a suitable state transformation, the system can be brought to the following form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix} \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{pmatrix}$$

$$C = \begin{bmatrix} 0 & C_1 & 0 & C_2 \end{bmatrix} \tag{4.15}$$

From this form, the following sets of modes can be identified:

- o uncontrollable modes as the eigenvalues of A_{33} and A_{44},
- o unobservable modes as the eigenvalues of A_{11} and A_{33},
- o unobservable - uncontrollable modes as the eigenvalues of A_{33}.

Note that an unobservable and uncontrollable mode λ does not need to be unobservable - uncontrollable, since λ can be an eigenvalue of both A_{11} and A_{44} without being an eigenvalue of A_{33}.

In this context we need geometric definitions of the mode concepts above. Let n denote the order of the system.

Definition 4.4. A complex number λ is an unobservable mode to the system S(A,B,C) iff $\ker(\lambda I-A) \cap \ker(C) \neq 0$. ■

Definition 4.5. A complex number λ is an uncontrollable mode to the system S(A,B,C) iff $\{\lambda I-A\} + \mathcal{B} \neq \mathbb{C}^n$. ■

Definition 4.6. A complex number λ is an uncontrollable - - unobservable mode to the system S(A,B,C) iff $\ker(\lambda I-A) \cap \ker(C) \not\subset \{\lambda I-A\} + \mathcal{B}$. ■

The validity of the definitions above is fairly easy to show. The reader is referred to Appendix 4B for a more detailed analysis. Note that Definition 4.5 implies that (A,B) being a controllable pair is equivalent to the matrix [sI-A,B] having full rank for all complex numbers s. The latter definition has been used by Rosenbrock [12] as a definition of controllability in a frequency domain context.

Introduce the following polynomial corresponding to the poles of the transfer function

$$d_\lambda(s) = \det(sI-A)$$

Below we give a series of results which show that uncontrollable and unobservable modes in the system S(A,B,C) occur as common elements between the poles and the invariant zeros, i.e. as common zeros of the polynomials $d_z(s)$ and $d_\lambda(s)$. This fact corresponds to "cancellations" in the transfer function approach.

For uncontrollable - unobservable modes according to Definition 4.6 the following can be shown.

<u>Theorem 4.6</u>. If λ is an uncontrollable - unobservable mode, then λ is a zero of both $d_\lambda(s)$ and $d_z(s)$.

In the proof of this theorem the following lemma is needed. A proof of the lemma can be found in Appendix 4A.

<u>Lemma 4.3</u>. If there is a nonzero vector $x \in \mathbb{C}^n$ such that

i) $x \in \mathcal{V}^M$

ii) $x \notin \mathcal{R}^M$

iii) $(A-\lambda I)x = 0$

then λ is a zero of the inverse ch.p. $d_z(s)$. ∎

<u>Proof of Theorem 4.6</u>. Introduce the subspaces

$$\mathcal{V}_\lambda = \ker(\lambda I-A) \cap \ker(C)$$

$$\mathcal{W}_\lambda = \{\lambda I-A\} + \mathcal{B}$$

Since λ is an uncontrollable - unobservable mode according to Definition 4.6, there exists a nonzero vector x such that

$$x \in \mathcal{V}_\lambda; \quad x \notin \mathcal{W}_\lambda; \quad (A-\lambda I)x = 0 \tag{4.16}$$

Since $\mathcal{V}_\lambda$ is A-invariant, it follows by the maximality of $\mathcal{V}^M$ that

$$\mathcal{V}_\lambda \subset \mathcal{V}^M \tag{4.17}$$

Moreover, since

$$\mathcal{W}_\lambda^\perp = \ker(\lambda^* I-A)^T \cap \ker(B^T)$$

we have by the same arguments

$$\mathcal{W}_\lambda^\perp \subset \mathcal{V}_*^M$$

i.e.

$$\mathcal{W}_\lambda \supset \{\mathcal{V}_*^M\}^\perp \supset \{\mathcal{V}_*^M\}^\perp \cap \mathcal{V}^M = \mathcal{R}^M \tag{4.18}$$

where the last equality follows from Theorem 4.1. Thus from (4.16), (4.17) and (4.18)

$$x \in \mathcal{V}^M$$

$$x \notin \mathcal{R}^M$$

$$(A-\lambda I)x = 0$$

The theorem then follows by Lemma 4.3. ■

In order to obtain the corresponding results for unobservable or uncontrollable modes, it is necessary to introduce certain invertibility conditions. It is not true in general that unobservable or uncontrollable modes occur as "common factors" between $d_\lambda(s)$ and $d_z(s)$ for noninvertible system. This will be illustrated by an example later.

For unobservable modes we have

Theorem 4.7. Assume the system S(A,B,C) is left invertible. If $\lambda \in \mathbb{C}$ is an unobservable mode then λ is a zero of both $d_z(s)$ and $d_\lambda(s)$.

Proof. By Definition 4.4 it follows that $\ker(\lambda I-A) \neq 0$. This implies there is a nonzero vector $x \in \mathbb{C}^n$ such that $(\lambda I-A)x = 0$, i.e. λ is an eigenvalue of A and thus $d_\lambda(\lambda) = 0$. To prove that $d_z(\lambda) = 0$ introduce the subspace $\mathcal{V}_\lambda$

$$\mathcal{V}_\lambda = \ker(\lambda I-A) \cap \ker(C)$$

This subspace satisfies $\ker(C) \supset \mathcal{V}_\lambda \neq 0$ since λ is an unobservable mode. Let $V_\lambda = [v_1\ v_2\ \dots\ v_q]$ be a basis matrix for $\mathcal{V}_\lambda$. Since $v_i \in \ker(\lambda I-A)$, we have $Av_i = \lambda v_i$. Thus $\mathcal{V}_\lambda$ is A-invariant and the ch.p. of $A|\mathcal{V}_\lambda$ becomes $(s-\lambda)^q$. By the maximal property of $\mathcal{V}^M$ it follows that $\mathcal{V}_\lambda \subset \mathcal{V}^M$, and we may write $\mathcal{V}^M = \mathcal{V}_\lambda \oplus \bar{\mathcal{V}}$ for some extension space $\bar{\mathcal{V}}$. Let P be a projection onto $\bar{\mathcal{V}}$ along $\mathcal{V}_\lambda$ and consider the map $L_1 = LP$ with $L \in \underline{L}^M$,

$$(A+BL_1)x = (A+BLP)x = Ax \in \mathcal{V}_\lambda \qquad \forall\ x \in \mathcal{V}_\lambda$$

which implies that $(A+BL_1)|\mathcal{V}_\lambda = A|\mathcal{V}_\lambda$, and the ch.p. of $(A+BL_1)|\mathcal{V}_\lambda$ becomes $(s-\lambda)^q$. Moreover, $L_1 \in \underline{L}^M$ since

$$(A+BL_1)\mathcal{V}^M = (A+BLP)(\mathcal{V}_\lambda \oplus \bar{\mathcal{V}}) = (A+BL)\bar{\mathcal{V}} + A\mathcal{V}_\lambda$$

$$\subset \mathcal{V}^M + \mathcal{V}_\lambda = \mathcal{V}^M$$

The subspaces $\mathcal{V}^M$ and $\mathcal{V}_\lambda$ are thus both $(A+BL_1)$-invariant.

Since $V_\lambda \subset V^M$, it follows by Lemma 4.2(i) that the ch.p. of $(A+BL_1)|V_\lambda$ divides the ch.p. of $(A+BL_1)|V^M$. By the invertibility assumption, $R^M = 0$, and

$$d_z(s) = d_v(s) = \text{ch.p. for } (A+BL_1)|V^M$$

Thus $(s-\lambda)^q$ divides $d_z(s)$, and $d_z(\lambda) = 0$. ■

The corresponding result for uncontrollable modes are now easily obtained by applying Theorem 4.7 to the system $S(A^T,C^T,B^T)$.

Corollary 4.1. Assume the system S(A,B,C) is right invertible. If $\lambda \in \mathbb{C}$ is an uncontrollable mode then λ is a zero of both $d_z(s)$ and $d_\lambda(s)$.

Proof. First note that λ is an unobservable mode to the system $S(A^T,C^T,B^T)$. Since S(A,B,C) is right invertible, $S(A^T,C^T,B^T)$ is left invertible. Applying Theorem 4.7 we conclude that λ is a zero of both $d_z^*(s)$ and $d_\lambda(s)$, where $d_z^*(s)$ is the inverse ch.p. for $S(A^T,C^T,B^T)$. Since $d_z^*(s) = d_z(s)$ according to Theorem 4.2, the corollary follows directly. ■

Unlike the single input-output case, a common pole and zero do not necessarily correspond to an uncontrollable or unobservable mode in the multivariable case. This can be seen from the following trivial example

$$G(s) = \begin{pmatrix} \frac{s+1}{(s+2)^2} & 0 \\ 0 & \frac{s+2}{(s+1)^2} \end{pmatrix}$$

However, for single input-output systems the following can be shown:

Corollary 4.2. Assume the system S(A,B,C) has a single input and output and is invertible. The system is completely controllable and observable if and only if $d_z(s)$ and $d_\lambda(s)$ have no zeros in common.

Proof. (if) Follows directly from Theorem 4.6 and Corollary 4.1, using the fact that the system is both left and right invertible.
(only if) Assume there is a complex number λ such that $d_z(\lambda) = d_\lambda(\lambda) = 0$. We intend to show that the system cannot be both controllable and observable under this assumption. Since λ is an eigenvalue of A, there is a nonzero vector x such that $(\lambda I-A)x = 0$. Moreover, since $d_z(\lambda) = 0$ and by the invertibility assumption $\mathcal{R}^M = 0$, there is a nonzero vector $v \in \mathcal{V}^M$ such that $(\lambda I-A-BL)v =$ $= 0$ with $L \in \underline{L}^M$. Assume first that $v = \alpha x$ for some scalar α. Then

$$v \in \ker(\lambda I-A) \cap \mathcal{V}^M \subset \ker(\lambda I-A) \cap \ker(C)$$

implying that the complex number λ is an unobservable mode. Assume instead that $v \neq \alpha x$. Let P be a projection onto v along x and consider $L_1 = LP$. We have

$$(\lambda I-A-BL_1)x = (\lambda I-A-BLP)x = (\lambda I-A)x = 0$$

$$(\lambda I-A-BL_1)v = (\lambda I-A-BLP)v = (\lambda I-A-BL)v = 0$$

Since $x \neq \alpha v$, this implies that $\ker(\lambda I-A-BL_1)$ and thereby also $\ker(\lambda^* I-A-BL_1)^T$ has dimension at least 2. Thus $\ker((\lambda^* I-A-BL_1)^T) \cap \ker(B^T) \neq 0$ since $\ker(B^T)$ has dimension n-1. The orthogonal complement of this condition is $\{\lambda I-A-BL_1\} + \mathcal{B} \neq \mathbb{C}^n$, and λ is an uncontrollable mode to the pair $(A+BL_1, B)$. Since the dimension of the uncontrollable subspace is unaffected by feedback, the pair (A,B) is not completely controllable. ■

Remark 4. We have used the fact that a system S(A,B,C) is completely observable and controllable if and only if it has no uncontrollable or unobservable mode. This has been proven elsewhere, cf. [2, 16].

Note that the invertibility assumptions are essential in Theorem 4.7 and its corollaries. This is illustrated by a simple example.

Example 4.2. Consider two systems $S_1(A_1, B_1, C_1)$ and $S_2(A_2, B_2, C)$ with

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad C = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

and

$$A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

The systems are described in block diagram form in Fig. 4.1.

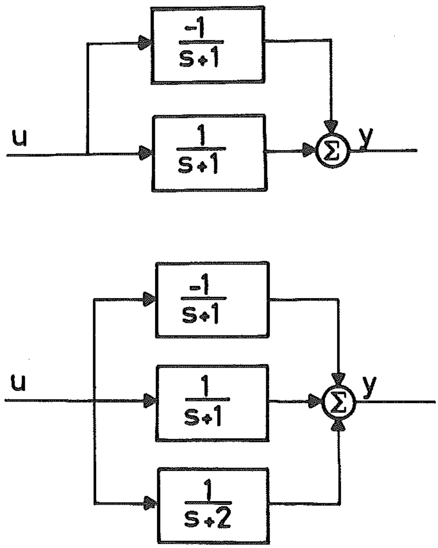

Fig. 4.1 - Block diagrams for the systems S_1 and S_2.

Their respective transfer functions are $G_1(s) = 0$ and $G_2(s) = 1/(s+2)$, i.e. S_2 is invertible while S_1 is not. It is easily verified that both systems contain an uncontrollable and unobservable mode in -1.0. This mode is, however, not uncontrollable - unobservable. Introduce a small perturbation ε of the pole in the upper block in both systems. The perturbed systems have the transfer functions:

$$G_1(s) = \frac{\varepsilon}{(s+1)(s+1+\varepsilon)} \qquad G_2(s) = \frac{\varepsilon(s+2) + (s+1)(s+1+\varepsilon)}{(s+2)(s+1)(s+1+\varepsilon)}$$

Letting $\varepsilon \to 0$, we see that cancellations will occur in $G_2(s)$ but not in $G_1(s)$. This is completely in agreement with Theorem 4.7 and its corollaries since S_2 is invertible and S_1 is not. Calculating the invariant zeros in a state space representation of S_1 and S_2 we obtain the same result.

$Z_1 = \emptyset$ (empty space) $\qquad Z_2 = \{-1.0, -1.0\}$ ■

4.4. Extension of Control and Measurement.

The invariant zeros have many properties in common with the ordinary set of zeros and the interpretations are in many cases similar. However, since the invariant zeros are a concept relating a group of inputs to a group of outputs, they also have properties that do not appear in the single input-output case. In fact, the invariant zeros are available for design purposes in the sense that adding new inputs and outputs to the system will change the inverse ch.p.

Let us first introduce a precise algebraic criterion for the somewhat loose statement "adding new inputs and outputs".

<u>Definition 4.7</u>. A system $S(A,B,C_e)$ is said to be an output extension of $S(A,B,C)$ if $\ker(C_e) \subset \ker(C)$. ■

and in the same way:

<u>Definition 4.8</u>. A system $S(A,B_e,C)$ is said to be an input extension of $S(A,B,C)$ if $\mathcal{B}_e \supset \mathcal{B}$. ■

Further interpretations of these criteria will be given later. Let it suffice to be mentioned here that Definition 4.7 corresponds to increasing the number of measure variables and Definition 4.8 to adding control variables. For notational convenience let $\mathcal{V}_e^M$, $\mathcal{R}_e^M$ and $\underline{L}_e^M$ be the concepts corresponding to Definition 4.1 for a system with an extended input or output vector.

<u>Theorem 4.8</u>. Let S(A,B,C) be a left invertible system with inverse ch.p. $d_z(s)$. The inverse ch.p. $d_z^e(s)$ of any output extension $S(A,B,C_e)$ divides $d_z(s)$.

<u>Proof</u>. Since $\ker(C_e) \subset \ker(C)$ by assumption, it follows that $\mathcal{V}_e^M \subset \mathcal{V}^M$ from the maximality of $\mathcal{V}^M$. Thus $\mathcal{V}^M = \hat{\mathcal{V}} \oplus \mathcal{V}_e^M$ for some extension space $\hat{\mathcal{V}}$. Let P_1 be a projection onto $\hat{\mathcal{V}}$ along $\mathcal{V}_e^M$ and P_2 a projection onto $\mathcal{V}_e^M$ along $\hat{\mathcal{V}}$. The map $L_1 = LP_1 + L_eP_2$, where $L \in \underline{L}^M$ and $L_e \in \underline{L}_e^M$, belongs to both the classes $\underline{L}_e^M$ and $\underline{L}^M$ since

$$(A+BL_1)\mathcal{V}_e^M \equiv (A+BLP_1+BL_eP_2)\mathcal{V}_e^M =$$

$$= (A+BL_eP_2)\mathcal{V}_e^M = (A+BL_e)\mathcal{V}_e^M \subset \mathcal{V}_e^M$$

and

$$(A+BL_1)\mathcal{V}^M \equiv (A+BLP_1+BL_eP_2)(\mathcal{V}_e^M \oplus \hat{\mathcal{V}}) =$$

$$= (A+BLP_1+BL_eP_2)\hat{\mathcal{V}} + (A+BLP_1+BL_eP_2)\mathcal{V}_e^M \subset$$

$$\subset (A+BL)\hat{\mathcal{V}} + (A+BL_e)\mathcal{V}_e^M \subset \mathcal{V}^M + \mathcal{V}_e^M = \mathcal{V}^M$$

where the properties of the projections P_1 and P_2 have been used. Since $\mathcal{V}_e^M \subset \mathcal{V}^M$, the ch.p. $d_v^e(s)$ of $(A+BL_1)|\mathcal{V}_e^M$ divides the ch.p. $d_v(s)$ of $(A+BL_1)|\mathcal{V}^M$, cf. Lemma 4.2. However, $\bar{d}_z^e(s) = \bar{d}_v^e(s)$ and $d_z(s) = d_v(s)$ by the invertibility assumption. Thus $\bar{d}_z^e(s)$ divides $d_z(s)$. ■

<u>Remark 5</u>. If the system S(A,B,C) is left (right) invertible, it follows trivially that any output (input) extension is also left (right) invertible.

The corresponding result for input extension is now easily obtained via the system $S(A^T,C^T,B^T)$.

Corollary 4.3. Let $S(A,B,C)$ be a right invertible system with inverse ch.p. $d_z(s)$. The inverse ch.p. $d_z^e(s)$ for any input extension $S(A,B_e,C)$ divides $d_z(s)$.

Proof. First notice that $\mathcal{B}_e \supset \mathcal{B}$ is equivalent to $\ker(B_e^T) \subset \ker(B^T)$ by taking orthogonal complements. Applying Theorem 4.8 to the systems $S(A^T,C^T,B_e^T)$ and $S(A^T,C^T,B^T)$ and using Theorem 4.2, the corollary follows directly. ∎

In order to interpret the results above, let us consider a specific design problem. Assume that $S(A,B,C)$ is a (left and right) invertible system with an equal number of inputs and outputs, i.e. $m = p$. Assume also that the system has an inverse ch.p. $d_z(s)$ with some zeros in the right half-plane, i.e. the system is difficult to control [10]. If some additional outputs $\tilde{y} = \tilde{C}x$ are selected, a new system $S(A,B,C_e)$ is obtained with output vector $y_e^T = [y^T \;\; \tilde{y}^T]$ and

$$y_e = C_e x = \begin{bmatrix} C \\ \tilde{C} \end{bmatrix} x$$

Since $\ker(C_e) \subset \ker(C)$, the new system is an output extension of the original one and Theorem 4.8 can be applied. The inverse ch.p. $d_z^e(s)$ for the new system is thus related to $d_z(s)$ as

$$d_z(s) = d_z^e(s)q(s)$$

for some polynomial $q(s)$. With a proper output extension the right half-plane zeros of $d_z(s)$ are contained amongst the zeros of $q(s)$, i.e. the system $S(A,B,C_e)$ is minimum phase. Observe that $y_e = x$ gives $\ker(C_e) = 0$ and thus $d_z^e(s) = 1$, implying that the right half-plane zeros can always be avoided by sufficient extension.

