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# Piecewise Linear Quadratic Optimal Control

Anders Rantzer and Mikael Johansson

**Abstract**—The use of piecewise quadratic cost functions is extended from stability analysis of piecewise linear systems to performance analysis and optimal control.

Lower bounds on the optimal control cost are obtained by semidefinite programming based on the Bellman inequality. This also gives an approximation to the optimal control law. An upper bound to the optimal cost is obtained by another convex optimization problem using the given control law.

A compact matrix notation is introduced to support the calculations and it is proved that the framework of piecewise linear systems can be used to analyze smooth nonlinear dynamics with arbitrary accuracy.

**Index Terms**—Nonlinear systems, optimal control, semidefinite programming.

## I. INTRODUCTION

A POWERFUL model class for nonlinear systems is the class of piecewise affine systems [18], [14]. Such systems arise naturally in many applications, for example in presence of saturations. Piecewise affine systems can also be used for approximation of other nonlinear systems.

A new framework for stability analysis of piecewise affine systems was developed in [9] and similar ideas were reported in [13]. It was suggested to search for piecewise quadratic Lyapunov functions using convex optimization. The approach covers polytopic Lyapunov functions (see [3] and the references therein) as a special case and is considerably more powerful than quadratic stability [4].

In this paper, the method is developed further, to treat performance analysis and optimal control. We show that several concepts from linear systems theory, such as observability Gramians, linear quadratic regulators, and  $L_2$  induced gains can be generalized using the framework of piecewise quadratic Lyapunov functions.

Quadratic control of piecewise linear systems was addressed earlier in [2]. The treatment there was based on backward solutions of Riccati differential equations, and the optimum had to be recomputed for each new final state. Computation of nonlinear  $L_2$  gain using the Hamilton–Jacobi–Bellman (HJB) equation has been done in [19] and [5]. Our use of convex optimization based on the HJB inequality is closely related to the optimality condition by [20] and leads to piecewise quadratic upper and lower bounds on the optimal cost function.

An important feature of our approach is that a local linear-quadratic analysis near an equilibrium point of a non-

linear system can be improved step by step, by splitting the state space into more regions, thereby increasing the flexibility in the nonlinearity description and enlarging the validity domain for the analysis. In principle, any smooth nonlinear system can be approximated to an arbitrary accuracy in this way, so the tradeoff between precision and computational complexity can be addressed directly.

This paper includes material from [16] and [8] is organized as follows. The basic setup for system representation and stability analysis is described in Section II. This analysis is refined in the next section, to estimate the transient properties of the system. Optimal control problems are studied in Section IV and applied to gain computations and other dissipation inequalities in Section V. Simplex partitions are discussed in Section VI and used to prove a converse theorem on existence of piecewise quadratic Lyapunov functions.

## II. STABILITY ANALYSIS

Consider piecewise affine systems of the form

$$\begin{cases} \dot{x} = a_i + A_i x + B_i u \\ y = c_i + C_i x + D_i u \end{cases} \quad \text{for } x \in X_i. \quad (1)$$

Here,  $\{X_i\}_{i \in I} \subseteq \mathbb{R}^n$  is a partition of the state space into a number of closed (possibly unbounded) polyhedral cells. The index set of the cells is denoted  $I$ . Let  $x(t) \in \cup_{i \in I} X_i$  be a continuous piecewise  $C^1$  function on the time interval  $[t_0, t_1]$ . We say that  $x(t)$  is a *trajectory* of the system (1), if for every  $t \in [t_0, t_1]$  such that the derivative  $\dot{x}(t)$  is defined, the equation  $\dot{x}(t) = A_i x(t) + a_i + B_i u(t)$  holds for all  $i$  with  $x(t) \in X_i$ . Note that the trajectory cannot stay on the boundary  $X_i \cap X_j$  unless it satisfies the differential equation for both  $i$  and  $j$  simultaneously. Hence, most kinds of sliding modes are excluded. Extensions to such cases are straightforward (see [10]) but will be omitted here for ease of presentation.

We let  $I_0 \subseteq I$  be the set of indices for the cells that contain origin and  $I_1 \subseteq I$  be the set of indices for cells that do not contain the origin. It is assumed that  $a_i = c_i = 0$ ,  $i \in I_0$ . For convenient notation, we introduce

$$\left[ \begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array} \right] = \left[ \begin{array}{cc|c} A_i & a_i & B_i \\ 0 & 0 & 0 \\ \hline C_i & c_i & D_i \end{array} \right] \quad \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}. \quad (2)$$

Then

$$\begin{cases} \dot{\bar{x}} = \bar{A}_i \bar{x} + \bar{B}_i u \\ y = \bar{C}_i \bar{x} + \bar{D}_i u \end{cases} \quad \text{for } x \in X_i.$$

The cells are polyhedrons, so we can construct matrices

$$\bar{E}_i = [E_i \quad e_i], \quad \bar{F}_i = [F_i \quad f_i]$$

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with  $e_i = 0$  and  $f_i = 0$  for  $i \in I_0$  and such that

$$\bar{E}_i \bar{x} \geq 0 \quad x \in X_i, i \in I \quad (3)$$

$$\bar{F}_i \bar{x} = \bar{F}_j \bar{x} \quad x \in X_i \cap X_j, i, j \in I. \quad (4)$$

The vector inequality  $z \geq 0$  means that each entry of  $z$  is non-negative. Construction of the constraint matrices  $E_i$  and  $F_i$  will be further discussed in Section VI. The following result on stability analysis was given in [9].

**Proposition 1 (Piecewise Quadratic Stability):** Consider symmetric matrices  $T$ ,  $U_i$  and  $W_i$ , such that  $U_i$ , and  $W_i$  have nonnegative entries, while  $P_i = F_i' T F_i$  for  $i \in I_0$  and  $\bar{P}_i = \bar{F}_i' T \bar{F}_i$  for  $i \in I$  satisfy

$$\begin{cases} 0 > A_i' P_i + P_i A_i + E_i' U_i E_i \\ 0 < P_i - E_i' W_i E_i \end{cases} \quad i \in I_0$$

$$\begin{cases} 0 > \bar{A}_i' \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i' U_i \bar{E}_i \\ 0 < \bar{P}_i - \bar{E}_i' W_i \bar{E}_i \end{cases} \quad i \in I_1.$$

Then  $x(t)$  tends to zero exponentially for every continuous piecewise  $C^1$  trajectory in  $\cup_{i \in I} X_i$  satisfying (1) with  $u \equiv 0$  for  $t \geq 0$ .

