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# INTEGRAL EQUATION METHODS AND NUMERICAL SOLUTIONS OF CRACK AND INCLUSION PROBLEMS IN PLANAR ELASTOSTATICS [*]

JOHAN HELSING[†] AND GUNNAR PETERS[‡]

**Abstract.** We present algorithms for the crack- and inclusion problem in planar linear elastostatics. The algorithms are based on new integral equations. For the pure crack problem the integral equations are of Fredholm's second kind. Our algorithms show great stability and allow for solutions to problems more complex than previously has been possible. Our results are orders of magnitudes more accurate than those of previous investigators, which rely on integral equations of Fredholm's first kind.

**Key words.** Cracks, composite materials, linear elasticity, integral equations of Fredholm type, effective elastic properties, stress intensity factors, numerical methods

**AMS subject classifications.** 73C02, 31A10, 45E05, 65R20

**1. Introduction.** Algorithms which accurately compute stress fields in loaded linearly elastic materials with cracks and inclusions are of great interest to the solid mechanics community. As an example experimental observations on unidirectional fiber reinforced composites have shown that the evolution and link up of matrix micro cracking and fiber-matrix debonding can result in complicated crack patterns which eventually lead to macroscopic fracture [8]. If this process could be modeled in a realistic and efficient way possibilities open up for optimization of composite material systems.

The stress field inside a loaded composite with cracks and inclusions is quite singular. For example, close to the tip of an open crack the stress field diverges like $1/\sqrt{s}$, where $s$ is the distance to the tip. If the crack is located at an interface the stress field exhibits an oscillatory singularity of the type $\cos(a \log s)/\sqrt{s}$, where $a$ is a real number.

Most algorithms currently in use for crack and inclusion problems are based either on the finite element method applied to the elastostatic PDE or on an integral equation reformulation which is solved using boundary elements, collocation, or a Nyström scheme. So far, no algorithm performs to full satisfaction. Difficulties include choice of integral equations and basis functions, ill-conditioning, discretization of singular integral kernels, strategies for adaptivity, imposition of periodic boundary conditions, and issues of storage and speed. Simplifying assumptions about crack and inclusion interactions are often made, even for simple geometries, to make the problems tractable. See, for example, Huang, Hu, and Chandra [7] where the authors compute the stress fields inside a weakly inhomogeneous dilute square array of circular elastic inclusions surrounded by straight cracks. The relative error is estimated to less than four per cent in an example with analytic solution.

This paper presents a robust algorithm for the crack and inclusion problem in the absence of interface cracks. We give

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- a reformulation of the integral equations appearing in a mixed crack and inclusion problem into a system of Fredholm equations of the second kind, that is both stable and easy to use in an efficient way
- a mathematical analysis of these equations
- particularities of efficient evaluation of the integral operators appearing in our algorithm
- examples showing that the use of these equations in numerical calculations improves accuracy and speed for complicated as well as for simple geometries.

It should be pointed out that, for crack problems, a first kind formulation is the usual choice; the evaluation of integrals involved in a second kind formulation is considered too difficult. Our reformulation does not suffer from this disadvantage. Also we show how the technical problems arising in calculating the integrals should be tackled. Apart from being an important step on the way to a general purpose algorithm for cracked composites, an algorithm for our restricted problem class is of interest in itself: cracks closely surrounding inhomogeneities can occur in composites in a number of ways, for example as a result of residual stress due to thermal mismatch or crystallographic transformation. By using a second kind integral equation formulation, rather than a first kind integral equation formulation or a superposition of analytical solutions where interactions are neglected, we are able to improve the accuracy in the solution for a well studied geometry by a factor of $10^9$, using IEEE double precision arithmetic.

The paper is organized as follows. Section 2 presents the basic integral equations. In §3 the nature of these equations on certain Hilbert spaces is discussed. Section 4 contains technical lemmas used in later sections. The second kind integral formulation is derived in §5, and in §6 for an alternative set of equations. Sections 7 and 8 give physical properties in terms of the solution of our integral equations. In §9 we show how to evaluate branches of analytic functions and singular expressions appearing in the integrals. Section 10 contains numerical results for several geometries.

