LUND UNIVERSITY

Fluids in motion

Johannesson, Björn

1998

[Link to publication](#)

Total number of authors:
1

Fluids in motion

Björn Johannesson

Fluids in motion

Björn Johannesson

1998

Abstract

1 Fluids in motion

Here all the essential principles within the continuum concept will be adopted to obtain the so-called Navier Stoke's equations. All five balance principles, i.e. balance of mass, linear and angular momentum, energy and the second axiom of thermodynamics will be considered. Furthermore the principles of constitutive theory will be utilized, that is, frame-indifference, material symmetry (isotropic materials only), and restrictions from the second axiom of thermodynamics.

1.1 Thermodynamic potentials

In order to give a detailed description of viscous fluids some thermodynamic definitions must be considered. The main reason to this is that a thermodynamic pressure is introduced in the description of a general fluid. This pressure differs in most cases from the mean mechanical pressure. Certain restriction on the thermodynamic pressure may be introduced which is restrictions imposed by the second axiom of thermodynamics. Here a method will be discussed where it is not necessary to impose special restriction on the thermodynamic pressure.

Continuum thermodynamics based on a caloric equation of state assumes that the local internal energy ε per unit mass is determined by the Thermodynamic state, specified by $n+1$ state variables ν_1, ν_2, ..., ν_n and η, where ν_j

are the *thermodynamic substate variables* and η is the specific entropy. The substate variables have mechanical or electromagnetical dimensions, but are otherwise left arbitrary in the general formulation. In the simplest case of a fluid pure substance there is only one substate variable, the specific volume v. In ideal elasticity we will have nine substate variables, the components of one of the strain or deformation tensors. In other cases the identification of the substate variables may be difficult.

The basic assumption of thermodynamics has been that in addition to the n substate variables ν_j, just one additional dimensionally independent scalar parameter suffices to determine the specific internal energy density ε. This assumes that there exists a *caloric equation of state*

$$\varepsilon = \varepsilon\left(\eta, \boldsymbol{\nu}, \mathbf{X}\right) \tag{1}$$

In any particular motion $\mathbf{x} = \mathbf{x}\left(\mathbf{X}, t\right)$, where $\mathbf{x}$ is the current *place* and $\mathbf{X}$ is the reference configuration (or equally the particle) and $\boldsymbol{\nu}$ is all the n substate variables. The possible dependence of $\mathbf{X}$ only includes the possibility of the caloric equation being different for different particles in inhomogeneous media. Often one do not explicitly include $\mathbf{X}$ in the caloric equation of state.

The *thermodynamic temperature* θ is defined by

$$\theta \equiv \left(\frac{\partial \varepsilon}{\partial \eta}\right)_{\nu} \tag{2}$$

where ε is the internal energy density and η is the entropy density. The properties $\boldsymbol{\nu}$ are the thermodynamic substate variables. The subscript indicates that the thermodynamic substate variables are held constant.

The so-called thermodynamic tensions τ_j are defined by

$$\tau_j \equiv \left(\frac{\partial \varepsilon}{\partial \nu_j}\right)_{\eta}; \qquad j = 1, 2, ..., n. \tag{3}$$

Thus, in any real or hypothetical change in the thermodynamic state of a given particle $\mathbf{X}$ one must have

$$d\varepsilon = \theta d\eta + \tau_j d\nu_j; \qquad \text{sum } j = 1 \text{ to } n. \tag{4}$$

This is known as a *Gibbs relation*, although Gibbs gave the equation only for the case of a fluid, whose only substate variable is the specific volume v. for the fluid the Gibbs relation takes the form

$$d\varepsilon = \theta d\eta - p dv; \quad \text{so that} \quad \theta = \left(\frac{\partial \varepsilon}{\partial \eta}\right)_v \quad \text{and} \quad -p = \left(\frac{\partial \varepsilon}{\partial v}\right)_{\eta} \tag{5}$$

2

where p is the thermodynamic pressure; the thermodynamic tension conjugate to the specific volume v is $-p$, as θ is conjugate to η. This p is not in general equal to the mean pressure $\bar{p}$ and its relation to any measured pressure for nonideal fluids in motion must be established. For a viscous fluid the mean pressure will be equal to the thermodynamic pressure only in two special cases. This will be discussed in detail in section xx.

From the caloric equation of state (1) and the definitions (2) and (3), it follows that the temperature and the thermodynamic tensions are functions of the thermodynamic state: For a given particle

$$\theta = \theta\left(\eta, \boldsymbol{\nu}\right); \qquad \tau_j = \tau_j\left(\eta, \boldsymbol{\nu}\right) \tag{6}$$

Assuming the first equation in (6) to be invertible to yield

$$\eta = \eta\left(\theta, \boldsymbol{\nu}\right) \tag{7}$$

and by substitute this into the caloric equation of state (1) to obtain an alternative form for the state equation

$$\varepsilon = \varepsilon\left(\theta, \boldsymbol{\nu}, \mathbf{X}\right) \tag{8}$$

Substituting equation (7) into the second equation (6), one obtain

$$\tau_j = \tau_j\left(\theta, \boldsymbol{\nu}, \mathbf{X}\right) \tag{9}$$

or assuming invertibility

$$\nu_j = \nu_j\left(\theta, \boldsymbol{\tau}, \mathbf{X}\right) \tag{10}$$

These last two equations are called *thermal equations of state*. (In most cases, the explicit dependence on the particle $\mathbf{X}$ will not be indicated.) The thermal equations of state resembles stress-strain relations, but some caution is necessary in interpreting the tensions as stresses and ν_j as strains (or as velocity gradients for a fluid). The difference between the thermodynamic pressure p and the mechanical pressure $\bar{p}$ is one reason to be careful.

With the above definition of the thermodynamic temperature and the thermodynamic tensions one can define the following thermodynamic potentials, four thermodynamic potentials are introduced, each useful for certain choice of independent state variables, as tabulated below

$$\varepsilon; \quad \text{Indep. Variab. } \eta, \nu_j \tag{11}$$

$$\psi = \varepsilon - \eta\theta; \quad \text{Indep. Variab. } \theta, \nu_j \tag{12}$$

$$h = \varepsilon - \tau_j\nu_j; \quad \text{Indep. Variab. } \eta, \tau_j \tag{13}$$

$$g = \varepsilon - \eta\theta - \tau_j\nu_j = h - \eta\theta; \quad \text{Indep. Variab. } \theta, \tau_j \tag{14}$$

3

where ψ is the Helmholt's free energy, h is the entalpy and g is the free entalpy or Gibbs function. By having the independent functions as

$$\varepsilon = \varepsilon(\eta, \nu_j) \tag{15}$$

$$\psi = \psi(\theta, \nu_j) \tag{16}$$

$$h = h(\eta, \tau_j) \tag{17}$$

$$g = g(\theta, \tau_j) \tag{18}$$

one can define the following partial derivative expressions

$$\theta = \left(\frac{\partial \varepsilon}{\partial \eta}\right)_\nu ; \qquad \tau_j = \left(\frac{\partial \varepsilon}{\partial \nu_j}\right)_\eta \tag{19}$$

$$\eta = -\left(\frac{\partial \psi}{\partial \theta}\right)_\nu ; \qquad \tau_j = \left(\frac{\partial \psi}{\partial \nu_j}\right)_\theta \tag{20}$$

$$\theta = \left(\frac{\partial h}{\partial \eta}\right)_\tau ; \qquad \nu_j = -\left(\frac{\partial h}{\partial \tau_j}\right)_\eta \tag{21}$$

$$\eta = -\left(\frac{\partial g}{\partial \theta}\right)_\tau ; \qquad \nu_j = -\left(\frac{\partial g}{\partial \tau_j}\right)_\theta \tag{22}$$

and the differentiation of (72)-(75) hence takes the forms

$$d\varepsilon = \theta d\eta + \tau_j d\nu_j \tag{23}$$

$$d\psi = -\eta d\theta + \tau_j d\nu_j \tag{24}$$

$$dh = \theta d\eta - \nu_j d\tau_j \tag{25}$$

$$dg = -\eta d\theta - \nu_j d\tau_j \tag{26}$$

The Helmholt's free energy ψ is the portion of the internal energy available for doing work at *constant* temperature. The entalpy h is the portion of the internal energy that can be released as heat when the thermodynamic tension are held *constant*.