**Remark 1:** In solving the inequalities of Theorem 1, it is advisable to first consider only the Lyapunov inequality (the one containing  $U_i$ ) in each region and ignore the positivity condition (the inequality containing  $W_i$ ). Once a solution to this reduced problem has been found, it remains to investigate whether or not the resulting piecewise quadratic function is nonnegative in the entire state space. This can be done in each region separately. If it turns out to be negative in some point, then no trajectory in  $\cup_{i \in I} X_i$  starting in this point can approach the origin as  $t \rightarrow \infty$ .  $\square$

The conditions of Proposition 1 assure that

$$V(x) = \bar{x}' \bar{P}_i \bar{x}, \quad x \in X_i, i \in I \quad (5)$$

is a Lyapunov function that is both decreasing and positive. Any level set of  $V(x)$  that is fully contained in the cell partition  $\cup_{i \in I} X_i$  is a region of attraction for the equilibrium  $x = 0$ . In particular, if  $\cup_{i \in I} X_i$  covers the whole state space, then the system is globally exponentially stable.

Proposition 1 can be used for systematic analysis of nonlinear systems based on piecewise approximations. A linear model valid locally around an equilibrium point can be refined by splitting the state space into more regions, each with different affine dynamics. Splitting a given partition also increases the flexibility of the piecewise quadratic Lyapunov function. The approach is illustrated in the following example.

**Example 1 (Piecewise Linear Analysis):** Simulations indicate that the following nonlinear system is stable:

$$\begin{aligned} \dot{x}_1 &= -2x_1 + 2x_2 + \text{sat}(x_1 x_2) x_1 \\ \dot{x}_2 &= -2x_1 - \text{sat}(x_1 x_2) (x_1 + 4x_2). \end{aligned}$$

We would like to verify global exponential stability of the origin by computing a piecewise quadratic Lyapunov function for the system. This will be done by writing the model as

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ -2 & 0 \end{bmatrix} x + p(x) \begin{bmatrix} 1 & 0 \\ -1 & -4 \end{bmatrix} x$$

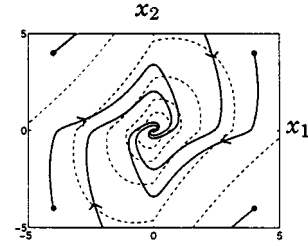


Fig. 1. Simulated trajectories (full) and level surfaces of the Lyapunov function (dashed).

and utilizing bounds of the form

$$p_{\min} \leq p(x) \leq p_{\max}.$$

We notice that a Lyapunov function candidate is decreasing along trajectories of the original system if it is decreasing for the two extremal systems obtained by replacing  $p(x)$  with  $p_{\min}$  and  $p_{\max}$ , respectively. The global bound  $-1 \leq p(x) \leq 1$  is, however, insufficient, since  $p(x) \equiv -1$  gives the unstable system

$$\dot{x} = \begin{bmatrix} -3 & 2 \\ -1 & 4 \end{bmatrix} x.$$

Instead, we can partition the state space into the four quadrants and note that

$$0 \leq \text{sat}(x_1 x_2) \leq 1$$

in the first and third quadrant and

$$-1 \leq \text{sat}(x_1 x_2) \leq 0$$

in the second and fourth quadrant. To assure stability of the system, we search for a piecewise quadratic Lyapunov function that is valid for the two extremal systems in each region. The numerical routines return the Lyapunov function with the level curves indicated in Fig. 1. This proves global exponential stability.  $\square$

### III. TRANSIENT ANALYSIS

The objective of this section is to refine the stability analysis by estimating the “transient integral”  $\int_{t=0}^{\infty} \bar{x}(t)' \bar{Q}_i \bar{x}(t) dt$  as a function of the initial state  $x(0)$ . Here,  $\bar{Q}_i$  denotes arbitrary symmetric matrices. It is assumed that  $\bar{x}' \bar{Q}_i \bar{x} = x' Q_i x$  for  $i \in I_0$ . The estimate can be obtained using a minor modification of the Lyapunov inequalities.

**Theorem 1 (Bound on Transient):** Let  $x(t)$  with  $x(0) \in X_{i_0}$ ,  $x(\infty) = 0$  be a continuous piecewise  $C^1$  trajectory of the system (1) with  $u \equiv 0$  and  $x(t) \in \cup_{i \in I} X_i$  for  $t \geq 0$ . Consider symmetric matrices  $T$  and  $U_i$ , such that  $U_i$  have nonnegative entries, while  $P_i = F_i' T F_i$  and  $\bar{P}_i = \bar{F}_i' T \bar{F}_i$  satisfy

$$\begin{aligned} 0 &> P_i A_i + A_i' P_i + Q_i + E_i' U_i E_i, & i \in I_0 \\ 0 &> \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{Q}_i + \bar{E}_i' U_i \bar{E}_i, & i \in I_1. \end{aligned}$$

Then

$$\int_0^{\infty} \bar{x}' \bar{Q}_i \bar{x} dt \leq \inf_{T, U_i} \bar{x}(0)' \bar{P}_{i_0} \bar{x}(0).$$

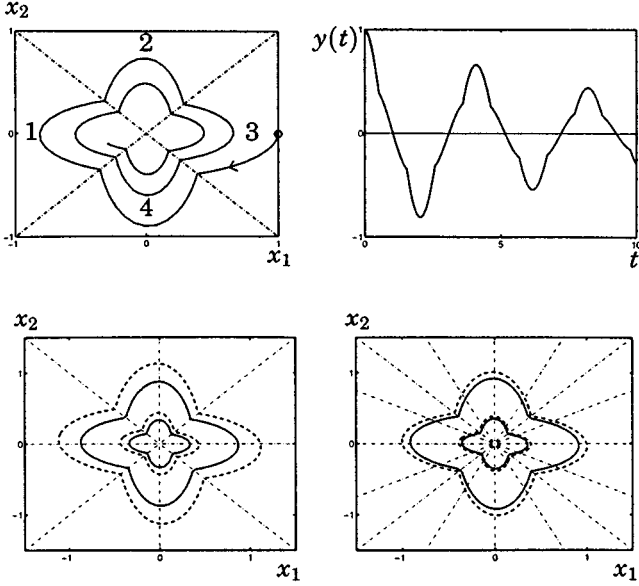


Fig. 2. Simulated state trajectory (left) and output (right).