**2. Basic equations.** Let $U$ denote the Airy stress function for a locally isotropic two dimensional linearly elastic material with cracks and inclusions. Since $U$ satisfies the biharmonic equation (outside the cracks and the inclusion interfaces), it can be represented as

$$(1) \qquad\qquad U = \Re\left\{ \bar{z}\phi + \chi \right\},$$

where $\phi$ and $\chi$ are possibly multivalued analytic functions outside the cracks and the inclusion interfaces and $\Re\{f\}$ denotes the real part of the function $f$. In the elasticity problem with external forces applied only at infinity, $\phi$ and $\chi$ are actually single valued, see paragraph 40 in [10]. For a thorough discussion of the complex variable approach to elasticity problems, see [10, 14, 13, 15]. For our purposes, and for the time being, it is sufficient to observe a few relations that link the complex potentials to quantities of physical interest: The displacement $(u, v)$ in the material satisfies

$$(2) \qquad\qquad u + iv = \left( \frac{1}{2\mu} + \frac{1}{\kappa} \right)\phi - \frac{1}{2\mu}\left( z\overline{\phi'} + \overline{\psi} \right),$$

where $\kappa$ is the two dimensional bulk modulus, $\mu$ is the two dimensional shear modulus, and $\psi = \chi'$. The integral of the traction $(t_x, t_y)$ along a curve $\Gamma(s)$ with normal

$(n_x, n_y)$ can be obtained from the relation

$$(3) \qquad \int_{\Gamma(s_0)}^{\Gamma(s)} (t_x + it_y)\mathrm{d}s = -\left.i\left(\phi + z\overline{\phi'} + \overline{\psi}\right)\right|_{s_0}^{s},$$

where $s$ denotes arclength. Complex differentiation of the expression (2) along the tangent to $\Gamma(s)$ gives

$$(4) \qquad \frac{\mathrm{d}}{\mathrm{d}z}(u + iv) = \left(\frac{1}{2\mu} + \frac{1}{\kappa}\right)\Phi - \frac{1}{2\mu}\left(\overline{\Phi} - \frac{\bar{n}}{n}z\overline{\Phi'} - \frac{\bar{n}}{n}\overline{\Psi}\right),$$

and differentiation with respect to arclength in (3) gives

$$(5) \qquad t_x + it_y = \Phi n + \overline{\Phi}n - z\overline{\Phi'}\bar{n} - \overline{\Psi}\bar{n},$$

where $n = n_x + in_y$, $\Phi = \phi'$, and $\Psi = \chi''$.

Consider now a material consisting of an infinite medium with elastic moduli $\kappa_1$ and $\mu_1$ which surrounds a number $N_c$ of cracks and a number $N_d$ of inclusions. The inclusions have elastic moduli $\kappa_2$ and $\mu_2$. We denote the cracks by $\Gamma^j$, $j = 1, \ldots, N_c$ and the interfaces of the inclusions $\Gamma^j$, $j = N_c + 1, \ldots, N_c + N_d$. The union of all cracks is $\Gamma_c$ and the union of all interfaces is $\Gamma_d$. The union of all cracks and interfaces is $\Gamma$. The starting point and the endpoint of crack $\Gamma^j$, the so-called crack tips, are denoted $\gamma_s^j$ and $\gamma_e^j$. The average strain is $\bar{\epsilon} = (\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy})$, and the average stress is $\bar{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$. We are going to compute stress intensity factors for the crack tips, effective elastic moduli, and the stress- and strain fields in the material subject to three different imposed average strains, namely $\bar{\epsilon}_I = (1, 0, 0)$, $\bar{\epsilon}_{II} = (0, 1, 0)$, and $\bar{\epsilon}_{III} = (0, 0, 1)$. Since the equations of elasticity are satisfied everywhere except for at $\Gamma$, and assuming all crack opening displacements are nonnegative, it remains only to solve the problem which consists of enforcing zero traction along the cracks and continuity of traction and displacement along the interfaces.

