Some Extensions of Poincaré-Bendixson Theory Applied to a Resonant Converter.

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Does distribution theory contain means for extending Poincaré–Bendixson theory?

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Abstract

We use the theory of distributions to extend the Poincaré–Bendixson theorem and the Bendixson criterion to piecewise Lipschitz continuous system possessing unique and continuous solutions. We demonstrate the use of these extensions by several examples that have recently appeared in the literature.

Keywords: Poincaré–Bendixson theorem; Distributions; Piecewise linear systems

1. Introduction

The study of piecewise linear systems has been essential for applications like control theory, electronics and automatic navigation systems, during the past decades. The formulation of a rich and satisfactory theory for such systems is of utmost importance. Yet, only a few attempts to treat such systems in a general and abstract mathematical setting has been made. Many papers that have appeared quite recently contain, for instance, explicit calculations in specific systems in order to estimate position and number of limit cycles in two-dimensional cases [4,6–9]. In this paper we suggest a new approach based on distrib-

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ution theory [5] in the two-dimensional case. We do not intend to make precise statements regarding the most general cases here, but our approach cover most cases that appear in the application areas mentioned above including optimal foraging theory in mathematical ecology [1]. Our paper is organized as follows. We formulate our two-dimensional setting and our generalized two basic theorems in Section 2. These two generalizations are the essence of the paper. In Section 3 we demonstrate the use of those theorems in several classical examples that contains many difficulties connected to differential equations with discontinuous right-hand sides. Any satisfactory theory must fully explain these examples. We attach figures to most of the examples giving the reader a rapid understanding of what ought to be explained. In Section 4, we give a short summary of our results and list some of their main implications.

2. Our settings and main theorems

We shall work with planar systems with discontinuous right-hand sides throughout this paper. We restrict the properties of the systems under consideration by four major assumptions. The purpose of this paper is to give a presentation of some new ideas, and for simplicity and clarity we do not formulate these ideas in their most general context.

We consider a planar autonomous system

$$\dot{x} = f(x). \tag{1}$$

(A1) Ω is an open domain in R^2, divided into a finite number of open sub-domains Ω_i, such that $\bigcup \bar{\Omega}_i = \bar{\Omega}$.

(A2) If $\bar{\Omega}_i$ and $\bar{\Omega}_j$ are not disjoint and $i \neq j$, then $\bar{\Omega}_i \cap \bar{\Omega}_j = \Gamma_{ij}$, where Γ_{ij} (joint boundaries) are piecewise smooth.

(A3) f is Lipschitz in all sub-domains Ω_i and possibly discontinuous along Γ_{ij} (also called discontinuity curves).

(A4) The vector field f defines a direction in each point in Ω. In particular, at every point of Γ_{ij} the vector field $f(x)$ specifies into which Ω_i the flow is directed.

The conditions (A3) and (A4) implies that the differential equation (1) has unique, continuous and piecewise smooth solutions in Ω. Note that (A4) gives strong restrictions on the possible discontinuities. In terms of Filippov [3] there are three kinds of sliding modes. We only allow transversal sliding mode, that is: the vector field is directed from one side to the other at the discontinuity curves. The solutions will pass the discontinuity curves in the field direction and we have uniqueness of solutions there. Attracting and repulsion sliding mode will be excluded.

Theorem 1 (Extension of the Poincaré–Bendixson theorem). *Consider the planar autonomous system* (1). *Let the conditions* (A1)–(A4) *be satisfied and let f be bounded in Ω. Suppose that K is a compact region in Ω, containing no fixed points of* (1). *If all solutions of* (1) *is in K, for all $t \geqslant t_0$, then* (1) *has a closed orbit in K.*

Remark. We show how to check the conditions of the above theorem at several classical examples.

The proof of the Poincaré–Bendixson theorem uses essentially the same steps as the original Poincaré–Bendixson theorem. We remind the reader about the fact that the direction of a Lipschitzian vector field changes continuously, and base our proof of that observation. The following lemma is true.