It is also possible to apply the reverse of the procedure above, i.e. to extend the control vector of $S(A,B,C)$ giving $S(A,B_e,C)$ with $\mathcal{B}_e \supset \mathcal{B}$. Analogously to the case of output extension, some additional inputs can be selected such that the right half-plane zeros of $d_z(s)$ are avoided. Practically, there are more severe restrictions on possible extensions of control than measurement in most applications.

In the view of the discussion above, it would be desirable to have explicit criteria on suitable input and output extensions in order to avoid a predefined set of zeros. Let $S(A,B,C)$ be a system whose inverse ch.p. is factorized into relatively prime polynomials as $d_z(s) = d_z^+(s)d_z^-(s)$. This factorization corresponds to a partition of the invariant zeros Z into two disjoint sets:

$$Z = \left\{z_1^+,\ z_2^+ \ \dots\ z_r^+;\ z_1^-,\ z_2^- \ \dots\ z_s^-\right\}$$

where $z_1^-, z_2^- \dots z_s^-$ are considered undesirable. For instance, the complex numbers z_i^- may be all the invariant zeros in the right half-plane. Criteria on suitable input and output extensions are given below.

<u>Theorem 4.9</u>. Assume $S(A,B,C)$ is a left invertible system with inverse ch.p. $d_z(s) = d_z^+(s)d_z^-(s)$ and with $d_z^+(s)$ and $d_z^-(s)$ being relatively prime. The inverse ch.p. $d_z^e(s)$ of an output extension $S(A,B,C_e)$ divides $d_z^+(s)$ if and only if for any $L \in \underline{L}^M$

$\ker(z_i^- I-A-BL) \cap \ker(C_e) = 0$

for all disjoint zeros z_i^- of $d_z^-(s)$.

<u>Proof</u>. First note that $d_z^e(s)$ divides $d_z(s) = d_z^+(s)d_z^-(s)$ by Theorem 4.8.

(if) We intend to show that $d_z^e(s)$ in fact divides $d_z^+(s)$, i.e. $d_z^e(z_i^-) \neq 0$ for all zeros z_i^- of $d_z^-(s)$. Assume then that $d_z^e(z_i^-) = 0$ for some zero z_i^-. Since both the systems are left invertible we have

$$d_z(s) = \text{ch.p. for } (A+BL)|V^M \text{ where } L \in \underline{L}^M$$

$$d_z^e(s) = \text{ch.p. for } (A+BL_e)|V_e^M \text{ where } L_e \in \underline{L}_e^M$$

If $d_z^e(z_i^-) = 0$, there is a nonzero vector $x_i \in V_e^M$ such that $(A+BL_e)x_i = x_i z_i^-$. By the maximal property of V^M, $V_e^M \subset V^M$, and thus $x_i \in V^M$. Write $V^M = V_i \oplus \{x_i\}$ for some extension space V_i. Let P_i be a projection onto $\{x_i\}$ along V_i and $\bar{P}_i$ a projection onto V_i along $\{x_i\}$. The map $L_1 = L_e P_i + L\bar{P}_i$ belongs to $\underline{L}^M$ since

$$(A+BL_1)V^M \equiv (A+BL_eP_i+BL\bar{P}_i)(V_i \oplus \{x_i\}) =$$

$$= (A+BL)V_i + (A+BL_e)\{x_i\} \subset$$

$$\subset V^M + \{x_i\} \subset V^M$$

Moreover,

$$(A+BL_1)x_i = (A+BL_eP_i+BL\bar{P}_i)x_i = (A+BL_e)x_i = x_i z_i^- \qquad (4.19)$$

Consider now an arbitrary $L \in \underline{L}^M$. Since the system is left invertible, $\mathcal{V}^M \cap \mathcal{B} = 0$. Let P be a projection onto $\mathcal{B}$ along $\mathcal{V}^M$. Using the fact that $(A+BL)x_i \in \mathcal{V}^M$ we have

$$0 = P(A+BL)x_i = PAx_i + BLx_i$$

$$0 = P(A+BL_1)x_i = PAx_i + BL_1x_i$$

A subtraction gives $BLx_i = BL_1x_i$, which implies that $Lx_i = L_1x_i$ since $\ker(B) = 0$ by our initial assumption. Thus from (4.19)

$$(A+BL)x_i = (A+BL_1)x_i = \bar{z}_ix_i$$

Implying that for any $L \in \underline{L}^M$

$$0 \neq x_i \in \ker(\bar{z}_iI-A-BL) \cap \mathcal{V}_e^M \subset \ker(\bar{z}_iI-A-BL) \cap \ker(C_e)$$

which is a contradiction. Thus $d_z^e(s)$ divides $d_z^+(s)$.

(only if) Conversely, assume there exists a nonzero vector v_i such that $v_i \in \ker(\bar{z}_iI-A-BL) \cap \ker(C_e)$. Then $v_i \in \mathcal{V}_e^M$ since $\mathcal{V}_e^M$ is maximal with respect to $\ker(C_e)$. Factorize $\mathcal{V}_e^M$ as $\mathcal{V}_e^M = \{v_i\} \oplus \bar{\mathcal{V}}_i$ and let P_i be a projection onto $\{v_i\}$ along $\bar{\mathcal{V}}_i$ and $\bar{P}_i$ a projection onto $\bar{\mathcal{V}}_i$ along $\{v_i\}$. The map $L_1 = LP_i + L_e\bar{P}_i$, where $L_e \in \underline{L}_e^M$, belongs to $\underline{L}_e^M$ since

$$(A+BL_1)\mathcal{V}_e^M \equiv (A+BLP_i+BL_e\bar{P}_i)(\{v_i\}\oplus\bar{\mathcal{V}}_i)$$

$$= (A+BL)\{v_i\} + (A+BL_e)\bar{\mathcal{V}}_i$$

$$\subset \{v_i\} + \mathcal{V}_e^M = \mathcal{V}_e^M$$

Moreover,

$$(A+BL_1)v_i = (A+BL)v_i = v_i\bar{z}_i$$

since $v_i \in \ker(z_i^- I-A-BL)$. The system $S(A,B,C_e)$ is left invertible, and thus z_i^- is a zero of $d_z^e(s)$. This implies that $d_z^e(s)$ does not divide $d_z^+(s)$ since $d_z^+(z_i^-) \neq 0$ (the polynomials $d_z^+(s)$ and $d_z^-(s)$ are assumed to be relatively prime). The theorem is thereby proven by contradiction. ■

The same result for control extension is given in the corollary below.

Corollary 4.4. Assume $S(A,B,C)$ is a right invertible system with inverse ch.p. $d_z(s) = d_z^+(s)d_z^-(s)$ and with $d_z^+(s)$ and $d_z^-(s)$ being relatively prime. The inverse ch.p. $d_z^e(s)$ of any input extension $S(A,B_e,C)$ divides $d_z^+(s)$ if and only if for any $L_* \in \underline{L}_*^M$

$$\{z_i^- I - A - L_*C\} + \mathcal{B}_e = \mathbb{C}^n \qquad (4.20)$$

for all disjoint zeros z_i^- of $d_z^-(s)$.

Proof. Observe that $\{z_i^- I - A - L_*C\} + \mathcal{B}_e = \mathbb{C}^n$ is equivalent to $\ker(z_i^- I-A^T-C^T L_*^T) \cap \ker(B_e^T) = 0$ by taking orthogonal complements. Apply Theorem 4.9 to the system $S(A^T, C^T, B^T)$ and use Theorem 4.2. ■

Remark 6. Observe that the conditions given in Theorem 4.9 and its corollary can easily be expressed in matrix terms. For instance, concerning output extension, let $L \in \underline{L}^M$ and calculate all the eigenvectors $v_1, v_2, \ldots, v_q$ of $A + BL$ corresponding to a certain invariant zero z^- (z^- is an eigenvalue of $A + BL$ according to Definition 4.3). An output extension $y_e = C_e x$ avoids z^- if and only if

$C_e v_i \neq 0 \qquad i = 1, 2, \ldots, q$

Exactly the same calculations occur in the case of input extension. ■

Theorem 4.9 is illustrated by an example.

Example 4.3. Consider a system S(A,B,C)

$$\dot{x} = \begin{pmatrix} -1 & -3 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = [1 \quad 0 \quad -1]$$

For this system

$$\mathcal{V}^M = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{Bmatrix} \qquad \mathcal{R}^M = 0$$

and an $L \in \underline{L}^M$ is given by

$$L = [1 \quad 4 \quad 1]$$

Since $\mathcal{R}^M = 0$, the system is left invertible, and in that case the invariant zeros Z equal the eigenvalues of $(A+BL)|\mathcal{V}^M$. Computationally Z can be obtained as the eigenvalues of $V_M^{\dagger}(A+BL)V_M$, where V_M is a basis matrix for $\mathcal{V}^M$ and $(\cdot)^{\dagger}$ denotes the pseudoinverse. In this case

$$(A+BL)|\mathcal{V}^M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus $d_z(s) = s^2 - 1$ and $Z = \{1,-1\}$. As a comparison, the transfer function of the system is

$$G(s) = \frac{s^2 - 1}{s^2 + s + 3s + 1}$$

The system is thus nonminimum phase. Let $d_z^+(s) = s + 1$ and $d_z^-(s) = s - 1$. The intention is to select an additional output $\tilde{y} = (c_1 \ c_2 \ c_3)x$ such that the extended system $S(A,B,C_e)$ with

$$y_e = \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = C_e x = \begin{pmatrix} 1 & 0 & -1 \\ c_1 & c_2 & c_3 \end{pmatrix} x$$

avoids the right half-plane zero $z^- = 1$. The eigenvector of $A + BL$ corresponding to the eigenvalue $z^- = 1$ is

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Using the remark above we see that the zero $z^- = 1$ is avoided if and only if $C_e v \neq 0$, i.e. if and only if

$$(c_1 \ c_2 \ c_3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq 0 \qquad \blacksquare$$

4.6. A Design Example.

The subspace notations used in this chapter can be transformed to matrix operations suitable for computations as is indicated in Appendix 3B of Chapter 3. In the examples below a computer program is utilized which calculates the invariant zeros given a state space description S(A,B,C) of the system. A specific process, a drum boiler, is analysed from an input-output view using the concepts of poles and zeros as defined above. This analysis clearly indicates that a multivariable viewpoint is needed in order to clearly understand the dynamical behaviour of this system.

A drum boiler.

Different types of models for a drum boiler are thoroughly described in [4]. Here we will use a fifth order model from [3].

The linearized equations for a boiler around a certain operating point can be written as

$$\dot{x} = Ax + Bu + Gv$$

$$y = Cx$$

where the state variables are

x_1 = drum pressure (bar)
x_2 = drum liquid level (m)
x_3 = drum liquid temperature ($^{\circ}C$)
x_4 = riser wall temperature ($^{\circ}C$)
x_5 = steam quality (%)

The control variables are

u_1 = heat flow to the risers (kJ/s)
u_2 = feedwater flow (kg/s)

and the disturbances are

v = load charges (bar)

Numerical values for A, B, C and G for a power station boiler with a maximum steam flow of about 350 t/h are calculated in [3]. The drum pressure is 140 bar and the operating point is 90% full load. From [3] we have

$$A = \begin{pmatrix} -0.129 & 0.000 & 0.396\times10^{-1} & 0.250\times10^{-1} & 0.191\times10^{-1} \\ 0.329\times10^{-2} & 0.000 & -0.779\times10^{-4} & 0.122\times10^{-3} & -0.621 \\ 0.718\times10^{-1} & 0.000 & -0.100 & 0.887\times10^{-3} & -0.385\times10^{1} \\ 0.411\times10^{-1} & 0.000 & 0.000 & -0.822\times10^{-1} & 0.000 \\ 0.361\times10^{-3} & 0.000 & 0.350\times10^{-4} & 0.426\times10^{-4} & -0.743\times10^{-1} \end{pmatrix}$$

$$B = \begin{pmatrix} 0.000 & 0.139\times10^{-2} \\ 0.000 & 0.359\times10^{-4} \\ 0.000 & -0.989\times10^{-2} \\ 0.249\times10^{-4} & 0.000 \\ 0.000 & -0.534\times10^{-5} \end{pmatrix} \qquad G = \begin{pmatrix} 0.995\times10^{-1} \\ -0.318\times10^{-2} \\ -0.232\times10^{-1} \\ 0.000 \\ -0.381\times10^{-3} \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where the output variables, drum level and drum pressure denote the main controlled variables. These are also the variables which usually are measured in the system.

In the boiler case, it is known that nonminimum phase behaviour appears between different pairs of input and output variables [4]. As we shall see, this does not mean that the process is nonminimum phase in a multivariable sense.

The open-loop poles, i.e. the eigenvalues of A, are

$$\Lambda = \{0.000,\ -0.060 \pm j \cdot 0.017,\ -0.086,\ -0.18\}$$

The invariant zeros can be calculated for different combinations of inputs and outputs. If u_1 and x_1 are considered as input and output respectively, the following set of invariant zeros are obtained

$$z_1 = \{0.000,\ -0.070,\ -0.106\}$$

Note that there is a common pole and zero at the origin, implying that $\lambda = 0.000$ is an uncontrollable or unobservable mode for this input-output pair, cf. Corollary 4.2.

If we instead consider u_1 as input and x_2 as output, the invariant zeros become

$$z_2 = \{0.022,\ -0.096,\ -0.689\}$$

As can be seen, there is a right half-plane zero implying nonminimum phase in this single loop. The invariant zeros associated with the multivariable system with (x_1, x_2) as output and (u_1, u_2) as input are, however,

$$z_3 = \{-0.065,\ -0.368\}$$

i.e. the system is minimum phase in a multivariable sense. Thus, the properties of a multivariable system need not to coincide with the properties of its single loops. A minimum phase appears between individual input and output variables.

Assume an additional measurement is selected, for instance drum liquid temperature x_3. The invariant zeros associated with the outputs (x_1, x_2, x_3) and the inputs (u_1, u_2) are

$z_4 = \emptyset$ (empty space)

i.e. the system has no zeros. Since we have made an output extension of the system, this is completely in agreement with Theorem 4.8.

The pole-zero configuration for the boiler in the case of measurement of drum level and drum pressure is shown in Fig. 4.2. In the single input-output case no trouble could be expected in controlling a system with such a pole-zero configuration. In fact, it is shown in Chapter 7 that this system can be controlled satisfactorily only using available measurement, i.e. by output feedback.

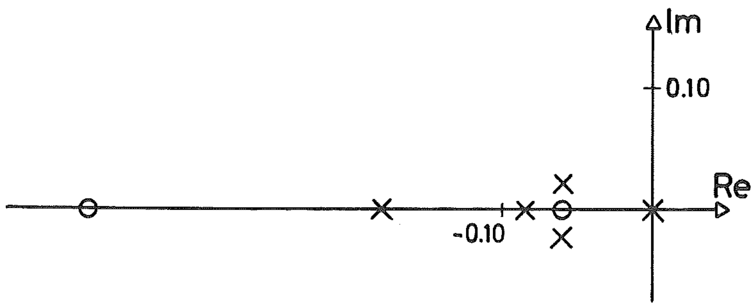

Fig. 4.2 - Pole-zero configuration for a boiler using measurement of drum level and drum pressure. x = pole, o = zero.

4.6. References.

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APPENDIX 4A - Proofs of Lemmas.

Proof of Lemma 4.1.

Introduce

$$\mathcal{W}_1 = \left[\mathcal{B} + A\left(\mathcal{D} \cap \ker(C)\right)\right] \cap \mathcal{V}^M \qquad \text{(A.1a)}$$

$$\mathcal{W}_2 = \left[\mathcal{B} + A\left(\mathcal{D} \cap \mathcal{V}^M\right)\right] \cap \mathcal{V}^M \qquad \text{(A.1b)}$$

Since $\mathcal{V}^M \subset \ker(C)$ by definition it follows that $\mathcal{D} \cap \ker(C) \supset \mathcal{D} \cap \mathcal{V}^M$. Thus $\mathcal{D} \cap \ker(C) = \mathcal{D} \cap \mathcal{V}^M \oplus \hat{\mathcal{V}}$ for some extension space $\hat{\mathcal{V}}$ and

$$\hat{\mathcal{V}} \cap \mathcal{V}^M = 0 \qquad \text{(A.2)}$$

otherwise the independence assumption in the factorization above is contradicted. The subspace $\mathcal{W}_1$ can then be written

$$\mathcal{W}_1 = \left(\mathcal{B} + A(\mathcal{D} \cap \mathcal{V}^M) + A\hat{\mathcal{V}}\right) \cap \mathcal{V}^M \qquad \text{(A.3)}$$

Comparing (A.3) and (A.1b) we conclude that $\mathcal{W}_1 \supset \mathcal{W}_2$. Assume then there is a vector $0_1 \neq z_1 \in \mathcal{W}_1$ and $z_1 \notin \mathcal{W}_2$. From (A.3)

$$z_1 = b_1 + Av_1 + A\hat{v}\ ; \qquad z_1 \in \mathcal{V}^M, \quad b_1 \in \mathcal{B},$$
$$v_1 \in \mathcal{D} \cap \mathcal{V}^M, \quad \hat{v} \in \hat{\mathcal{V}} \qquad \text{(A.4)}$$

Observe that $\hat{v} \neq 0$, otherwise $z_1 \in \mathcal{W}_2$ and the initial assumption on z_1 is contradicted. Since $v_1 \in \mathcal{V}^M$ and $A\mathcal{V}^M \subset \mathcal{V}^M + \mathcal{B}$, we have

$$Av_1 = z_2 + b_2\ ; \qquad z_2 \in \mathcal{V}^M, \quad b_2 \in \mathcal{B} \qquad \text{(A.5)}$$

A substitution of (A.5) into (A.4) gives $A\hat{v} = z_1 - z_2 - b_1 - b_2 = z + b$ where $z = -z_1 - z_2 \in \mathcal{V}^M$ and $b = - b_1 - b_2 \in \mathcal{B}$. Consider then the subspace $\bar{\mathcal{V}} = \mathcal{V}^M \oplus \{\hat{v}\}$ ($\hat{v}$ does not belong to $\mathcal{V}^M$ according to (A.2)). We obtain

$$A\bar{\mathcal{V}} = A\mathcal{V}^M + A\{\hat{v}\} \subset \mathcal{V}^M + \mathcal{B} + \mathcal{V}^M + \mathcal{B}$$

$$= \mathcal{V}^M + \mathcal{B} \subset \bar{\mathcal{V}} + \mathcal{B}$$

and $\bar{\mathcal{V}}$ is (A,B)-invariant. Moreover, $\hat{v} \in \ker(C)$ by assumption and thus $\bar{\mathcal{V}} \subset \ker(C)$. Since $\bar{\mathcal{V}} \supset \mathcal{V}^M$, the maximal property of $\mathcal{V}^M$ is contradicted and there is no vector z_1 with the property above. Thus $\mathcal{W}_1 = \mathcal{W}_2$. ■

Proof of Lemma 4.2.

The proof of (i) and (ii) are straightforward. Only (iii) will be proven here.

