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Karlsson, Anders

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Physical Limitations of Antennas in a Lossy Medium

Anders Karlsson

***Abstract*—The dissipated power and the directivity of antennas in a homogeneous, lossy medium are systematically analyzed in this paper. The antennas are ideal and located inside a lossless sphere. In the lossy space outside the sphere, the electromagnetic fields are expanded in a complete set of vector wave functions. The radiation efficiency, the directivity, and the power gain are defined for antennas in a lossy medium, and the optimal values of these quantities are derived. Simple relations between the maximal number of ports, or channels, an antenna can use and the optimal directivity and gain of the antenna are presented.**

***Index Terms*—Antenna theory, lossy systems.**

I. Introduction

In some applications there is a need for wireless communication with devices in lossy materials. A conductive medium is a low-pass filter for the electromagnetic waves, and one is then often forced to use low frequencies, or equivalently, long wavelengths. If the space for the antenna is limited it results in an antenna that is small compared to the wavelength. The drawback is that small antennas in lossy materials consume much power, due to the ohmic losses in the near-zone of the antenna. Hence, the design of the antenna and the choice of frequency are delicate problems, where two power loss mechanisms with counteracting frequency dependences are involved. This power problem is addressed in this paper.

Antennas in lossy materials are found in various areas. In geophysical applications underground antennas are used, e.g., in bore holes. In marine technology antennas are used for communication with underwater objects. In medical applications there is an increased usage of wireless communication with implants. Implants, e.g., pacemakers, have limited power supply and it is important to use power efficient antennas.

Some of the results in this paper are based on the results obtained by Chu [4] and Harrington [8], who investigated physical limitations for antennas in free space. Chu derived the optimal value of the directivity and the optimal value of the ratio between the directivity and the Q-value of omni-directional antennas and Harrington derived the corresponding results for general antennas. There are a number of other articles that address the optimization of the Q-value of an antenna, cf. [5], [7], and [12].

For a lossy material it is the dissipated power, rather than the Q-value, that is the most important quantity in the design of an antenna. In this paper, the radiation efficiency, the directivity, and the power gain of antennas are defined and studied for the simplified geometry where the antenna is enclosed in a lossless sphere. The optimal values of these three quantities are the main results in this paper. The optimal value of the directivity is shown to be related to the maximum number of ports, or channels, of the antenna, a result that holds also in a lossless medium. It is emphasized that in a lossy medium the magnetic dipole is the most radiation efficient antenna, a well known and important result, cf. [11].

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The author is with the Department of Electroscience, Lund Institute of Technology, S-221 00 Lund, Sweden (e-mail: anders.karlsson@es.lth.se).

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II. Preliminaries

The antennas are confined in a spherical, lossless region, $r < a$ denoted V_{int}. They are idealized in the sense that there are no ohmic losses in V_{int}. The volume $r > a$ is denoted V_{ext} and is an infinite, homogeneous, conducting medium with a complex permittivity

$$\varepsilon = \varepsilon_0 \varepsilon_r - j\frac{\sigma}{\omega} \tag{2.1}$$

where the time-dependence $e^{j\omega t}$ is assumed. The corresponding wave number is denoted k and is given by

$$k = \omega\sqrt{\mu\varepsilon}. \tag{2.2}$$

The permeability $\mu = \mu_0\mu_r$ is assumed to be real. The wave impedance in V_{ext} reads

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}. \tag{2.3}$$

III. General Antennas in Conducting Media

In the exterior region V_{ext} the electric field is expanded in spherical vector waves $\boldsymbol{u}_{\tau\kappa ml}(\boldsymbol{r})$, also referred to as partial waves. These waves satisfy Maxwell's equations and are complete on a spherical surface. The details of the spherical vector waves are given in Appendix A. The expansion reads

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{l=1}^{\infty}\sum_{m=0}^{l}\sum_{\kappa=\text{e/o}}\sum_{\tau=1}^{2} a_{\tau\kappa ml}\boldsymbol{u}_{\tau\kappa ml}(\boldsymbol{r}). \tag{3.1}$$

The corresponding magnetic field is given by the induction law

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{r}) &= \frac{j}{\omega\mu}\sum_{l=1}^{\infty}\sum_{m=0}^{l}\sum_{\kappa=\text{e/o}}\sum_{\tau=1}^{2} a_{\tau\kappa ml}\nabla\times\boldsymbol{u}_{\tau\kappa ml}(\boldsymbol{r})\\ &= \frac{jk}{\omega\mu}\sum_{l=1}^{\infty}\sum_{m=0}^{l}\sum_{\kappa=\text{e/o}}\sum_{\tau=1}^{2} a_{\tau\kappa ml}\boldsymbol{u}_{\tau'\kappa ml}(\boldsymbol{r})\end{aligned} \tag{3.2}$$

where $\tau' = 3 - \tau$. Here, $\tau = 1, 2$ is the index for the two different wave types (TE and TM), $\kappa = \mathrm{e}$ for waves that are even with respect to the azimuthal angle ϕ and $\kappa = \mathrm{o}$ for the waves that are odd with respect to $\phi, l = 1, 2 \ldots$ is the index for the polar direction, and $m = 0, \ldots l$ is the index for the azimuthal angle. For $m = 0$ only the partial waves with $\kappa = \mathrm{e}$ are nonzero, cf., (A.2). The expansion in (3.1) covers all possible types of time harmonic sources inside V_{int}.