A standard starting point for crack and inclusion problems is to work with (4) and (5) and to represent the uppercase potentials $\Phi$ and $\Psi$ as Cauchy-type integrals:

$$(6) \qquad \Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\tau - z)} + \frac{\alpha}{2},$$

and

$$(7) \qquad \Psi(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{\overline{\Omega(\tau)\rho(\tau)}\mathrm{d}\bar{\tau}}{(\tau - z)} - \frac{1}{2\pi i}\int_\Gamma \frac{\bar{\tau}\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\tau - z)^2} + \beta,$$

where $\Omega(z)$ is an unknown density on $\Gamma$ and $\rho(z)$ is a weight function which on crack $\Gamma^j$ is given by

$$(8) \qquad \rho(z) = ((z - \gamma_s^j)(z - \gamma_e^j))^{-\frac{1}{2}},$$

and on the interfaces by

$$(9) \qquad \rho(z) = 1.$$

In (8) the weight function $\rho(z)$ is the limit from the right (relative to the orientation of the crack) of the branch given by a branch cut along $\Gamma^j$ and

$$(10) \qquad \lim_{z \to \infty} z\rho(z) = 1.$$

*Remark* 1. The constants $\alpha$ and $\beta$ in (6) and (7) represent the forcing terms at infinity in our formulation. The two constants take the values $\kappa_1$ and $-\mu_1$ for strain $\bar{\epsilon}_{\mathrm{I}}$, the values $\kappa_1$ and $\mu_1$ for strain $\bar{\epsilon}_{\mathrm{II}}$, and the values $0$ and $2i\mu_1$ for strain $\bar{\epsilon}_{\mathrm{III}}$. For imposed average stresses $\bar{\sigma}_{\mathrm{I}} = (1,0,0)$, $\bar{\sigma}_{\mathrm{II}} = (0,1,0)$, and $\bar{\sigma}_{\mathrm{III}} = (0,0,1)$ the constants take the values $\alpha = 1/2$ and $\beta = -1/2$, $\alpha = 1/2$ and $\beta = 1/2$, and $\alpha = 0$ and $\beta = i$. Thus, the constant $\alpha$ can always be assumed to be real, while $\beta$ is either a real or an imaginary number.

Once $\Phi$ is assumed to take the form (6), the expression (7) for $\Psi$ enforces continuity of traction across the cracks and the interfaces. The requirements of zero traction along the cracks, continuity of displacement along the interfaces, and closure of each crack lead to an integral equation for $\Omega(z)$:

$$(11) \qquad (M_1 - M_3)\,\Omega(z) = \frac{\bar{n}}{n}\bar{\beta} - \alpha, \qquad z \in \Gamma_{\mathrm{c}},$$

$$(12) \qquad (I + d_1 M_1 + d_2 M_3)\,\Omega(z) = -d_1\alpha - d_2\frac{\bar{n}}{n}\bar{\beta}, \qquad z \in \Gamma_{\mathrm{d}},$$

$$(13) \qquad Q\Omega = 0.$$

Here $M_1$ and $M_3$ are integral operators given by

$$(14) \qquad M_1\Omega(z) = \frac{1}{\pi i}\int_\Gamma \frac{\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\tau - z)},$$

and

$$(15) \qquad M_3\Omega(z) = \frac{1}{2\pi i}\left[\int_\Gamma \frac{\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\tau - z)}\right.$$

$$\left. + \frac{\bar{n}}{n}\int_\Gamma \frac{\Omega(\tau)\rho(\tau)\mathrm{d}\tau}{(\bar{\tau} - \bar{z})} + \int_\Gamma \frac{\overline{\Omega(\tau)\rho(\tau)}\mathrm{d}\bar{\tau}}{(\bar{\tau} - \bar{z})} + \frac{\bar{n}}{n}\int_\Gamma \frac{(\tau - z)\overline{\Omega(\tau)\rho(\tau)}\mathrm{d}\bar{\tau}}{(\bar{\tau} - \bar{z})^2}\right],$$

and $Q$ is an operator from $\Gamma$ into the complex linear space $\mathbb{C}^{N_{\mathrm{c}}}$ whose $j$th component is given by

$$(16) \qquad Q_j\Omega = \frac{1}{\pi i}\int_{\Gamma^j} \Omega(\tau)\rho(\tau)\mathrm{d}\tau, \qquad j = 1, ..., N_{\mathrm{c}}.$$