It can be shown that the mass density in the reference configuration ρ_{oR} can be related as

$$\frac{\rho_{oR}}{\rho} = \frac{dV_{oR}}{dV} = |\det \mathbf{F}| \tag{27}$$

where $\mathbf{F}$ is the deformation gradient. It is also possible define a stress tensor related to the reference configuration $\mathbf{T}^{oR}$ as

$$\mathbf{T}^{oR} = \frac{\rho_{oR}}{\rho} \mathbf{F}^{-1} \mathbf{T} = |\det \mathbf{F}| \mathbf{F}^{-1} \mathbf{T} \tag{28}$$

4

or

$$\mathbf{T} = \frac{\rho}{\rho_{oR}}\mathbf{F}\mathbf{T}^{oR} \tag{29}$$

Loosely speaking, the transformation (28) accounts for that the direction of the traction force on a surface (contributing to a stress state $\mathbf{T}^{oR}$ in the undeformed state) changes its direction in the deformed state. When the deformation gradients are small $\mathbf{T}^{oR} \approx \mathbf{T}$ i.e. $\rho_{oR} \approx \rho$. But, in general one may for example have a viscous fluid which mass density from a reference state differs significantly from the actual mass density in the deformed state.

The so-called stress power $\mathrm{tr}(\mathbf{TD})$ can be formulated as

$$\mathrm{tr}\left(\mathbf{TD}\right) = \mathrm{tr}\left(\mathbf{TL}\right) = \mathrm{tr}\left(\mathbf{T}\dot{\mathbf{F}}\mathbf{F}^{-1}\right) \tag{30}$$

since $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ and due to $\mathbf{L} = \mathbf{D} + \mathbf{W}$ where $\mathbf{W}$ is skew symmetric with zero element in the diagonal. And further

$$\mathrm{tr}\left(\mathbf{T}\dot{\mathbf{F}}\mathbf{F}^{-1}\right) = \mathrm{tr}\left(\frac{\rho}{\rho_{oR}}\mathbf{F}\mathbf{T}^{oR}\dot{\mathbf{F}}\mathbf{F}^{-1}\right) = \frac{\rho}{\rho_{oR}}\mathrm{tr}\left(\mathbf{T}^{oR}\dot{\mathbf{F}}\right) \tag{31}$$

since $\mathbf{F}\mathbf{F}^{-1} = \mathbf{1}$. That is

$$\mathrm{tr}\left(\mathbf{TD}\right) = \frac{\rho}{\rho_{oR}}\mathrm{tr}\left(\mathbf{T}^{oR}\dot{\mathbf{F}}\right) \tag{32}$$

One may choose the stress state $\mathbf{T}^{oR}$ by identifying the thermodynamic tensions τ_j as nine components denoted τ where τ is identified with the deformation gradient $\mathbf{F}$ which is a natural choice since $\mathbf{F}$ relates the mass density ρ_{oR} and ρ. That is one may write

$$\tau = \frac{1}{\rho_{oR}}\mathbf{T}^{oR} \tag{33}$$

where ρ_{oR} has been included, and ν is identified as

$$\nu = \mathbf{F} \tag{34}$$

Using the partial derivative expression (19b), i.e. $\tau_j = \left(\partial\varepsilon/\partial\nu_j\right)_\eta$ where each of the nine components are arranged into the format of τ, as

$$\tau = \frac{1}{\rho_{oR}}\mathbf{T}^{oR} = \left(\partial\varepsilon/\partial\nu\right)_\eta = \left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \tag{35}$$

5

where the choice (33) and (34) has been adopted.

Consider, also, the stress tensor $\mathbf{T}$ defined as consisting of two parts, one equilibrium part $\mathbf{T}^E$ which is independent of the velocity gradient $\mathbf{D}$ and a dynamical part $\mathbf{T}^D$ which is dependent on the velocity gradient, i.e. $\mathbf{T}^E$ and $\mathbf{T}^D$ are independent, and related as

$$\mathbf{T} = \mathbf{T}^E + \mathbf{T}^D \tag{36}$$

Noting, further that $\mathbf{T} = \rho/\rho_{oR}\mathbf{F}\mathbf{T}^{oR}$, i.e. (29), therefore the stress tensor $\mathbf{T}^E$ can be formulated as

$$\mathbf{T}^E = \mathbf{T} = \rho/\rho_{oR}\mathbf{F}\mathbf{T}^{oR} = \rho\mathbf{F}\left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \tag{37}$$

during a situation where $\mathbf{T}^D = \mathbf{0}$.

Dealing only with a symmetric stress tensor $\mathbf{T}$ the energy balance can be written

$$\rho\dot{\varepsilon} = \mathrm{tr}\,(\mathbf{TL}) - \mathrm{div}\,(\mathbf{q}) + \rho r \tag{38}$$

From the 'equilibrium' thermodynamic relation we choose (23), i.e.

$$d\varepsilon = \theta d\eta + \tau_j d\nu_j \tag{39}$$

writing this with the differential d exchanged by the material derivative as

$$\dot{\varepsilon} = \theta\dot{\eta} + \sum_{j=1}^{n}\tau_j\dot{\nu}_j \tag{40}$$

The nine components in the last term on the left-hand side are arranged with help from the certain choice for τ_j, i.e. (35) and ν_j, i.e. (34), as

$$\sum_{j=1}^{n}\tau_j\dot{\nu}_j = \left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \cdot \dot{\mathbf{F}} \tag{41}$$

Combining (38), (40), and (36) to yield

$$\rho\theta\dot{\eta} = \mathrm{tr}\left(\left(\mathbf{T}^E + \mathbf{T}^D\right)\mathbf{L}\right) - \mathrm{div}\,(\mathbf{q}) + \rho r - \rho\sum_{j=1}^{n}\tau_j\dot{\nu}_j \tag{42}$$

Using, the expression (41) in (42) and noting that $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, gives

$$\rho\theta\dot{\eta} = \mathrm{tr}\left(\left(\rho\mathbf{F}\left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{T}^D\mathbf{L}\right)\right) - \mathrm{div}\,(\mathbf{q}) + \rho r - \rho\left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \cdot \dot{\mathbf{F}} \tag{43}$$

since $\mathbf{F}\mathbf{F}^{-1} = \mathbf{1}$ i.e. $\operatorname{tr}\left(\rho\mathbf{F}\left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \dot{\mathbf{F}}\mathbf{F}^{-1}\right) = \operatorname{tr}\left(\rho\left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \dot{\mathbf{F}}\right)$. Further, by noting that $\operatorname{tr}\left(\rho\left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \dot{\mathbf{F}}\right) = \rho\left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \cdot \dot{\mathbf{F}}$ it is clear that the contribution from the thermodynamic tension $\rho\left(\partial\varepsilon/\partial\mathbf{F}\right)_\eta \cdot \dot{\mathbf{F}}$ cancels due to the 'equilibrium' stress power $\operatorname{tr}\left(\mathbf{T}^E\mathbf{L}\right)$ being equal with an opposite sign. That is (43) reduces to

$$\rho\theta\dot{\eta} = \operatorname{tr}\left(\mathbf{T}^D\mathbf{L}\right) - \operatorname{div}\left(\mathbf{q}\right) + \rho r \tag{44}$$

or

$$\rho\theta\dot{\eta} = \operatorname{tr}\left(\mathbf{T}^D\mathbf{D}\right) - \operatorname{div}\mathbf{q} + \rho r \tag{45}$$

since $\operatorname{tr}\left(\mathbf{T}^D\mathbf{L}\right) = \operatorname{tr}\left(\mathbf{T}^D\mathbf{D}\right)$. That is

$$\frac{r}{\theta} = \dot{\eta} - \frac{1}{\rho\theta}\operatorname{tr}\left(\mathbf{T}^D\mathbf{D}\right) + \frac{1}{\rho\theta}\operatorname{div}\left(\mathbf{q}\right) \tag{46}$$

The second axiom of thermodynamics i.e.

$$\dot{\eta} \geq r/\theta - \frac{1}{\rho}\operatorname{div}\left(\mathbf{q}/\theta\right) \tag{47}$$

where γ is the entropy production, can be rewritten as

$$\gamma = \dot{\eta} - r/\theta + \frac{1}{\rho\theta}\operatorname{div}\mathbf{q} - \frac{\mathbf{q}}{\rho\theta^2}\cdot\operatorname{grad}\theta \geq 0 \tag{48}$$

where the identity

$$\frac{1}{\rho}\operatorname{div}\left(\mathbf{q}/\theta\right) = \frac{\mathbf{q}}{\rho\theta^2}\cdot\operatorname{grad}\theta - \frac{1}{\rho\theta}\operatorname{div}\mathbf{q} \tag{49}$$

has been used.