A. Classification

Antennas that radiate partial waves with $\tau = 1$ are referred to as magnetic antennas, since the reactive part of their radiated complex power is positive, i.e., inductive. Antennas radiating partial waves with $\tau = 2$ are referred to as electric antennas, since they are capacitive when they are small compared to the wavelength.

The expansion coefficients $a_{\tau\kappa ml}$ in the expansion (3.1) can theoretically be altered independently of each other. Hence, each partial wave corresponds to an independent port of the antenna. The maximum number of ports, or channels, an antenna can use is then equal to the maximum number of partial waves the antenna can radiate.

The following classification of antennas is used in this paper:

Partial wave antenna—antenna that radiates only one partial wave $(\tau\kappa ml)$. The antenna has one port.

Magnetic multipole antenna of order l An antenna that radiates partial waves with $\tau = 1$ and index l. The maximum number of ports is $N_{l\mathrm{port}} = 2l + 1$.

Electric multipole antenna of order l An antenna that radiates partial waves with $\tau = 2$ and index l. The maximum number of ports is $N_{l\mathrm{port}} = 2l + 1$.

Magnetic antenna of order $l_{\max}$ An antenna that radiates partial waves with $\tau = 1$ and with $l = 1, \ldots l_{\max}$. The maximum number of ports is $N_{\mathrm{port}} = l_{\max}(l_{\max} + 2)$.

Electric antenna of order $l_{\max}$ An antenna that radiates partial waves with $\tau = 2$ and with $l = 1, \ldots l_{\max}$. The maximum number of ports is $N_{\mathrm{port}} = l_{\max}(l_{\max} + 2)$.

Combined antenna of order $l_{\max}$ An antenna that radiates partial waves with $\tau = 1, 2$ and $l = 1, \ldots l_{\max}$. The maximum number of ports is $N_{\mathrm{port}} = 2l_{\max}(l_{\max} + 2)$.

B. Rotation of an Antenna

If an antenna is rotated, the new set of radiated partial waves is determined by the rotational matrix for the vector waves, cf. [3]. That matrix is diagonal in the index τ and in the index l, but not in the other two indices κ and m. Thus, a magnetic multipole antenna of index l is still a magnetic multipole antenna of index l, after it is rotated. This type of invariance under rotation is true for all types of antennas in Section III-A, except for the partial wave antenna. The invariance is utilized in Section IV to determine the optimal values of the directivity and power gain. A partial wave antenna that radiates the partial wave $(\tau\kappa ml)$, is under a rotation transformed to an antenna that radiates several partial waves $(\tau\kappa' m' l)$, where κ' can be both e, o, and m can take the values $0, \ldots, l$.

C. The Power Flow

The complex power $S(a)$ radiated from an antenna is given by

$$S(a) = \frac{1}{2} \oint_{S_a} (\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})) \cdot \hat{r}\, dS \tag{3.3}$$

where S_a is the surface $r = a, \hat{r} = \boldsymbol{r}/|\boldsymbol{r}|$ is the radial unit vector, and $\boldsymbol{H}^*(\boldsymbol{r})$ is the complex conjugate of the magnetic field. The complex power is decomposed as

$$S(a) = P(a) + 2j\omega(W_m(a) - W_e(a)). \tag{3.4}$$

The active part of the power $P(a)$ is the power dissipated in the region V_{ext}, whereas W_m and W_e are the time averages of the stored magnetic and electric energies in the exterior region.

The impedance Z and admittance Y of the antenna are related to the complex power $S(a)$ by the power relation

$$S(a) = \frac{1}{2} Z|I|^2 = \frac{1}{2} Y^* |V|^2 \tag{3.5}$$

where I and V are the complex current and voltage that feeds the antenna, respectively. The star denotes complex conjugate. For a nonideal antenna the powers inside V_{int} should be added to the left-hand side of (3.5).