Despite appearances, the operator $M_3$ is Hilbert-Schmidt when defined on a certain weighted $L^2$ space along $\Gamma$, while the operator $M_1$ is to be interpreted in the Cauchy principal value sense. The weighted $L^2$ space used will be discussed in §3. The constants $d_1$ and $d_2$ of (13) are given by

$$d_1 = \left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right) \Big/ \left(\frac{1}{\mu_2} + \frac{1}{\kappa_2} + \frac{1}{\mu_1} + \frac{1}{\kappa_1}\right),$$

and

$$(17) \qquad d_2 = \left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right) \Big/ \left(\frac{1}{\mu_2} + \frac{1}{\kappa_2} + \frac{1}{\mu_1} + \frac{1}{\kappa_1}\right).$$

*Remark* 2. Our constants $d_1$ and $d_2$ are simply related to bimaterial parameters introduced by other investigators. For example, in terms of the parameters $a$, $b$, and

$c$ of equation (3.12) in Sherman [18] we have $d_1 = b/a$ and $d_2 = -c/a$. In terms of the parameters $\alpha$ and $\beta$ in Dundurs [2] we have $d_1 = \beta$ and $d_2 = \alpha - \beta$.

A less common option for crack and inclusion problems, but one that has shown to be numerically more efficient for the pure inclusion problem [4], is to work with (2) and (3) and to represent the lowercase potentials $\phi$ and $\psi$ in the form

$$(18) \qquad \phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\tau)\mathrm{d}\tau}{\rho(\tau)(\tau - z)} + \frac{\alpha z}{2},$$

and

$$(19) \qquad \psi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\tau)\mathrm{d}\bar{\tau}}{\rho(\tau)(\tau - z)} - \frac{1}{2\pi i} \int_\Gamma \frac{\overline{\omega(\tau)}\mathrm{d}\tau}{\overline{\rho(\tau)}(\tau - z)}$$

$$- \frac{1}{2\pi i} \int_\Gamma \frac{\bar{\tau}\omega(\tau)\mathrm{d}\tau}{\rho(\tau)(\tau - z)^2} + \beta z,$$

where $\omega(z)$ is an unknown density related to the density $\Omega(z)$ through differentiation. We will return to this option in §6.

**3. Hilbert space formulation of the problem.** To make a more precise mathematical formulation of the integral equations (11–13) we denote $\nu(z) = |\rho(z)|$ and define the weighted $L^2$ space

$$L^2(\Gamma, \nu) = \{f : \int_\Gamma |f(z)|^2 \nu(z)|dz| < \infty\},$$

where $|dz|$ is the arclength measure on the curve $\Gamma$. The system (11–13) will be solved in the space $L^2(\Gamma, \nu)$. Since $\nu(z) = 1$ on $\Gamma_\mathrm{d}$ we have the orthogonal decomposition

$$(20) \qquad L^2(\Gamma, \nu) = L^2(\Gamma_\mathrm{c}, \nu) \oplus L^2(\Gamma_\mathrm{d}),$$

where $L^2(\Gamma_\mathrm{d})$ is the space of square integrable functions with respect to the arclength measure on $\Gamma_\mathrm{d}$. When $\Gamma$ is at least $C^{1,\alpha}$, $\alpha > 0$, we can use standard Hilbert-Schmidt theory and the fact that $\nu$ is a Muckenhoupt weight, see [12], to prove that on the Hilbert space $L^2(\Gamma, \nu)$ the operators $M_3$ and $Q$ as well as the operator $M_4^*$ defined in §4 are bounded. Furthermore $M_3$ is compact, while the operator $M_1$ is unbounded. The structure of $M_1$ will be further discussed in §5.

It follows from the general theory in [13] that the system (11–13) has a unique solution and the analysis in §5 of this paper shows that this solution lies in $L^2(\Gamma, \nu)$.

For future references we make the following definitions. Write $\bar{c} = (c_1, \ldots, c_{N_c})$ for a vector in $\mathbb{C}^{N_c}$ and define the embedding $e : \mathbb{C}^{N_c} \to L^2(\Gamma_\mathrm{c}, \nu)$ by

$$(21) \qquad e(\bar{c})(z) = c_j, \qquad z \in \Gamma^j$$
$$e(\bar{c})(z) = 0, \qquad \text{otherwise.}$$

If we denote by $E_1$ the kernel of $Q$ and by $E_2$ the image of the embedding $e$ then it follows from Lemma 3 that

$$(22) \qquad L^2(\Gamma_\mathrm{c}, \nu) = E_1 + E_2,$$

and the system of equations in (11–13) is equivalent to solving (11) and (12) in the closed subspace $E_1$. This will be discussed further in §5.