*Proof:* It follows directly from the two inequalities that

$$0 \geq \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{Q}_i + \bar{E}_i' U_i \bar{E}_i, \quad i \in I.$$

Multiplying this inequality from left and right by  $\bar{x}$  and removing the nonnegative terms including  $U_i$  gives

$$0 \geq 2\bar{x}(t)' \bar{P}_{i(t)} \bar{A}_{i(t)} \bar{x}(t) + \bar{x}(t)' \bar{Q}_i \bar{x}(t)$$

where  $i(t)$  is chosen so that  $X_{i(t)} \ni x(t)$ . Integration from  $t = 0$  to  $t = \infty$  gives the desired result.  $\square$

In particular, upper and lower bounds on the “output energy” are obtained from Theorem 1 as

$$\begin{aligned} -\inf \bar{x}(0)' \bar{P}_{i_0}^l \bar{x}(0) &\leq \int_0^\infty |y(t)|^2 dt \\ &\leq \inf \bar{x}(0)' \bar{P}_{i_0}^u \bar{x}(0) \end{aligned} \quad (6)$$

where  $\{\bar{P}_i^l\}_{i \in I}$  satisfy the conditions of Theorem 1 with  $\bar{Q}_i = -\bar{C}_i' \bar{C}_i$  and  $\{\bar{P}_i^u\}_{i \in I}$  satisfy them with  $\bar{Q}_i = \bar{C}_i' \bar{C}_i$ .

*Example 2 (Transient in Flower Example):* Consider the piecewise linear system with the cell partition shown in Fig. 2 (left) and dynamics given by the matrices

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix}$$

and  $C_i = [1 \ 0]$ ,  $i \in I$ . The trajectory of a simulation with initial value  $x_0 = (1, 0)'$  moves toward the origin in a flower-like trajectory, as shown in Fig. 2 (left). The corresponding output is shown in Fig. 2 (right). This output has the total energy  $\int_0^\infty |y|^2 dt = 1.88$ , while (6) with the initial cell partition gives the bounds  $0.60 \leq \int_0^\infty |y|^2 dt \leq 2.50$ .

A possible reason for the gap between the bounds is that the level curves of the cost function cannot be well approximated by piecewise quadratic functions. To improve the bounds, we introduce more flexibility in the approximation by repeatedly splitting every cell in two. This simple-minded refinement procedure, illustrated in Fig. 3, is repeated three times yielding the

$$\begin{aligned} &\text{Minimize} && \int_0^\infty l(x, u) dt \\ &\text{subject to} && \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(0) = x_0 \end{cases} \end{aligned}$$

Fig. 3. Upper (full) and lower (dashed) bounds on the storage function computed in Example 2. The bounds get increasingly tight when we move from 8 cells (left) to 16 cells (right).

TABLE I  
LOWER AND UPPER BOUNDS OBTAINED  
FROM (6) FOR EXAMPLE 2

Number of Cells	Lower bound	Upper bound
4	0.60	2.50
8	1.33	2.18
16	1.65	1.98
32	1.78	1.88

bounds shown in Table I. Note that the bounds on the output energy optimized for the initial state  $(1, 0)$  match closely over the whole state space, giving good estimates of the output energy also for other initial states. The computation time for the final partition is comparable to the computation time for a simulation giving the same accuracy.  $\square$

It should be noted that systems with discontinuous dynamics require special attention in analysis and simulation. All simulation examples in this paper were performed in Omsim [1] with special treatment of discrete events.

In duality with transient estimation, which can be viewed as an observability problem, one may also consider reachability. The problem is then to estimate the input energy  $\int_0^\tau |u(t)|^2 dt$  that is needed to reach a certain state  $x(\tau)$  starting from  $x(0) = 0$ . However, rather than the reachability problem, we will next consider a more general class of optimal control problems.

#### IV. PIECEWISE LINEAR QUADRATIC OPTIMAL CONTROL

Consider the following general form of optimal control problem:

$$\begin{aligned} &\text{Minimize} && \int_0^\infty l(x, u) dt \\ &\text{subject to} && \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(0) = x_0. \end{cases} \end{aligned}$$

It is well known that the optimal cost  $V^*(x_0)$  for this problem can be characterized in terms of the HJB equation

$$0 = \inf_u \left( \frac{\partial V^*}{\partial x} f(x, u) + l(x, u) \right). \quad (7)$$

Lower bounds on the optimal cost are obtained by integrating the corresponding inequality

$$0 \leq \frac{\partial V}{\partial x} f(x, u) + l(x, u), \quad \forall x, u. \quad (8)$$

Assuming that  $x(\infty) = 0$ , we get

$$V(x_0) - V(0) = - \int_0^\infty \frac{\partial V}{\partial x} f(x, u) dt \leq \int_0^\infty l(x, u) dt.$$

Hence, every  $V$  that satisfies (8) gives a lower bound. In fact, the maximization of  $V(x_0) - V(0)$  subject to (8) is a convex optimization problem in  $V$  with an infinite number of constraints parameterized by  $x$  and  $u$ . Under fairly general conditions [20], the supremum of such lower bounds is equal to the optimal value of the control problem. The objective of this section is to show how the maximization of the lower bound can be done numerically in terms of piecewise quadratic functions.

Let us consider the case where  $f$  is piecewise linear and  $L$  is piecewise quadratic. The control problem is to bring the system to  $x(\infty) = 0$  from an arbitrary initial state  $x(0)$ , while limiting the cost

$$J(x_0, u) = \int_0^\infty (\bar{x}' \bar{Q}_i \bar{x} + u' R_i u) dt.$$

Here  $i(t)$  is defined so that  $x(t) \in X_{i(t)}$ . Under the assumption that

$$\bar{Q}_i = \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for } i \in I_0 \quad (9)$$

this can be done in analogy with the previous results as follows.