The complex power radiated from a combined antenna of order $l_{\max}$ follows from (A.4) and (A.5), and from (3.1)–(3.3)

$$\begin{aligned} S(a) = -\frac{j}{2\eta^*} a^2 \sum_{l=1}^{l_{\max}} \sum_{m=-l}^{l} \sum_{\kappa=\mathrm{e/o}} \Bigg(& |a_{1\kappa ml}|^2 h_l(ka) \\ & \times \left(h_l'(ka) + \frac{1}{ka} h_l(ka) \right)^* - |a_{2\kappa ml}|^2 h_l^*(ka) \\ & \times \left(h_l'(ka) + \frac{1}{ka} h_l(ka) \right) \Bigg). \end{aligned} \tag{3.6}$$

The complex powers of the other types of antennas in Section III-A are special cases of (3.6). The normalized complex power, $S(a)/|S(a)|$, of multipole antennas of order l depends only on the indices τ and l. If the transmitted complex power of such an antenna is denoted $S_{\tau l}(a)$ and the corresponding impedance is denoted $Z_{\tau l}$ then

$$\begin{cases} \dfrac{\eta^* S_{1l}(a)}{|S_{1l}(a)|} = \dfrac{\eta S_{2l}^*(a)}{|S_{2l}(a)|} \\ \dfrac{\eta^* Z_{1l}}{|Z_{1l}|} = \dfrac{\eta Z_{2l}^*}{|Z_{2l}|} \end{cases} \tag{3.7}$$

where η is the wave impedance.

D. Asymptotic Values of S and Z

When $|k|a \to \infty$ the asymptotic behavior of the Hankel functions, (A.6), implies that the asymptotic values of the radiated complex power, cf., (3.6) and of the impedance, cf., (3.5), are

$$\frac{S(a)}{|S(a)|} = \frac{Z}{|Z|} = \frac{\eta}{|\eta|}. \tag{3.8}$$

As $ka \to 0$ the limiting values of the Hankel functions yield

$$\frac{S(a)}{|S(a)|} = \frac{Z}{|Z|} = \begin{cases} j & \text{for a magnetic antenna} \\ -j\dfrac{\eta^2}{|\eta|^2} & \text{for an electric antenna.} \end{cases} \tag{3.9}$$

The asymptotic values in (3.8) and (3.9) are valid for all of the antennas in Section III-A. The values are illustrated in Fig. 1.

In the rest of the pape,r only active power will be considered and for this reason the term power is understood to mean active power.

E. Far-Field and Directivity

The far-field amplitude is defined as

$$\boldsymbol{F}(\theta,\phi) = \lim_{|k|r\to\infty} \boldsymbol{E}(\boldsymbol{r}) kr e^{jkr}. \tag{3.10}$$

The far-field amplitude of a combined antenna of order $l_{\max}$ is given by the asymptotic values of the spherical Hankel functions, cf., (A.6)

$$\boldsymbol{F}(\theta,\phi) = \sum_{l=1}^{l_{\max}} \sum_{m=0}^{l} \sum_{\kappa=\mathrm{e/o}} \sum_{\tau=1}^{2} a_{\tau\kappa ml} j^{l+2-\tau} \boldsymbol{A}_{\tau\kappa ml}(\theta,\phi). \tag{3.11}$$

The far-field amplitude of the antenna corresponding to (3.1) is thus completely defined by the coefficients $a_{\tau\kappa ml}$.

The directivity is defined in the same way as for a lossless medium, cf., [10]. The directivity of the combined antenna is obtained from the far-field amplitude of the antenna and from the orthogonality of the vector spherical harmonics, cf., (3.12), as shown in (A.4), at the bottom of the page, where $|\boldsymbol{F}(\theta,\phi)|^2 = \boldsymbol{F}(\theta,\phi)\cdot\boldsymbol{F}^*(\theta,\phi)$, and where max is with respect to θ and ϕ. Hence, also the directivity is completely defined by the expansion coefficients. The far-field amplitudes and the directivities of the other antennas in Section III-A follow from (3.11) and (3.12).

F. Radiation Efficiency and Power Gain

For antennas in a lossless space the radiation efficiency, η_{eff}, is defined as the ratio of the power radiated from the antenna to the power put into the antenna. This definition is not applicable here since the antenna is ideal and hence, the efficiency would be one. A possible alternative definition for an ideal antenna in a lossy material is the quotient $P(r)/P(a)$, where $P(a)$ is the power radiated from the antenna and $P(r)$ is the power radiated through a spherical surface of radius r. That ratio indicates how much of the power fed to the antenna is radiated in the far-zone. In the far-zone