*Remark* 3. We note that, depending on the right-hand side in equations (11) and (12), the actual solution in general has additional smoothness, but the $L^2(\Gamma, \nu)$ formulation of the problem facilitates the theoretical considerations.

**4. Some identities.** We define the following auxiliary operators on $L^2(\Gamma, \nu)$.

$$(23) \qquad M_1^* \Omega(z) = \frac{1}{\pi i} \int_{\Gamma^j} \frac{\rho(\tau)\Omega(\tau)\mathrm{d}\tau}{(\tau - z)}, \qquad z \in \Gamma^j, \quad j = 1, \ldots, N_\mathrm{c},$$

and

$$(24) \qquad M_4^* \Omega(z) = \frac{1}{\pi i} \int_{\Gamma^j} \frac{\Omega(\tau)\mathrm{d}\tau}{\rho(\tau)(\tau - z)}, \qquad z \in \Gamma^j, \quad j = 1, \ldots, N_\mathrm{c}.$$

The following lemma restates some facts, proved in paragraph 27 of [10], in a form compatible with our formulation of the problem.

LEMMA 1.

$$(25) \qquad M_1^* \circ M_4^* = Id_{L^2(\Gamma_\mathrm{c}, \nu)},$$

$$(26) \qquad M_4^* \circ M_1^* \, \Omega = Id_{L^2(\Gamma_\mathrm{c}, \nu)} \, \Omega - e(\bar{c}), \qquad \Omega \in L^2(\Gamma_\mathrm{c}, \nu),$$

$$(27) \qquad M_1^* \circ e = 0,$$

where the vector $\bar{c} \in \mathbb{C}^{N_\mathrm{c}}$ depends linearly on $\Omega$.

Next we consider integrals of the type

$$(28) \qquad \int_{\Gamma^j} \rho(z) f(z) dz, \qquad j = 1, \ldots, N_\mathrm{c},$$

where $\rho(z)$ is as in (8) and $f(z)$ is analytic. Remember that we use the limit from the right of the branch of $\rho(z)$ whose branch cut is along $\Gamma^j$ itself. Next we will prove that if, when we vary the path $\Gamma^j$, the branch cut defining $\rho(z)$ moves along with $\Gamma^j$, then the integral in (28) depends only on the endpoints.

LEMMA 2. *Let $U$ be a simply connected domain in $\mathbb{C}$ and let $f(z)$ be analytic in $U$. If we define the complex line integral*

$$(29) \qquad \int_C \rho(z) f(z) dz$$

*as in the preceding paragraph, i.e., with the branch cut for $\rho(z)$ along the simple curve $C \subset U$, then this integral depends only on the endpoints of $C$.*

*Proof.* Fix the endpoints $\gamma_\mathrm{s}$ and $\gamma_\mathrm{e}$ and let $C$ be any simple curve in $U$ connecting $\gamma_\mathrm{s}$ and $\gamma_\mathrm{e}$. First we consider a small perturbation $\delta C$ of this curve. Simple arguments show that there exists a branch of $\rho(z)$ that is analytic in a simply connected neighborhood $V \subset U$ containing both $C$ and $C + \delta C$ and that coincides with the limits from the right of the appropriate branches used in defining the integrals along $C$ and $C + \delta C$, respectively. The independence of the path of the integral under such a small perturbation therefore follows from the Cauchy theorem. Next we note that any simple path connecting $\gamma_\mathrm{s}$ and $\gamma_\mathrm{e}$ can be obtained from $C$ by a continuous deformation. The lemma follows.  ☐

LEMMA 3. *Let $Q_j$ denote the projection onto the $j$th component of the operator $Q$. Then*

$$(30) \qquad Q \circ e = Id_{\mathbb{C}^{N_\mathrm{c}}},$$

$$(31) \qquad Q_j(z) = \frac{\gamma_\mathrm{s}^j + \gamma_\mathrm{e}^j}{2},$$

$$(32) \qquad Q \circ M_4^* = 0.$$

*Proof.* To prove (30) it suffices to prove that for a single crack $\Gamma^j$

$$(33) \qquad \frac{1}{i\pi} \int_{\Gamma^j} \rho(z)dz = 1 \,.$$

Using Lemma 2 this integral can be reduced to a standard integral in one real variable which equals the number 1. Equation (31) reads

$$(34) \qquad \frac{1}{i\pi} \int_{\Gamma^j} z\,\rho(z)dz = \frac{\gamma_{\mathrm{s}}^j + \gamma_{\mathrm{e}}^j}{2} \,,$$

which is proved in a similar manner.