Write $\mathcal{V}_2 = \hat{\mathcal{V}}_2 \oplus \mathcal{V}_2 \cap \mathcal{V}_3$ and $\mathcal{V}_3 = \hat{\mathcal{V}}_3 \oplus \mathcal{V}_2 \cap \mathcal{V}_3$ for some extension spaces $\hat{\mathcal{V}}_2$ and $\hat{\mathcal{V}}_3$. Since $\mathcal{V}_4 = \mathcal{V}_2 \cap \mathcal{V}_3$ we have also $\mathcal{V}_1 = \mathcal{V}_2 + \mathcal{V}_3 = \hat{\mathcal{V}}_2 \oplus \mathcal{V}_4 \oplus \hat{\mathcal{V}}_3$. Introduce the corresponding basis matrices:

$$V_2 = [\hat{V}_2 \quad V_4] \qquad V_3 = [\hat{V}_3 \quad V_4] \qquad V_1 = [\hat{V}_2 \quad V_4 \quad \hat{V}_3]$$

Since $\mathcal{V}_2$ and $\mathcal{V}_4$ are A-invariant:

$$AV_2 = [A\hat{V}_2 \quad AV_4] = [A\hat{V}_2 \quad V_4A_{22}] = [\hat{V}_2 \quad V_4]\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \qquad (A.6)$$

where $A_{22} = A|\mathcal{V}_4$ and thus $\hat{d}_2(s) = \det(sI-A_{11})$. Using (A.6) and the fact that $\mathcal{V}_3$ is also A-invariant

$$AV_1 = [A\hat{V}_2 \quad AV_4 \quad A\hat{V}_3] = [\hat{V}_2 \quad V_4 \quad \hat{V}_3]\begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}$$

Since $[V_4 \quad \hat{V}_3]$ is a basis matrix for $\mathcal{V}_3$ it follows that $d_1(s) = \hat{d}_2(s)d_3(s)$. ■

Proof of Lemma 4.3.

Let x be a vector such that

$$x \in \mathcal{V}^M; \quad x \notin \mathcal{R}^M; \quad (A-\lambda I)x = 0$$

and let $L \in \underline{L}^M$. Since $\mathcal{R}^M \subset \mathcal{V}^M$, we can then write

$$\mathcal{V}^M = \mathcal{V} \oplus \{x\} \oplus \mathcal{R}^M \tag{A.7}$$

for some extension space $\mathcal{V}$. Let P be a projection onto $\mathcal{V} \oplus \mathcal{R}^M$ along $\{x\}$ and define $L_1 = LP$. Then $L_1 \in \underline{L}^M$ since

$$\begin{aligned}(A+BL_1)\mathcal{V}^M &= (A+BLP)(\mathcal{V}\oplus\mathcal{R}^M) + (A+BLP)\{x\} \\ &\subset (A+BL)\mathcal{V}^M + A\{x\} \\ &\subset \mathcal{V}^M + \{x\} = \mathcal{V}^M\end{aligned}$$

Moreover,

$$(A+BL_1)x = Ax = \lambda x \tag{A.8}$$

Introduce a basis matrix for $\mathcal{V}^M$ according to (A.7) as

$$V_M = [V, x, R_M]$$

Since $\mathcal{V}^M$, x and $\mathcal{R}^M$ are all $(A+BL_1)$-invariant, we have using (A.8)

$$(A+BL_1)V_M = (A+BL_1)[V,x,R_M]$$

$$= [V,x,R_M]\begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & \lambda & 0 \\ A_{31} & 0 & A_{22} \end{pmatrix}$$

Thus from Definition 4.3, $d_z(s) = (s-\lambda)\det(sI-A_{11})$. ■

APPENDIX 4B - Mode Controllability and Mode Observability.

Definition 4.4 (Unobservable modes)

λ is an unobservable mode to the system S(A,B,C) if

$$\ker(\lambda I-A) \cap \ker(C) \neq 0$$

i.e. if there exists a nonzero vector w such that

$$(\lambda I-A)w = 0$$

$$Cw = 0$$

Since w is a right eigenvector to A, the validity of the definition is clear. ■

Definition 4.5 (Uncontrollable modes)

λ is an uncontrollable mode to the system if

$$\{\lambda I-A\} + \mathcal{B} \neq C^n$$

i.e. if

$$(\{\lambda I-A\} + \mathcal{B})^{\perp} \neq 0$$

This condition implies that λ is uncontrollable if there exists a nonzero vector v such that

$$v^*(\lambda I-A) = 0$$

$$v^*B = 0$$

Since v is a left eigenvector to A, the validity of the definition is clear. ∎

Definition 4.6 (Uncontrollable - unobservable modes)

λ is an uncontrollable - unobservable mode if

$$\ker(\lambda I-A) \cap \ker(C) \not\subset \{\lambda I-A\} + \mathcal{B} \qquad \text{(A.9)}$$

Note first that it follows that

$$\ker(\lambda I-A) \cap \ker(C) \neq 0$$

$$\{\lambda I-A\} + \mathcal{B} \neq \mathbb{C}^n$$

implying that λ must be an uncontrollable and unobservable mode. It then remains to show that λ is in fact an eigenvalue to A_{33} in the canonical decomposition (4.15).

The condition (A.9) is satisfied if there is a nonzero vector w such that

$$w \in \ker(\lambda I-A) \cap \ker(C)$$

$$w \notin \{\lambda I-A\} + \mathcal{B}$$

or alternatively if there are nonzero vectors v and w such that

$$w \in \ker(\lambda I-A) \cap \ker(C)$$

$$v \in \left(\{\lambda I-A\} + \mathcal{B}\right)^{\perp}$$

$$v^*w = 1$$

Note that w and v satisfy

$$(\lambda I-A)w = 0; \quad Cw = 0$$
$$v^*(\lambda I-A) = 0 \quad v^*B = 0 \qquad (A.10)$$

Consider now the system S(A,B,C) and the transformation $z = Tx$ with

$$T = [Q,w]$$

where Q is a basis matrix for $\{v\}^\perp$. The inverse must be of the form

$$T^{-1} = \begin{pmatrix} P \\ \cdots \\ v^* \end{pmatrix}$$

where P is a basis matrix for $\{w\}^\perp$. This follows by the fact that $T^{-1}T = I$.

Using (A.10), we have

$$T^{-1}AT = \begin{pmatrix} A_{11} & 0 \\ 0 & \lambda \end{pmatrix} \qquad T^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

$$CT = [C_1 \quad 0]$$

From this form, the validity of the definition is clear. ∎

5. THE SERVO PROBLEM.

In Chapter 2 the combined regulator and servo problem was discussed in fairly general terms. The control system configuration in Fig. 2.7 was suggested as a starting point for the design in this class of problems. In this configuration the inverse system is used to achieve the ideal servo as an identity mapping between the command inputs and the controlled outputs. The problem of nonunique inverses for multivariable systems with a different number of inputs and outputs was solved in Chapter 3 by the minimal right inverse and the properties of its spectrum.

Consider the following linear system model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ z &= Hx \end{aligned} \qquad (5.1)$$

where distinction has been made between the controlled outputs y and the measured outputs z. In Chapter 2, the control system was assumed to consist of a servo part u_r and a regulating part Δu according to

$$u = u_r + \Delta u = S^{-1} y_r + R(z_r - z) \qquad (5.2)$$

where S^{-1} is a right inverse of S(A,B,C) and R a dynamical system. This chapter will be devoted to a more detailed analysis of the servo problem as expressed by (5.2).

Reference values z_r of the measured variables must be generated and fed into the regulator. The reference values correspond to an idealized situation with no model error and no disturbances. Two ways to generate z_r will be described below.

The second problem considered is that of unstable inverses. Note that if the minimal right inverse is unstable, there is no stable inverse to the system. Such systems are called nonminimum phase in classical terminology and are known to be difficult to control, see e.g. [6]. This property is very apparent in the control system (5.2). If the inverse S^{-1} is unstable, the reference input u_r will be unbounded and consequently the control collapses. In this case the exact unstable inverse must be approximated by a neighbouring stable system. It is shown how such "approximative" inverses can be obtained using internal stabilizing feedback on the inverse system.

5.1. Reference Values.

The problem of reference values is a trivial one if the measured variables z form a subset of the controlled variables y, i.e. when $\ker(H) \supset \ker(C)$ in (5.1). Here the situation when this is not the case will be analyzed. One way to attack this kind of problems is to introduce post compensators in order to make the system square. In this context frequency domain techniques have been used, see e.g. [1, 4]. However, using the post compensator approach it is not quite clear which variables are controlled in the servo sense. In this section the problem will be approached according to the principles suggested by (5.2).

Applying the reference input u_r to (5.1) one gets

$$\begin{aligned} \dot{x}_r &= Ax_r + Bu_r = Ax_r + BS^{-1}y_r \\ y_r &= Cx_r \\ z_r &= Hx_r \end{aligned} \tag{5.3}$$

which defines the reference values z_r. The problem is thus to generate z_r given u_r and y_r.

Reference values generated by state space model.

The formulation of the problem above is identical to the problem treated in observer theory, see e.g. [3], even if it is here given another interpretation. Using the standard observer approach, an insensitive reconstruction of z_r is obtained by the following model.

$$\dot{x}_r = Ax_r + Bu_r + K(y_r - Cx_r) = (A-KC)x_r + Bu_r + Ky_r \qquad (5.4)$$
$$z_r = Hx_r$$

where K should be chosen so that the matrix A-KC has all its eigenvalues in the open left half-plane Re(s) < 0. If the model (5.4) is used to generate z_r, an auxiliary system of the same dynamical order as the original system is needed. The order of the model, however, can be reduced by standard techniques [3], if the fact that the reference output y_r is already known is utilized. In this case a reduced order observer of order n-p, where p is the number of controlled outputs, will be sufficient. Note that the auxiliary dynamics is not a part of the feedback loop, but is used to generate the desired trajectory under reference changes. The control system configuration in Fig. 2.7 with z_r generated by an observer is shown in Fig. 5.1.

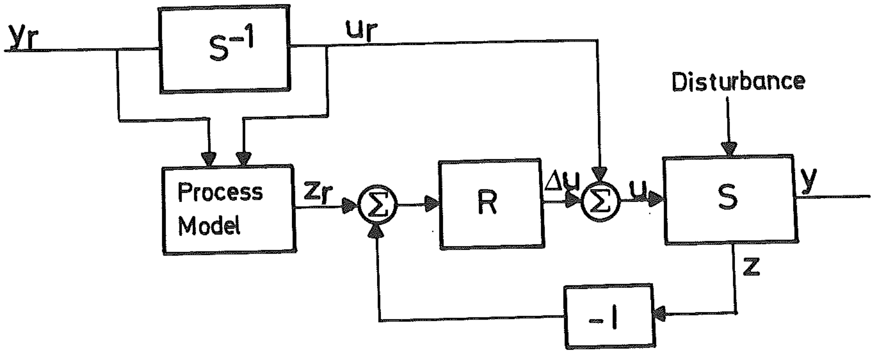

Fig 5.1 - A control system configuration for regulators and servos with the reference values z_r generated by a state space model.

Reference values generated by the inverse system.

It is also possible to use properties of the minimal right inverse S^{-1} in order to obtain z_r. This way of reconstructing z_r is more satisfactory since no auxiliary dynamics is needed. The reconstruction will contain pure differentiators, but the total number is always less than the number of pure differentiators in the inverse itself. This means that the final order of the servo part only depends on the approximations or the model for the desired input-output behaviour. This question has been discussed in more detail in Section 2.2.

Consider the specific inverse which is used to generate the reference input u_r in (5.2). Assume that this inverse is a minimal right inverse constructed according to the rules described in Section 3.4.

Consider the steps (3.34 - 39). In these steps different operations are made upon the original system which leaves its state, which in this case is x_r, unaltered up to (3.38). Thus from (3.35-36)

$$x_r = T_M^{*-1}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

where T_M^* is defined by (3.33). If v_1 and v_2 can be generated, the reference value of the state is known. By comparing (3.40a) and (3.37), it is immediately clear that, cf Remark 15,

v_2 = state of the minimal right inverse (3.40a)

It remains to reconstruct v_1. Apply u_r to (3.39),

$$w_1 = \hat{C}_1 y_r$$

It is then necessary to relate v_1 and w_1. This was done in the proof of Lemma 3.5 in Appendix 3A, cf. (A.7) and (A.8) with $n_1 = q$. Thus

$$v_1 = \hat{C}_1 y_r + M(p)u_1$$

where

$$u_1 = (pI-\bar{A}_{11})\,\hat{C}_1 y_r + \bar{L}^*_{M1} y_r - \bar{A}_{12} v_2$$

$$M(p) = \sum_{i=0}^{q-1} M_i p^i \tag{5.5}$$

$$M_{q-1} = \bar{B}_1 N_q \;; \qquad M_{i-1} = \bar{A}_{11} M_i + \bar{B}_1 N_i \quad i=1,2,\ldots,q-1$$

and N_i are the coefficient matrices in N(p). Summarizing, the reference value x_r is generated by

$$x_r = T_M^{*-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{5.6}$$

where

$$\begin{aligned} v_1 &= M(p)u_1 + \hat{C}_1 y_r \\ v_2 &= \text{state of the minimal right inverse} \end{aligned} \tag{5.7}$$

The control system configuration with reference values generated from the inverse is shown in Fig. 5.2.

Finally a simple example is given to illustrate the ideas above.

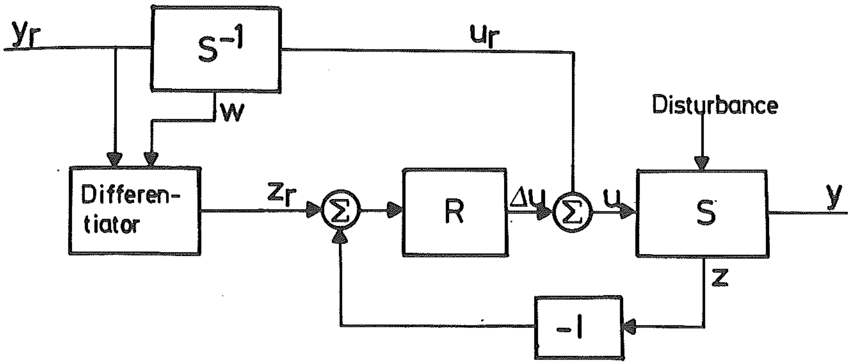

Fig 5.2 - A control system configuration for regulators and servos with the reference values z_r generated using the state of the inverse system.

Example 5.1. Consider

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ -2 & -1 \end{pmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \tag{5.8}$$

The desired responses for command inputs are assumed to be described by the nonlinear model shown in Fig. 5.3. This model implies that the system should respond as

K/(s+K) for moderate changes in the command input. The nonlinear term guarantees that the time derivatives of the output y never exceed a certain value. Moreover, the system is assumed to be controlled by a state feedback regulator.

The control system thus has the form

$$u = u_r + \Delta u = S^{-1}S_m v + L(x_r - x)$$

where L is a state feedback matrix and v denotes the command input. The model S_m is nonlinear

$$\dot{x}_m = f_m(x_m, v)$$
$$y_r = x_m \qquad (5.9)$$

The purpose is to generate the reference input u_r and the reference value x_r as is indicated above.

A minimal right inverse of (5.8) is constructed as described in Section 3.3. Using the same terminology as in Section 3.3, we have

$$\nu_*^M = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

If the system is transformed by $v = T_M^* x$ where

$$T_M^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

the structure (3.37) is obtained, i.e.

$$\dot{v}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} v_1 + \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} u - \begin{pmatrix} 3 \\ 0 \end{pmatrix} y$$

$$\dot{v}_2 = - v_2 + 4y$$

$$y = [0 \quad 1] v_1$$

The polynomial operator N(p) becomes

$$N(p) = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

The minimal system inverse is then obtained by substitution into (3.40), i.e.

$$\dot{w} = - w + 4y_r$$

$$u_r = \begin{pmatrix} 1 \\ -1 \end{pmatrix} w + \begin{pmatrix} -3 \\ p+3 \end{pmatrix} y_r$$

If the model (5.9) for y_r is applied, the following dynamical equations for the reference input u_r are obtained

$$\dot{w} = - w + 4x_m$$

$$\dot{x}_m = f_m(x_m, v)$$

$$u_r = \begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} w \\ x_m \end{pmatrix} + \begin{pmatrix} 0 \\ f(x_m, v) \end{pmatrix}$$

$$y_r = x_m$$

The reference values x_r on the state are obtained by mere substitution into (5.5-7). Thus

$$v_1 = \hat{C}_1 y_r = \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_m$$

$$v_2 = w$$

and

$$x_r = T_M^{*-1} v = \begin{pmatrix} x_m \\ 0 \\ w-x_m \end{pmatrix}$$

Note that no pure differentiators occur in this case since the degree of N(p) is zero.

Summing up, the control system which satisfies the desired goals is given by the second order system

$$u = u_r + L(x_r - x)$$

where

$$\dot{w} = -w + 4x_m$$

$$\dot{x}_m = f_m(x_m, v)$$

$$u_r = \begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} w \\ x_m \end{pmatrix} + \begin{pmatrix} 0 \\ f(x_m, v) \end{pmatrix}$$

$$x_r = \begin{pmatrix} x_m \\ 0 \\ w-x_m \end{pmatrix}$$

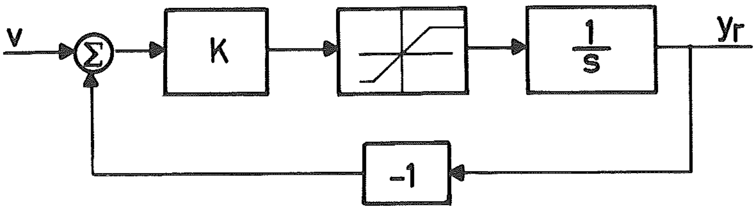

Fig. 5.3 - Model for desired input-output behaviour in Example 5.1.

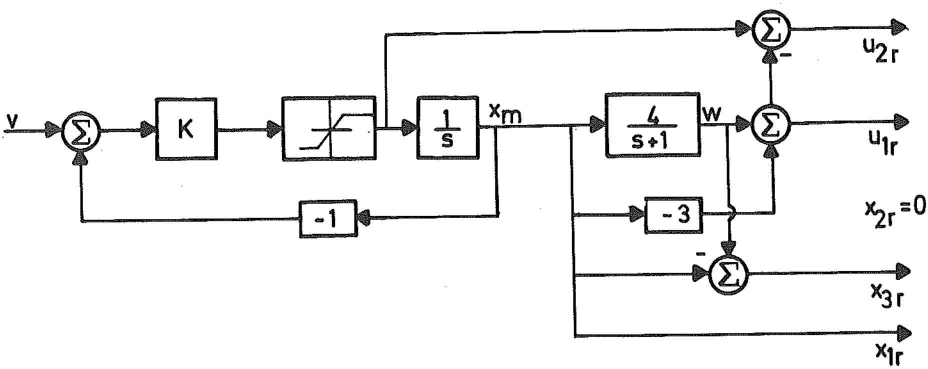

Fig. 5.4 - Auxiliary servo system needed to generate u_r and x_r in Example 5.1.

With this control system, the system behaves as the model S_m for command inputs with state feedback regulation around the desired trajectory x_r.

The auxiliary system needed to generate x_r and u_r is shown in Fig. 5.4. ■

5.2. Systems with Unstable Inverses.

Right invertible systems with any invariant zeros in the right half-plane according to Definition 4.3 have no stable right inverse. This follows directly from Theorem 4.3 and the uniqueness of the spectrum of a minimal right inverse, cf. Corollary 3.3. Unstable inverses are unacceptable in the control system (5.2). The servo part u_r is given by

$$u_r = S^{-1} y_r \tag{5.10}$$

or more precisely, using the representation (3.12)

$$\begin{aligned} \dot{w} &= \hat{A}w + \hat{B}y_r \\ u_r &= N_1(p)w + N_2(p)y_r \end{aligned} \tag{5.11}$$

If the right inverse is unstable, i.e. if the matrix $\hat{A}$ has some eigenvalues in the right half-plane, the servo part u_r will be unbounded unless the desired output y_r has some very special form. It is thus necessary to replace the exact inverse by a suitable stable approximate $\hat{S}$ in (5.10).