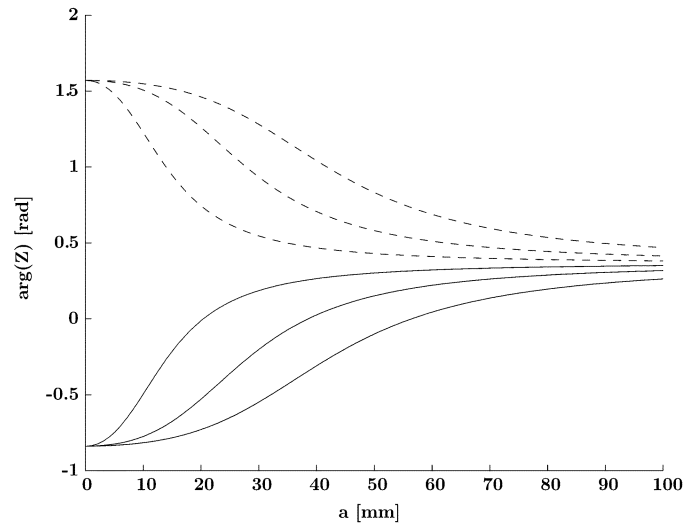

Fig. 1. Argument of the impedance, $\arg(Z)$ of magnetic multipole antennas of order $l = 1$ (lower dashed line), 2 (middle dashed), and 3 (upper dashed) and for the electric multipole antennas of order $l = 1$ (upper solid line), 2 (middle solid), and 3 (lower solid). The frequency is 400 MHz, $\varepsilon_r = 50$, and $\sigma = 1$ S/m, corresponding to an argument of the wave impedance $\arg(\eta) = 0.37$ rad. The asymptotic values in (3.8) and (3.9) are reached when $a \to 0$ and ∞, respectively.

$$P(r) = \mathrm{Re}\left\{\frac{1}{2|k|^2\eta}\right\} \sum_{l=1}^{l_{\max}} \sum_{m=0}^{l} \sum_{\kappa=\mathrm{e/o}} \sum_{\tau=1}^{2} |a_{\tau\kappa ml}|^2 e^{-2|\mathrm{Im}\{k\}|r} \tag{3.13}$$

for a combined antenna of order $l_{\max}$, as seen from (3.6) and (A.6). The radiated powers of the other types of antennas are special cases of this expression. In order to have a definition of radiation efficiency that is independent of the radius r, the following dimensionless quantity is used

$$\eta_{\text{eff}} = \frac{P^0}{P(a)} \tag{3.14}$$

where

$$P^0 = \mathrm{Re}\left\{\frac{1}{2|k|^2\eta}\right\} \sum_{l=1}^{l_{\max}} \sum_{m=0}^{l} \sum_{\kappa=\mathrm{e/o}} \sum_{\tau=1}^{2} |a_{\tau\kappa ml}|^2. \tag{3.15}$$

The radiated power at a distance r from an antenna is expressed in terms of the radiation efficiency and the input power, $P(a)$, as $P(r) = P(a)\eta_{\text{eff}} e^{-2|\mathrm{Im}\{k\}|r}$. The notation η_{eff} is in accordance with most antenna literature. It should not be confused with the notation η for the wave impedance.

$$\begin{aligned} D &= \frac{4\pi|\boldsymbol{F}(\theta,\phi)|^2_{\max}}{\int |\boldsymbol{F}(\theta,\phi)|^2\, d\Omega} \\ &= \frac{4\pi\left|\sum_{l=1}^{l_{\max}}\sum_{m=0}^{l}\sum_{\kappa=\mathrm{e/o}}\sum_{\tau=1}^{2} a_{\tau\kappa ml} j^{l+2-\tau}\boldsymbol{A}_{\tau\kappa ml}(\theta,\phi)\right|^2_{\max}}{\sum_{l=1}^{l_{\max}}\sum_{m=0}^{l}\sum_{\kappa=\mathrm{e/o}}\sum_{\tau=1}^{2}|a_{\tau\kappa ml}|^2} \end{aligned} \tag{3.12}$$

The asymptotic value of η_{eff} for large radii a is

$$\eta_{\text{eff}} = e^{2|\text{Im}\{k\}|a}, \quad \text{as } a \to \infty \tag{3.16}$$

which is in agreement with the graph of the efficiency in Fig. 2. The radiation efficiencies of multipole antennas of order l are denoted $\eta_{\text{eff}\tau l}$ and according to (3.6), (3.14), and (3.15) they read as in (3.17), shown at the bottom of the page.

The product of the directivity and the radiation efficiency, $D\eta_{\text{eff}}$, is proportional to the quotient of the maximum power flow density in the far-zone and the input power to the antenna. It is referred to as the power gain of the antenna and the notation

$$G = D\eta_{\text{eff}} \tag{3.18}$$

is adopted. This definition is in concordance with the power gain of antennas in lossless media, also referred to as the maximum value of the gain, cf. [2]. The notations G_1 and G_2 are below used for the power gains of magnetic and electric antennas, respectively.