**Theorem 2 (Lower Bound on Optimal Cost):** Assume existence of symmetric matrices  $T$  and  $U_i$ , such that  $U_i$  have non-negative entries, while  $P_i = F_i' T F_i$  and  $\bar{P}_i = \bar{F}_i' T \bar{F}_i$  satisfy

$$\begin{aligned} 0 &< \begin{bmatrix} P_i A_i + A_i' P_i + Q_i - E_i' U_i E_i & P_i B_i \\ B_i' P_i & R_i \end{bmatrix} & i \in I_0 \\ 0 &< \begin{bmatrix} \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{Q}_i - \bar{E}_i' U_i \bar{E}_i & \bar{P}_i \bar{B}_i \\ \bar{B}_i' \bar{P}_i & R_i \end{bmatrix} & i \in I_1. \end{aligned}$$

Then, every continuous piecewise  $C^1$  trajectory  $x(t) \in \cup_{i \in I} X_i$  of (1) with  $x(\infty) = 0$ ,  $x(0) = x_0 \in X_{i_0}$  satisfies

$$J(x_0, u) \geq \sup_{T, U_i} \bar{x}_0' \bar{P}_{i_0} \bar{x}_0.$$

**Remark 2:** Theorem 2 can be readily modified to handle the case of input constraints of the form

$$\begin{aligned} G_i x + H_i u &\geq 0 & \text{for } i \in I_0 \\ \bar{G}_i \bar{x} + \bar{H}_i u &\geq 0 & \text{for } i \in I. \end{aligned}$$

The first inequality condition then becomes

$$\begin{aligned} 0 &< \begin{bmatrix} P_i A_i + A_i' P_i + Q_i & P_i B_i \\ B_i' P_i & R_i \end{bmatrix} \\ &\quad - \begin{bmatrix} E_i & 0 \\ G_i & H_i \end{bmatrix}' U_i \begin{bmatrix} E_i & 0 \\ G_i & H_i \end{bmatrix} \end{aligned}$$

and the second is analogous.  $\square$

**Proof of Theorem 2:** It follows directly from the two matrix inequalities in Theorem 2 that

$$0 \leq \begin{bmatrix} \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{Q}_i - \bar{E}_i' U_i \bar{E}_i & \bar{P}_i \bar{B}_i \\ \bar{B}_i' \bar{P}_i & R_i \end{bmatrix} \quad i \in I.$$

Multiplying from left and right by  $(\bar{x}, u)$  and removing the non-negative terms including  $U_i$  gives

$$\begin{aligned} 0 &\leq 2\bar{x}' \bar{P}_i (\bar{A}_i \bar{x} + \bar{B}_i u) + \bar{x}' \bar{Q}_i \bar{x} + u' R_i u \\ &= \frac{d}{dt} (\bar{x}' \bar{P}_i \bar{x}) + \bar{x}' \bar{Q}_i \bar{x} + u' R_i u. \end{aligned}$$

Integration from 0 to  $\infty$  gives the desired result.  $\square$

Theorem 2 gives a lower bound on the minimal value of the cost function  $J$ . Upper bounds are obtained by studying specific control laws. Consider the control law obtained by the minimization

$$\min_u \left( \frac{\partial V}{\partial x} f(x, u) + l(x, u) \right). \quad (10)$$

If  $V$  satisfies the HJB equation (7), then every minimizing control law is optimal. In particular,  $V^*$  has the decay rate given by  $-l(x, u^*)$ , which is typically negative, so  $V^*$  may serve as a Lyapunov function to prove that the optimal control law is stabilizing.

However, if only (8) holds, for example as a result of solving the matrix inequalities in Theorem 2, then there is no guarantee that the control law minimizing (10) is even stabilizing. Still, the minimization problem is the starting point for definition of control laws that will be used in our further analysis.

Exact minimization of the expression (10) without input constraints can be done analytically in analogy with ordinary linear quadratic control, using the notation

$$\begin{aligned} L_i &= -R_i^{-1} B_i' P_i \\ \bar{L}_i &= -R_i^{-1} \bar{B}_i' \bar{P}_i \\ A_i &= A_i + B_i L_i \\ \bar{A}_i &= \bar{A}_i + \bar{B}_i \bar{L}_i \\ Q_i &= Q_i + P_i B_i R_i^{-1} B_i' P_i \\ \bar{Q}_i &= \bar{Q}_i + \bar{P}_i \bar{B}_i R_i^{-1} \bar{B}_i' \bar{P}_i. \end{aligned} \quad (11)$$

The minimizing control law can then be written as

$$u(t) = \bar{L}_i \bar{x}, \quad x \in X_i.$$

This control law is simple but may be discontinuous and give rise to sliding modes. For simplex partitions, to be described in detail later, this difficulty can be avoided as follows. First design control vectors  $u_i$  for the grid-points of the partition, then use linear interpolation between these vectors to define linear state feedback laws  $u = \bar{L}_i \bar{x}$  inside the simplices. In this way, no sliding modes are created and the design approach can also be used in the case of state constraints.

Once a stabilizing piecewise linear control law has been designed, an upper bound of the optimal cost is obtained from Theorem 1.

**Example 3 (LQ Control of an Inverted Pendulum):** Consider the following simple model of an inverted pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.1x_2 + \sin(x_1) + u. \end{aligned} \quad (12)$$

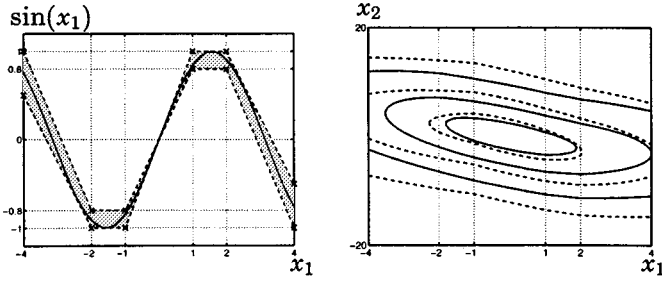


Fig. 4. The left figure shows piecewise affine bounds on the system nonlinearity. The right figure shows lower (dashed) and upper (solid) bounds on the optimal cost.