A class of stable approximative inverses.

The discussion above implies that it is impossible to generate the desired output exactly. Trade-off must thus be accepted. The desired output y_r must be replaced by a neighbouring function $\hat{y}_r$ which can be produced by a bounded input $\hat{u}_r$. Consider the choice

$$\hat{y}_r = \hat{L}w + \hat{G}y_r \tag{5.12}$$

where w denotes the state of the minimal right inverse. To obtain the corresponding reference input $\hat{u}_r$, $\hat{y}_r$ is applied to the right inverse (5.11) giving

$$\begin{aligned} \dot{w} &= (\hat{A}+\hat{B}\hat{L})w + \hat{B}\hat{G}y_r \\ \hat{u}_r &= \left(N_1(p) + N_2(p)\hat{L}\right)w + N_2(p)\hat{G}y_r \end{aligned} \tag{5.13}$$

Since $(\hat{A},\hat{B})$ always is a controllable pair for minimal right inverses, cf. Lemma 3.4, it is possible to stabilize the inverse by an appropriate feedback matrix $\hat{L}$. Note also that the system (5.13) has y_r as input and $\hat{u}_r$ as output. If $\hat{S}$ denotes the input-output operator for the system (5.13), it thus follows that

$$S\hat{S}y_r = \hat{y}_r \tag{5.14}$$

Since the purpose is to choose $\hat{L}$ and $\hat{G}$ in (5.12) such that $\hat{y}_r$ is a neighbouring function of y_r, it is plausible to regard the system $\hat{S}$ as a stable approximation of the exact inverse S^{-1}. The approximative inverse is illustrated in Fig. 5.5.

By different choices of stabilizing $\hat{L}$ and $\hat{G}$ in (5.12) it is generated

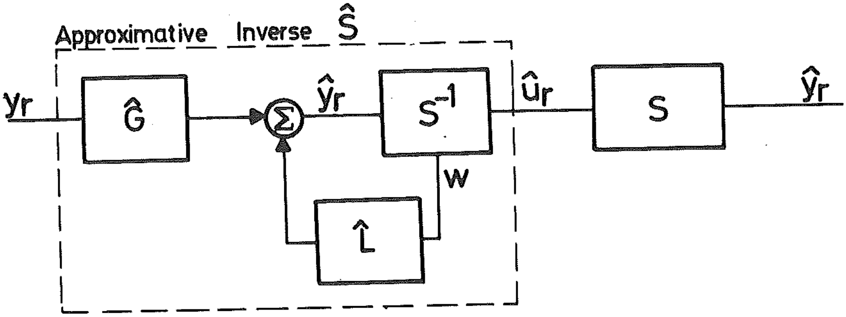

Fig. 5.5 - An illustration of stable approximative inverses obtained by internal stabilization on the exact inverse.

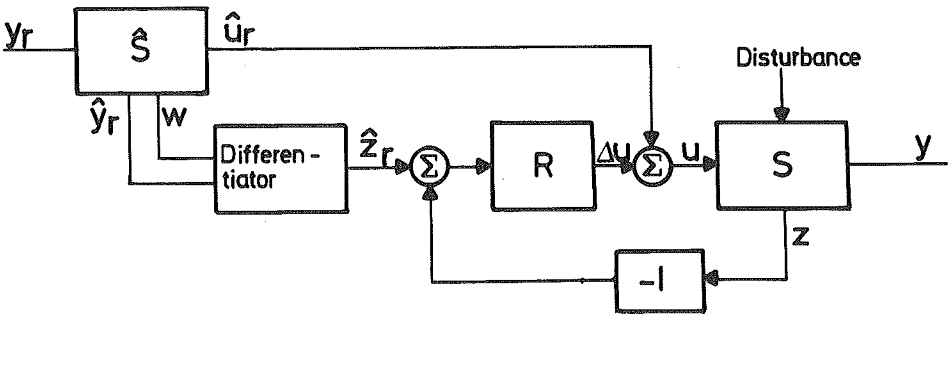

Fig. 5.6 - A control system configuration for regulators and servos using a stable approximative inverse. Note that $\hat{y}_r$ is an available signal, cf. Fig.

(i) a class of neighbouring functions $\hat{y}_r$ (5.12) to a desired output y_r. All $\hat{y}_r$ within this class are such that they can be produced by a stable reference input $\hat{u}_r$.

(ii) a class of stable approximations (5.13) to the exact right inverse.

The stable approximation (5.13) should replace the exact inverse in the control system (5.2). The corresponding reference values $\hat{z}_r$ are generated exactly as in (5.7) if y_r is replaced by $\hat{y}_r = \hat{L}w + \hat{G}y_r$.

The total control system with a stable approximative inverse is shown in Fig. 5.6. The following natural question arises: Is there a "best" choice of approximate within the class described above? This question is discussed below.

Minimum energy stabilization.

In order to perform a more detailed analysis on the problem of finding a "best" stable approximation to a given inverse, it is necessary to restrict the possible choices of y_r. Let the desired output y_r be described by a linear model of the form

$$\dot{x}_m = A_m x_m \qquad x_m(0) = x_{m0}$$
$$y_r = C_m x_m \tag{5.15}$$

A large class of important cases are included in (5.15), e.g. step functions or ramp functions into a linear model. Note that the initial state of the model must be chosen so that the output y_r is enough differentiable at the initial point $t = 0$. The reason is, of course, that y_r is differentiated when applied to the inverse.

Let y_r be the desired output and write the neighbouring function $\hat{y}_r$ as

$$\hat{y}_r = y_r + \Delta y \tag{5.16}$$

Since it is desirable that $\hat{y}_r$ is "near" the desired output y_r, we should obviously select $\hat{y}_r$ so that Δy is small over the whole time interval. Introduce the criterion

$$\int_0^\infty \Delta y^T Q_2 \Delta y dt \qquad Q_2 > 0 \tag{5.17}$$

Consider now $\hat{u}_r$ which is produced by the right inverse with $\hat{y}_r$ as input. Taking the model (5.15) into account, the following set of equations are obtained from (5.11)

$$\dot{w} = \hat{A}w + \hat{B}C_m x_m + \hat{B}\Delta y \tag{5.18a}$$

$$\dot{x}_m = A_m x_m \tag{5.18b}$$

$$\hat{u}_r = N_1(p)w + N_2(p)\hat{y}_r \tag{5.18c}$$

where $\hat{A}$ is unstable since the minimal right inverse is assumed to be unstable.

Let w_1 be defined by

$$w_1 = w + Px_m \tag{5.19}$$

where P is a solution of

$$PA_m - \hat{A}P + \hat{B}C_m = 0 \tag{5.19'}$$

The existence of a solution to this linear matrix equation is guaranteed if the eigenvalues of A_m and $\hat{A}$ are disjoint, [2] p. 225. This is therefore assumed for convenience. The transformed system has the form

$$\dot{w}_1 = \hat{A}w_1 + \hat{B}\Delta y \qquad w_1(0) = Px_{m0} \tag{5.20a}$$

$$\dot{x}_m = A_m x_m \qquad x_m(0) = x_{m0} \tag{5.20b}$$

$$\hat{u}_r = N_1(p)w_1 + N_2(p)\hat{y}_r - N_1(p)Px_m \tag{5.20c}$$

where Δy is to be chosen so that the unstable modes of $\hat{A}$ do not influence $\hat{u}_r$. It is then adequate to consider

$$\dot{w}_1 = \hat{A}w_1 + \hat{B}\Delta y \qquad w_1(0) = Px_m(0) \tag{5.21}$$

Note that Δy is formally an input signal to (5.21). The problem of choosing the "best" Δy which minimizes the criterion (5.17) simultaneously stabilizing (5.21) is thus equivalent to the problem of finding a control which stabilizes a given system with minimum control energy. This problem is treated in detail in [5]. There it is shown that there exists a specific L_{ME} such that

$$\Delta y = L_{ME}w_1 \tag{5.22}$$

minimizes (5.17) within the class of all stabilizing feedback controls to (5.21). It is described in [5] how L_{ME} can be calculated by forming the Euler matrix. The "best" choice of $\hat{y}_r$ is thus

$$\hat{y}_r = y_r + \Delta y = L_{ME}w + y_r + L_{ME}Px_m \tag{5.23}$$

where the last equality follows from (5.19). Can this y_r be produced by a realizable $\hat{u}_r$? Observe that $\hat{y}_r$ in (5.23) contains a feedforward term from the state of the model. This means that $\hat{y}_r$ is not necessarily differentiable at the initial point t = 0 even if this is the case for the desired output y_r. This fact is illustrated in Fig. 5.7, where it is assumed that the system is initially at rest.

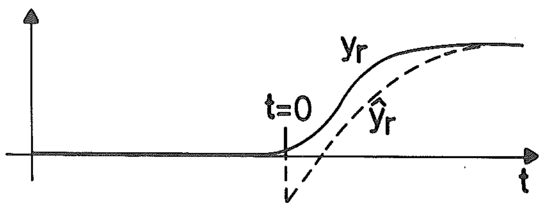

Fig. 5.7 - Possible shape of y_r and $\hat{y}_r$ for a certain choice of $x_m(0)$.

If $\hat{y}_r$ is as described in Fig. 5.7 the corresponding reference input $\hat{u}_r$ given by (5.18c) contains dirac pulses at t = 0. It can thus be concluded that the "optimal" solution (5.23) is not realizable in general. A heuristic solution can, however, be given based upon the following two facts.

- The feedback part L_{ME} in (5.23) depends only on $\hat{A}$, $\hat{B}$ and Q_2, i.e. only on the inverse and the criterion. The stabilizing feedback L_{ME} on the inverse is thus independent of the model for y_r.

o In the servo problem, the steady state behaviour is often of importance. It is thus plausible to require that $\hat{y}_r$ coincides with the desired output y_r at least in steady state.

The discussion above leads to the following heuristic choice of $\hat{y}_r$

$$\hat{y}_r = L_{ME}w + \hat{G}y_r \tag{5.24}$$

where $\hat{G}$ is to be chosen so that $\hat{y}_r = y_r$ in steady state. The latter specification gives

$$\hat{G} = I + L_{ME}\hat{A}^{-1}\hat{B} \tag{5.25}$$

where it has been assumed that the inverse does not contain an eigenvalue at the origin. In fact, it can be seen from (5.19) and (5.23), that (5.24) and (5.25) are the optimal solutions for the model

$$\dot{y}_r = 0$$

i.e. when the desired output is a step.

Besides from the interpretations above, the heuristic choice has the advantage of only being dependent on the inverse. The corresponding stable approximation $\hat{S}_{ME}$ of the exact inverse becomes

$$\begin{aligned} \dot{w} &= (\hat{A}+\hat{B}L_{ME})w + \hat{B}(I+L_{ME}\hat{A}^{-1}\hat{B})y_r \\ \hat{u}_r &= \big(N_1(p) + N_2(p)L_{ME}\big)w + N_2(p)(I+L_{ME}\hat{A}^{-1}\hat{B})y_r \end{aligned} \tag{5.26}$$

The feedback matrix L_{ME} has an interesting effect on the eigenvalues of $\hat{A}$. Assume that spectrum of the minimal right inverse, i.e. the eigenvalues of $\hat{A}$, are

$$Z = \left\{z_1^+,\ z_2^+,\ \ldots,\ z_s^+;\ z_1^-,\ \ldots,\ z_r^-\right\}$$

where z_i^+ and z_i^- denote the eigenvalues in the left and right half-plane respectively. The eigenvalues of $\hat{A}+\hat{B}L_{ME}$ then becomes [5]

$$\hat{Z} = \left\{z_1^+,\ \ldots,\ z_s^+;\ -z_1^-,\ \ldots,\ -z_r^-\right\}$$

The unstable modes z_i^- of $\hat{A}$ are thus reflected in the imaginary axis while the stable modes of $\hat{A}$ remain unchanged, cf. Fig. 5.8.

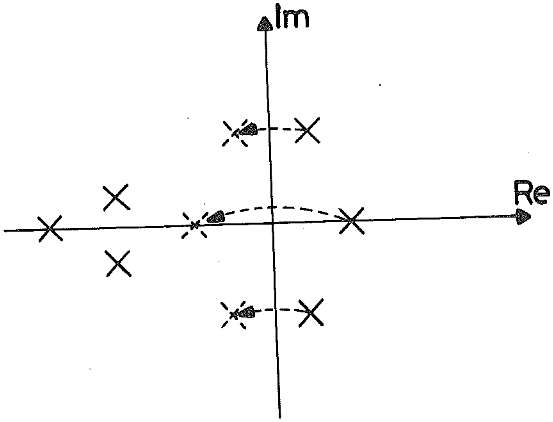

Fig. 5.8 - Illustration of how modes of the inverse system change under minimal energy stabilization.

A frequency domain interpretation.

Let S(A,B,C) be a system with one input and one output. If the system is nonminimum phase, its transfer function can be written

$$G(s) = \frac{q^{+}(s)q^{-}(s)}{p(s)} \qquad (5.27)$$

where the zeros of $q^{+}(s)$ and $q^{-}(s)$ lie in the left and right half-plane respectively. The inverse system

$$G^{-1}(s) = \frac{p(s)}{q^{+}(s)q^{-}(s)} \qquad (5.28)$$

is unstable in this case. What is the "best" stable approximate $\hat{G}_{ME}(s)$ of $G^{-1}(s)$? The choice (5.26) implies that the steady state gain of the inverse is preserved while the unstable modes are reflected in the imaginary axis as illustrated in Fig. 5.8. This is achieved by an internal stabilizing state feedback on the inverse. Therefore p(s) in (5.28) remains unchanged. Thus

$$\hat{G}_{ME}(s) = \frac{p(s)}{q^{+}(s)q^{-}(-s)} \qquad (5.29)$$

is the approximative inverse obtained by (5.26). The system $\hat{G}_{ME}(s)$ is the exact inverse of

$$\tilde{G}(s) = \frac{q^{+}(s)q^{-}(-s)}{p(s)} \qquad (5.30)$$

The system (5.30) is the minimum phase system corresponding to (5.27). Thus for single input single output nonminimum phase systems, the "best" stable approximation of the exact inverse is obtained by inverting the corresponding minimum phase system. The combined servo and regulator system is thus given by

$$u = \hat{G}_{ME}(s)y_r + R(\hat{z}_r - z)$$

The trade-off, i.e. the difference between the desired output and the "best" neighbouring output, becomes

$$\Delta y_r = \hat{y}_r - y_r = \left\{G(s)\hat{G}_{ME}(s) - 1\right\}y_r =$$

$$= \left\{\frac{q^-(s)}{q^-(-s)} - 1\right\}y_r =$$

$$= \frac{(q^-(s) - q^-(-s))}{q^-(-s)}\, y_r \qquad (5.31)$$

Observe that this disscussion refers to the special choice (5.24) and (5.25) which is "optimal" only in the case when the desired output is a step.

The section is concluded by two examples.

<u>Example 5.2</u>. Consider the system

$$G(s) = \frac{K(\alpha - s)}{p(s)}$$

The stable approximative inverse (5.30) is

$$\hat{G}_{ME}(s) = \frac{p(s)}{K(\alpha + s)}$$

The trade-off becomes according to (5.31)

$$\Delta y = \hat{y}_r - y_r = -\frac{2s}{s+\alpha}\, y_r$$

From this expression an interesting conclusion can be made. The trade-off is larger, the nearer the imaginary axis the right half-plane zero is situated. For two systems with pole-zero configurations as shown in Fig. 5.9, the system (a) is more difficult to control than system (b) in the servo sense. Note that $\hat{A}^{-1}$ in (5.25) does not exist if $\alpha=0$. This is due to the fact that the actual output is always zero in steady state for bounded inputs if $\alpha=0$.

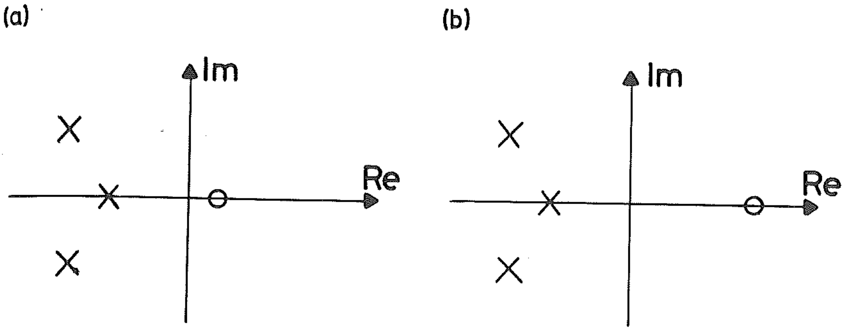

Fig. 5.9 - Pole-zero configurations discussed in Example 5.2.
o zeros
x poles ■

Example 5.3. Consider a system S(A,B,C) with

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{5.32}$$

The transfer function for this system is

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

It has been shown that this system is difficult to control, see Rosenbrock [6]. The invariant zeros for the system are, cf. Example 4.1,

$$z = \{1.0\}$$

The system is thus nonminimum phase in the multivariable sense and therefore the minimal right inverse is unstable. A minimal right inverse of the system is

$$\begin{aligned} \dot{w} &= w + \begin{bmatrix} -4 & 4 \end{bmatrix} y \\ u &= \begin{pmatrix} 2 \\ -2 \end{pmatrix} w + \begin{pmatrix} -5-p & 6+2p \\ 5+p & -5-p \end{pmatrix} y \end{aligned} \tag{5.33}$$

The inverse can be constructed as shown in Section 3.4. Since the computational details have already been demonstrated in earlier examples, they are omitted here.