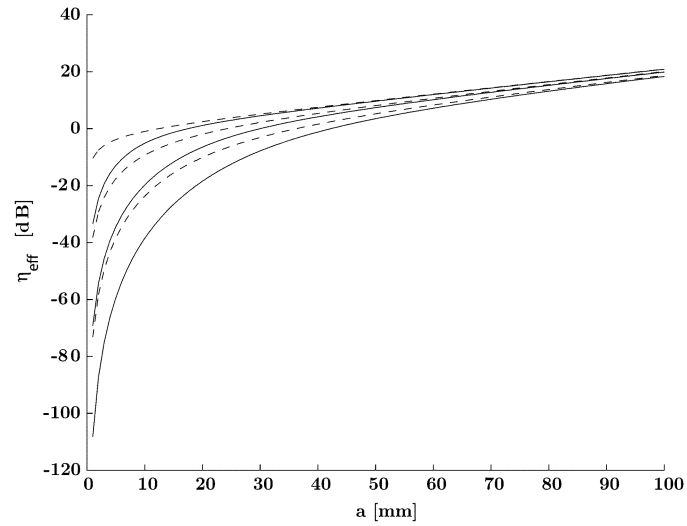

Fig. 2. Radiation efficiency, η_{eff}, of six different multipole antennas. The dashed curves are for the magnetic antennas and the solid curves are for electric antennas. The curves are for $l = 1$ (upper), $l = 2$ (middle), and $l = 3$ (lower). These curves emphasize that the magnetic dipole is the most radiation efficient antenna. When $a = 5$ mm the magnetic dipole is approximately 10 dB more efficient than the electric dipole. The frequency is 400 MHz, $\varepsilon_r = 50$, and $\sigma = 1$ S/m.

IV. Optimization

Optimization of an antenna is in this context to find the amplitudes $a_{\tau lm\kappa}$ of the radiated partial waves such that a specified quantity is optimized. The techniques used by Chu and Harrington, cf. [4] and [8] can be used to derive the optimal (i.e., maximal) values of the directivity and of the power gain, G, of general spherical antennas in a lossy medium. Harrington showed that the optimal value of the directivity for a combined antenna of order l in vacuum is $l(l+2)$, i.e., half of the number of ports for the antenna. That proof holds also for conductive media. For convenience a derivation of the optimal directivity, analogous to the one given by Harrington, is given in Appendix B. The other derivations are left to the reader. Optimization of the radiation efficiency is to minimize the power fed to an antenna for a given power flow in the far-zone, regardless of the directivity. Optimization of the directivity is to maximize the power flow density in one direction in the far-zone, for a given total power flow in the far-zone, regardless of the power fed to the antenna. Optimization of the power gain is to maximize the power flow density in one direction in the far-zone, for a given power fed to the antenna.

For a lossy material it can be shown from (3.17) that the optimal value of the radiation efficiency, η_{eff}, for any antenna is the one obtained for a magnetic dipole. This is seen in Fig. 2.

In Appendix B it is shown that the optimal directivity of an electric or magnetic antenna of order $l_{\max}$ is

$$D_{\tau\text{opt}} = \frac{l_{\max}(l_{\max}+2)}{2} = \frac{N_{\text{port}}}{2}.$$

The corresponding value for a combined antenna of order $l_{\max}$ is $D_{\text{opt}} = l_{\max}(l_{\max}+2)$.

For magnetic and electric multipole antennas of order l the corresponding results for the directivities D_{1l} and D_{2l}, respectively, are

$$D_{\tau l\text{opt}} = \frac{2l+1}{2} = \frac{N_{l\text{port}}}{2}. \tag{4.1}$$

The relation

$$D_{\tau\text{opt}} = \sum_{l=1}^{l_{\max}} D_{\tau l\text{opt}} \tag{4.2}$$

is a result of the fact that sets of partial waves of different index l are independent of each other. It is notable that the optimal value of the directivity of an electric or a magnetic multipole antenna of order one, i.e., a dipole antenna, is 1.5. This value is the same as the directivity of each partial wave antenna of order one. For higher order antennas the directivity of a partial wave antenna of order l is always smaller than the maximum directivity of the multipole antenna of order l.

$$\eta_{\text{eff}\tau l} = \begin{cases} -\dfrac{\text{Re}\{\eta\}}{\text{Re}\left\{j|k|^2a^2\eta h_l(ka)\left(h_l'(ka)+\frac{1}{ka}h_l(ka)\right)^*\right\}} & \text{magnetic antenna} \\ \dfrac{\text{Re}\{\eta\}}{\text{Re}\left\{j|k|^2a^2\eta h_l^*(ka)\left(h_l'(ka)+\frac{1}{ka}h_l(ka)\right)\right\}} & \text{electric antenna} \end{cases}. \tag{3.17}$$

The optimal directivity of a general antenna that consists of a combination of N independent partial wave antennas, where the maximum order of any of the antennas is $l_{\max}$, has a lower and upper bound

$$\frac{N}{2} \leq D_{\text{opt}} \leq l_{\max}(l_{\max}+2).$$

Equality is only achieved for a combined antenna of order $l_{\max}$. Notice also that two times the optimal value of the directivity is an upper bound for the number of independent ports an antenna can have.