Lemma 1. *Because f is Lipschitz in $K \cap \Omega_i$ there exists $\varepsilon > 0$ such that for all $(x, y), (x', y')$ in Ω_i and $|x - x'| < \varepsilon$, $|y - y'| < \varepsilon$ implies that $\wedge(f(x, y), f(x', y')) < \pi/4$ (where $\wedge$ is the angle between the vector fields).*

Proof of Theorem 1. Take (a, b) an arbitrary point in K, $\bar{B}_\varepsilon = [a - \varepsilon, a] \times [b - \varepsilon, b]$ a compact box, and $B_\varepsilon = K \cap \bar{B}_\varepsilon$. If a discontinuity curve crosses such a box, divide it into sub-boxes separated by the discontinuity curves: $B_\varepsilon = \bigcup B_{\varepsilon,i}$, where $B_{\varepsilon,i} = B_\varepsilon \cap \bar{\Omega}_i$. Now K can be covered by a finite number of boxes such that in every box f is Lipschitz and Lemma 1 holds. Choose one of the boxes B_ε such that a trajectory which starts at a point A in B_ε, returns to B_ε at a point B and the line through A and B is a transversal (to the trajectories of (1) inside B_ε). Now all trajectories cross this transversal in the same direction. The trajectory connecting A with B and the segment from B to A along the transversal form a Jordan curve. For the rest of the proof we refer to the proof of the Poincaré–Bendixson theorem. □

Calculus with distributions turn out to be most important when finding upper bounds on the number of limit cycles for systems with discontinuous right-hand sides, as the following examples will show.

Theorem 2 (Extension of the original Bendixson criterion). *Consider the planar autonomous system* (1). *Let the conditions* (A1)–(A4) *be satisfied and let f be bounded in the simply connected region Ω and C^1 in Ω_i. If* div f (*the divergence of f calculated in distribution sense*) *is of the same sign and is not identically zero in Ω, then* (1) *has no closed orbit in Ω.*

Proof. Since the right-hand side f is defined piecewise we have $f = f_i$, $(x, y) \in \Omega_i$. Let χ_{Ω_i} be the characteristic function of Ω_i; then $f = \sum_i f_i \cdot \chi_{\Omega_i}$. Let $f_i = (g_i, h_i)$ so $f = \sum_i (g_i \cdot \chi_{\Omega_i}, h_i \cdot \chi_{\Omega_i})$; this implies

$$\operatorname{div} f = \sum_i \bigl(\partial_x(g_i \cdot \chi_{\Omega_i}) + \partial_y(h_i \cdot \chi_{\Omega_i})\bigr) = \sum_i \bigl(\operatorname{div} f_i \cdot \chi_{\Omega_i} + \langle f_i, \operatorname{grad} \chi_{\Omega_i}\rangle\bigr),$$

div f is defined in the sense of distribution theory and contains, in this case, Dirac pulses and therefore is in $L^1(\Omega)$. Now take a closed, continuous and piecewise smooth curve γ in Ω. Let D denote the inside region of γ and $T = (\dot{x}, \dot{y})$ is the tangent vector of γ; then $N = (-\dot{y}, \dot{x})$ is its normal vector. Consider the part of the line integral $\int_\gamma \langle f, N\rangle\, ds$ in Ω_i, where we have the system $(\dot{x}, \dot{y}) = (g_i(x, y), h_i(x, y))$ and t is in some interval Δ_i. If we use t as parameter this part becomes

$$\int_\gamma \langle f_i \cdot \chi_{\Omega_i}, N\rangle\, ds = \int_{\Delta_i} (-g_i \cdot \dot{y} + h_i \cdot \dot{x})\, dt = \int_{\Delta_i} (-g_i \cdot h_i + h_i \cdot g_i)\, dt = 0.$$

The line integral along the closed curve γ now becomes

$$\int_\gamma \langle f, N\rangle\, ds = \sum_i \int_\gamma \langle f_i \cdot \chi_{\Omega_i}, N\rangle\, ds = 0.$$

According to Hörmander [5] the Gauss–Green formula

$$\int_\gamma \langle f, N\rangle\, ds = -\iint_D \operatorname{div} f\, dx\, dy$$

holds for $f \in C_0^0(\Omega)$ and $\operatorname{div} f \in L^1(D)$. In our case we have $f \in L^1(\Omega)$, this implies that there exists $g \in C_0^0(\Omega)$ such that $\int_\gamma |\langle f, N\rangle - \langle g, N\rangle|\, ds < \varepsilon$.

Then

$$\left| \int_\gamma \langle f, N\rangle\, ds - \int_\gamma \langle g, N\rangle\, ds \right| \leqslant \int_\gamma \bigl|\langle f, N\rangle - \langle g, N\rangle\bigr|\, ds < \varepsilon,$$

so the Gauss–Green formula holds for $f \in L^1(\Omega)$.