Introduce the criterion

$$\int_0^\infty \Delta y^T \Delta y dt$$

$$\Delta y = \hat{y}_r - y_r$$

The minimum energy stabilizing feedback matrix to the system (5.33) is given by

$$L_{ME} = \begin{pmatrix} 0.25 \\ -0.25 \end{pmatrix} \tag{5.34a}$$

and $\hat{G}$ (5.25)

$$\hat{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0.25 \\ -0.25 \end{pmatrix} \cdot 1 \cdot [-4 \quad 4] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.34b}$$

The stable approximative inverse S_{ME} (5.26) becomes then

$$\dot{w} = - w + [4 \quad -4]y_r$$

$$\hat{u}_r = \frac{1}{4}\begin{pmatrix} -3p-3 \\ 2p+2 \end{pmatrix} w + \begin{pmatrix} 6+2p & -5-p \\ -5-p & 5+p \end{pmatrix} y_r = \begin{pmatrix} 3+2p & -2-p \\ -3-p & 3+p \end{pmatrix} y_r \tag{5.35}$$

where the reduction procedure in (3.43) has been used in the last equality. Note that the approximative inverse contains no dynamics. This is due to the fact that a feedback has been applied on the inverse system which reflects the zero in -1 onto +1. Since the original system contains a pole in +1, there has thus occurred a "cancellation". The trade-off, i.e. the difference between $\hat{y}_r$ and y_r, becomes

$$\Delta y = \hat{y}_r - y_r = L_{ME}w + (\hat{G}-I)y_r$$

Using transfer functions (5.34) gives

$$\Delta y(s) = \begin{pmatrix} -\dfrac{s}{s+1} & \dfrac{s}{s+1} \\ \\ \dfrac{s}{s+1} & -\dfrac{s}{s+1} \end{pmatrix} y_r \tag{5.36}$$

Assume that the input-output behaviour is described by a decoupled model of the form

$$y_r = \begin{pmatrix} \frac{\alpha}{s+\alpha} & 0 \\ 0 & \frac{\beta}{s+\beta} \end{pmatrix} v \qquad (5.37)$$

The control system with a state feedback regulator L becomes

$$u = \hat{u}_r + \Delta u = \hat{S}_{ME} y_r + L(\hat{x}_r - x) \qquad (5.38)$$

By some straightforward calculations which has been demonstrated in earlier examples, it follows that

$$\begin{pmatrix} \dot{w} \\ \dot{x}_m \end{pmatrix} = \begin{pmatrix} -1 & 4 & -4 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} w \\ x_m \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & 0 \\ 0 & \beta \end{pmatrix} v \qquad (5.39a)$$

$$\hat{u}_r = \begin{pmatrix} 0 & 3-2\alpha & \beta-2 \\ 0 & -3+\alpha & 3-\beta \end{pmatrix} \begin{pmatrix} w \\ x_m \end{pmatrix} + \begin{pmatrix} 2\alpha & -\beta \\ -\alpha & \beta \end{pmatrix} v \qquad (5.39b)$$

$$\hat{x}_r = \begin{pmatrix} 0.25 & 2 & -1 \\ -0.5 & -1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} w \\ x_m \end{pmatrix} \qquad (5.39c)$$

which gives the complete control system. The state feedback regulator is calculated by linear quadratic control theory in this case

$$L = \begin{pmatrix} 17.5 & 8.8 & 2.1 \\ 13.1 & 6.5 & 11.4 \end{pmatrix}$$

The responses for the open loop system is shown in Fig.5.10 a-b and the corresponding responses for the closed loop system in Fig.5.10 c-d, where the trade-off is indicated.

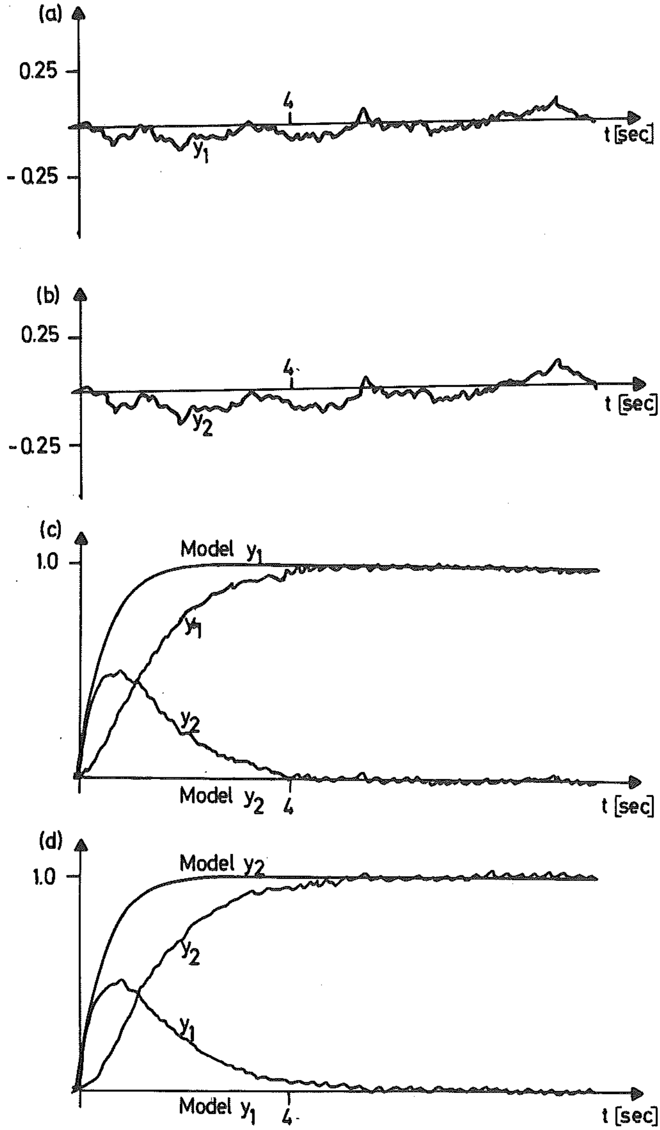

Fig. 5.10 - Responses for the system in Example 5.3 with added process disturbance.

(a) Open-loop system u=0

(b) Open-loop system u=0

(c) Closed-loop system with unit step on v_1.

(d) Closed-loop system with unit step on v_2.

A disturbance has been added to the system in order to demonstrate that the controlled system might have good disturbance rejection, even if the responses for command inputs are slow due to nonminimum phase properties. Since the regulator and servo parts are "separated" by (5.38), the regulation efficiency for disturbances in the system depends only on the choice of L in (5.38). ■

5.3. References.

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6. REGULATION IN THE PRESENCE OF CONSTANT DISTURBANCES.

In the design of control systems for industrial processes there is often inadequate knowledge of the disturbances in the process. This can be partially overcome at a design stage, by designing the controllers so that accurate control is achieved for the most relevant frequencies. Many real process disturbances contain considerable constant or slowly varying components. To ensure accurate steady-state operation in such cases, it will be necessary to supplement the controllers with some integral action, removing steady-state errors. Unlike the single input-output case, however, the insertion of integrations in the feedback loop can be done in several alternative ways in the multivariable case.

In this chapter a complete theory will be presented for output regulation to zero in the presence of step disturbances with a known input space. Simple necessary and sufficient conditions are given, that ensure no steady-state errors and stability of the regulated system. A simple design scheme for such regulators is discussed, where the "proportional" and "integral" parts can be designed separately. The design scheme also exploits the freedom in the choice of integrators in the system. The developed theory shows, that the feedback and feedforward problem can be treated within one framework with similar computations. The proposed design scheme also has the advantage of producing a small number of integrators, in a sense the minimal number.

Similar problems have recently been considered in [2, 10] and [12]. In [12] a solution is given to the asymptotic "zeroing the output" problem in a geometric framework. The actual design problem is, however, not discussed. In [10] the system is assumed to be reduced to a certain minimal form. Necessary and sufficient conditions are then given in terms of the reduced system. In this chapter the solvability conditions are expressed in the actual system matrices and appear as natural steps in the design procedure. Our approach will also, in most cases, result in a control with a less number of integrators, especially if the number of disturbance sources are few in comparison with the number of controlled variables. This is not merely an academic question, since each introduced integrator will cause additional phase retardation, thus making the system harder to stabilize.

This chapter should also be viewed in relation to estimation theory and classical PI control. The classical approach is to integrate the whole output vector. Of course, this is the only solution if the disturbance sources and thereby their input spaces are completely unknown. However, the system may be difficult to stabilize or even unstabilizable. Solvability conditions for this kind of control are given in [2, 4, 13].

Estimation theory in connection with feedforward control can also be used [3, 10]. In [1, 6, 7] the problem has been considered as an extension of ordinary linear quadratic control theory.

The chapter is organized as follows. In Section 6.1 the regulation problem is consisely formulated. Solvability conditions and a constructive algorithm are given in Section 6.2. In Section 6.3 the feedforward and integral feedback problem is discussed. A design example concludes the chapter.

6.1. System Description and Problem Formulation.

Consider a linear time invariant system in state space form

$$\dot{x} = Ax + Bu + Gv$$
$$y = Cx + Du + Fv \qquad (6.1)$$

where x denotes the n-vector of states, u the m-vector of control inputs, v the r-vector of disturbance inputs and y the p-vector of controlled outputs. A, B, G, C, D, and F are assumed to be constant matrices of compatible dimensions. Moreover, it will be assumed that the disturbances v are constant but unknown. Thus v can be described by

$$\dot{v} = 0 \qquad v(0) = v_0 \qquad (6.2)$$

The output regulation problem is then to find a control u that may depend on x such that the following two conditions are satisfied:

1^o $y(t) \to 0$ when $t \to \infty$ for all possible unknown values on v.

2^o The closed loop system is asymptotic stable.

Of course, these are not the only desired properties of a regulator system. This question will be further discussed in connection with the design problem in Section 6.3.

In controlling the system (6.1) complete state feedback will be used. Assuming the system is observable, this can be overcome in the usual way by attaching a state estimator to the measured outputs [5, 9]. If a simpler control system is desired, the reduction technique of the next chapter may be used, avoiding the estimator system.

6.2. Output Regulation.

In this section some simple necessary and sufficient conditions are given that guarantee output regulation to zero in the presence of step disturbances and stability of the regulated system. These conditions are stated in some different ways, to shed additional light on the problem. The relation between the feedback and feedforward problem will be clearly explained and a constructive algorithm is given.

An introductory lemma.

Before considering the main problem described in the previous section, let us analyse the corresponding feedforward problem. Assume that the disturbances v are measured. Find a control of the form

$$u = Lx - Nv \tag{6.3}$$

where L and G are matrices of appropriate dimensions, such that the specifications 1° and 2° in Section 6.1 are satisfied. Apply the control (6.3) to (6.1) and let $t \to \infty$. Since the time derivates become zero in steady-state, we have

$$\begin{aligned} 0 &= (A+BL)x_s - BNv + Gv \\ y_s &= (C+DL)x_s - DNv + Fv \end{aligned} \tag{6.4}$$

where the subscript s denotes the steady state value. Solving (6.4) for y_s, we obtain

$$y_s = \left((C+DL)(A+BL)^{-1}B - D\right)Nv - \left((C+DL)(A+BL)^{-1}G - F\right)v$$

The output shall be zero in steady irrespective of the constant disturbance v. There must thus be a matrix N in (6.3) such that

$$\left((C+DL)(A+BL)^{-1}B - D\right)N = (C+DL)(A+BL)^{-1}G - F \qquad (6.5)$$

A question which then naturally arises is the following. Does the existence of a matrix N which satisfies (6.5) depend on the specific choice of stabilizing L in (6.3)? That this is not so is stated in the following introductory lemma.

Lemma 6.1. Assume there exists a pair (Ω_*, N_*) such that

$$[(C+D\Omega_*)(A+B\Omega_*)^{-1}B - D]N_* = (C+D\Omega_*)(A+B\Omega_*)^{-1}G - F \qquad (6.6)$$

Then to every Ω such that $\det(A+B\Omega) \neq 0$ there exists an N such that the pair (Ω, N) also satisfies (6.6). ■

From this lemma we see that the question whether the equation (6.5) has a solution or not does not depend on the choice of feedback matrix L in (6.3). This is an important fact in the design problem, since it means that the feedback term L and the feedforward term N can be designed separately.

Necessary and sufficient conditions.

Introduce matrices M_u and M_v as

$$M_u = (C+D\Omega)(A+B\Omega)^{-1}B - D$$

$$M_v = (C+D\Omega)(A+B\Omega)^{-1}G - F \qquad (\text{(}$$

where Ω is any matrix such that $\det(A+B\Omega) \neq 0$.

Interpretations of the matrices M_u and M_v are given in the discussion on the feedforward problem above. The matrix Ω is introduced only to cover the case when the open system already contains an integrator. If the matrix A contains no eigenvalue at the origin, for instanc if the open loop system is asymptotic stable, then we could select $\Omega = 0$.

Necessary and sufficient conditions for the existence of a stable feedback control which removes the steady state errors are given in the following theorem.

Theorem 6.1. Consider the disturbed system (6.1). There exists a control u which only depends on x such that

1° $y(t) \to 0$ when $t \to \infty$ for all unknown constant value on v,

2° the closed loop system is asymptotic stable,

if and only if the pair (A,B) is stabilizable and there is a matrix N such that

$$M_u N = M_v \qquad (6.$$

Proof. (only if) From condition 2° it immediately follows that the pair (A,B) must be stabilizable. Let u_* be any control such that conditions 1° and 2° are satisfied. Write

$$u_* = \Omega x - w_*$$

where Ω is chosen such that $\det(A+B\Omega) \neq 0$. The closed loop system becomes

$$\dot{x} = (A+B\Omega)x - Bw_* + Gv \qquad (6.9a)$$

$$y = (C+D\Omega)x - Dw_* + Fv \qquad (6.9b)$$

Since the closed loop system is asymptotic stable with control u_* it follows that

$$\dot{x}_s = 0$$

where the subscript s indicates the steady state value $(t \to \infty)$. Thus solving y_s from (6.9)

$$y_s = (C+D\Omega)(A+B\Omega)^{-1}Bw_* - Dw_* - (C+D\Omega)(A+B\Omega)^{-1}Gv + Fv =$$

$$= M_u w_* - M_v v = 0$$

The last equality is satisfied for all $v \in R^r$ only if

$$\{M_u\} \supset \{M_v\} \qquad \{\cdot\} \text{ range space}$$

or equivalently only if there is a matrix N such that

$$M_u N = M_v$$

and necessity is clear.

(if) Sufficiency is shown by construction. Since (A,B) is a stabilizable pair there is an Ω_0 such that the spectrum of $A+B\Omega_0$ lies in the open half-plane $Re(s) < 0$. Thus $\det(A+B\Omega_0) \neq 0$ and according to Lemma 6.1 a matrix N_0 can be found such that

$$[(C+D\Omega_0)(A+B\Omega_0)^{-1}B - D]N_0 = (C+D\Omega_0)(A+B\Omega_0)^{-1}G - F$$

From (6.4) with $\Omega = \Omega_0$

$$M_u N_0 = M_v \tag{6.10}$$

Let $q = \text{rank}(M_v) \leq p$. Then there is a nonsingular $p\times p$ matrix W such that

$$WM_v \doteq \begin{pmatrix} P \\ --- \\ Q \end{pmatrix} M_v = \begin{pmatrix} PM_v \\ ----- \\ 0 \end{pmatrix} \begin{matrix} q \text{ rows} \\ \\ p-q \text{ rows} \end{matrix} \tag{6.11}$$

Let M be an arbitrary $q\times q$ matrix with its spectrum in $Re(s) < 0$. It is then claimed that the following control satisfies the conditions 1° and 2° of the theorem

$$u = L_0 x + L_1 \int^t Py(\sigma)d\sigma \tag{6.12}$$

with

$$L_1 = - N_0(PM_v)^{-r}M \tag{6.13a}$$

$$L_0 = \Omega_0 - L_1 P(C+D\Omega_0)(A+B\Omega_0)^{-1} \tag{6.13b}$$

where $(\cdot)^{-r}$ denotes a right inverse.

The right inverse exists in this case since PM_v has full row rank by construction (6.11). Introduce

$$z = \int^{t} Py(\sigma)d\sigma$$

Then $\dot{z} = PCx + PDu + PFv$ and the closed loop system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A+BL_0 & BL_1 \\ P(C+DL_0) & PDL_1 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} G \\ PF \end{pmatrix} v$$

$$y = (C+DL_0 \quad DL_1) \begin{pmatrix} x \\ z \end{pmatrix} + Fv$$

Applying the transformation

$$\begin{pmatrix} x \\ z' \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ K & I_q \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

with

$$K = - P(C+D\Omega_0)(A+B\Omega_0)^{-1} \tag{6.14}$$

we obtain using (6.10) and (6.13)

$$\begin{pmatrix} \dot{x} \\ \dot{z}' \end{pmatrix} = \begin{pmatrix} A+B\Omega_0 & BL_1 \\ 0 & M \end{pmatrix} \begin{pmatrix} x \\ z' \end{pmatrix} + \begin{pmatrix} G \\ PF+KG \end{pmatrix} v \tag{6.15}$$

$$y = (C+D\Omega_0 \quad DL_1) \begin{pmatrix} x \\ z' \end{pmatrix} + Fv$$

and 2^o is certainly true since the spectra of $A+B\Omega_0$ and M lie in the open half-plane $Re(s) < 0$.

To prove 1° we will just calculate the steady state value y_s of y from (6.15). First notice that

$$PF + KG = - P(C+D\Omega_0)(A+B\Omega_0)^{-1}G + PF = - PM_v$$

Since v is constant, $\dot{x}_s = 0$ and $\dot{z}_s' = 0$. Solving x_s and z_s' from (6.15) we obtain

$$x_s = \left(- (A+B\Omega_0)^{-1}G - (A+B\Omega_0)^{-1}BL_1M^{-1}PM_v\right)v$$

$$z_s' = M^{-1}PM_v v$$

Thus

$$y_s = (C+D\Omega_0)x_s + DL_1z_s' + Fv =$$

$$= \Big(- (C+D\Omega_0)(A+B\Omega_0)^{-1}G - (C+D\Omega_0)(A+B\Omega_0)^{-1}BL_1M^{-1}PM_v +$$

$$+ DL_1M^{-1}PM_v + F\Big)v =$$

$$= (-M_uL_1M^{-1}PM_v-M_v)v$$

Substitute L_1 (6.13a) and use (6.10)

$$y_s = \left(M_uN_0(PM_v)^{-r}PM_v - M_v\right)v =$$

$$= \left(M_v(PM_v)^{-r}PM_v - M_v\right)v$$

Consider then Wy_s where W is given by (6.11)

$$Py_s = P\left(M_v(PM_v)^{-r}PM_v - M_v\right)v = (PM_v-PM_v)v = 0$$

and

$$Qy_s = Q\left(M_v(PM_v)^{-r}PM_v - M_v\right)v =$$

$$= QM_v[(PM_v)^{-r}PM_v - I]v = 0$$

Thus $Wy_s = 0$ and since W is nonsingular $y_s = 0$. ∎

<u>Remark 1</u>. The condition (6.5) is equivalent to

$$\{M_u\} \supset \{M_v\} \tag{6.16}$$

One way to test (6.8) and (6.15) is the following. Compute

$$S = I - M_u M_u^{\dagger}$$

where $M_u^{\dagger}$ is the pseudoinverse of M_u [8, 11]. Then P is the orthogonal projection onto $\{M_u\}^{\perp}$. The conditions (6.8) or (6.16) are then satisfied if and only if

$$SM_v = 0.$$

A reasonable test quantity for computational purposes is

$$\sigma = \frac{\| SM_v \|}{\| M_v \|}$$

where $\| \cdot \|$ denotes a matrix norm, e.g. $\| M \| = (\mathrm{tr}(MM^T))^{\frac{1}{2}}$. If solutions to (6.8) exist, one is given by

$$N = M_u^{\dagger} M_v$$ ∎

From the discussion on the corresponding feedforward problem above, cf. with (6.5) and Lemma 6.1, the following corollary follows directly.

Corollary 6.1. The feedback problem can be solved if and only if the feedforward problem can be solved. ■

Step responses.

Theorem 6.1 can also be expressed in terms of step responses to the system (6.1). Let $u = Lx + u$ be a stabilizing control to (6.1). If the system already is asymptotic stable this step may be omitted. The closed loop system becomes

$$\begin{aligned} \dot{x} &= (A+BL)x + Bu_0 + Gv \\ y &= (C+DL)x + Du_0 + Fv \end{aligned} \tag{6.17}$$

Let $Y_u = [y_u^1 \;\; y_u^2 \;\; \ldots \;\; y_u^m]$ denote the steady state responses of (6.17) for m linearly independent step inputs $U = [u_0^1 \;\; u_0^2 \;\; \ldots \;\; u_0^m]$ and $v = 0$. Similarly, let $Y_v = [y_v^1 \;\; y_v^2 \;\; \ldots \;\; y_v^r]$ denote the steady state responses for r linearly independent disturbance inputs $V = [v^1 \;\; v^2 \;\; \ldots \;\; v^r]$.