Next the optimal power gain G is presented. Using the same method as in Appendix B the optimal values of G_τ for a magnetic antenna and an electric antenna of order $l_{\max}$ can be derived

$$G_{\tau\text{opt}} = \sum_{l=1}^{l_{\max}} \frac{2l+1}{2}\eta_{\text{eff}\tau l} = \sum_{l=1}^{l_{\max}} D_{\tau l\text{opt}}\eta_{\text{eff}\tau l} = \sum_{l=1}^{l_{\max}} G_{\tau l\text{opt}} \tag{4.3}$$

where $\eta_{\text{eff}\tau l}$ and $D_{\tau l\text{opt}}$ are given by (3.17) and (4.1), and $G_{\tau l\text{opt}}$ is the optimal power gain of a multipole antenna of order l. The optimal value of the power gain of a combined antenna of order $l_{\max}$ equals the sum of the optimal gains of the electric and the magnetic antenna of order $l_{\max}$, i.e.

$$G_{\text{opt}} = G_{1\text{opt}} + G_{2\text{opt}}. \tag{4.4}$$

According to (3.16), the asymptotic values as $a \to \infty$ are

$$\lim_{a\to\infty} G_{\text{opt}} = e^{2|\text{Im}\{k\}|a} l_{\max}(l_{\max}+2) = e^{2|\text{Im}\{k\}|a}\frac{N_{\text{port}}}{2}. \tag{4.5}$$

V. Numerical Examples

From the formulas in this paper it is straightforward to write short programs that illustrate the difference between the antennas in Section III-A. The three graphs given here are for antennas at 400 MHz, located in a material that is similar to muscles in a body. The conceivable application is implanted devices with wireless communication, even though the infinite lossy region V_{ext} is somewhat unrealistic. The conductivity is $\sigma = 1$ S/m and the relative permittivity is $\varepsilon_r = 50$. In Fig. 1 the phase of the impedance of six different multipole antennas is plotted as a function of the radius a. The argument of the wave impedance of the material in V_{ext} is 0.37 radians. It is seen that the asymptotic values in (3.8) and (3.9) are approached for large and small values of a, respectively. In Fig. 2 the radiation efficiency $\eta_{\text{eff}\tau l}$ is given as a function of a for the same six multipole antennas. The figure clearly shows that for a small radius a the magnetic dipole is the most efficient antenna. For a radius $a = 1$ mm it is more than 20 dB more efficient than the electric dipole, and 30 dB better than the magnetic quadrupole ($l = 2$). In Fig. 3, the power gain $G_{\tau\text{opt}}$ is plotted for electric and magnetic antennas with $l_{\max} = 1, 2$, and 3. One always obtains a larger gain by adding higher order multipoles, but for small antennas the improvement compared to the dipole antennas is negligible. Graphs like that in Fig. 3 indicate what order, $l_{\max}$, one should use for an electric or magnetic antenna. In that way they also indicate the number of useful ports of the antenna.

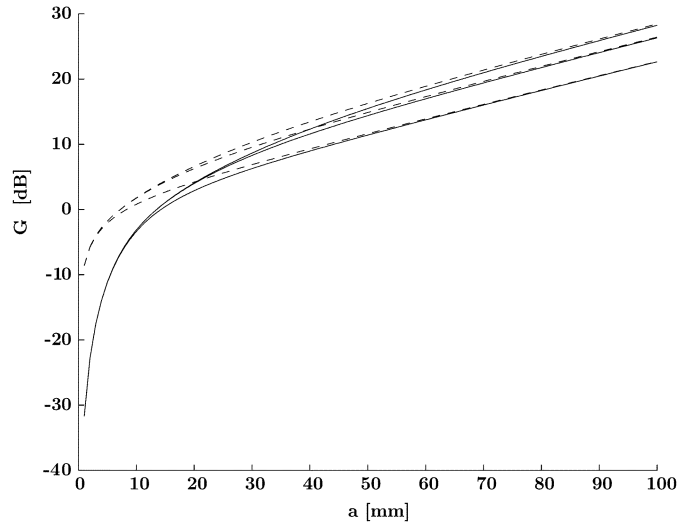

Fig. 3. Power gain $G_{\tau\text{opt}}$ of three magnetic and three electric antennas. The dashed curves are for the magnetic antennas and the solid curves for the electric antennas. The curves are for $l_{\max} = 1$ (lower), $l_{\max} = 2$ (middle), and $l_{\max} = 3$ (upper). The frequency is 400 MHz, $\varepsilon_r = 50$, and $\sigma = 1$ S/m.

VI. Conclusion

The main results in the paper are the optimal values of the radiation efficiency, the directivity, and the power gain of antennas confined in a lossless sphere. Only ideal antennas are treated in this paper. Real antennas have ohmic losses in the wires that reduce the radiation efficiency as well as the power gain. However, that power problem is associated with the actual antenna design and is out of the scope of this paper. A comprehensive study of the design of antennas in lossy materials is found in [9].