By eliminating the term r/θ between (45) and (48) to obtain

$$\gamma = \frac{1}{\rho\theta}\operatorname{tr}\left(\mathbf{T}^D\mathbf{D}\right) - \frac{\mathbf{q}}{\rho\theta^2}\cdot\operatorname{grad}\theta \geq 0 \tag{50}$$

or equally

$$\rho\theta\gamma = \operatorname{tr}\left(\mathbf{T}^D\mathbf{D}\right) - \frac{\mathbf{q}}{\theta}\cdot\operatorname{grad}\theta \geq 0 \tag{51}$$

It is clear that the following two conditions is sufficient to assure that the inequality is satisfied

$$\rho\theta\gamma_{mech.} = \operatorname{tr}\left(\mathbf{T}^D\mathbf{D}\right) \geq 0 \tag{52}$$

7

$$\rho\theta\gamma_{cond.} = -\frac{\mathbf{q}}{\theta} \cdot \operatorname{grad}\theta \geq 0 \qquad (53)$$

By considering the stress power $\operatorname{tr}\left(\mathbf{T}^D\mathbf{D}\right) \geq 0$ only, it is concluded that $\mathbf{T}^E$ as defined in (37) can be chosen as $\mathbf{T}^E = -p\mathbf{I}$, where p being a thermodynamic pressure. This means that the thermodynamic pressure can be assumed to be recoverable not contributing to any dissipation since the inequality (52) do not include the effect of the term $\operatorname{tr}\left(\mathbf{T}^E\mathbf{D}\right)$.

1.2 Deviatoric stress

In turns out that it is advantageous to separate the mechanical pressure from the stress tensor when it comes to derive the equations for a fluid. Therefore the concept of deviatoric stress will be introduced. The mean normal 'mechanical' pressure is defined as

$$-\bar{p} = \tfrac{1}{3}\left(T_{11} + T_{22} + T_{33}\right) = \tfrac{1}{3}\operatorname{tr}\mathbf{T} \qquad (54)$$

where

$$-\bar{p}\mathbf{I} = \begin{bmatrix} -\bar{p} & 0 & 0 \\ 0 & -\bar{p} & 0 \\ 0 & 0 & -\bar{p} \end{bmatrix} \qquad (55)$$

Recall that the components in the stress tensor $\mathbf{T}$ can be represented as

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \qquad (56)$$

The deviatoric stress tensor $\mathbf{T}'$ is defined by the stress tensor $\mathbf{T}$ and the mechanical pressure $\bar{p}$, as

$$\mathbf{T}' = \begin{bmatrix} T_{11} + \bar{p} & T_{12} & T_{13} \\ T_{21} & T_{22} + \bar{p} & T_{23} \\ T_{31} & T_{32} & T_{33} + \bar{p} \end{bmatrix} \qquad (57)$$

Therefore, the relation between the stress tensor $\mathbf{T}$ and the deviatoric stress tensor $\mathbf{T}'$ is

$$\mathbf{T}' = \mathbf{T} + \bar{p}\mathbf{I} \qquad (58)$$

which follows from the definitions (54) and (57).

Another deviator often used in fluid mechanics is the definition

$$\mathbf{D}' = \mathbf{D} - \tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} \qquad (59)$$

The physical significance of the second term on the right hand side of this expression will be analyzed further in section xx.

1.3 Restrictions imposed by material frame-indifference

It was shown in previous section that an assumption for the stress tensor cannot be arbitrary chosen due to the principle of frame-indifference (or objectivity of space). For example the constitutive function for the stress; $\mathbf{T} = \mathbf{f}\left(\mathbf{D}, \mathbf{W}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \theta, \mathrm{grad}\theta, \rho\right)$ cannot hold due to the frame-indifference postulate, since $\mathbf{W}$, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ do not behave indifferent under a change of frame. That is the above proposed relation for the stress tensor $\mathbf{T}$ must, in this case, be reduced to $\mathbf{T} = \mathbf{f}\left(\mathbf{D}, \theta, \mathrm{grad}\theta, \rho\right)$.

It can, also, be shown that $\mathrm{grad}\theta$ cannot be included as an independent constitutive variable due to the restrictions imposed by the second axiom of thermodynamics. This has already been pointed out in section xx. Indeed, the independence of $\mathrm{grad}\theta$ when determining the stress is not the only restriction imposed by the entropy inequality when studying viscous fluids, therefore this subject will be discussed further in section xx.

1.4 Restrictions imposed by material symmetry, isotropy

It was shown in section xx that an isotropic material function, e.g. $\mathbf{T} = \mathbf{f}_m\left(\mathbf{D}\right)$, where the material function $\mathbf{f}_m$ is independent of the reference configuration, i.e. $\mathbf{f}_m\left(\mathbf{D}\right)$ is an isotropic material function, the stress tensor $\mathbf{T}$ can only be related to the symmetric part of the velocity gradient as

$$\mathbf{T} = \mathbf{f}_m\left(\mathbf{D}\right) = \alpha_1\mathbf{I} + \alpha_2\mathbf{D} + \alpha_3\mathbf{D}^2 \qquad (60)$$

where α_1, α_2 and α_3 are material parameters which can be functions of invariant measures only. In section xx three different invariant measures where discussed i.e. $\beta_1 = \mathrm{tr}\mathbf{D}$, $\beta_2 = \tfrac{1}{2}\left(\mathrm{tr}\mathbf{D}\right)^2 - \tfrac{1}{2}\mathbf{D}^2$ and $\beta_3 = \det(\mathbf{D})$. The values β_1, β_2 and β_3 takes the same values independently of any orthogonal transformation since they are invariants.

By introducing the thermodynamic pressure p (which not should be confused with the mechanical pressure $\bar{p}$ which is due to the fluid having a motion) and the two material coefficients λ and μ characterizing the viscosity

of the fluid, as

$$\alpha_1 = -p + \lambda\left(\mathrm{tr}\mathbf{D}\right); \quad \text{and} \quad \alpha_2 = 2\mu; \quad \text{and} \quad \alpha_3 = 0 \tag{61}$$

where only one invariant is invoked, that is, $\mathrm{tr}\mathbf{D}$. The constitutive relation for the stress tensor $\mathbf{T}$ in (60) then becomes

$$\mathbf{T} = \mathbf{f}_m\left(\mathbf{D}\right) = -p\mathbf{I} + \lambda\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\mathbf{D} \tag{62}$$

.This is the so-called *Navier-Poisson law of a Newtonian fluid*. The simplest case of (62) is due to newton (1687). The three-dimensional case for a in-compressible fluid where obtained by molecular models by Navier (1821) and by Poisson (1831) in general. The continuum theory is due to St. Venant (1843) and Stokes (1845).