Corollary 6.2. There is a control u which only depends on x such that the conditions 1° and 2° of Theorem 6.1 are satisfied if and only if there is a matrix W such that

$$Y_u W = Y_v \tag{6.18}$$

Remark 3. Note that Corollary 6.2 gives a possibility to test the solvability condition in Theorem 6.1 by performing experiments on the system, without knowing the exact model.

A constructive algorithm.

Finally the construction problem is considered. The sufficiency part of the proof of Theorem 6.1 in fact gives an algorithm for constructing a control of the form

$$u = L_0 x + L_1 \int^t Py(\sigma)d\sigma \tag{6.19}$$

This algorithm will always produce a solution if one exists. The algorithm is summarized in the form of a theorem.

Theorem 6.2. If there exists any control u such that conditions 1° and 2° of Theorem 6.1 are satisfied, it can always be taken to be of the form (6.19) where L_0, L_1 and P are constructed as follows:

1° Choose Ω_0 to stabilize $A+B\Omega_0$.

2° Determine N such that $M_u N = M_v$ where M_u and M_v are given by (6.7) with $\Omega = \Omega_0$.

3° Let $q = \text{rank}(M_v)$. Find a $q\times p$ matrix P such that $\text{rank}(PM_v) = q$.

4° Choose Ω_1 to stabilize $PM_v\Omega_1$.

5° Finally

$$L_1 = -N\Omega_1$$

$$L_0 = \Omega_0 - L_1 P(C+D\Omega_0)(A+B\Omega_0)^{-1}$$

and the closed loop eigenvalues equal the joint spectra of the matrices $A+B\Omega_0$ and $PM_v\Omega_1$.

Proof. Replace $(PM_v)^{-r}M$ by Ω_1 in (6.13a). The proof then follows the proof of Theorem 6.1. ■

Remark 4. Notice that the number of introduced integrators equals $q = \text{rank } M_v \leq \min(p,r)$ where p = number of controlled output and r = number of disturbance inputs. In fact the developed theory results in a design scheme that in a sense compounds classical PI control and estimation theory in connection with feedforward control into one simple algorithm, always giving a compensator with equal or lower dynamical order than any of these.

6.3. The Design Problem.

In this section we will use the results from the previous section to discuss the integral-feedback and feedforward design problem. In doing so the disturbance vector could be separated into two subvectors $v^T = [v_1^T \;\; v_2^T]$ where v_1 is assumed to be measured and v_2 is not. However, Corollary 6.1 and Theorem 6.2 show that the two problems can be handled by the same algorithm. Therefore, we will not separate between measured and non-measured disturbance inputs, but just indicate the extra computations that are needed for integral-feedback control.

Feedback and feedforward designs.

We will follow the algorithm given in Theorem 6.2 and outline how a practical design could be made based upon this algorithm. In some of the steps below somewhat heuristic arguments will be used. These are, however, verified by the design example in the next section.

1° First find an Ω_0 such that $A+B\Omega_0$ is asymptotic stable. In this step the "proportional" action in the controller is designed, and the spectrum of $A+B\Omega_0$ will essentially determine the regulation efficiency in the presence of disturbances of white noise character. The ordinary steady state linear quadratic control law may be a good choice for Ω_0.

2° Calculate

$$\begin{aligned} M_u &= (C+D\Omega_0)(A+B\Omega_0)^{-1}B - D \\ M_v &= (C+D\Omega_0)(A+B\Omega_0)^{-1}G - F \end{aligned} \tag{6.20}$$

and solve for N

$$M_u N = M_v \tag{6.21}$$

If the last equation cannot be solved, then the problem cannot be solved in its original statement, i.e. output regulation to zero cannot be achieved (Theorem 6.1). The natural approach is then to solve (6.21) in a least square sense, i.e. to minimize $\| M_u N - M_v \|$. The minimum will be given by

$$N = M_u^{\dagger} M_v \tag{6.22}$$

where $M_u^\dagger$ stands for the pseudoinverse of M_u. Efficient computational algorithms for such inverses exist, see [8] or [11]. If feedforward control is desired the design is now complete and the control becomes

$$u = \Omega_0 x - Nv$$

If integral-feedback control is desired proceed to the next step.

3° Select a $q \times p$ matrix P such that $\mathrm{rank}\{PM_uN\} = q$, where $q = \mathrm{rank}\{M_uN\}$. In this step a maximal number of independent vectors from the row space of M_uN is selected. This selection then determines P. Since P will determine the set of integrated outputs, the freedom in the selection above can be used so that the most "essential" output variables are integrated. Thus, if y_j is considered to be the most essential output variable, one can start the selection above with the j:th row in M_uN. Notice that the number of integrators becomes $q = \mathrm{rank}\{M_uN\} = \mathrm{rank}\{M_v\} \leq \min(p,r)$ where p is the number of controlled variables and r is the number of disturbance inputs.

4° Choose Ω_1 to stabilize $(PM_uN)\Omega_1$. For instance let $\Omega_1 = (PM_uN)^\dagger M$ where M has some desired spectrum. In this step the "integral" action in the controller is designed in the sense that the spectrum of M essentially will determine the setting times for the step disturbances.

5° The integral-feedback control becomes

$$u = L_0 x + L_1 \int^t Py(\sigma)d\sigma \tag{6.23}$$

where P is given above and

$$L_1 = - N\Omega_1 \qquad (6.24)$$

$$L_0 = \Omega_0 - L_1 P(C+D\Omega_0)(A+B\Omega_0)^{-1} \qquad (6.25)$$

6.4. A Design Example.

To illustrate the results of the chapter, we will perform the calculations for a specific process, a boiler. For this purpose a conversative design and simulation program, SYNPAC, has been used, which has been developed at the Division of Automatic Control in Lund.

The linearized equations for a boiler around a certain operating point can be written

$$\dot{x} = Ax + Bu + Gv$$

$$y = Cx$$

where the state variables are

x_1 = drum pressure (bar)

x_2 = drum liquid level (m)

x_3 = drum liquid temperature (^oC)

x_4 = riser wall temperature (^oC)

x_5 = steam quality (%)

The control variables are

u_1 = heat flow to the risers (kJ/s)

u_2 = feed water flow (kg/s)

and the disturbances are

$v(t)$ = load changes (bar)

Numerical values of A, B, C and G for a power station boiler with a maximum steam flow of about 350 t/h are calculated in [3] and are shown in Section 4.6. The drum pressure is 140 bar. The operating point is 90% full load. The intention is to find a control law of the form

$$u = L_0 x + L_1 \int^{t} Py(s)ds \tag{6.26}$$

such that the steady state deviations in y due to step changes in the load v are removed. Following the design scheme in Section 6.3 we have:

1° A state feedback $u = \Omega_0 x$ is found such that the regulation efficiency is satisfactory for disturbances of white noise character. This is done by linear quadratic control theory in this case. An appropriate steady state linear quadratic control law is given by [3]

$$\Omega_0 = \begin{pmatrix} -0.668\times10^{4} & -0.418\times10^{6} & -0.135\times10^{4} & -0.137\times10^{4} & 0.175\times10^{7} \\ -0.803\times10^{1} & -0.908\times10^{3} & -0.486 & -0.816 & 0.431\times10^{4} \end{pmatrix} \tag{6.27}$$

2° Calculate M_u and M_v. Since D = 0 and F = 0 we have

$$M_u = C(A+B\Omega_0)^{-1}B = \begin{pmatrix} -0.446\times10^{-4} & -0.205\times10^{-1} \\ -0.390\times10^{-6} & -0.921\times10^{-3} \end{pmatrix}$$

$$M_v = C(A+B\Omega_0)^{-1}G = \begin{pmatrix} -0.766 \\ \\ 0.106\times10^{-1} \end{pmatrix}$$

Since $M_uN = M_v$ can be solved, the conditions in Theorem 6.1 are satisfied and

$$N = \begin{pmatrix} 0.994\times10^4 \\ \\ -0.157\times10^2 \end{pmatrix}$$

3° Calculate rank$\{M_v\} = 1$. Find an 1×3 matrix P such that $PM_v = PM_uN$ has rank one. Any matrix $P = [p_1 \ p_2]$ such that

$$[p_1 \quad p_2]\begin{pmatrix} -0.766 \\ \\ 0.106\times10^{-1} \end{pmatrix} \neq 0$$

will work. Take for instance $P = [-0.766 \quad 0.106\times10^{-1}]$.

4° We have $PM_uN = PM_v = 0.586$. A 1×1 (scalar) matrix Ω_1 shall be selected so that $0.586\times\Omega_1$ equals some prescribed eigenvalue λ, i.e.

$$\Omega_1 = \lambda/0.586$$

This eigenvalue determines the setting time for the step disturbance.

5° The control is given by (6.26) where

$$L_1 = -N\Omega_1 = \begin{pmatrix} -0.994\times10^4 \\ \\ 0.138\times10^2 \end{pmatrix}\lambda/0.586$$

$$L_0 = \Omega_0 - L_1 PC(A+B\Omega_0)^{-1} =$$

$$= \begin{pmatrix} -0.668\times10^4 & -0.418\times10^6 & -0.135\times10^4 & -0.137\times10^4 & 0.175\times10^7 \\ -0.803\times10^1 & -0.908\times10^3 & -0.486 & -0.816 & 0.431\times10^4 \end{pmatrix} -$$

$$- \lambda/0.586 \begin{pmatrix} 0.994\times10^4 \\ -0.157\times10^2 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 0.557\times10^1 & -0.631\times10^2 & 0.197\times10^1 & 0.137\times10^1 & 0.321\times10^3 \end{pmatrix}$$

The eigenvalues of the enlarged system are the eigenvalues of $A+B\Omega_0$ and λ. Simulations (Fig. 6.1) have been made for two values of λ, -0.02 and -0.10, where the first value corresponds to a slow absorption of the step disturbance and the second to a faster. The control law (6.26) is compared with the proportional control $u = \Omega_0 x$ with Ω_0 as in (6.27). The effect of different choices of λ is clearly illustrated in Fig. 6.1.

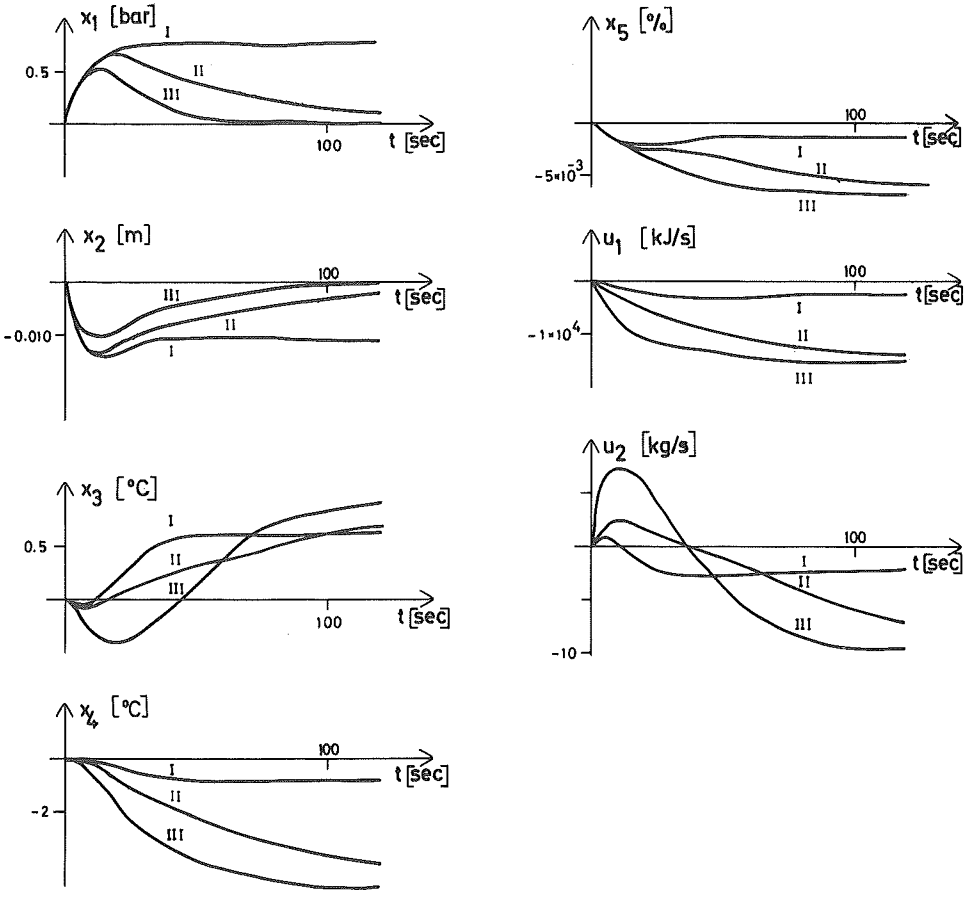

Fig. 6.1 - Time responses for a drum boiler with a load change of 1 bar.

I ... control law (6.26)

II .. control law (6.25) with $\lambda = -0.02$

III .. control law (6.25) with $\lambda = -0.10$

6.5. References.

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APPENDIX 6A - Proof of Lemma 6.1.

Rewrite (6.5) in the following way:

$$0 = \big((C+D\Omega_*)(A+B\Omega_*)^{-1}B - D\big)N_* - (C+D\Omega_*)(A+B\Omega_*)^{-1}G + F =$$

$$= CX_* + DU_* + F \qquad (A.1)$$

where

$$X_* = -(A+B\Omega_*)^{-1}(G-BN_*) \qquad (A.2)$$

$$U_* = -\Omega_*(A+B\Omega_*)^{-1}(G-BN_*) - N_* = \Omega_*X_* - N_* \qquad (A.3)$$

Moreover let

$$N = N_* + (\Omega_*-\Omega)(A+B\Omega_*)^{-1}(G-BN_*) \qquad (A.4)$$

and write

$$W = \big((C+D\Omega)(A+B\Omega)^{-1}B - D\big)N - (C+D\Omega)(A+B\Omega)^{-1}G + F =$$

$$= CX + DU + F \qquad (A.5)$$

where

$$X = -(A+B\Omega)^{-1}(G-BN) \qquad (A.6)$$

$$U = -\Omega(A+B\Omega)^{-1}(G-BN) - N = \Omega X - N \qquad (A.7)$$

Then from (A.3), (A.4) and (A.7)

$$U - U_* = \Omega X - N - \Omega_*X_* + N_* =$$

$$= \Omega X - (\Omega_*-\Omega)(A+B\Omega_*)^{-1}(G-BN_*) - \Omega_*X_* =$$

$$= \Omega X + (\Omega_*-\Omega)X_* - \Omega_*X_* = \Omega(X-X_*) \qquad (A.8)$$

where the third equality follows from (A.2). Moreover from (A.2) and (A.6)

$$0 = (A+B\Omega_*)X_* + G - BN_* = AX_* + BU_* + G$$

$$0 = (A+B\Omega)X + G - BN = AX + BU + G$$

Taking differences between the last two expressions we obtain

$$0 = A(X-X_*) + B(U-U_*)$$

and using (A.8)

$$(A+B\Omega)(X-X_*) = 0$$

Since $A+B\Omega$ is nonsingular it follows that $X = X_*$ and from (A.8) $U = U_*$. Then comparing (A.1) and (A.5) we see that $W = 0$ and the pair (Ω, N) where N is given by (A.4) satisfies (6.6). ■

7. A DESIGN SCHEME FOR INCOMPLETE STATE OR OUTPUT FEEDBACK.

The concept of state feedback plays an important role in existing control theory for linear systems. Linear quadratic control theory [1] and pole assignment theory [20, 21] are two well-known examples. Unfortunately the whole state vector is, however, rarely available for measurement. Even if it was available a state feedback control would sometimes result in far too complex control systems. The standard way to bypass these difficulties is to measure only a limited set of outputs and reconstruct the complete state vector using a Kalman filter [2] or an observer [14]. The result is, however, still somewhat unsatisfactory since the reconstruction by itself might produce high order dynamics in the control function.

These facts justify the demands for simpler or suboptimal control policies. Practical constraints on the feedback system must be considered. A limited number of measurements is one obvious constraint. In large systems consisting of several coupled subprocesses, such as power systems, there may be a desire to control the system with local feedbacks on the different processes, eventually with the addition of a small number of interconnections. There are, however, no rational ways to design such hierarchical control schemes. Another example is diagonally controlled systems where the design philosophy is the classical one with each input variable controlling a single output variable.

A few methods exist to treat problems of these types. The use of dynamic feedback [4, 9, 17] has the same disadvantages as the observer approach above, i.e. the control may be unnecessarily complex. The problem can also be tackled by direct optimization [11, 12, 13, 16]. However, this technique does not seem to be practical when applied to large systems. A special version of modal control [5, 20] has also been used in this class of problems. Quite recently frequency domain methods have been developed which extend the classical Nyquist criteria to multivariable systems. A survey of these results can be found in [15, 19]. These criteria seem, however, to be difficult to use for large systems with several inputs and outputs. There is one considerable difference between the approach in this chapter and the frequency domain techniques. In this chapter we start with an "optimal" solution which is made suboptimal by imposing constraints in the control structure. In the frequency domain approach one attempts to successively improve the solution from an initial guess by varying the gains and the structure of the control.

In this chapter a state feedback control is used as the starting point. This is quite a realistic assumption, since there are straightforward methods to find such controllers even for fairly large systems. See for instance [1] and [20]. The step taken is then to fit this control into another "similar" control with a predefined structure. The idea behind this fit is to make it as accurate as possible on the eigenspace corresponding to a dominant set of eigenvalues to the closed loop system. It is illustrated by examples that satisfactory controllers may be obtained in this way after a few iterations. It should be noticed that the method does not depend on how the state feedback controller is obtained. The reduction technique is thus applicable to any method that results in a linear feedback from the state.

Notice that this reduction procedure is a rational way of designing hierarchical control systems. Sometimes it is not possible to control the system satisfactorily by output feedback only. In such cases the reduction scheme can be used to find controllers of PD types, where the derivative term will give additional information about the state of the system and thus make the system easier to stabilize.

Numerical examples are included to illustrate different aspects of the scheme. In a more full scale example, a drum boiler, it is shown that the feedback from all five states can be replaced by the feedback from two measured outputs. In this case it is possible to avoid the Kalman filter, proposed for the reconstruction of the state, without any significant decrease in performance. Another example, a three-machine power system, which is considered in [3], shows that the design scheme is also applicable to fairly large systems.

In large systems the computational effort is of importance. The major computational burden in this case lies on an initial eigenvalue - eigenvector calculation, which corresponds to approximately $8n^3$ operations. An additional eigenvalue calculation may have to be done to check if the reduced control law has an acceptable degree of stability.

This method could be an effective tool for the design of multivariable controllers in an interactive mode.

Statement of the problem.

Consider a linear time invariant system in state space form

$$\dot{x} = Ax + Bu \tag{7.1}$$

where x is the n-vector of states and u is the m-vector of control inputs. A and B are real-valued matrices of compatible dimensions. Moreover, assume that a state feedback controller

$$u = Lx + v \tag{7.2}$$

where v is some external input and L an m×n matrix, is found such that the system (7.1) with the controller (7.2) has the desired properties.

In controlling the system (7.1) we will set certain constraints on the feedback system. The intention is then to "reduce" the control law (7.2) such that these constraints are satisfied. Two specific types of constraints will be considered corresponding to different degrees of complexity in the control function, cf. [11]. These definitions should cover a large variety of practical constraints that might be imposed on the structure of a feedback system.