To prove (32) we have to prove that for a single crack $\Gamma^j$

$$(35) \qquad \int_{\Gamma^j} \rho(z)dz \int_{\Gamma^j} \frac{\Omega(\tau)d\tau}{\rho(\tau)(\tau - z)} = 0 \,.$$

Changing the order of integration in the left hand side of (35) gives

$$(36) \qquad \int_{\Gamma^j} \frac{\rho(z)dz}{(\tau - z)} \int_{\Gamma^j} \frac{\Omega(\tau)d\tau}{\rho(\tau)}$$

which is equal to 0 because of statement (27) of Lemma 1. $\square$

LEMMA 4. *For $j = 1, \ldots, N_{\mathrm{c}}$ we have*

$$(37) \qquad M_4^* \circ e(\bar{c})(z) = c_j\Big(z - \frac{\gamma_{\mathrm{s}}^j + \gamma_{\mathrm{e}}^j}{2}\Big), \qquad z \in \Gamma^j \,.$$

*Proof.* It suffices to prove that for a single crack $\Gamma^j$

$$(38) \qquad \frac{1}{i\pi} \int_{\Gamma^j} \frac{d\tau}{\rho(\tau)(\tau - z)} = z - \frac{\gamma_{\mathrm{s}}^j + \gamma_{\mathrm{e}}^j}{2}, \qquad z \in \Gamma^j \,.$$

With $Q_j$ as in Lemma 3 this integral can be written

$$(39) \quad \frac{1}{i\pi} \int_{\Gamma^j} \frac{\rho(\tau)(\tau - \gamma_{\mathrm{s}}^j)(\tau - \gamma_{\mathrm{e}}^j)d\tau}{(\tau - z)} = \frac{1}{i\pi} \int_{\Gamma^j} \frac{\rho(\tau)(\tau - z)(\tau - (\gamma_{\mathrm{s}}^j + \gamma_{\mathrm{e}}^j - z))d\tau}{(\tau - z)}$$
$$= Q_j(\tau - (\gamma_{\mathrm{s}}^j + \gamma_{\mathrm{e}}^j - z))$$
$$= Q_j(\tau) - (\gamma_{\mathrm{s}}^j + \gamma_{\mathrm{e}}^j) + z \,.$$

where we have used (27) of Lemma 1 and (30) of Lemma 3. Using (31) this proves the lemma. $\square$

We may now give a more precise formulation of (26) that uses the operators defined in §2 and §3.

LEMMA 5.

$$(40) \qquad M_4^* \circ M_1^* = Id_{L^2(\Gamma_{\mathrm{c}}, \nu)} - e \circ Q.$$

*Proof.* If we apply $Q$ to the left in (26) and use Lemma 3 we obtain

$$(41) \qquad Q\Omega = \bar{c} \,.$$

Applying the embedding $e$ to the left in (41) gives $e \circ Q(\Omega) = e(\bar{c})$, and the lemma is proved. $\square$

**5. Fredholm integral equations of the second kind.** We argue that the system (11–13) is not the best formulation for numerical calculations.

Actually, equation (12) is morally a Fredholm equation of the second kind. By this we mean that the perturbation of the identity is, if not compact, small in the operator norm and therefore equation (12) is well suited for numerical calculations as it stands.

We consider now in some detail the system given by equations (11) and (13). Denote $M_1^\flat$ for the part of the operator $M_1$ that appears in equation (11). First we note that the operator $M_1^\flat$ acting on $\Omega$ is unbounded. In fact $M_1^\flat = M_1^* + M_1^0$ where $M_1^*$ is defined in (23), and the remaining part $M_1^0$ is a compact operator on $L^2(\Gamma, \nu)$. The unbounded operator $M_1^*$ describes the self-interaction for the cracks. Since the spectrum of $M_1^\flat$ is unbounded a direct discretization of $M_1^\flat$ leads to numerical instabilities. Secondly we need to solve the integral equations in (11–12) and the constraint (13) simultaneously. The first of these problems is the more serious. Numerical application of a suitable preconditioner is a possible way out, but cost and storage requirement associated with such calculations is high: With $n_j$ discretization points on crack $\Gamma^j$ the cost of computing a reasonable preconditioner is proportional to $\sum n_j^3$ [6]. Furthermore, some accuracy will be lost. Here we will show that the system given by (11) and (13) is equivalent to a Fredholm integral equation of the second kind on the Hilbert space $L^2(\Gamma, \nu)$, that is well suited for accurate numerical calculations.