The purpose with the optimal values of the radiation efficiency, the directivity, and the power gain is to give the antenna designer relative measures and theoretical limitations of the properties of antennas. Optimization of the radiation efficiency of an antenna is to minimize the dissipated power for a given power flow in the far-zone. The most radiation efficient antenna is the magnetic dipole. The radius a of the sphere should be as large as possible.

Optimization of the directivity of an antenna is to maximize the power flow density in one direction in the far-zone for a given total power flow in the far-zone. For an electric antenna or magnetic antenna of order $l_{\max}$ the optimal directivity is $l_{\max}(l_{\max}+2)/2$ and the amplitudes of the radiated partial waves are given by (B.5). The maximum number of ports the antenna can use is twice the optimal directivity. The optimal value of the directivity is independent of frequency and of the material in V_{ext}. In theory one can achieve any directivity, even for small antennas, by a suitable choice of $l_{\max}$. However, for a small antenna the dissipated losses increase very rapidly with $l_{\max}$ and it costs a lot of power to obtain high directivity.

Optimization of the power gain of an antenna is to maximize the power flow density in one direction for a given input power to the antenna. For an electric antenna or magnetic antenna of order $l_{\max}$ the optimal power gain is given by (4.3). The power gain increases with increasing $l_{\max}$. A graph like that in Fig. 3 indicates the most suitable value of $l_{\max}$.

Appendix A
Vector Waves

The definition of spherical vector waves can be found in different textbooks, e.g., [6] and [8]. In this paper they are defined using vector spherical harmonics, cf. [1]

$$\begin{aligned}\boldsymbol{A}_{1\kappa ml}(\theta,\phi) &= \frac{1}{\sqrt{l(l+1)}}\nabla\times(\boldsymbol{r}Y_{\kappa ml}(\theta,\phi))\\ \boldsymbol{A}_{2\kappa ml}(\theta,\phi) &= \frac{1}{\sqrt{l(l+1)}}r\nabla Y_{\kappa ml}(\theta,\phi)\\ \boldsymbol{A}_{3\kappa ml}(\theta,\phi) &= \hat{r}Y_{ml}(\theta,\phi).\end{aligned} \tag{A.1}$$

The following definition of the spherical harmonics is used:

$$Y_{\kappa ml}(\theta,\phi) = \sqrt{\frac{\varepsilon_m}{2\pi}}\sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix} \tag{A.2}$$

where $\varepsilon_m = 2-\delta_{m0}$ and κ, m, l take the values

$$\kappa = \begin{Bmatrix}\mathrm{e}\\ \mathrm{o}\end{Bmatrix},\quad m=0,1,2,\ldots,l,\quad l=0,1,\ldots. \tag{A.3}$$

In the current application the index l will never take the value 0, since there are no monopole antennas. The vector spherical harmonics constitute an orthogonal set of vector function on the unit sphere

$$\int_\Omega \boldsymbol{A}_{\tau n}(\theta,\phi)\cdot\boldsymbol{A}_{\tau' n'}(\theta,\phi)\,d\Omega = \delta_{\tau\tau'}\delta_{nn'} \tag{A.4}$$

where the integration is over the unit sphere and where $n = \kappa ml$. The outgoing divergence-free spherical vector waves are defined by (A.5), show at the bottom of the page, where $h_l(kr) = h_l^{(2)}(kr)$ is the spherical Hankel function of the second kind. The asymptotic behavior in the far-zone and the limiting values in the near-zone of the spherical Hankel functions are

$$h_l^{(2)}(kr) \to \begin{cases} j^{l+1}\dfrac{e^{-jkr}}{kr} & \text{when } |k|r\to\infty\\ j(2l+1)!!(kr)^{-(l+1)} & \text{when } |k|r\to 0.\end{cases} \tag{A.6}$$

Appendix B
Optimal Directivity

The optimization problems of finding the maximum value the directivity, D is a multivariable optimization problem. First assume the following function of N variables

$$F(q_1,\ldots q_N) = \frac{\left(\sum_{n=1}^N q_n\alpha_n\right)^2}{\sum_{n=1}^N (q_n)^2\gamma_n} \tag{B.1}$$

where α_n are given real numbers and γ_n are given positive real numbers. This function has a maximum when all of its first order derivatives with respect to $q_i, i=1,\ldots N$ are zero. That leads to the following relations for the variables

$$q_n = \frac{\alpha_n}{\gamma_n}\frac{\gamma_1}{\alpha_1}q_1. \tag{B.2}$$

The corresponding maximum value of F is

$$F(q_1,\ldots q_N)_{\max} = \sum_{n=1}^N \frac{\alpha_n^2}{\gamma_n}. \tag{B.3}$$