In order to simplify the notations we will use stars (*) to indicate properties associated with the reduced control laws.

The simplest kind of constraint is to permit output feedback. Let y = Cx denote the output of (7.1) where C is a

7.1. Preliminaries.

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In order to simplify the notations we will use stars (*) to indicate properties associated with the reduced control laws.

The simplest kind of constraint is to permit output feedback. Let y = Cx denote the output of (7.1) where C is a real r×n matrix. A control of the form

$$u = K^*Cx + v \tag{7.3}$$

will be referred as a control with a <u>single constrained feedback structure</u>.

A more complex structure is obtained if the i:th input component vector is restricted to be a function of certain specified outputs. Let $y_i = C_i x$, $i = 1, 2, \ldots, q$, denote q sets of output variables to (7.1) where C_i is an $r_i \times n$ matrix. Moreover, let $u^T = [u_1^T \; u_2^T \; \ldots \; u_q^T]$ be a partition of the control vector into an appropriate set of q subvectors. A control of the form

$$u_i = K_i^* C_i x + v_i \qquad i = 1, 2, \ldots, q \tag{7.4}$$

will be referred as a control with a <u>multiple constrained feedback structure</u>. It is easily verified that local as well as hierarchical types of control systems are included in this formulation. Notice that the control (7.3) is a special case of (7.4) with q = 1. An illustration of the two concepts is given in Fig. 7.1 and Fig. 7.2.

A common way to do the kind of reductions considered here is to simply neglect those entries of the state feedback matrix that are "small" in comparison with the others.

There are, however, several difficulties involved in such a procedure, and it requires frequently a fairly deep understanding of the process dynamics. Moreover, there is no rational way to "compensate" the remaining entries for the approximations made. The approach of this paper will instead be to construct a certain subspace of the state space where the reduction is made. In this way the "compensating" problem is avoided and converted to the problem of finding the appropriate subspace.

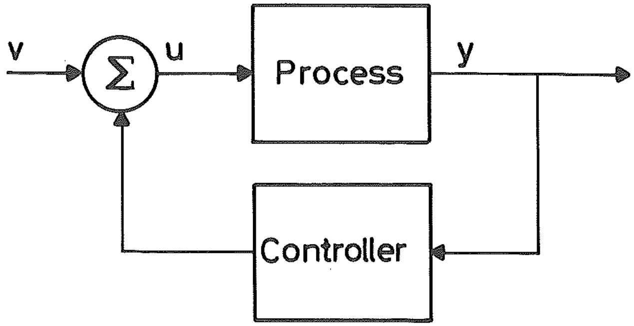

Fig. 7.1 - A system controlled via single constrained feedback structure.

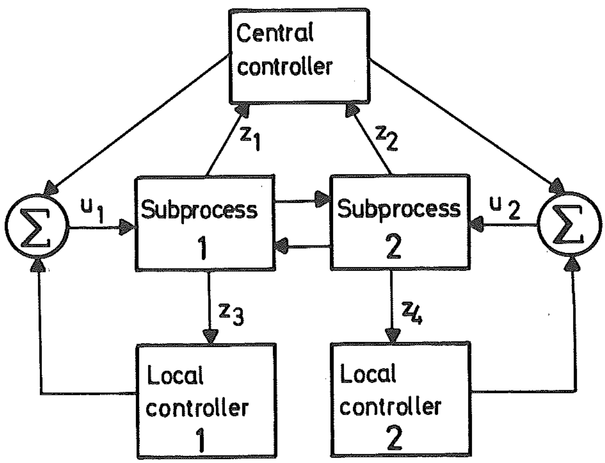

Fig. 7.2 - Two coupled systems controlled via multiple constrained feedback structure, where $y_1^T = [z_1^T \; z_2^T \; z_3^T]$ and $y_2^T = [z_1^T \; z_2^T \; z_3^T]$ in (7.4).

Invariant eigenspaces.

Let A be an arbitrary n×n matrix and let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ be a given subset of eigenvalues to A.

Assume that Λ is a symmetric set, i.e. if $\lambda \in \Lambda$ then also $\bar{\lambda} \in \Lambda$, where the bar indicates complex conjugation. First consider the case when A is cyclic, i.e. there are n linearly independent eigenvectors to A. Then an invariant subspace is simply obtained from the eigenvectors $a_1, a_2, \ldots, a_p$ corresponding to Λ, i.e.

$$Q = [a_1 \; a_2 \; \ldots \; a_p] \tag{7.5}$$

is a basis matrix for the eigenspace.

If A is non-cyclic the concept of generalized eigenvectors is introduced. Let λ_i be an eigenvalue of multiplicity $\sigma_i > 1$. The generalized eigenvectors $a_i^1, a_i^2, \ldots, a_i^{\sigma_i}$ corresponding to λ_i are then defined as the nontrivial solutions of

$$(A-\lambda_i I)a_i^1 = 0$$

$$(A-\lambda_i I)a_i^k = a_i^{k-1}; \qquad k = 2, 3, \ldots, \sigma_i$$

The basis for the eigenspace is then constructed according to the following rule. If a_i^ℓ is selected, then a_i^k, $k = 1, 2, \ldots, \ell-1$, must also be selected as members of the basis if an invariant subspace is to be obtained. In this way an invariant eigenspace can be constructed corresponding to any set of eigenvalues to A.

Finally, observe that since Λ is assumed to be a symmetric set and A is assumed to be real, a real basis for the eigenspace is obtained by taking

$$Q = [a_1\ a_2\ \dots\ a_s;\ \mathrm{Re}\{a_{s+1}\}\ \mathrm{Im}\{a_{s+1}\}\ \mathrm{Re}\{a_{s+2}\}\dots] \qquad (7.6)$$

where $a_1, a_2, \dots, a_s$ are assumed to be real and a_{s+1}, $a_{s+2}, \dots, a_p$ are assumed to be complex. For any pair $\lambda, \bar{\lambda}$ belonging to Λ then choose Re{a}, Im{a} as members of the basis where a is the eigenvector corresponding to λ. In this way complex arithmetic is avoided in the sequel, and is only needed in the eigenvector calculation.

Pseudo inverses.

Let M be an arbitrary real matrix. The pseudo inverse $M^\dagger$ of M is then defined by the following four conditions:

1° $M^\dagger M M^\dagger = M^\dagger$

2° $M \cdot M^\dagger M = M$

3° $M^\dagger M$ is symmetric

4° $M M^\dagger$ is symmetric

It is shown in [18] that $M^\dagger$ is uniquely defined by these conditions. Numerical algorithms exist to find such inverses, see for instance [10].

The pseudo inverse has some nice properties in minimization on inner product spaces. Consider the equation

$$Mx = y$$

which shall be solved for x. Then $x_0 = M^\dagger y$ has the following properties:

1° x_0 minimizes $\| Mx - y \|$ where $\| \cdot \|$ denotes the ordinary euclidian quadratic norm,

2° amongst the possible candidates for the minimum of $\| Mx - y \|$, x_0 is the one that minimizes $\| x \|$.

7.2. Control Approximation.

Assume that a state feedback control is given. This control is then replaced with a "similar" control with a predefined feedback structure. It is shown that this can be done in such a way that a certain number of eigenvalues remain invariant (mode preservation). Since there is an upper bound on the number of invariant eigenvalues a different reduction is also given which minimizes a weighted shift of the eigenvalues (mode weighting). Controls of derivative types and other dynamic compensators will be considered at the end of the section.

Mode preservation.

Consider the system (7.1) with the control (7.2). The closed loop system becomes

$$\dot{x} = (A+BL)x + Bv \tag{7.7}$$

We will attempt to replace the control (7.2) with a similar control of the multiple constrained form (7.4). For this control the closed loop system becomes

$$\dot{x} = \left(A + \sum_{i=1}^{q} B_i K_i^* C_i\right)x + Bv \tag{7.8}$$

where $B = [B_1 \ B_2 \ \dots \ B_q]$ is a partition of the input matrix compatible with the partition of the control vector in (7.4). Moreover, the reduced control law shall be selected so that some dominant properties of (7.7) are preserved in (7.8).

Partition the state feedback matrix as

$$L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_q \end{pmatrix}$$

where L_i is $m_i \times n$. Then if $K_i C_i = L_i$ have solutions K_i^* for $i = 1, 2, \dots, q$, the exact and the reduced control laws would be identical. However, such solutions rarely exist, and therefore approximations must be made. The following theorem describes one rational way to do such approximations.

Theorem 7.1. Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_p\}$ be a symmetric set of eigenvalues to $A+BL$ and let Q be a real basis matrix for the corresponding eigenspace. Then if

$$K_i C_i Q = L_i Q \tag{7.9}$$

have solutions K_i^* for $i = 1, 2, \dots, q$, then Λ is also a set of eigenvalues to

$$A + \sum_{i=1}^{q} B_i K_i^* C_i$$

Moreover, if $T = [Q \ \ \hat{Q}]$ where the columns of $\hat{Q}$ are any set of vectors that extend the columns of Q to a basis in R^n then

$$T^{-1}(A+BL)T = \begin{pmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{22}^0 \end{pmatrix} \tag{7.10}$$

and

$$T^{-1}\left(A + \sum_{i=1}^{q} B_i K_i^* C_i\right)T = \begin{pmatrix} A_{11}^0 & A_{12}^0 + \sum_{i=1}^{q} B_i^1 \Delta L_i^* \hat{Q} \\ 0 & A_{22}^0 + \sum_{i=1}^{q} B_i^2 \Delta L_i^* \hat{Q} \end{pmatrix} \tag{7.11}$$

where

$$\Delta L_i^* = K_i^* C_i - L_i \qquad T^{-1}B_i = \begin{pmatrix} B_i^1 \\ B_i^2 \end{pmatrix}$$

<u>Proof</u>. Introduce $A_0 = A+BL$ and

$$A_0^* = A + \sum_{i=1}^{q} B_i K_i^* C_i$$

From (7.9) we have

$$\left(A + \sum_{i=1}^{q} B_i K_i^* C_i\right)w = \left(A + \sum_{i=1}^{q} B_i L_i\right)w = (A+BL)w \tag{7.12}$$

for any $w \in \{Q\}$. Since $\{Q\}$ is A_0 invariant by construction, it follows from (7.12) that $\{Q\}$ is also A_0^* invariant and $A_0Q = A_0^*Q$. Let the columns of $\hat{Q}$ be any set of vectors that extend the columns of Q to a basis in R^n. Choose $T = [Q \quad \hat{Q}]$ and write

$$T^{-1} = \begin{pmatrix} V \\ \hat{V} \end{pmatrix}$$

We then have

$$T^{-1}A_0T = \begin{pmatrix} VA_0Q & VA_0\hat{Q} \\ 0 & \hat{V}A_0\hat{Q} \end{pmatrix} = \begin{pmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{22}^0 \end{pmatrix} \tag{7.13}$$

$$T^{-1}A_0^*T = \begin{pmatrix} VA_0^*Q & VA_0^*\hat{Q} \\ 0 & \hat{V}A_0^*\hat{Q} \end{pmatrix} = \begin{pmatrix} A_{11}^0 & VA_0^*\hat{Q} \\ 0 & \hat{V}A_0^*\hat{Q} \end{pmatrix} \tag{7.14}$$

The set of eigenvalues of A_{11}^0 equals Λ. From (7.14) it then follows that Λ is also a set of eigenvalues to A_0^*. Moreover, we have

$$VA_0^*\hat{Q} = VA_0\hat{Q} + V\left(\sum_{i=1}^{q} B_iK_i^*C_i - L\right)\hat{Q} =$$

$$= A_{12}^0 + V\sum_{i=1}^{q} B_i(K_i^*C_i - L_i)\hat{Q} = A_{12}^0 + \sum_{i=1}^{q} B_i^1 \Delta L_i^*\hat{Q}$$

and in the same way

$$\hat{V}A_0^*\hat{Q} = A_{22}^0 + \sum_{i=1}^{q} B_i^2 \Delta L_i^*\hat{Q} \quad \blacksquare$$

Remark 1. A real basis for the eigenspace can be constructed from the eigenvectors as was described in Section 7.1, cf. (7.6).

Remark 2. A comparison between the matrices (7.10) and (7.11) clearly illustrates the kind of approximations that are made. The upper left block corresponding to eigenvalues Λ are identical in both systems. The remaining blocks are changed by an amount depending on ΔL_i^*, i.e. the difference between the exact and the reduced control laws.

Remark 3. The remaining eigenvalues of A+BL are different from those of $A+BK^*C$. Observe, however, that the effect of the approximations is only localized to the part of A+BL that contains the less dominant modes. The case when the approximations still cause an unacceptable change in the system is covered below.

Remark 4. Theorem 7.1 also yields an algorithm for pole assignment via output feedback. It has been shown in [5] that if rank C = r, then a symmetric set of r eigenvalues may be "almost" freely assigned. If a state feedback matrix L has been found so that r eigenvalues to the closed loop system take some prescribed values Theorem 7.1 may be used to find a corresponding output feedback matrix (assuming (7.9) is solvable).

Mode weighting.

The condition that (7.9) shall be solvable for K_i gives an upper bound on the number of eigenvalues that can be held fixed. This bound mostly equals r_i, i.e. the number of measured variables. One trivial exception is C = L, where K = I preserves all the eigenvalues. It may, however, still happen that some of the remaining eigenvalues move to undesired locations in the complex plane. The solution to this problem is to include a larger number of eigenvalues looking for least square solutions of (7.9).

Introduce the matrix norm

$$\| M \| = \left(tr\{MM^T\}\right)^{1/2}$$

valid for an arbitrary real matrix M.

Consider first the case when there is more than one solution to (7.9). Let R_i, i = 1, 2, ..., q, be nonsingular $r_i \times r_i$ matrices. Then one solution is given by

$$K_i^* = L_i Q(R_i^{-1} C_i Q)^{+} R_i^{-1} \qquad (7.15)$$

Moreover, this solution is the one that minimizes the norm $\| K_i \; R_i \|$, i.e. a solution with small feedback gains is selected. The matrices R_i are used to scale the output variables.

Consider now the opposite case when there is no solution of (7.9). We may then attempt to minimize the norm

$$\| (K_i C_i Q - L_i Q) W \| \qquad (7.16)$$

where W is a nonsingular p×p matrix. In fact the minimum is obtained by taking

$$K_i^* = L_i QW(C_i QW)^\dagger$$

Now remember the special choice of basis that is made in (7.6), i.e.

$$Q = [a_1 \; a_2 \; \dots \; a_s; \; \mathrm{Re}\{a_{s+1}\} \; \mathrm{Im}\{a_{s+1}\} \; \mathrm{Re}\{a_{s+2}\} \; \dots] \qquad (7.17)$$

where a_k is the eigenvector corresponding to λ_k. If we choose $W = \mathrm{diag}(w_1, w_2, \dots, w_p)$ where $w_k \neq 0$, (7.16) may be rewritten as

$$\| (K_i C_i Q - L_i Q)W \|^2 = \sum_{k=1}^{p} w_k^2 |K_i C_i a_k - L_i a_k|^2 \qquad (7.18)$$

From the last expression we see that a successive increase in w_k causes a successive better fit of the eigenvalue λ_k in the closed loop system, cf. Theorem 7.1. In this way W may be interpreted as a weighting matrix for the eigenvalues we desire to hold fixed. This point is further clearified by examples later.

Finally we observe that (7.15) and (7.16) may be combined to

$$K_i^* = L_i QW(R_i^{-1} C_i QW)^\dagger R_i^{-1} \qquad (7.19)$$

Proportional and derivative control.

In some cases acceptable degree of stability cannot be achieved by output feedback only. The classical way to bypass this difficulty is to include derivatives of the outputs in the feedback loop.

We will now permit a control of the form

$$u = K_1^* y + K_2^* P\dot{y} \tag{7.20}$$

where P is a given m×r matrix and y = Cx. Only the single constrained case will be considered. The extension to the multiple constrained case is straightforward. In classical control terms the control (7.20) is of PD-type. The derivative term will set some constraints on the quality of the measured signals, especially the presence of high frequency noise. This kind of control has, however, turned out to be successful in many applications.

By some simple manipulations the control (7.20) is transformed to the standard form (7.3). Using (7.1) we have

$$u = K_1^* y + K_2^* PC\dot{x} = K_1^* Cx + K_2^* PC(Ax+Bu)$$

Assuming $I - K_2^* PCB$ is invertible the last expression may be solved for u

$$u = (I-K_2^* PCB)^{-1} K_1^* Cx + (I-K_2^* PCB)^{-1} K_2^* PCAx =$$

$$= \hat{K}_1 Cx + \hat{K}_2 PCAx \tag{7.21}$$

Now defining a new output vector $\hat{y}$ as

$$\hat{y} = \hat{C}x = \begin{pmatrix} C \\ PCA \end{pmatrix} x \qquad (7.22)$$

The equation (7.21) can then be rewritten as

$$u = [\hat{K}_1 \ \ \hat{K}_2]\hat{C}x = \hat{K}\hat{C}x \qquad (7.23)$$

The previous results can now be used to find an appropriate $\hat{K}$. The feedback gains in (7.20) are then calculated as

$$K_1^* = (I-K_2^*PCB)\hat{K}_1 \qquad (7.24)$$

$$K_2^* = \hat{K}_2(I+PCB\hat{K}_2)^{-1} \qquad (7.25)$$

The benefit of this kind of control is apparent from (7.22) and (7.23). By having a larger portion of the state available we are also, in view of the reduction technique above, able to keep a larger number of eigenvalues fixed. Moreover, if rank $\{\hat{C}\} = n$ then the reduced and the exact control laws become identical.

Compensators.

The derivative term above is one example of a dynamic compensator which is introduced to improve the feedback properties of the system. More general types of such compensators are sometimes desired. It is shown below that dynamic compensators are easily introduced into the design scheme described above.

Let S_c be a dynamic system cascaded with the original system as is shown in Fig. 7.3.

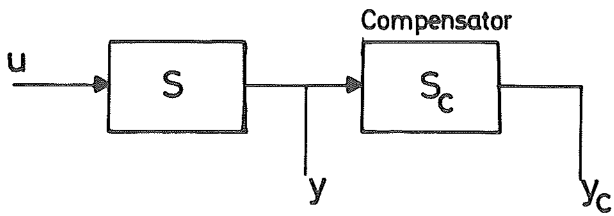

Fig. 7.3 - A system with dynamic compensation.

As before, we assume that the control, which is to be approximated, is a state feedback control on the original system, i.e.

$$u = Lx \tag{7.26}$$

Let the dynamics of S_c be described by

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c u_c \\ y_c &= C_c x_c + D_c u_c \end{aligned} \tag{7.27}$$

The total system, including both the systems S and S_c cascaded as shown in Fig. 7.3, becomes

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} A & 0 \\ B_c C & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

$$\begin{pmatrix} y \\ y_c \end{pmatrix} = \begin{pmatrix} C & 0 \\ D_c C & C_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} \tag{7.28}$$

The state feedback control (7.26) can be written

$$u = [L \quad 0] \begin{pmatrix} x \\ x_c \end{pmatrix} \tag{7.29}$$

The purpose is now to approximate u with another control only using feedback from the available outputs which are y and y_c. We have thus arrived at the same problem formulation as above. The difference is that, by the compensator outputs y_c, the number of available outputs have increased and consequently more eigenvalues can be held fixed.