This implies that

$$\iint_D \operatorname{div} f\, dx\, dy = 0.$$

This is a contradiction, because $\operatorname{div} f$ never changes signs in Ω and this proves the theorem. □

Remark. It is not a trivial problem to calculate derivatives in sense of distributions, but according to theorems in [5] regarding multiplication and composition of distributions we can use the familiar laws.

3. Examples

Example 1 (Branicky [2]). In this example we consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + (100 - 90\lambda)y + 90(2\lambda - 1) \cdot y \cdot (\mathrm{H}(x) + \mathrm{H}(y) - 2\mathrm{H}(x)\mathrm{H}(y)) \\ -(90\lambda + 10)x - y + 90(2\lambda - 1) \cdot x \cdot (\mathrm{H}(x) + \mathrm{H}(y) - 2\mathrm{H}(x)\mathrm{H}(y)) \end{pmatrix},$$

where H is the Heavyside function and $0 \leqslant \lambda \leqslant 1$. The right-hand side f is in C^1 in each quadrant. Let f_c be the C^1-part of f, the divergence of f_c is $\operatorname{div} f_c = -2$. According to the classical Bendixson criterion this would mean that this system has no closed orbit. The origin is in fact the only fixed point of the system. Let $\lambda_0 = 0.4825443328\ldots$ be the unique solution of the transcendent equation

$$\frac{(10-9\lambda)^4}{(10+81\lambda(1-\lambda))^2} \cdot e^{-\frac{2\pi}{\sqrt{1000+8100\lambda(1-\lambda)}}} = 1$$

and note that f is in C^1 when $\lambda = 1/2$.

A careful analysis of the system reveals that all other orbits except for the fixed point are closed orbits if $\lambda = \lambda_0$. For $0 \leqslant \lambda < \lambda_0$ all orbits spiral outwards from the origin and for $\lambda_0 < \lambda < 1$ the origin is globally asymptotically stable. A calculation of the divergence in sense of distributions confirm that we can use Theorem 2 here. In fact,

$$\operatorname{div} f(x, y) = -2 - 90(2\lambda - 1)\big(|y| \cdot \delta(x) + |x| \cdot \delta(y)\big),$$

from this we can tell:

If $1/2 \leqslant \lambda \leqslant 1$ then $\operatorname{div} f < 0$. According to Theorem 2 this means that the system has no closed orbit, which does not contradict the result above.

Example 2 (Giannakopoulos and Pliete [4]). Consider the planar system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + y + b_1 \cdot \operatorname{sgn}(x) \\ -p \cdot x + b_2 \cdot \operatorname{sgn}(x) \end{pmatrix}, \quad \text{where } p > \frac{1}{4}. \tag{$*$}$$

Giannakopoulos and Pliete [4] concluded after a careful investigation that a necessary condition for the existence of closed orbits of $(*)$ is $b_1 > 0$. In fact Theorem 2 above can be used in order to reduce the algebra here, and we demonstrate the use of it below. We continue by calculating the divergence of $(*)$ and get

$$\operatorname{div} f(x, y) = -1 + 2b_1 \cdot \delta(x)$$

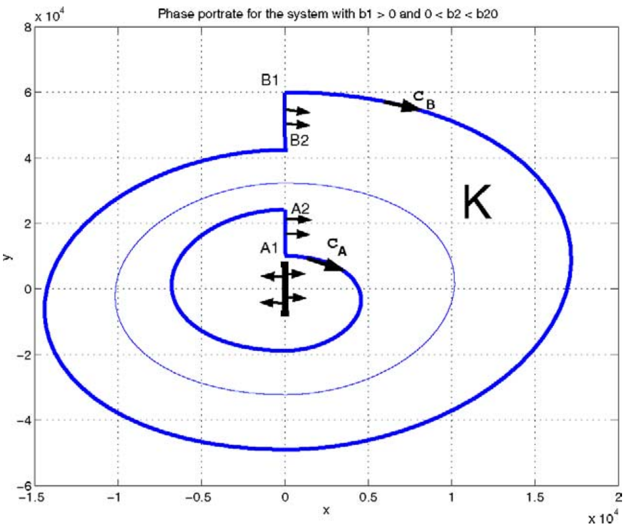

Fig. 1. Limit cycle, Example 2.

concluding that $b_1 > 0$ leads to a sign-change of the divergence at the y-axis. Thus, a necessary condition for closed orbits and limit cycles is $b_1 > 0$.