In some application it is of interest to assume constitutive relations for the stress tensor $\mathbf{T}$ by an isotropic function $\mathbf{f}_m$ which is dependent on two different second order tensors. This can be illustrated as

$$\mathbf{T} = \mathbf{f}_m\left(\mathbf{D}, \mathbf{N}\right) \tag{63}$$

where $\mathbf{D}$ and $\mathbf{N}$ are symmetric second order tensors. It turns out that the exact dependency of this type of isotropic constitutive relation must be related as

$$\begin{aligned}
\mathbf{T} = \ & \alpha_1\mathbf{I} + \alpha_2\mathbf{D} + \alpha_3\mathbf{D}^2 + \alpha_4\mathbf{N} + \alpha_5\mathbf{N}^2 \\
& + \alpha_6\left(\mathbf{ND} + \mathbf{DN}\right) + \alpha_7\left(\mathbf{N}^2\mathbf{D} + \mathbf{DN}^2\right) \\
& + \alpha_8\left(\mathbf{ND}^2 + \mathbf{D}^2\mathbf{N}\right)
\end{aligned} \tag{64}$$

due to the assumed material symmetry (isotropy). The material parameters $\alpha_{1...8}$ can only be dependent on invariants of $\mathbf{D}$ and $\mathbf{N}$ and on other scalar properties.

Moreover, an isotropic vector-valued function $\mathbf{f}_m\left(\mathbf{D}, \mathbf{v}\right)$ of a symmetric tensor $\mathbf{D}$ and a vector $\mathbf{v}$ has the representation

$$\mathbf{f}_m\left(\mathbf{D}, \mathbf{v}\right) = \left(\alpha_1\mathbf{I} + \alpha_2\mathbf{D} + \alpha_3\mathbf{D}^2\right)\mathbf{v} \tag{65}$$

where α_1, α_2 and α_3 are simultaneous scalar function of the invariants.

1.5 Obtaining the Navier-Poisson law in an alternative manner

It turns out that it can be illustrative to used the tensor index notation to show the approximations which are involved in reaching the Navier-Poissons law. It should be observed, however, that nothing new is introduced into the model.

The stress is assumed to be determined by the thermodynamic pressure p and the velocity gradient $\mathbf{D}$, as

$$\mathbf{T} = -p\mathbf{I} + \hat{\mathbf{C}}(\mathbf{D}); \quad T_{ij} = -p\delta_{ij} + \hat{C}_{ijkl}D_{kl} \qquad (66)$$

Where $\hat{\mathbf{C}}$ is a fourth order tensor, i.e. $\hat{C}_{ijkl}$.

The fourth order tensor $\hat{\mathbf{C}}$ is explicitly assumed to be isotropic, that is any change of reference configuration will leave $\hat{\mathbf{C}}$ unaffected.. Alternatively, an isotropic tensor is one whose rectangular Cartesian components are unchanged by any orthogonal transformation of the coordinate axes. A trivial example is the zero tensor of any order. All tensors order zero (scalars) are isotropic, but there are no isotropic first order tensors (vectors) except the zero vector. The unit tensor $\mathbf{I}$, whose components are given in any rectangular Cartesian system by the Kronecker delta δ_{ij}, is isotropic, and it can therefore be proved that this and scalar multiples of it are the only nontrivial second-order isotropic tensors. (Malvern)

The most general fourth-order *isotropic* tensor $\hat{C}_{ijkl}$ has Cartesian components of the form

$$\hat{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + v\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) \qquad (67)$$

where the material parameter scalars λ, μ and v have the same value in all coordinate systems.

It is possible to show that the isotropic tensor $\hat{C}_{ijkl}$ must have the symmetry condition $\hat{C}_{ijkl} = \hat{C}_{jikl}$ when the stress tensor is symmetric, i.e. when $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$. That is

$$T_{ij} = T_{ji}; \quad \mathbf{T} = \mathbf{T}^{\mathrm{T}} \quad \text{one must have} \quad \hat{C}_{ijkl} = \hat{C}_{jikl} \qquad (68)$$

The velocity gradient $\mathbf{D}$ is symmetric by definition. One may therefore choose yet another symmetry condition, namely $\hat{C}_{ijkl} = \hat{C}_{ijlk}$

$$D_{ij} = D_{ji}; \quad \mathbf{D} = \mathbf{D}^{\mathrm{T}} \quad \text{one may choose} \quad \hat{C}_{ijkl} = \hat{C}_{ijlk} \qquad (69)$$

11

compare xx for proof.. In either case, if $\hat{C}_{ijkl}$ is required to be symmetric in either ij or kl, one must have $\upsilon = 0$.

Hence, the most general fourth-order isotropic tensor with Cartesian components symmetric in either ij or kl has the components of the form

$$\hat{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{70}$$

Then this is substituted into (66b) the constitutive equation takes the form

$$T_{ij} = -p\delta_{ij} + \left(\lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right)D_{kl} \tag{71}$$

i.e.

$$T_{ij} = -p\delta_{ij} + \lambda D_{mm}\delta_{ij} + 2\mu D_{ij} \tag{72}$$

or in the direct tensor notation

$$\mathbf{T} = -p\mathbf{I} + \lambda\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\mathbf{D} \tag{73}$$

That is, expression (62) and (73) are identical.

1.6 Relation between the thermodynamic and mechanical pressure

Here a physical interpretation of the thermodynamic and mechanical pressure and its relation to each others will be discussed. To show this relation the deviatoric measures of the stress tensor $\mathbf{T}$ and the symmetric part of the velocity gradient $\mathbf{D}$ will be used.

The deviator form of the stress tensor $\mathbf{T}$ denoted $\mathbf{T}'$ is defined as

$$\mathbf{T}' = \mathbf{T} + \bar{p}\mathbf{I} \tag{74}$$

where $\bar{p}$ is the mechanical pressure defined in (54). In the same manner one can define a deviatoric part of $\mathbf{D}$ as

$$\mathbf{D}' = \mathbf{D} - \tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} \tag{75}$$

where $\mathbf{D}'$ denotes the deviator of the symmetric part of the velocity gradient $\mathbf{D}$.

Now consider the Navier-Poisson law rewritten by replacing some terms with its corresponding deviators, as.

$$\mathbf{T} = \mathbf{T}' - \bar{p}\mathbf{I} = -p\mathbf{I} + \lambda\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\left(\mathbf{D}' + \tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I}\right) \tag{76}$$

12

where the term $2\mu\mathbf{D}$ has been replaced by $2\mu\left(\mathbf{D}' + \frac{1}{3}(\text{tr}\mathbf{D})\mathbf{I}\right)$ by use of the definition (75). Further, rearrangement of (76) yields

$$\mathbf{T}' = (\bar{p}-p)\,\mathbf{I} + \lambda\,(\text{tr}\mathbf{D})\,\mathbf{I} + \tfrac{2}{3}\mu\,(\text{tr}\mathbf{D})\,\mathbf{I} + 2\mu\mathbf{D}' \tag{77}$$

or equally

$$\mathbf{T}' = (\bar{p}-p)\,\mathbf{I} + \left(\lambda + \tfrac{2}{3}\mu\right)(\text{tr}\mathbf{D})\,\mathbf{I} + 2\mu\mathbf{D}' \tag{78}$$

This expression can be simplified further by using the relations $\text{tr}\mathbf{D}' = 0$ and $\text{tr}\mathbf{T}'=0$. To validate these conditions consider the trace of $\mathbf{D}$ together with the definition (75), as

$$\text{tr}\mathbf{D} = \text{tr}\mathbf{D}' + \text{tr}\left(\tfrac{1}{3}(\text{tr}\mathbf{D})\,\mathbf{I}\right) \tag{79}$$

where

$$\text{tr}\left(\tfrac{1}{3}(\text{tr}\mathbf{D})\,\mathbf{I}\right) = \text{tr}\mathbf{D} \tag{80}$$

that is

$$\text{tr}\mathbf{D} = \text{tr}\mathbf{D}' + \text{tr}\mathbf{D} \tag{81}$$

i.e.

$$\text{tr}\mathbf{D}' = 0 \tag{82}$$

To verify $\text{tr}\mathbf{T}'=0$ consider the definition (54) and (58) combined, as

$$\mathbf{T}' = \mathbf{T} - \tfrac{1}{3}(\text{tr}\mathbf{T})\,\mathbf{I} \tag{83}$$