We are interested in applying the proposed technique to find a feedback control that brings the pendulum from rest at the stable equilibrium  $(\pi, 0)'$  to the upright position  $(0, 0)'$  while minimizing the criterion

$$J(x, u) = \int_{t=0}^{\infty} 4x_1^2(t) + 4x_2(t)^2 + u^2 dt.$$

A piecewise linear model of (12) can be constructed by finding piecewise affine bounds on the system nonlinearity  $\sin(x_1)$ . For the purpose of this example, we divide the interval  $[-4, 4]$  into five segments and compute the bounds illustrated in Fig. 4 (left). This description of the system nonlinearity induces the partition shown by dotted vertical lines in Fig. 4 (right). The partition can be viewed as a simplex partition in the  $x_1$  variable, while  $x_2$  is independent of the partition. We apply Theorem 2 to compute a lower bound on the achievable performance as  $J(x_0, u) \geq 15.2$ .

It is easy to verify that the closed-loop system obtained by applying the control law (11) is stable and has no attractive sliding modes. Theorem 1 can now be applied to compute the upper bound on the performance to be  $J(x_0, u) \leq 16.6$ . We conclude that both the optimal and the computed control law satisfy

$$15.2 \leq J(x_0, u) \leq 16.6.$$

The level surfaces of the upper and lower bounds on the value function is shown in Fig. 4 (right). Although the bounds are valid for all initial values within the estimated region of attraction, they match most closely for the optimized initial value. In addition, the computed control law is evaluated on the pendulum model (12) by simulation. The value of the loss function computed in this way is  $J(x_0, u) = 15.4$ .  $\square$

## V. GAIN AND DISSIPATION INEQUALITIES

As another application of the central idea, we shall compute bounds on the  $L_2$ -induced gain of a piecewise linear system as well as other dissipation inequalities. After verification of stability, for example using Proposition 1, an upper bound for the gain can be obtained as follows.

*Theorem (Upper Bound on  $L_2$  Gain):* Suppose there exist symmetric matrices  $T$ ,  $U_i$ , and  $W_i$  such that  $U_i$ , and  $W_i$  have

nonnegative entries, while  $P_i = F_i' T F_i$  and  $\bar{P}_i = \bar{F}_i' T \bar{F}_i$  satisfy

$$0 > \begin{bmatrix} P_i A_i + A_i' P_i + C_i' C_i + E_i' U_i E_i & P_i B_i \\ B_i' P_i & -\gamma^2 I \end{bmatrix} \quad \text{for } i \in I_0$$

$$0 > \begin{bmatrix} \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{C}_i' \bar{C}_i + \bar{E}_i' U_i \bar{E}_i & \bar{P}_i \bar{B}_i \\ \bar{B}_i' \bar{P}_i & -\gamma^2 I \end{bmatrix} \quad \text{for } i \in I_1.$$

Then every continuous piecewise  $C^1$  trajectory  $x$  with  $\int_0^\infty (|x|^2 + |u|^2) dt < \infty$ ,  $x(0) = 0$  satisfies

$$\int_0^\infty |y|^2 dt \leq \gamma^2 \int_0^\infty |u|^2 dt.$$

The best upper bound on the  $L_2$ -induced gain is achieved by minimizing  $\gamma$  subject to the constraints defined by the inequalities.

*Proof:* It follows as in the proof of Theorem 2 that

$$0 \geq \frac{d}{dt} (\bar{x}' \bar{P}_i \bar{x}) + |y|^2 - \gamma^2 |u|^2.$$

Integration from 0 to  $\infty$  gives the desired inequality.  $\square$

In analogy with the previous section, it is possible to compute a lower bound from an explicit control law  $u = \bar{L}_i \bar{x}$ . A candidate for such a control law is obtained by maximizing the expression

$$2\bar{x}' \bar{P}_i (\bar{A}_i + \bar{B}_i u) + |\bar{C}_i \bar{x}|^2 - \gamma^2 |u|^2$$

with respect to  $u$ , where  $\bar{P}_i$  and  $\gamma$  come from the upper bound computation. Simulating the system with this control law and comparing the input and output norms gives a lower bound on the  $L_2$  gain.

*Example 4 (Analysis of a Saturated Control System):* Consider the control system shown in Fig. 5. The output of the system  $G_1(s)$  is subject to a unit saturation. The closed-loop dynamics is piecewise affine, with three cells induced by the saturation limits  $u = \pm 1$ . We set  $r = 0$  and estimate the  $L_2$  induced gain from the disturbance  $d$  to the output  $y$ . With the transfer functions

$$G_1(s) = \frac{s+7}{16s^2+s+2}, \quad G_2(s) = \frac{s-3}{4s^2+3s+12}$$

we obtain the results shown in Table II. Here “Lure function” means a Lyapunov function of the form  $V(x) = x' P x + \eta \int_0^{Cx} \text{sat}(s) ds$  and “IQC for monotonic nonlinearities” means a gain estimate computed based on [21] using the toolbox [12]. A lower bound on the  $L_2$  gain, computed in the linear region, is equal to 2.36.  $\square$

The results on gain computation can be generalized in a natural way to validation or invalidation of other dissipation inequalities for the nonlinear system. More precisely, suppose

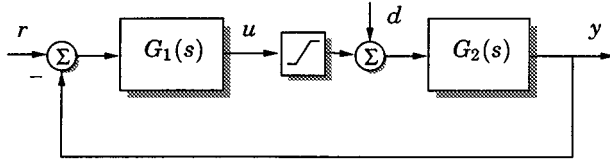


Fig. 5. Saturated control system.

TABLE II  
VARIOUS UPPER BOUNDS ON  $L_2$  GAIN

Method	Gain Bound
Quadratic Lyapunov function	No solution
Lure function with positive $\eta$	No solution
Lure without constraints on $\eta$	17.4
IQC for monotonic nonlinearities	2.41
Piecewise quadratic Lyapunov function	2.38

there exist symmetric matrices  $T$ ,  $U_i$  and  $W_i$  such that  $U_i$ , and  $W_i$  have nonnegative entries, while  $\bar{P}_i = \bar{F}_i' T \bar{F}_i$  satisfy

$$0 \geq \begin{bmatrix} \bar{C}_i' & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} \bar{C}_i & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{E}_i' U_i \bar{E}_i & \bar{P}_i \bar{B}_i \\ \bar{B}_i' \bar{P}_i & 0 \end{bmatrix}$$

for  $i \in I$ . Then, the trajectories of (1) satisfy the dissipation inequality

$$V(x(t_0)) - V(x(t_1)) \geq \int_{t_0}^{t_1} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}' M \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} dt$$

where  $V$  is defined by (5).