How shall then the compensator be chosen? In state space terminology, a compensator is introduced to bring more information about the state of the system. From above we see that the system closed by state feedback is

$$\begin{pmatrix} A+BL & 0 \\ B_c C & A_c \end{pmatrix}$$

i.e. the eigenvalues of the enlarged system equals the eigenvalues of A+BL and the eigenvalues of the compenstor A_c. The eigenvalues of A_c will not remain fixed during the approximation and should therefore be chosen far into the left half-plane. A reasonable choice of compensator is to cascade "important" output variables with systems of the form $s/(s+\alpha)$ where α is large enough. Another possibility is to choose S_c as a low order state estimator obtained from a reduced model of the system S.

Examples.

Finally we will give some examples to illustrate the ideas of this section. A more farreaching example is considered in the next section.

Example 7.1.

$$\dot{x} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u$$

$$u = \begin{pmatrix} -5 & -1 \\ 2 & -5 \end{pmatrix} x + v$$

The closed loop system becomes

$$\dot{x} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v$$

and the closed koop eigenvalues equal $\lambda_1 = -3$ and $\lambda_2 = -5$. Assume we shall hold $\lambda_1 = -3$ fixed. The eigenvector corresponding to λ_1 is

$$a_1 = 1/\sqrt{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Assume we permit a control of the form

$$u = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} x + v$$

This control is then of the muliple constrained typ (7.4). The feedback structure becomes

$$u_1 = k_1(1 \quad 0)x + v_1$$

$$u_2 = k_2(0 \quad 1)x + v_2$$

Solving (7.9) for i = 1,2 we have

$$u = \begin{pmatrix} -6 & 0 \\ 0 & -3 \end{pmatrix} x + v$$

The eigenvalues of

$$A + \sum_{i=1}^{q} B_i k_i^* C_i$$

becomes $\nu_1 = -3$ and $\nu_2 = -4$. ■

<u>Example 7.2</u>.

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} x$$

$$u = \begin{pmatrix} -2 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} x + v$$

The closed loop system is

$$\dot{x} = \begin{pmatrix} -2 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} v$$

The eigenvalues of A+BL are $\lambda_{1,2} = -1 \pm j$ and $\lambda_3 = -2$.

Assume we permit a feedback of the form

$$u = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} y + v$$

First we attempt to keep the eigenvalues $\lambda_{1,2} = -1 \pm j$ fixed. The corresponding eigenvectors are

$$a_1 = \begin{pmatrix} 0 \\ -0.5 \\ 1 \end{pmatrix} + j \begin{pmatrix} 0.5 \\ -0.5 \\ 0 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 0 \\ -0.5 \\ 1 \end{pmatrix} - j \begin{pmatrix} 0.5 \\ -0.5 \\ 0 \end{pmatrix}$$

The basis matrix Q for the eigenspace is then selected according to (7.6) as

$$Q = \begin{pmatrix} 0 & 0.5 \\ -0.5 & -0.5 \\ 1 & 0 \end{pmatrix}$$

Solving (7.9) for K we have

$$K_I^* = \begin{pmatrix} 0.67 & 0.33 \\ -0.33 & -1.67 \end{pmatrix}$$

and the eigenvalues of $A+BK_I^*C$ are $\nu_{1,2} = -1\pm j$ as desired and $\nu_3 = 1.33$. The third mode has become unstable and therefore we include also this mode looking for least square solutions according to (7.18) and (7.19).

The eigenvector corresponding to $\lambda_3 = -2$ is

$$a_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The basis matrix for the eigenspace then becomes

$$Q = \begin{pmatrix} 0 & 0.5 & 1 \\ -0.5 & -0.5 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We choose the weighting matrix W as W = diag(1,1,0.1), where a relatively small weight has been laid on λ_3. Using (7.19) we now have

$$K_{II}^* = \begin{pmatrix} 0.64 & 0.32 \\ -0.32 & -1.66 \end{pmatrix}$$

and the closed loop eigenvalues are $\nu_{1,2} = -0.995\pm0.998j$ and 1.29. The weight on the third mode was obviously too small. Take instead W = diag(1,1,0.8). We then obtain

$$K_{III}^* = \begin{pmatrix} -0.30 & -0.15 \\ 0.15 & -1.42 \end{pmatrix}$$

The closed loop eigenvalues are now $\nu_{1,2} = -0.57\pm1.14j$ and $\nu_3 = -0.73$, which is considered to be satisfactory in this case. The solution was obtained after a few iterations by successively altering the weighting factors. In the general case there is no guarantee that a satisfactory solution can be obtained. However, if a satisfactory solution is difficult to obtain by altering the weighting factors, this indicates that the system is difficult to control with the prescribed feedback structure. ■

7.3. A Design Example.

A computer program for control reduction has been written based upon Theorem 7.1 and the least square solution (7.19). This program has been used to find simple control strategies for a boiler. The starting point is here a linear quadratic control law, which is used to fit a certain feedback structure. By simulations we show that a reduction can be made without any significant decrease in performance. In fact, the responses of the reduced control system are very similar to the responses of the system controlled by complete state feedback.

The linearized equations for a boiler around a certain operating point can be written

$$\dot{x} = Ax + Bu + Gv$$
$$y = Cx$$

where the state variables are

x_1 = drum pressure (bar)

x_2 = drum level (m)

x_3 = drum liquid temperature ($^{\circ}C$)

x_4 = riser wall temperature ($^{\circ}C$)

x_5 = steam quality

The control variables are

u_1 = heat flow to the risers (kJ/s)

u_2 = feedwater flow (kg/s)

and the disturbances are

v(t) = load changes

Numerical values of A, B, C and G for a power station boiler with a maximum steam flow of about 350 t/h are calculated in [8] and are shown in Section 4.6. The drum pressure is 140 bar. The operating point is 90% of full load.

A state feedback matrix can be calculated using linear quadratic theory. In [8] it is shown that the following feedback matrix gives satisfactory responses.

$$L = \begin{pmatrix} -0.668\times10^4 & -0.418\times10^6 & -0.136\times10^4 & -0.137\times10^4 & 0.175\times10^7 \\ -0.803\times10^1 & -0.908\times10^3 & -0.486 & -0.815 & 0.431\times10^4 \end{pmatrix} \quad (7.30)$$

The intention is now to replace the control u = Lx with a simpler control using only output feedback, i.e.

$$u = K^*y = K^*Cx$$

where K^* shall be properly chosen. In Section 4.6 the boiler was analysed from an input-output point of view using the poles and zeros of the system. It was shown that the system has a favourable pole-zero configuration, cf. Fig. 4.2. Therefore it can be expected that the boiler can be satisfactorily controlled by output feedback.

The eigenvalues of A+BL are

$$\lambda_1 = -0.490\times10^{-1}$$

$$\lambda_{2,3} = -0.755\times10^{-1}\pm j\cdot0.511\times10^{-1}$$

$$\lambda_{4,5} = -0.141\pm j\cdot0.170\times10^{-1}$$

and they are shown in Fig. 7.4a.

First we attempt to include only the three eigenvalues $\lambda_{1,2,3}$ of A+BL having the least real part. Somewhat arbitrarily we choose the corresponding weights as W = = diag(1,1,1). Using (7.19) we have

$$K_I^* = \begin{bmatrix} 0.924\times10^4 & -0.347\times10^6 \\ 0.403\times10^2 & -0.827\times10^3 \end{bmatrix} \tag{7.31}$$

The eigenvalues of $A+BK_I^*C$ are shown in Fig. 7.4b. We observe that the relative damping of the neglected pair $\lambda_{4,5} = -0.141\pm0.017$ has decreased in the reduced control system. In order to increase the damping we include also $\lambda_{4,5}$ in the solution and choose the weighting factors as W = diag(1,1,1,0.2,0.2), where smaller weights have been laid on $\lambda_{4,5}$. The least square solution (7.19) becomes now

$$K_{II}^* = \begin{bmatrix} 0.569\times10^3 & -0.286\times10^6 \\ 0.870\times10^1 & -0.601\times10^3 \end{bmatrix} \tag{7.32}$$

and the eigenvalues of $A+BK_{II}^*C$ are as shown in Fig. 7.4c. As can be seen the damping of the second complex pair has increased, but at the expense that the rightmost eigenvalue has moved somewhat nearer the imaginary axis. A further iteration with W = diag(1,1,1,0.5,0.5) gives

$$K^*_{III} = \begin{bmatrix} -0.265\times10^4 & -0.263\times10^6 \\ -0.167\times10^1 & -0.528\times10^3 \end{bmatrix} \qquad (7.33)$$

The corresponding eigenvalue configuration is shown in Fig. 7.4d.

Simulations show that K^*_{II} is the most satisfactory choice in this case. The output feedback matrix can be compared with the corresponding elements in the state feedback matrix (7.30) (the two leftmost columns). As can be seen the feedback gains are slightly less in K^*_{II}, but of the same magnitude. However, the relations between the individual feedback gains differ considerably. This is due to the fact that compensations have been made in K^*_{II} for the remaining columns in L.

In Fig. 7.5-6 the system is simulated with control laws (7.30) and (7.32). Fig. 7.5 shows the responses for an initial condition in drum level of 0.02 m and Fig. 7.6 the same responses for an initial condition in drum pressure of 1 bar. As can be seen the difference between the exact and the reduced control laws is astonishingly small, indicating that a control only using feedback from the measured variables will be sufficient in this case.

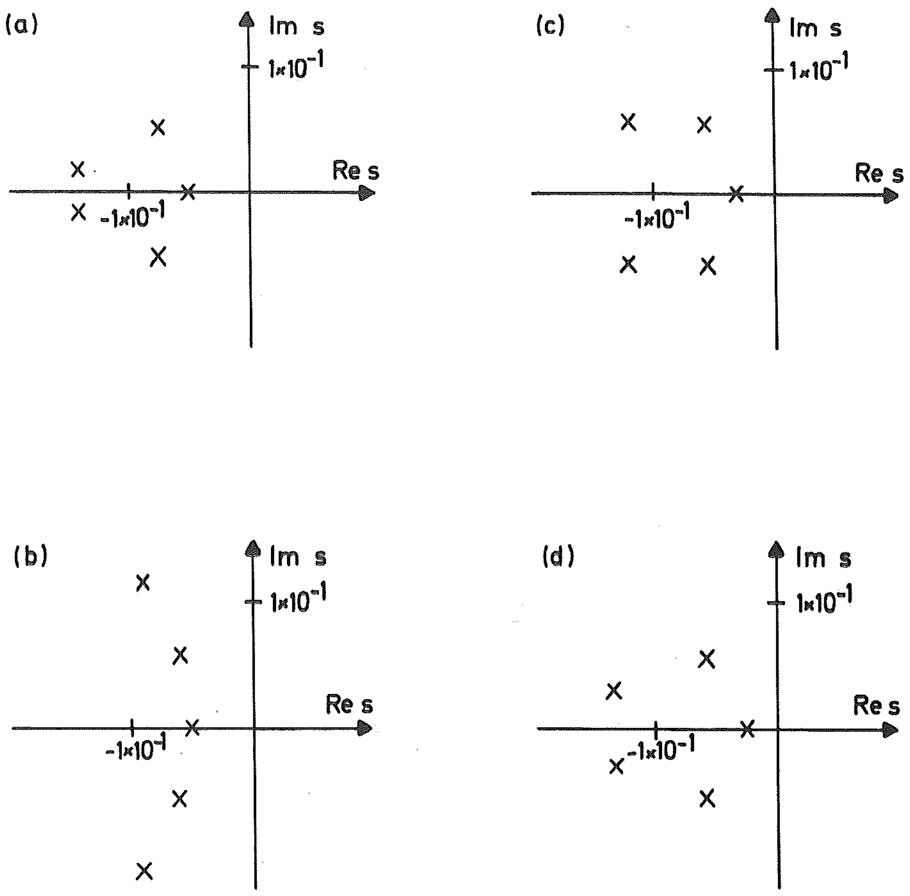

Fig. 7.4 - The pole configurations for the exact and the reduced control laws.

(a) exact control law (7.30)

(b) reduced control law (7.31)

(c) reduced control law (7.32)

(d) reduced control law (7.33)

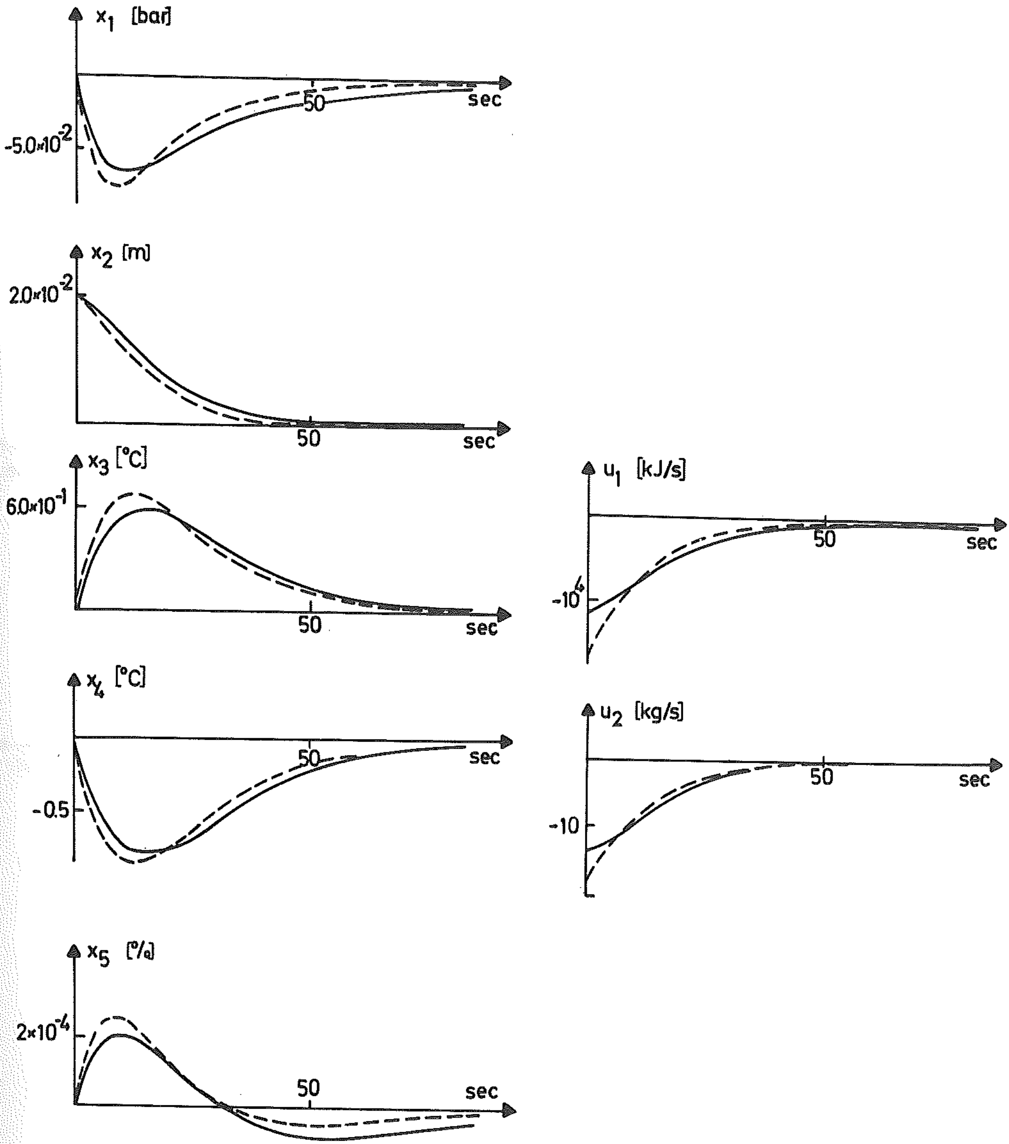

Fig. 7.5 - Responses for an initial condition in drum level of 0.02 m.

------- exact control law

——— reduced control law

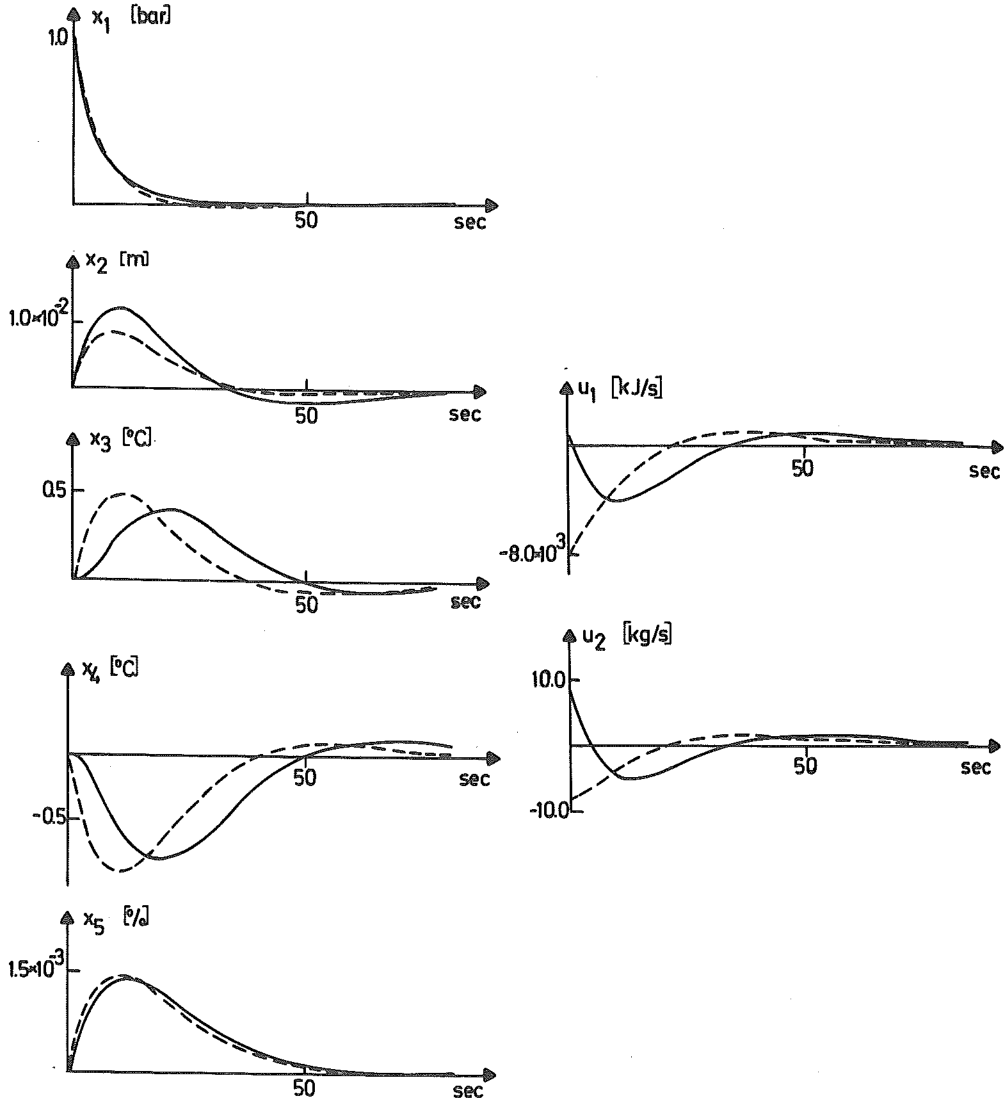

Fig. 7.6 - Responses for an initial condition in drum pressure of 1.0 bar.

------- exact control law

——— reduced control law

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