The trace operating on this expression is

$$\text{tr}\mathbf{T}' = \text{tr}\mathbf{T} - \text{tr}\left(\tfrac{1}{3}(\text{tr}\mathbf{T})\,\mathbf{I}\right) \tag{84}$$

Noting that the identity

$$-\text{tr}\left(\tfrac{1}{3}(\text{tr}\mathbf{T})\,\mathbf{I}\right) = -\text{tr}\mathbf{T} \tag{85}$$

holds. That is (84) and (85) combines to yield

$$\text{tr}\mathbf{T}' = 0 \tag{86}$$

By using the Navier-Poisson law expressed with the deviatoric stress $\mathbf{T}'$, i.e. equation (78), together with (86), yields

$$\text{tr}\mathbf{T}' = \text{tr}\left((\bar{p}-p)\,\mathbf{I} + \lambda\,(\text{tr}\mathbf{D})\,\mathbf{I} + \tfrac{2}{3}\mu\,(\text{tr}\mathbf{D})\,\mathbf{I} + 2\mu\mathbf{D}'\right) = 0 \tag{87}$$

13

i.e.

$$\mathrm{tr}\,(\bar{p}-p)\,\mathbf{I} + \mathrm{tr}\lambda\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + \mathrm{tr}\tfrac{2}{3}\mu\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + \mathrm{tr}2\mu\mathbf{D}' = 0 \tag{88}$$

From (82) one conclude that $\mathrm{tr}2\mu\mathbf{D}' = 0$, and expression (88) therefore simplifies to

$$\mathrm{tr}\,(\bar{p}-p)\,\mathbf{I} + \mathrm{tr}\lambda\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + \mathrm{tr}\tfrac{2}{3}\mu\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} = 0 \tag{89}$$

Noting also that the following identities holds

$$\mathrm{tr}\,((\bar{p}-p)\,\mathbf{I}) = 3\,(\bar{p}-p) \tag{90}$$

$$\mathrm{tr}\,(\lambda\,(\mathrm{tr}\mathbf{D})\,\mathbf{I}) = 3\lambda\,(\mathrm{tr}\mathbf{D}) \tag{91}$$

$$\mathrm{tr}\,\left(\tfrac{2}{3}\mu\,(\mathrm{tr}\mathbf{D})\,\mathbf{I}\right) = 2\mu\,(\mathrm{tr}\mathbf{D}) \tag{92}$$

By (90), (91) and (92) expression (89) takes the form

$$3\,(\bar{p}-p) + 3\lambda\,(\mathrm{tr}\mathbf{D}) + \mu\,(\mathrm{tr}\mathbf{D}) = 0 \tag{93}$$

i.e.

$$(\bar{p}-p) + \lambda\,(\mathrm{tr}\mathbf{D}) + \tfrac{2}{3}\mu\,(\mathrm{tr}\mathbf{D}) = 0 \tag{94}$$

or equally

$$(\bar{p}-p) = -\left(\lambda + \tfrac{2}{3}\mu\right)(\mathrm{tr}\mathbf{D}) \tag{95}$$

By combining (78), i.e.

$$\mathbf{T}' = (\bar{p}-p)\,\mathbf{I} + \lambda\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + \tfrac{2}{3}\mu\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + 2\mu\mathbf{D}', \tag{96}$$

with (95), gives

$$\mathbf{T}' = -\left(\lambda + \tfrac{2}{3}\mu\right)(\mathrm{tr}\mathbf{D})\,\mathbf{I} + \lambda\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + \tfrac{2}{3}\mu\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + 2\mu\mathbf{D}' \tag{97}$$

From (97) it then follows that

$$\mathbf{T}' = 2\mu\mathbf{D}' \tag{98}$$

holds.

The relation between the mechanical pressure $\bar{p}$ and the thermodynamic pressure p is the expression (95). This relation can be formulated by use of the so-called *bulk viscosity* κ by writing

$$\bar{p} = p - \left(\lambda + \tfrac{2}{3}\mu\right)(\mathrm{tr}\mathbf{D}) = p - \kappa\,(\mathrm{tr}\mathbf{D}) \tag{99}$$

14

where $\kappa = \lambda + \frac{2}{3}\mu$ is the bulk viscosity.

By considering the balance of mass written as

$$\dot{\rho} + \rho \text{div}\dot{\mathbf{x}} = \dot{\rho} + \rho \text{tr}\mathbf{D} = 0 \tag{100}$$

i.e.

$$\text{tr}\mathbf{D} = -\frac{\dot{\rho}}{\rho} \tag{101}$$

the relation between $\bar{p}$ and p can be further simplified by combining (99) and (101) to yield

$$\bar{p} = p + \kappa \frac{\dot{\rho}}{\rho} \tag{102}$$

Equation (102) shows that the mean mechanical pressure $\bar{p}$ equals the thermodynamic pressure p if and only if one of the following two conditions is satisfied

$$\text{tr}\mathbf{D} = 0 \quad (\text{i.e., } \dot{\rho} = 0) \quad or \quad \kappa \equiv \lambda + \frac{2}{3}\mu = 0 \tag{103}$$

which is, also, a physical interpretation of the invariant $\text{tr}\mathbf{D}$. The special choice $\lambda + \frac{2}{3}\mu = 0$ is called the Stokes condition.

1.7 Restriction imposed by the second axiom of thermodynamics

When dealing with mechanical stresses in solids or fluids it turns out that it is convenient to write the second axiom of thermodynamics in terms of the entropy η and the Helmholt's free energy ψ, as

$$-\rho\eta\dot{\theta} - \rho\dot{\psi} - \text{grad}\theta \cdot \mathbf{q}/\theta + \text{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) \geq 0 \tag{104}$$

compare the derivation of the balance principles in chapter xx. Due to $\mathbf{T}$ being symmetric the relation $\text{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) = \text{tr}(\mathbf{T}\mathbf{L})$ holds, furthermore it turns out that $\text{tr}(\mathbf{T}\mathbf{L}) = \text{tr}(\mathbf{T}\mathbf{D})$ holds, which is due to the spin tensor $\mathbf{W}$ (note that $\mathbf{L} = \mathbf{D} + \mathbf{W}$) being skew-symmetric having zeros in the diagonal, compare chapter xx. That is, the entropy inequality (104) can be written as

$$-\rho\eta\dot{\theta} - \rho\dot{\psi} - \text{grad}\theta \cdot \mathbf{q}/\theta + \text{tr}\left(\mathbf{T}\mathbf{D}\right) \geq 0 \tag{105}$$

when the stress tensor is symmetric, which is the case, compare chapter xx.

Due to the frame-indifference principle, the velocity gradient $\mathbf{L}$, the velocity $\dot{\mathbf{x}}$, and the acceleration $\ddot{\mathbf{x}}$ cannot be included as constitutive independent

properties. Due to the equiprecense principle it is also proposed that all constitutive dependent properties should be assumed to depend on the same quantities. The following choice will be tested

$$\psi = \psi(\theta, \rho, \mathbf{D}, \mathbf{g}); \quad \eta = \eta(\theta, \rho, \mathbf{D}, \mathbf{g}) \tag{106}$$
$$\mathbf{q} = \mathbf{q}(\theta, \rho, \mathbf{D}, \mathbf{g}); \quad \mathbf{T} = \mathbf{T}(\theta, \rho, \mathbf{D}, \mathbf{g})$$

Consider, first, a differentiation of ψ i.e.

