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1971

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Åström, K. J. (1971). *Dynamics of Concentration Variations in a Laminar Tube Flow*. (Research Reports TFRT-3028). Department of Automatic Control, Lund Institute of Technology (LTH).

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DYNAMICS OF CONCENTRATION VARIATIONS IN LAMINAR TUBE FLOW.

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REPORT 7102(B) APRIL 1971
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Dynamics of concentration variations in laminar tube flow

1. INTRODUCTION

The dynamics of concentration variations in a tube with a stationary laminar flow will be investigated. It is assumed that the concentration across the inlet is constant. This concentration is taken as the input to the system. The average concentration over the cross section of the tube at a distance a downstream from the inlet is taken as the output.

2. A MATHEMATICAL MODEL

Consider a stationary laminar flow in a circular tube with radius r_0. Let r denote the distance from the center of the tube and x the distance downstream from the tube inlet.

Let $c(r,x,t)$ be the concentration at a point (r,x) at time t. If the flow is laminar we have

$$c(r,x,t+h) = c(r,x-hv(r),t) \tag{1}$$

where $v(r)$ is the velocity at a distance r from the center of the tube. The concentration thus satisfies the following partial differential equation

$$\frac{\partial c}{\partial t} = -v(r)\frac{\partial c}{\partial x} \tag{2}$$

Since the concentration at $x=0$ is assumed independent of r we get

$$c(r,0,t) = u(t) \tag{3}$$

where u is the input. The output is the average concentration at $x = a$ i.e.

$$(4) \qquad y(t) = 2r_0^{-2} \int_0^{r_0} rc(r,a,t)dt$$

The dynamics of the system can thus be represented by the equations (2), (3) and (4). The state is the concentration in $0 \leq r \leq r_0$, $0 \leq x \leq a$.

3. INPUT- OUTPUT RELATIONS

Introducing (1) into (3) we find that the system can be characterized by the input output relation

$$(5) \qquad Y(t) = 2r_0^{-2} \int_0^{r_0} r\, c(r,\, 0,\, t-\frac{a}{v(r)})dr =$$

$$= 2r_0^{-2} \int_0^{r_0} r\, u(t- \frac{a}{v(r)})dr$$

Taking Laplacetransforms we find

$$Y(s) = 2r_0^{-2} \int_0^{r_0} r\, e^{-s \frac{a}{v(r)}} dr\, U(s)$$

The transfer function is thus given by

$$(6) \qquad G(s) = 2r_0^{-2} \int_0^{r_0} re^{-s \frac{a}{v(r)}} dr =$$

$$\int_{a/N_0}^{\infty} e^{-st}\, d(r_0^{-1} v^{-1}(a/t))^2$$

where the last equality is obtained by changing the variables $\frac{a}{v(s)} \rightarrow t$ and v^{-1} denotes the inverse function.

Now observe the relation

$$G(s) = \int_0^\infty e^{-st} h(t)dt$$

and we find that the step response of the system is given by

$$(7) \qquad H(t) = \begin{cases} 0 & t < a/v_0 \\ [r_0^{-1} v^{-1}(a/t)]^2 & t \geq a/v_0 \end{cases}$$

Notice that the equation (6) can also be obtained by taking Laplace transform of (2), (3) and (4).

The equation (7) can be derived directly by assuming that the system is initially of rest and making a unit step change. The concentration after the step is shown in Fig. 1.

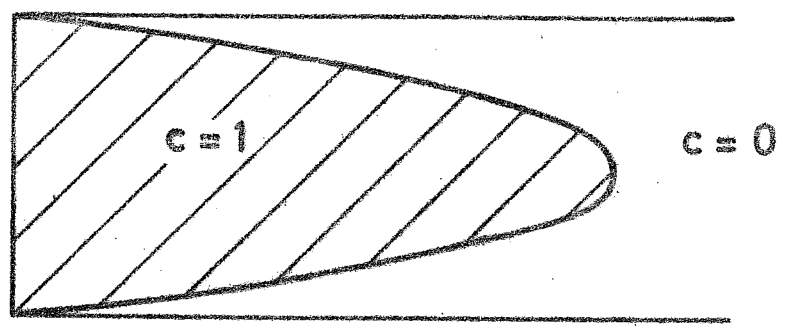

Fig. 1 Concentration after a unit step.

For x=a we thus find that the concentration equals 1 for $r \leq r_1$ where

$$tv(r_1) = a$$

i.e. $r_1 = v^{-1}(\frac{a}{t})$

The average concentration of x = a is thus

$$y(t) = \begin{cases} 0 & t < a/v_0 \\ [r_o^{-1} v^{-1}(a/t)]^2 & t \geq a/v_0 \end{cases}$$

which is identical to (7)

4. A SPECIAL CASE

Now consider a special case. Assume

$$(8) \quad v(r) = v_o \, [1 - (r/r_0)^n]$$

Then

$$r = r_0 \, [1 - \frac{v(r)}{v_0}]^{1/n}$$

Hence

$$(9) \quad H(t) = \begin{cases} 0 & t < a/v_0 \\ (1 - \frac{a}{v_0 t})^{2/n} & t \geq a/v_0 \end{cases}$$

In Fig. 2 we show the step responses for different values of n.

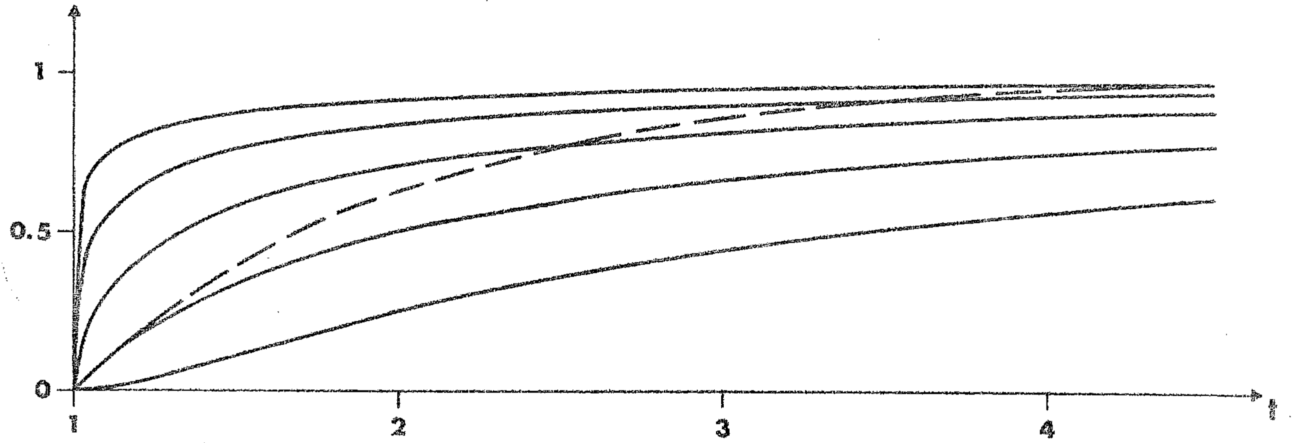

Fig. 2 Step responses (9) for $n = 1,2,4,8$ samt 16. For comparisions we also show the response of a delayed first order system.

For $n = 2$ we get in particular

$$H(t) = \begin{cases} 0 & t < a/v_0 \\ 1 - \dfrac{a}{v_0 t} & t \geq a/v_0 \end{cases}$$