Also integral quadratic constraints with a frequency-dependent weight instead of the constant matrix  $M$  can be verified in the same way, by first introducing a state space realization of the weight and include these dynamics in the system description.

The opposite problem to validation is invalidation. In analogy with the lower bound bound on the  $L_2$  gain, this can be done by simulation, for example using a piecewise linear control law defined in term of the matrices  $\bar{P}_i$ , by minimization of the expression

$$\begin{bmatrix} y \\ u \end{bmatrix}' M \begin{bmatrix} y \\ u \end{bmatrix} + 2\bar{x}' \bar{P}_i (\bar{A}_i \bar{x} + \bar{B}_i u)$$

with respect to  $u$ .

## VI. SIMPLEX PARTITIONS

So far, we did not pay much attention to the partitioning of the state space and the specification of the matrices  $\bar{E}_i$  and  $\bar{F}_i$ . For a piecewise linear system, with a given state space partition for the dynamics, it is natural to do the initial analysis using the same partition for the Lyapunov function or loss function. However, there are many examples where a more refined partition is needed for the analysis. The purpose of this section is to introduce some convenient concepts for this purpose and discuss their properties.

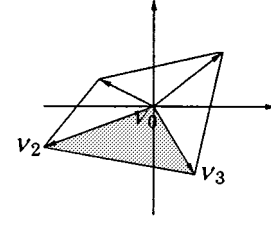


Fig. 6. Simplex partition of a compact domain of the state space.

An  $n$ -dimensional polytope is defined as the convex hull of a finite number of corner points in  $\mathbb{R}^n$ . It is called an  $n$ -dimensional simplex if the number of corner points is  $n + 1$ . Note that any polytope which is not a simplex can be partitioned into two polytopes, each with fewer corners than the original one. Repeating this procedure eventually generates a partition of the original polytope into simplices; see Fig. 6.

A simple and flexible way to partition the state space is to divide it in to simplices. An important advantage with simplex partitions is that arbitrary functions values at the grid points can be interpolated by continuous functions that are linear (or quadratic) in each simplex. This property makes it possible to approximate arbitrary continuous functions over simplex partitions, but is not valid for polytopic partitions in general. In this section we define a parameterization that supports the interpolation in simplices.

Let  $X \subset \mathbb{R}^n$  be a polytope with the simplex partition  $X = \bigcup_{i \in I} X_i$ , where all the simplices have nonempty interior and  $x = 0$  is a simplex vertex. Let  $\nu_0, \nu_1, \dots, \nu_p$  with  $\nu_0 = 0$  be the collection of vertices and define

$$\mathcal{V} = [\nu_0 \quad \dots \quad \nu_p] \\ \bar{\mathcal{V}} = [\bar{\nu}_0 \quad \dots \quad \bar{\nu}_p].$$

Then, each  $\bar{x}$  has a unique representation as a convex combination  $\bar{x} = \sum_{k=0}^p z_k \bar{\nu}_k$  with  $\sum_k z_k = 1$ ,  $z_k \geq 0$  for all  $k$  and  $z_k \neq 0$  if and only if  $\nu_k \in X_i$ . Define  $z = [0 \ z_1 \ \dots \ z_p]'$ . Then

$$x = \mathcal{V}z, \quad x \in X_i \quad i \in I \\ \bar{x} = \bar{\mathcal{V}}z, \quad x \in X_i \quad i \in I_1.$$

For each simplex  $X_i$ , define an extraction matrix  $\mathcal{E}_i \in \mathbb{R}^{(p+1) \times (n+1)}$  of  $X_i$  as follows. The  $k$ th row of  $\mathcal{E}_i$  is zero for all  $k$  such that  $\nu_k \notin X_i$  and the remaining rows of  $\mathcal{E}_i$  are equal to the rows of an identity matrix.

The extraction matrix then has the property that  $z = \mathcal{E}_i' \bar{\mathcal{E}}_i' z$  for all  $z$  corresponding to  $x \in X_i$ . In addition, the matrix  $\bar{\mathcal{V}} \mathcal{E}_i$  is invertible, due to the nonempty interior of  $X_i$ . Let  $\bar{E}_i$  and  $\bar{F}_i$  be defined by

$$\bar{F}_i = [0 \quad I_p] \mathcal{E}_i (\bar{\mathcal{V}} \mathcal{E}_i)^{-1} \quad (13)$$

$$\bar{E}_i = \mathcal{E}_i' \begin{bmatrix} 0 \\ \bar{F}_i \end{bmatrix} \quad (14)$$

for all  $i \in I$ . Then (3) and (4) are implied by the following proposition.



*Proposition 2:*

$$\begin{aligned}\bar{E}_i \bar{x} &= \mathcal{E}'_i z \\ \bar{F}_i \bar{x} &= [z_1 \quad \cdots \quad z_p]'\end{aligned} \quad \text{for } x \in X_i, i \in I. \quad (15)$$

In particular,  $e_i = 0$  and  $f_i = 0$  for  $i \in I_0$ .

*Proof:* Let  $\bar{z} = [z_0 \quad z_1 \quad \cdots \quad z_p]'$ . Then

$$\begin{aligned}\bar{x} &= \bar{V}\bar{z} = \bar{V}\mathcal{E}_i\mathcal{E}'_i\bar{z} \\ \bar{z} &= \mathcal{E}_i\mathcal{E}'_i\bar{z} = \mathcal{E}_i(\bar{V}\mathcal{E}_i)^{-1}\bar{x} \\ \bar{F}_i\bar{x} &= [0 \quad I_p]\bar{z} = [z_1 \quad \cdots \quad z_p]' \\ \bar{E}_i\bar{x} &= \mathcal{E}'_i \begin{bmatrix} 0 \\ \bar{F}_i \end{bmatrix} \bar{x} = \mathcal{E}'_i z.\end{aligned}$$