First we consider the simplified system

$$(42) \qquad M\Omega = \begin{pmatrix} M_1^* & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} \Omega_c \\ \Omega_d \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$
$$Q\Omega = 0.$$

If we define $M_4^*$ as in (24) then Lemmas 1 and 3 imply that the solution to (42) is given by

$$(43) \qquad \Omega = \begin{pmatrix} M_4^* & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Returning to the system given by (11) and (13) this can be viewed as a small perturbation of (42);

$$(44) \qquad (M + K)\Omega = b,$$
$$(45) \qquad Q\Omega = 0,$$

and the operator in (43) serves as a parametrix for (44) that also projects out the solution obeying the constraint (45). This viewpoint gives us the following simplification of the system.

THEOREM 6. *Let $M_4^*$ be as above. Then the system given by (11) and the constraint (13) is equivalent to the following single Fredholm integral equation of the second kind.*

$$(46) \qquad \left(Id_{L^2(\Gamma_c)} + M_4^* \circ (M_1^0 - M_3)\right) \Omega(z) = M_4^*(\frac{\bar{n}}{n}\bar{\beta} - \alpha), \qquad z \in \Gamma_c.$$

*Proof.* That (46) is a Fredholm equation of the second kind follows from the compactness of $(M_1^0 - M_3)$ and the fact that $M_4^*$ is bounded on $L^2(\Gamma, \nu)$. If we apply

$M_4^*$ on the left in (11) and use Lemma 5 then we obtain

$$(47) \qquad \left(Id_{L^2(\Gamma_c,\nu)} + M_4^* \circ (M_1^0 - M_3) - e \circ Q\right) \Omega = M_4^*(\frac{\bar{n}}{n}\bar{\beta} - \alpha).$$

From (47) and the constraint (13) we obtain (46). In the other direction we apply $M_1^*$ to the left in (46) to obtain (11). Also using the fact that $Q \circ M_4^* = 0$ we obtain (13) by applying $Q$ to the left in (46). $\square$

In conclusion we end up with the following system of integral equations.

$$(48) \qquad \left(Id_{L^2(\Gamma_c,\nu)} + M_4^* \circ (M_1^0 - M_3)\right) \Omega(z) = M_4^*(\frac{\bar{n}}{n}\bar{\beta} - \alpha), \qquad z \in \Gamma_c,$$

$$(49) \qquad \left(Id_{L^2(\Gamma_d)} + d_1 M_1 + d_2 M_3\right) \Omega(z) = -d_1\alpha - d_2\frac{\bar{n}}{n}\bar{\beta}, \qquad z \in \Gamma_d.$$

As we shall see in § 10, equations (48–49), as well as their companion system of equations (56–57) given in §6, allow for very accurate numerical solutions of complex situations.

As far as we know, a second kind Fredholm integral equation formulation for the crack problem has not been used before in the context of a general numerical algorithm. On the contrary, both Erdogan, Gupta and Cook [3] in a much cited paper and Parton and Perlin [16] recommend numerical solution of integral equations of the first kind. Recently, however, Chen and Hasebe [1] used a combination of a second kind integral equation and a free-space solution on analytic form to solve a problem involving three well separated cracks in the shape of circular arcs.

Since we use the parametrix in (43) instead of inverting the full operator $M_1$ as described in Muskhelishvili [13] the evaluation of the singular integral operators appearing, for example $M_4^*$, simplifies considerably. In §9 we discuss the particularities of these calculations.

**6. An alternative formulation.** We now return to the representation (18) and (19) for the lowercase potentials $\phi$ and $\psi$ respectively. These are just a reformulation of (6) and (7), which gives the the unique existence stated below.