Now consider electric antennas and magnetic antennas of order $l_{\max}$ and let D_1 and D_2 denote the corresponding directivities. Without loss of generality the maximum power flow density is assumed to be in some direction given by the spherical angles θ_d and ϕ_d, and in that direction the polarization of the corresponding wave is assumed to be parallel to some unit vector $\hat{\beta}_d$. The optimal value of the directivity is independent of the angles θ_d and ϕ_d, and of the vector $\hat{\beta}_d$, due to the invariance under rotation described in Section III-B. If $q_n = a_{\tau\kappa ml}j^{l+2-\tau}$, then α_n and γ_n are identified as the real quantities

$$\begin{aligned}\alpha_n &= 2\sqrt{\pi}\hat{\beta}_d\cdot\boldsymbol{A}_{\tau\kappa ml}(\theta_d,\phi_d)\\ \gamma_n &= 1.\end{aligned} \tag{B.4}$$

According to (B.2), the optimal directivity is obtained when

$$a_{\tau\kappa ml} = -j^{-l-1}\frac{\hat{\beta}_d\cdot\boldsymbol{A}_{\tau\kappa ml}(\theta_d,\phi_d)}{\hat{\beta}_d\cdot\boldsymbol{A}_{\tau e01}(\theta_d,\phi_d)}a_{\tau e01}. \tag{B.5}$$

The optimal value of the directivity is given by (B.3)

$$D_{\tau\mathrm{opt}} = 4\pi\sum_{l=1}^{l_{\max}}\sum_{m=0}^{l}\sum_{\kappa=\mathrm{e/o}}(\hat{\beta}_d\cdot\boldsymbol{A}_{\tau\kappa ml}(\theta_d,\phi_d))^2. \tag{B.6}$$

Next, $\hat{\beta}_d$ is expressed in terms of $\hat{\theta}_d$ and $\hat{\phi}_d$ as $\hat{\beta}_d = \cos\beta\hat{\theta}_d + \sin\beta\hat{\phi}_d$. Since $D_{\tau\mathrm{opt}}$ is independent of β one may integrate (B.6) in β from 0 to 2π. The result is

$$D_{\tau\mathrm{opt}} = 2\pi\sum_{l=1}^{l_{\max}}\sum_{m=0}^{l}\sum_{\kappa=\mathrm{e/o}}(\boldsymbol{A}_{\tau\kappa ml}(\theta_d,\phi_d))^2 \tag{B.7}$$

$$\begin{cases}\boldsymbol{u}_{1n}(\boldsymbol{r}) = h_l(kr)\boldsymbol{A}_{1n}(\theta,\phi)\\ \boldsymbol{u}_{2n}(\boldsymbol{r}) = \dfrac{1}{k}\nabla\times(h_l(kr)\boldsymbol{A}_{1n}(\theta,\phi))\\ \qquad\quad = h_l'(kr)\boldsymbol{A}_{2n}(\theta,\phi) + \dfrac{1}{kr}h_l(kr)(\boldsymbol{A}_{2n}(\theta,\phi) + \sqrt{l(l+1)}\boldsymbol{A}_{3n}(\theta,\phi))\end{cases} \tag{A.5}$$

where $(\boldsymbol{A})^2 = \boldsymbol{A}\cdot\boldsymbol{A}$. Furthermore, $D_{\tau\mathrm{opt}}$ is also independent of θ_d and ϕ_d, and the relation above can be integrated over the unit sphere. The orthonormality of the vector spherical harmonics, (A.4), results in

$$D_{\tau\mathrm{opt}} = \frac{l_{\max}(l_{\max}+2)}{2}. \tag{B.8}$$

This is in accordance with the result in [8].

Notice that that (B.5) and (B.6) can be generalized. Assume a general antenna consisting of N independent partial wave antennas that are to be fed so that the directivity function $D(\theta,\phi)$, cf. [10], is optimized in a prescribed direction and with a prescribed polarization of the radiated wave. Then a slight modification of (B.5) and (B.6) gives the amplitudes of the antennas and the value of the optimal directivity function. It also follows that the mean value of the optimal directivity function, with respect to β, θ and ϕ, is $N/2$. Hence, the optimal value of the directivity is always greater than or equal to $N/2$.

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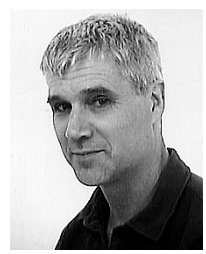

Anders Karlsson was born in 1955, Gothenburg, Sweden. He received the M.Sc. and Ph.D. degrees from Chalmers University of Technology, Gothenburg, Sweden, in 1979 and 1984, respectively.

Since 2000, he has been a Professor at the Department of Electroscience, Lund University, Lund, Sweden. His research acivities comprehend scattering and propagation of waves, inverse problems, and time-domain methods. Currently, he is involved in projects concerning propagation of light in blood, wireless communication with implants, and design of passive components on silicon.