The last column of  $\bar{F}_i$ , denoted  $f_i$ , is identical to the  $z$  that corresponds to  $\bar{x} = [0 \quad \cdots \quad 0 \quad 1]' = \bar{v}_0$ . Hence  $f_i = [0 \quad \cdots \quad 0]'$ .  $\square$

*Remark 3:* In applications, it is often advantageous to extend the  $\bar{F}_i$  further, so that  $\bar{F}_i\bar{x} = [z_1 \quad \cdots \quad z_p \quad x_1 \quad \cdots \quad x_n]'$ .  $\square$

#### A. Polyhedral Regions

As a generalization of “polytope,” that also allows corners at infinity,  $X_i \subset \mathbb{R}^n$  is called a *polyhedron*, if every  $x \in X_i$  can be written as

$$x = \sum_{k=0}^q z_k \nu_k + \sum_{k=q+1}^p z_k \nu_k \quad (16)$$

with  $z_k \geq 0$ ,  $\sum_{k=0}^q z_k = 1$ . The vectors  $\nu_1, \dots, \nu_q$  are finite vertices, while  $\nu_{q+1}, \dots, \nu_p$  define vertex directions at infinity. A *generalized simplex* is a polyhedron with  $p = n$ .

Let  $X = \cup_{i \in I} X_i$  be a polyhedron partitioned into generalized simplices, each with nonempty interior. Let the partition be given by the finite vertices  $\nu_0, \nu_1, \dots, \nu_q$  with  $\nu_0 = 0$  and the infinite vertex directions  $\nu_{q+1}, \dots, \nu_p$ . Then, with  $\mathcal{V}, \bar{V}, \bar{E}_i$  and  $\bar{F}_i$  defined as in the previous subsection, all the earlier statements remain valid, except that the identity  $\sum_k z_k = 1$  does not include terms with  $k > q$ .

#### B. Partitioning a Subspace of the State Space

In some cases, it is natural to partition only a subspace of the state space. This can be done conveniently by replacing  $x$  with  $Cx$  for some matrix  $C \in \mathbb{R}^{m \times n}$  everywhere in the discussion of simplex partitions. Then  $\nu_0, \dots, \nu_p \in \mathbb{R}^m$

$$Cx = \mathcal{V}z, \quad x \in X_i \quad i \in I$$

and Proposition 2 holds with

$$\begin{aligned}\bar{F}_i &= \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix} \mathcal{E}_i(\bar{V}\mathcal{E}_i)^{-1} \begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix} \\ \bar{E}_i &= \mathcal{E}'_i \bar{F}_i.\end{aligned}$$

## VII. APPROXIMATION OF SMOOTH SYSTEMS

One motivation for the study of piecewise linear systems is that they can be used to approximate smooth nonlinear systems. The purpose of this section is to show how the approximation error can be explicitly taken into account, in order to generate formal results also for smooth systems. Moreover, we prove a converse result for smooth nonlinear systems on the existence and computability of piecewise quadratic Lyapunov functions.

In [7], it was suggested that upper and lower bounds of the smooth nonlinearity are used in each polyhedral region. Stability of the original system follows if it is possible to find a Lyapunov function that is valid for the bounding systems in all regions. Another good alternative, particularly for multivariable nonlinearities, is to use a norm bound of the approximation error in the following manner.

*Theorem 4:* Let  $x(t)$  be a piecewise  $C^1$  trajectory of the system  $\dot{x} = f(x)$  and assume that

$$|f(x) - A_i x - a_i| \leq \varepsilon_i |x|, \quad i \in I.$$

If there exist numbers  $\gamma_i > 0$ , symmetric matrices  $U_i$  and  $V_i$  with nonnegative entries, and a symmetric matrix  $T$  such that  $\bar{P}_i = \bar{F}'_i T \bar{F}_i$  and  $P_i = F'_i T F_i$  satisfy

$$E'_i U_i E_i < P_i < \gamma_i I \quad (17)$$

$$-E'_i V_i E_i > A'_i P_i + P_i A_i + 2\varepsilon_i \gamma_i I \quad (18)$$

for  $i \in I_0$  and

$$\bar{E}'_i U_i \bar{E}_i < \bar{P}_i < \gamma_i I \quad (19)$$

$$-\bar{E}'_i V_i \bar{E}_i > \bar{A}'_i \bar{P}_i + \bar{P}_i \bar{A}_i + 2\varepsilon_i \gamma_i I \quad (20)$$

for  $i \in I_1$ , then  $x(t)$  tends to zero exponentially.

*Proof:* Define

$$\bar{V}(x) = \bar{x}' \bar{P}_i \bar{x}, \quad x \in X_i, i \in I. \quad (21)$$

The inequalities (17) and (19) imply that

$$c_1 |x|^2 \leq \bar{V}(x) \leq c_2 |x|^2$$

for some  $c_1, c_2 > 0$ . Let the approximation error be

$$\tilde{a}_i(x) = \begin{bmatrix} f(x) - A_i x - a_i \\ 0 \end{bmatrix} \quad x \in X_i, i \in I.$$

Then, (18) and (17) together with the assumption  $|\tilde{a}_i(x)| \leq \varepsilon_i |x|$  imply that

$$\begin{aligned}\frac{d}{dt} \bar{V}(x) &= \bar{x}'(\bar{P}_i \bar{A}_i + \bar{A}'_i \bar{P}_i) \bar{x} + 2\bar{x}' \bar{P}_i \tilde{a}_i(x) \\ &< -2(\varepsilon_i \gamma_i + \delta) |x|^2 + 2\gamma_i |x| \cdot |\tilde{a}_i(x)| \quad (22)\end{aligned}$$

$$\leq -\delta |x|^2 \leq -\delta \bar{V}(x) / c_2 \quad (23)$$

for some  $\delta > 0$ . This proves the exponential decay.  $\square$

Theorem 4 quantifies the trade off between computational effort and precision in the analysis. If no solution to the above inequalities is found, one may refine the state space partition for the piecewise linear system approximation and the piecewise quadratic Lyapunov function, and try again.

It is natural to ask how restrictive this approach is, compared to a theorem based on arbitrary continuous Lyapunov functions. The answer is given by the following result, showing that in principle, whenever a Lyapunov function exists, there also exists a solution to the relevant matrix inequalities.