$$\dot{\psi} = \frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \rho}\dot{\rho} + \frac{\partial \psi}{\partial \mathbf{D}} \cdot \dot{\mathbf{D}} + \frac{\partial \psi}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} \tag{107}$$

$$-\rho\eta\dot{\theta} - \rho\left(\frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \rho}\dot{\rho} + \frac{\partial \psi}{\partial \mathbf{D}} \cdot \dot{\mathbf{D}} + \frac{\partial \psi}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}}\right) - \mathrm{grad}\theta \cdot \mathbf{q}/\theta + \mathrm{tr}(\mathbf{TD}) \geq 0 \tag{108}$$

$$-\rho\dot{\theta}\left(\eta + \frac{\partial \psi}{\partial \theta}\right) - \rho\frac{\partial \psi}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} - \rho\frac{\partial \psi}{\partial \rho}\dot{\rho} - \rho\frac{\partial \psi}{\partial \mathbf{D}} \cdot \dot{\mathbf{D}} - \mathrm{grad}\theta \cdot \mathbf{q}/\theta + \mathrm{tr}(\mathbf{TD}) \geq 0 \tag{109}$$

Since the temperature change $\dot{\theta}$ must be allowed to be arbitrary in a general model one usually define the thermodynamic relation

$$\frac{\partial \psi}{\partial \theta} = -\eta \tag{110}$$

Furthermore, it will be explicitly assumed that Helmholt's free energy ψ cannot depend on the temperature gradient $\mathbf{g}$ and on the symmetrical part of the velocity gradient $\mathbf{D}$, i.e.

$$\rho\frac{\partial \psi}{\partial \mathbf{g}} = 0; \qquad \rho\frac{\partial \psi}{\partial \mathbf{D}} = 0. \tag{111}$$

This 'choice' is done because it is realized that it is very difficult to satisfy the inequality (109) when the terms $\dot{\mathbf{g}}$ and $\dot{\mathbf{D}}$ are allowed to be arbitrary. So far the dependency of the Helmholt's free energy ψ is reduced to $\psi = \psi(\theta, \rho)$ and also due to (110) one must have $\eta = \eta(\theta, \rho)$. Using this dependency on ψ and the thermodynamic relation (110) the terms in the inequality which still must be dealt with is

$$-\rho\frac{\partial \psi}{\partial \rho}\dot{\rho} - \mathrm{grad}\theta \cdot \mathbf{q}/\theta + \mathrm{tr}(\mathbf{TD}) \geq 0 \tag{112}$$

where $\psi = \psi(\theta, \rho)$ and where $\partial \psi/\partial \theta = -\eta$ must hold.

16

From (101) we have

$$\dot{\rho} = -\rho \mathrm{tr}\mathbf{D} \tag{113}$$

Combining (112) and (113), yields

$$\rho^2 \frac{\partial \psi}{\partial \rho} \mathrm{tr}\mathbf{D} - \mathrm{grad}\theta \cdot \mathbf{q}/\theta + \mathrm{tr}\left(\mathbf{TD}\right) \geq 0 \tag{114}$$

$$\mathbf{T} = -p\mathbf{I} + \mathbf{f}_m\left(\mathbf{D}\right); \tag{115}$$
$$\mathbf{T} = \mathbf{T}^E + \mathbf{T}^D \tag{116}$$

where $\mathbf{T}^E$ is the equilibrium pressure defined as $\mathbf{T}^E = -p\mathbf{I}$ which is independent of velocity gradients, and $\mathbf{T}^D$ is the dynamic stress which is determined by the velocity gradients.

$$\mathrm{tr}\left(\left(\rho^2 \frac{\partial \psi}{\partial \rho} + \mathbf{T}\right)\mathbf{D}\right) - \mathrm{grad}\theta \cdot \mathbf{q}/\theta \geq 0 \tag{117}$$

or with

$$\mathrm{tr}\left(\left(\rho^2 \frac{\partial \psi}{\partial \rho} + \mathbf{T}^E\right)\mathbf{D}\right) + \mathrm{tr}\left(\mathbf{T}^D\mathbf{D}\right) - \mathrm{grad}\theta \cdot \mathbf{q}/\theta \geq 0 \tag{118}$$

Since $\mathbf{D}$ must be allowed to be arbitrary the equilibrium pressure $\mathbf{T}^E$ can be defined by the thermodynamic relation

$$\mathbf{T}^E = -\rho^2 \frac{\partial \psi}{\partial \rho} \mathbf{I} \tag{119}$$

This is reasonable since the mechanical dissipation hardly can be active for a equilibrium pressure. Due to the pressure $\mathbf{T}^E$ being defined as $\mathbf{T}^E = -p\mathbf{I}$ the thermodynamic relation (119) can be expressed as

$$p = \rho^2 \frac{\partial \psi}{\partial \rho} \tag{120}$$

where Helmholt's free energy ψ can be a function of the temperature θ en mass density ρ only, i.e. $\psi = \psi\left(\theta, \rho\right)$. That is, the Helmholt's free energy is not 'allowed' to depend on, for example, the temperature gradient. Due to the restricted dependence on ψ and the thermodynamic relation (120) for the thermodynamic pressure, it is directly concluded that the equilibrium pressure only can depend on the same quantities, i.e.

$$\mathbf{T}^E = \mathbf{T}^E\left(\theta, \rho\right) \tag{121}$$

17

The term including the heat flux in the inequality i.e. $-\text{grad}\theta \cdot \mathbf{q}/\theta$ can be assured to fullfil the inequality (120) by choice of an odd function of $\mathbf{g}$, where $\mathbf{g} = \text{grad}\theta$, such as

$$\mathbf{q}\left(\theta, \rho, \mathbf{D}, \mathbf{g}\right) = -\xi\left(\theta, \rho, \mathbf{D}\right) \text{grad}\theta \qquad (122)$$

where the material parameter $\xi\left(\theta, \rho, \mathbf{D}\right)$ must be non-negative. This means that thermal dissipation will be included in the model having the magnitude

$$\varphi_{diss.}^{therm.} = \text{grad}\theta \cdot \left(\xi\left(\theta, \rho, \mathbf{D}\right) \text{grad}\theta\right)/\theta \qquad (123)$$

So far our restriction and thermodynamic relations imposed by the second axiom of thermodynamics are

$$\begin{aligned}
\psi &= \psi\left(\theta, \rho\right); & \eta &= \eta\left(\theta, \rho\right); \\
\mathbf{q} &= \mathbf{q}\left(\theta, \rho, \mathbf{D}, \mathbf{g}\right); & \mathbf{T}^E &= \mathbf{T}^E\left(\theta, \rho\right) \\
\mathbf{T}^D &= \mathbf{T}^D\left(\theta, \rho, \mathbf{D}, \mathbf{g}\right); & \xi\left(\theta, \rho, \mathbf{D}\right) &\geq 0
\end{aligned} \qquad (124)$$

and the introduced thermodynamic relations are

$$\frac{\partial\psi}{\partial\theta} = -\eta; \quad \text{and} \quad \mathbf{T}^E = p\mathbf{I} = \rho^2\frac{\partial\psi}{\partial\rho}\mathbf{I} \qquad (125)$$

The only term left to be analyzed in the inequality (118) is $\text{tr}\left(\mathbf{T}^D\mathbf{D}\right) \geq 0$ where the constitutive dependent stress was assumed to have the general dependency $\mathbf{T}^D = \mathbf{T}^D\left(\theta, \rho, \mathbf{D}, \mathbf{g}\right)$. One possible choice is simply

$$\mathbf{T}^D = \lambda\left(\text{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\mathbf{D} \qquad (126)$$

where λ and μ can be functions of the temperature θ and the mass density ρ. The temperature gradient $\mathbf{g}$ is omitted by two main reasons.. The first is due to the second axiom of thermodynamics, since it is realized that the condition $\text{tr}(\mathbf{TD}) \geq 0$ is very difficult to satisfy when $\mathbf{T}$ is allowed to be a function of $\mathbf{g}$ mainly due to $\mathbf{g}$ being an odd function and therefore the symmetric part of the velocity gradient $\mathbf{D}$ cannot be arbitrary. The second reason is due to the assumed material symmetry condition (isotropy). It has been previously pointed out that the only symmetric first order tensor is the zero vector, that is the assumed material isotropy also reject the proposed dependence of a temperature gradient on the stress. Indeed, and isotropic material assumption can be obtained by letting the stress tensor depend on

18

two different second order tensors as $\mathbf{T} = \mathbf{f}_m\left(\mathbf{D}, \mathbf{N}\right)$, i.e. compare (64), where the tensor $\mathbf{N}$ can be chosen as: $\mathbf{N} \equiv \mathbf{g} \otimes \mathbf{g}$. However, there will still be serious problems involved in satisfying the second axiom of thermodynamics for this special choice.