We continue by formulating a sufficient condition for limit cycles and demonstrating the use of Theorem 1. We assume that the parameters are in the range $b_1 > 0$ and $0 < b_2 < b_2^0$. The system has two fixed points (cf. [4]) and a discontinuity line at the y-axis. Now we construct two closed paths c_A and c_B, see Fig. 1.

c_A: a trajectory from A_1 to A_2, then along the y-axis from A_2 to A_1. Choose A_1 so large that the fixed points are inside γ_A and that the repulsion sliding mode interval $[-b_1, b_1]$ at the y-axis is inside c_A as well.

c_B: a trajectory from B_1 to B_2, then along the y-axis from B_2 to B_1.

Note that the location of A_1, A_2 and B_1, B_2 depend on the parameter range. According to the vector fields there is only transversal sliding mode outside $[-b_1, b_1]$ at the discontinuity line (the y-axis). Let K be the annular region between c_A and c_B including the boundary. Then K is a compact set and the conditions of Theorem 1 are satisfied. Then Theorem 1 implies that the system has a closed orbit in K, which coincides with [4].

Example 3. Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x - h \cdot y + k \cdot H(y) \end{pmatrix}.$$

This implies that $\operatorname{div} f(x, y) = -h + k \cdot \delta(y)$. If $h > 0$ and $k > 0$, we have a necessary condition for existence of closed orbits. Of course this does not imply a closed orbit, but direct calculations show that there is a unique limit cycle if $0 < h < 2$ and $k > 0$.

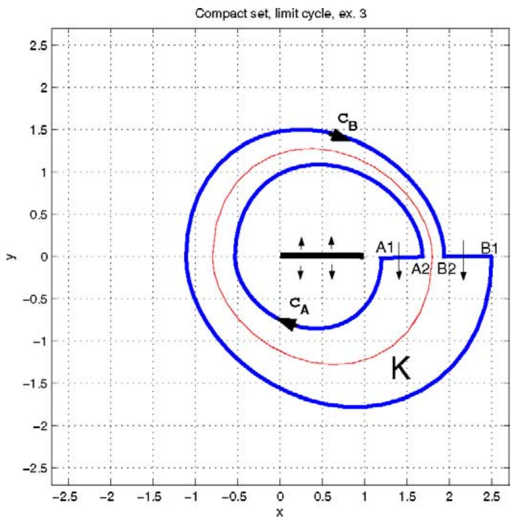

Fig. 2. Limit cycle, Example 3.

We can use Theorem 1 to prove existence of a closed orbit, for the same parameter range. The fixed points are $(0, 0)$ and $(k, 0)$ and the discontinuity line is the x-axis. The direction of the vector fields gives repulsion sliding mode in the interval $[0, k]$ and transversal sliding mode elsewhere at the x-axis. We choose a starting point $(r_0, 0)$ with $r_0 > k$ and follow the trajectory until it intersects the positive x-axis again at $(r_1, 0)$. Elementary calculations gives

$$r_1 = r_0 \cdot e^{-2\pi h/\omega} + k \cdot \left(1 + e^{-\pi h/\omega}\right), \quad \text{where } \omega = \sqrt{4 - h^2}.$$

Put $r_0' = \frac{k}{1-e^{-\pi h/\omega}}$, this implies that $r_0' > k$ and $r_1 - r_0 = (1 - e^{-2\pi h/\omega}) \cdot (r_0' - r_0)$.

We have $r_1 > r_0$ if $r_0 < r_0'$ and $r_1 < r_0$ if $r_0 > r_0'$. Construct a compact set K, as the annular region between the two closed paths c_A and c_B including the boundary, see Fig. 2.

c_A: a trajectory from A_1 to A_2 and the x-axis from A_2 to A_1.
c_B: a trajectory from B_1 to B_2 and the x-axis from B_2 to B_1.

The set K now satisfies the conditions of Theorem 1, and it follows that the system has a closed orbit in K. The closed orbit will of course always pass through the point $(r_0', 0)$.

Example 4. Let $\Omega_1 = \{(x, y);\ x^2 + y^2 < 1\}$, $\Omega_2 = \{(x, y);\ 1 < x^2 + y^2 < 4\}$ and $\Omega_3 = \{(x, y);\ x^2 + y^2 > 4\}$.