**Theorem 5:** Let  $f \in C^1(X, \mathbb{R}^n)$ , where  $X$  is a bounded invariant polytope for the system  $\dot{x} = f(x)$ . If the system is globally exponentially stable on  $X$ , then for every sufficiently refined simplex partition  $X = \cup_{i=1}^N X_i$  with  $A_i = (\partial f / \partial x)(\nu_i)$  and  $\bar{E}_i, \bar{F}_i$  defined by (13), (14), there exists a solution  $\gamma_i, U_i, V_i$ , and  $T$  to the inequalities (17)–(20).

*Proof:* First note that by a standard converse Lyapunov theorem, see [11, Th. 3.12] for example, there exists a  $C^1$  Lyapunov function  $V(x)$  that satisfies

$$c_1|x|^2 \leq V(x) \leq c_2|x|^2 \quad (24)$$

$$\begin{aligned} \frac{\partial V}{\partial x} f(x) &\leq -c_3|x|^2 \\ \left| \frac{\partial V}{\partial x} \right| &\leq c_4|x| \end{aligned} \quad (25)$$

for some positive constants  $c_1, c_2, c_3, c_4$ . The function  $V$  can be approximated by a function  $\bar{V}$  of the form (21) by letting  $\bar{P}_i = \bar{F}_i' T \bar{F}_i$  with  $\bar{F}_i$  defined by (15) and

$$\begin{aligned} T &= [1 \ \cdots \ 1]' [V(\nu_1) \ \cdots \ V(\nu_p)] / 2 \\ &\quad + [V(\nu_1) \ \cdots \ V(\nu_p)]' [1 \ \cdots \ 1] / 2. \end{aligned}$$

This choice makes  $\bar{V}$  quadratic in all regions that contain the origin and affine in all others. Moreover

$$\bar{V}(\nu_i) = T_{ii} := V(\nu_i), \quad i \in I$$

so  $\bar{V}$  and  $\partial \bar{V} / \partial x$  become arbitrarily accurate approximations of  $V$  and  $\partial V / \partial x$  as the partition is refined.

In the regions that do not contain the origin,  $\bar{V}$  and  $\partial \bar{V} / \partial x$  are affine, so the inequalities (24) and (25) imply that

$$\begin{aligned} \varepsilon_i &\leq \bar{x}' \bar{P}_i \bar{x} & x \in X_i \\ -\varepsilon_i &\geq \bar{x}' (\bar{A}_i' \bar{P}_i + \bar{P}_i \bar{A}_i + \varepsilon_i I) \bar{x} & x \in X_i \end{aligned}$$

for sufficiently small  $\varepsilon_i > 0$ . By Farkas lemma [17], this is equivalent to existence of vectors  $u_i$  and  $w_i$  with positive coefficients such that

$$\begin{aligned} \varepsilon_i &\leq \bar{x}' \bar{P}_i \bar{x} - u_i' \bar{E}_i \bar{x} & x \in \mathbb{R}^n \\ -\varepsilon_i &\geq \bar{x}' (\bar{A}_i' \bar{P}_i + \bar{P}_i \bar{A}_i + \varepsilon_i I) \bar{x} - w_i' \bar{E}_i \bar{x} & x \in \mathbb{R}^n. \end{aligned}$$

Let  $e = [1 \ \cdots \ 1]'$  and note that  $e' \bar{E}_i = [0 \ \cdots \ 0 \ 1]$ . With  $U_i = (e u_i' + u_i e') / 2 + \varepsilon_i (e e' - I) / 2$  it follows that

$$\begin{aligned} \bar{x}' \bar{E}_i' U_i \bar{E}_i \bar{x} &= u_i' \bar{E}_i \bar{x} + \text{var} \varepsilon_i / 2 - \varepsilon_i |\bar{E}_i \bar{x}|^2 / 2 \\ &\leq \bar{x}' \bar{P}_i \bar{x} - \varepsilon_i / 2 - \varepsilon_i |\bar{E}_i \bar{x}|^2 / 2 & x \in \mathbb{R}^n \\ &\leq \bar{x}' (\bar{P}_i - \mu_i I) \bar{x} \end{aligned}$$

for sufficiently small  $\mu_i > 0$ . The analog choice of  $W_i$  gives

$$-\bar{x}' \bar{E}_i' W_i \bar{E}_i \bar{x} \geq \bar{x}' (\bar{A}_i' \bar{P}_i + \bar{P}_i \bar{A}_i + \mu_i I) \bar{x}, \quad x \in \mathbb{R}^n.$$

Hence (19), (20) holds with appropriate values of  $\varepsilon$  and  $\gamma_i$ .

Finally, every region that contains the origin must have a boundary surface in common with a region that does not. On this surface, the  $\bar{V}$  is again affine, so the same argument can be applied to construct  $U_i$  and  $W_i$  such that

$$\begin{aligned} x' E_i' U_i E_i x &\leq x' (P_i + \mu_i I) x \\ -x' E_i' W_i E_i x &\geq x' (A_i' P_i + P_i A_i + \mu_i I) x \end{aligned}$$

for all  $x$  on the hyperplane defined by the boundary surface. This implies (17)–(18), so the proof is complete.  $\square$

## VIII. CONCLUSIONS

A flexible and powerful approach to computational analysis and optimization of nonlinear control systems has been developed, using a combination of piecewise linear system descriptions and piecewise quadratic Lyapunov functions and loss functions.

Local analysis of nonlinear systems near an equilibrium is usually done based on linearization. The linear approximation is good close to the equilibrium and there is a powerful theory for control and performance analysis of linear systems. However, as the region of investigation is extended, it becomes desirable to take the nonlinear effects more explicitly into account. Using the framework of this paper this can be done incrementally. Starting from the purely linear analysis, one can add more and more partitions of the state space in order to extend the investigated region of state space, piece by piece.

Last, but not least, it should be emphasized that the main limiting factor for the application of the ideas in this paper is the issue of computational complexity. For a fixed number of regions, the complexity grows polynomially with the state dimension. However, the necessary number of regions often grows rapidly with dimension, resulting in exponential complexity growth anyway. This is indeed a subject for future research, and further ideas based on linear programming are presented in [15]. Furthermore, the combination of upper and lower bounds sometimes can be used to reduce the need for grid refinement and increase the computational speed considerably.

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