Due to the above restrictions imposed by material symmetry and due to the second axiom of thermodynamics, the dynamical part of the stress tensor $\mathbf{T}^D$ will be of the reduced form

$$\mathbf{T}^D = \mathbf{T}^D\left(\theta, \rho, \mathbf{D}\right) \tag{127}$$

It turn out that the material parameters λ and μ in (126) cannot be chosen arbitrary due to the second axiom of thermodynamics. The following discussion will be devoted to this certain restriction for λ and μ.

Consider the stress power $\mathrm{tr}(\mathbf{TD})$ written as

$$\mathrm{tr}\left(\mathbf{TD}\right) = \mathrm{tr}\left(\left(-p\mathbf{I}\lambda\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\mathbf{D}\right)\mathbf{D}\right) \tag{128}$$

where (73) has been used. The symmetric part of the velocity gradient $\mathbf{D}$ can be decomposed by help from the deviator $\mathbf{D}'$, as

$$\mathbf{D} = \mathbf{D}' + \tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} \tag{129}$$

which is (59) repeated. Noting also that the use of (73) and (58) combines to yield

$$\mathbf{T} = \mathbf{T}' - \bar{p}\mathbf{I} = -p\mathbf{I} + \kappa\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\mathbf{D}' \tag{130}$$

Combining (128) and (129), gives

$$\mathrm{tr}\left(\mathbf{TD}\right) = \mathrm{tr}\left(\left(-p\mathbf{I} + \kappa\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\mathbf{D}'\right)\left(\mathbf{D}' + \tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I}\right)\right) \tag{131}$$

i.e.

$$\begin{aligned}\mathrm{tr}\left(\mathbf{TD}\right) =\ & \mathrm{tr}\left(-p\mathbf{ID}' + \kappa\left(\mathrm{tr}\mathbf{D}\right)\mathbf{ID}' + 2\mu\mathbf{D}'\mathbf{D}'\right) \\ & + \mathrm{tr}\left(-p\mathbf{I}\tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + \kappa\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I}\tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\mathbf{D}'\tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I}\right)\end{aligned} \tag{132}$$

By observing that $\mathrm{tr}\mathbf{D}' = 0$, i.e. compare the derivation leading to (82), and also, in general, that $\mathrm{tr}\left(\mathbf{D}'\mathbf{D}'\right) \neq 0$, one is left with the terms

$$\mathrm{tr}\left(\mathbf{TD}\right) = \mathrm{tr}\left(-p\mathbf{I}\tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + \kappa\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I}\tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\mathbf{D}'\mathbf{D}'\right) \tag{133}$$

19

By noting that the following three identities holds

$$-\mathrm{tr}\left(p\mathbf{I}\tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I}\right)=-p\mathrm{tr}\mathbf{D} \tag{134}$$

$$\mathrm{tr}\left(\kappa\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I}\tfrac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I}\right)=\kappa\left(\mathrm{tr}\mathbf{D}\right)^{2} \tag{135}$$

$$\mathrm{tr}\left(2\mu\mathbf{D}'\mathbf{D}'\right)=2\mu\mathrm{tr}\left(\mathbf{D}'\mathbf{D}'\right) \tag{136}$$

it is clear that (133) simplifies to

$$\mathrm{tr}\left(\mathbf{TD}\right)=-p\mathrm{tr}\mathbf{D}+\kappa\left(\mathrm{tr}\mathbf{D}\right)^{2}+2\mu\mathrm{tr}\left(\mathbf{D}'\mathbf{D}'\right) \tag{137}$$

Due to the second axiom of thermodynamics the stress power must be greater or equal to zero, i.e. $\mathrm{tr}(\mathbf{TD}) \geq 0$. To examine this condition for the thermodynamic pressure p the inequality (114) will be considered i.e.

$$\rho^{2}\frac{\partial\psi}{\partial\rho}\mathrm{tr}\mathbf{D}-\mathrm{grad}\theta\cdot\mathbf{q}/\theta+\mathrm{tr}\left(\mathbf{TD}\right)\geq 0 \tag{138}$$

Combining (137) and (138) gives

$$\rho^{2}\frac{\partial\psi}{\partial\rho}\mathrm{tr}\mathbf{D}+\left(-p\mathrm{tr}\mathbf{D}+\kappa\left(\mathrm{tr}\mathbf{D}\right)^{2}+2\mu\mathrm{tr}\left(\mathbf{D}'\mathbf{D}'\right)\right)-\mathrm{grad}\theta\cdot\mathbf{q}/\theta\geq 0 \tag{139}$$

And it is again concluded that the thermodynamic pressure satisfies the second axiom of thermodynamic as long as the Helmholz's free energy is defined from

$$\left(\rho^{2}\frac{\partial\psi}{\partial\rho}-p\right)\mathrm{tr}\mathbf{D}\geq 0 \tag{140}$$

where $\mathrm{tr}\mathbf{D}$ is arbitrary, i.e.

$$p=\rho^{2}\frac{\partial\psi}{\partial\rho} \tag{141}$$

which has already been noticed.. The term $-\mathrm{grad}\theta\cdot\mathbf{q}/\theta$ has also been tackled which resulted in the restriction for the constitutive relation for $\mathbf{q}$, i.e. the relation (120).

At last we are ready to consider the last requirement imposed by the second axiom of thermodynamics. That is the terms $\kappa\left(\mathrm{tr}\mathbf{D}\right)^{2}$ and $2\mu\mathrm{tr}\left(\mathbf{D}'\mathbf{D}'\right)$ in the inequality (139). For a fluid one generally accept that dissipation can occur due to internal friction. That is, one is interested in satisfying the condition

$$\varphi_{mech.}^{dissip.}=\kappa\left(\mathrm{tr}\mathbf{D}\right)^{2}+2\mu\mathrm{tr}\left(\mathbf{D}'\mathbf{D}'\right)\geq 0 \tag{142}$$

Since both the terms $\text{tr}\mathbf{D}$ and the deviator $\mathbf{D}'$ is quadratic terms all negative values are canceled out. Therefore it is sufficient to assure that the material parameters taking non-negative values, i.e.

$$\mu \geq 0 \quad \text{and} \quad \kappa \geq 0 \tag{143}$$

or equally

$$\mu \geq 0 \quad \text{and} \quad \lambda + \tfrac{2}{3}\mu \geq 0 \tag{144}$$

This means that

$$\lambda \geq -\tfrac{2}{3}\mu \quad \text{and} \quad \mu \geq 0 \tag{145}$$

Since the $\varphi_{mech.}^{dissip.}$ is nonnegative this part of the stress power never contributes to an increase of the kinetic energy of the system. The result that the increase of the kinetic energy of the system is equal to the mechanical power input minus the total stress power, is due to Stokes in 1851. That is, $P^{input} - \int_V \text{tr}(\mathbf{TD}) = \frac{\partial}{\partial t} \int_V \frac{1}{2}\rho \dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$, compare previous chapters.

1.8 Navier-Stokes equations

By using the derived expressions in previous section the Navier-Stokes equations will be developed.