Consider the piecewise linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad (x, y) \in \Omega_1,$$
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad (x, y) \in \Omega_2,$$

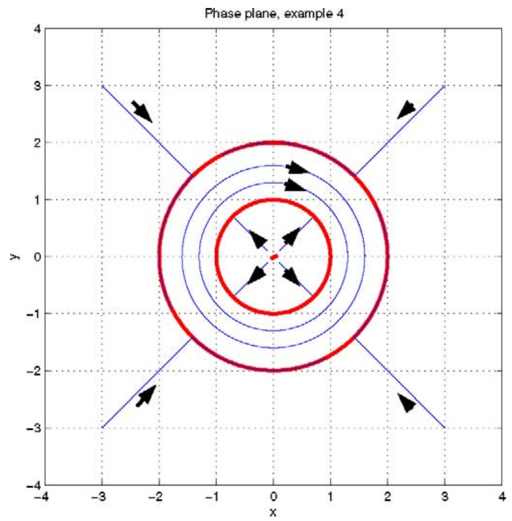

Fig. 3. Phase plane, Example 4.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}, \quad (x, y) \in \Omega_3.$$

The only fixed point of the system is origin, which is unstable. The phase portrait is shown in Fig. 3.

We have attracting sliding mode along the circles and consequently the conditions of the local uniqueness of solutions are not satisfied. The solutions are indeed not unique at this circles. Either forward or backward uniqueness is broken. Thus the conditions of Theorem 1 are not satisfied. But if we let $K = \{(x, y);\ 1 < r^2 \leqslant x^2 + y^2 \leqslant R^2 < 4\}$, then K is a compact set satisfying the conditions and there is a closed orbit in K. In fact according to the phase portrait there is a infinite number of closed orbits in this annulus.

After some nontrivial calculations, the divergence of the right-hand side is

$$\begin{aligned} \operatorname{div} f(x, y) = 2 - 2 \cdot \mathrm{H}(x^2 + y^2 - 1) - 2 \cdot \mathrm{H}(x^2 + y^2 - 4) - 2 \cdot \delta(x^2 + y^2 - 1) \\ - 8 \cdot \delta(x^2 + y^2 - 4). \end{aligned}$$

We conclude that $\operatorname{div} f(x, y) = 2$ in the simply connected region Ω_1. According to Theorem 2 the system has no closed orbits in Ω_1 and that coincides with the phase portrait.

4. Summary

In this paper we have formulated extensions of the Poincaré–Bendixson theorem and the Bendixson criterion that in principle could be applied to piecewise nonlinear systems. Current literature that is concerned with ordinary differential equations possessing piecewise continuously differentiable right-hand sides have mainly considered the piecewise linear case. In order to demonstrate the usefulness of our extensions we have chosen several (piecewise linear) examples that have occurred previously in the literature and to which our theorems can be applied. When our theorems are applied we must start by using Heavyside functions to describe the right-hand sides of the systems. The derivatives of such functions will then usually contain Dirac pulses. It is not a trivial matter to calculate quantities like the divergence of the vector field in distribution sense, but once such an expression has been calculated correctly, our extension of the Bendixson criterion may reveal interesting qualitative properties of the system. The application of our extension of the Poincaré–Bendixson theorem does not include nontrivial calculations of distributions. The application of the theorem requires construction of a compact set without attracting or repulsion sliding mode inside. The main advantage of this extension is thus a possibility to avoid tedious explicit calculations of the trajectories (when possible) in order to prove existence of limit cycles in systems of ordinary differential equations possessing discontinuities in their right-hand sides.

We have demonstrated the usefulness of our extensions on several examples that have appeared recently in the literature. We begin by analyzing an example brought out by Branicky [2] demonstrating that classical two-dimensional qualitative theory does not extend to discontinuous systems, but where our extensions of the theory give accurate explanations of the qualitative behavior of the system.

Our second example was analyzed in detail by Giannakopoulos and Pliete [4] through extensive explicit calculations of the trajectories of the system involved. We show how parts of this example can be managed by less extensive calculations. The prize we pay is that we have to be able to make calculus with distributions available for broader audiences. In addition we provide two examples showing explicitly the construction of relevant compact regions in our extension of the Poincaré–Bendixson theorem.

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