The mass balance is

$$\dot{\rho} + \rho\text{div}(\dot{\mathbf{x}}) = 0 \tag{146}$$

The momentum balance can be written

$$\rho\ddot{\mathbf{x}} = \text{div}\,\mathbf{T} + \rho\mathbf{b}; \qquad \mathbf{T} = \mathbf{T}^{\mathrm{T}} \tag{147}$$

and the energy balance as

$$\rho\dot{\varepsilon} = \text{tr}\,(\mathbf{TD}) - \text{div}\,(\mathbf{q}) + \rho r \tag{148}$$

One may choose the following general assumptions for the fluid

$$\begin{aligned} \varepsilon &= \varepsilon\,(\theta,\rho)\,; \\ \mathbf{T}^E &= \mathbf{T}^E\,(\theta,\rho)\,; \quad \mathbf{T}^D = \mathbf{T}^D\,(\theta,\rho,\mathbf{D}) : \\ \mathbf{q} &= \mathbf{q}\,(\theta,\rho,\mathbf{D},\mathbf{g}) \end{aligned} \tag{149}$$

Due to the second axiom of thermodynamics the following choice is made for the heat flux

$$\mathbf{q} = -\xi\,(\theta,\rho,\mathbf{D})\,\text{grad}\theta \tag{150}$$

where
$$\xi\left(\theta,\rho,\mathbf{D}\right)\geq \tag{151}$$

And the dynamical part of the stress tensor is the assumption
$$\mathbf{T}^{D}=\lambda\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I}+2\mu\mathbf{D} \tag{152}$$

where
$$\lambda\geq-\tfrac{2}{3}\mu \quad\text{and}\quad \mu\geq0 \tag{153}$$

according to the second axiom of thermodynamics.

And for the thermodynamic pressure one may choose
$$\mathbf{T}^{E}=-p\mathbf{I} \tag{154}$$

where p can be assumed to be recoverable and therefore not subjected to any restrictions from the second axiom of thermodynamics, compare previous sections.

The caloric equation of state is the assumption
$$\varepsilon=C\left(\rho\right)\theta \tag{155}$$

Noting that we have the following relation between material derivatives and the spatial derivatives

$$\ddot{\mathbf{x}}=\frac{\partial\dot{\mathbf{x}}}{\partial t}+[\mathrm{grad}\,\dot{\mathbf{x}}]\,\dot{\mathbf{x}} \tag{156}$$

$$\dot{\rho}=\frac{\partial\rho}{\partial t}+\mathrm{grad}\,(\rho)\cdot\dot{\mathbf{x}} \tag{157}$$

$$\dot{\varepsilon}=\frac{\partial\varepsilon}{\partial t}+\mathrm{grad}\,(\varepsilon)\cdot\dot{\mathbf{x}} \tag{158}$$

Hence, the balance laws can be written as

$$\frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\dot{\mathbf{x}})=0 \tag{159}$$

$$\rho\frac{\partial\dot{\mathbf{x}}}{\partial t}+\rho\,[\mathrm{grad}\,\dot{\mathbf{x}}]\,\dot{\mathbf{x}}=\mathrm{div}\,\mathbf{T}+\rho\mathbf{b} \tag{160}$$

$$\rho\frac{\partial\varepsilon}{\partial t}+\rho\mathrm{grad}\,(\varepsilon)\cdot\dot{\mathbf{x}}=\mathrm{tr}\,(\mathbf{TD})-\mathrm{div}\,(\mathbf{q})+\rho r \tag{161}$$

22

Due to $\mathbf{T} = \mathbf{T}^E + \mathbf{T}^D$ the stress is constituted as

$$\mathbf{T} = -p\mathbf{I} + \lambda\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + 2\mu\mathbf{D} \tag{162}$$

Note that the symmetric part of the velocity gradient is $\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right)$, i.e. the expression for $\mathbf{D}$ is equally $\mathbf{D} = \frac{1}{2}\left(\mathrm{grad}\,\dot{\mathbf{x}} + (\mathrm{grad}\,\dot{\mathbf{x}})^{\mathrm{T}}\right)$.

By combining the assumptions (162), (155) and (150) with the balance principles (159), (160) and (161), and also noting that the restrictions imposed by the second axiom of thermodynamics, i.e. (151) and (153) must hold, one obtain the following governing equations:

The mass density is governed by the mass balance equation, i.e.

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\dot{\mathbf{x}}) = 0 \tag{163}$$

The velocity fields can be calculated with the momentum balance equation where the constitutive relation fro the stress has been inserted to yield the so-called generalized Navier-Stokes equation for fluids with bulk viscosity, i.e.

$$\rho\frac{\partial \dot{\mathbf{x}}}{\partial t} + \rho\,[\mathrm{grad}\,\dot{\mathbf{x}}]\,\dot{\mathbf{x}} = \mathrm{div}\,\left(-p\mathbf{I} + \lambda\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + 2\mu\mathbf{D}\right) + \rho\mathbf{b} \tag{164}$$

The temperature field is governed by the energy equation together with the constitutive relation for the heat flux and the constitutive relation for the caloric equation of state, i.e.

$$\begin{aligned}
\rho C\frac{\partial \theta}{\partial t} + \rho C\mathrm{grad}\,(\theta)\cdot\dot{\mathbf{x}} &= -p\mathrm{tr}\mathbf{D} + \kappa\,(\mathrm{tr}\mathbf{D})^2 + 2\mu\mathrm{tr}\left(\mathbf{D}'\mathbf{D}'\right) \\
&\quad -\mathrm{div}\,(\lambda\mathrm{grad}\theta) + \rho r
\end{aligned} \tag{165}$$

These three equations, i.e. (163), (164) and (165), constitute the so-called Navier-Stokes equations.

1.9 Incompressible fluids

An incompressible fluid is defined as a fluid having constant mass density, i.e.

$$\dot{\rho} = 0 \tag{166}$$

By considering the mass balance, that is

$$\dot{\rho} + \rho \operatorname{div}(\dot{\mathbf{x}}) = 0 \qquad (167)$$

It is clear that incompressibility, i.e. $\dot{\rho} = 0$, requires that

$$\operatorname{div}(\dot{\mathbf{x}}) = 0 \qquad (168)$$

From (113) we also conclude that

$$\operatorname{tr}\mathbf{D} = -\frac{\dot{\rho}}{\rho} \qquad (169)$$

An incompressible fluid therefore must fulfill

$$\operatorname{tr}\mathbf{D} = 0 \qquad (170)$$

The generalized Navier-Stokes equation (164) therefore reduces to

$$\rho\frac{\partial \dot{\mathbf{x}}}{\partial t} + \rho\left[\operatorname{grad} \dot{\mathbf{x}}\right]\dot{\mathbf{x}} = \operatorname{div}\left(-p\mathbf{I} + 2\mu\mathbf{D}\right) + \rho\mathbf{b} \qquad (171)$$

due to (170), which is the so-called Navier-Stokes equation for a incompressible fluid.

1.10   Compressible fluid with no bulk viscosity

The generalized Navier-Stokes equation (164) can be approximated by assuming the so-called Stokes condition which is a relation between the two material coefficients λ and μ, as

$$\lambda + \tfrac{2}{3}\mu = 0 \qquad (172)$$

which means that the thermodynamic pressure and the mean mechanical pressure is identical since the bulk viscosity κ is identical to zero in this case, compare (102).

From the generalized Navier-Stokes equation (164) and the assumption (172), one obtain a description of a compressible fluid with no bulk viscosity, i.e.

$$\rho\frac{\partial \dot{\mathbf{x}}}{\partial t} + \rho\left[\operatorname{grad} \dot{\mathbf{x}}\right]\dot{\mathbf{x}} = \operatorname{div}\left(-p\mathbf{I} - \tfrac{2}{3}\mu\left(\operatorname{tr}\mathbf{D}\right)\mathbf{I} + 2\mu\mathbf{D}\right) + \rho\mathbf{b} \qquad (173)$$

by dividing this with ρ, as

$$\frac{\partial \dot{\mathbf{x}}}{\partial t} + \left[\operatorname{grad} \dot{\mathbf{x}}\right]\dot{\mathbf{x}} = \operatorname{div}\left(-\frac{p}{\rho}\mathbf{I} - \tfrac{2}{3}k\left(\operatorname{tr}\mathbf{D}\right)\mathbf{I} + 2k\mathbf{D}\right) + \mathbf{b} \qquad (174)$$

where $k = \mu/\rho$, which is the so-called *kinematic viscosity*.