Once $\phi$ is assumed to take the form (18), the expression (19) for $\psi$ enforces the continuity of the integral of traction across the cracks and across the interfaces. The requirements of zero traction along the cracks and of continuity of displacement along the interfaces lead to the following system of integral equations for $\omega(z)$.

$$(50) \qquad M_4\omega(z) - M_2\omega(z) = -\alpha z - \bar{\beta}\bar{z} + e(\bar{c}), \qquad z \in \Gamma_c,$$

$$(51) \qquad (I + d_1 M_4 + d_2 M_2)\,\omega(z) = -d_1\alpha z + d_2\bar{\beta}\bar{z}, \qquad z \in \Gamma_d,$$

where $\bar{c} \in \mathbb{C}$ is a vector of constants of integration (one for each crack). Using the splitting given in equation (22) and the fact the image of $M_4$, the kernel of $Q$, and the subspace $E_1$ coincide we obtain the following consistency condition for $\omega$ and $\bar{c}$.

$$(52) \qquad -Q \circ M_2\omega(z) = Q\left(-\alpha z - \bar{\beta}\bar{z}\right) + \bar{c}.$$

The operators $M_4$ and $M_2$ are integral operators given by

$$(53) \qquad M_4\omega(z) = \frac{1}{\pi i} \int_\Gamma \frac{\omega(\tau)\mathrm{d}\tau}{\rho(\tau)(\tau - z)},$$

and

$$(54) \qquad M_2\omega(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\tau)}{\rho(\tau)}\mathrm{d}\left[\log\frac{\tau - z}{\bar{\tau} - \bar{z}}\right] + \frac{1}{2\pi i} \int_\Gamma \overline{\frac{\omega(\tau)}{\rho(\tau)}}\mathrm{d}\left[\frac{\tau - z}{\bar{\tau} - \bar{z}}\right].$$

Equation (51), for the pure inclusion problem, is equivalent to the formulation used by Sherman [18] and Greengard and Helsing [4]. Equation (50), for the pure crack problem, appears to be new. That the system (50–52) has a unique solution follows from the result of Muskhelishvili [13] cited in §3 and the fact that this system is a reformulation of (11–13).

Again (51) will be used as it stands while (50) and (52) will be transformed into a Fredholm equation of the second kind. This is the content of Theorem 7 below. In analogy with the notation in §5 we write $M_4^\flat = M_4^* + M_4^0$ for the part of $M_4$ that appears in equation (50). Here $M_4^*$ is given in (24).

Here the situation is a bit more complicated than in §3 and §5. We define $L^2(\nu_1, \Gamma)$ as the $L^2$ space relative to the weight $\nu_1 = |1/\rho(z)|$. Using the fact that also $1/|\rho(z)|$ is a Muckenhoupt-weight we see that $M_4$ is bounded while $M_2$ is compact on this space. The operator $M_1$ is still unbounded but we have the following. $M_4 : L^2(\nu_1, \Gamma) \to L^2(\nu, \Gamma)$ and $M_1 : L^2(\nu, \Gamma) \to L^2(\nu_1, \Gamma)$ are bounded while $M_4^0 : L^2(\nu_1, \Gamma) \to L^2(\nu, \Gamma)$ and $M_2 : L^2(\nu_1, \Gamma) \to L^2(\nu, \Gamma)$ are compact. These statements together with the Lemmas in §4 allow us to prove the following theorem.

THEOREM 7. *Let $M_1^*$ be as above. Then the system (50) and (52) is equivalent to the following Fredholm integral equation of the second kind on $L^2(\nu_1, \Gamma)$.*

$$(55) \qquad \left(Id_{L^2(\Gamma_c)} + M_1^* \circ (M_4^0 - M_2)\right) \omega(z) = -M_1^*(\alpha z + \bar{\beta}\bar{z}), \qquad z \in \Gamma_c.$$

*Proof.* Applying $M_1^*$ to the left in (50) and observing that $M_1^* \circ e = 0$ gives (55). That this is a Fredholm equation follows from the boundedness of $M_1^*$ and compactness of $M_4^0 - M_2$ on the appropriate spaces as stated in the preceding paragraph. Applying $M_4^*$ to the left in 55 gives us back equation(50) where $\bar{c}$ has to be determined from the consistency condition (52). $\square$

This gives us the following alternative system of integral equations.

$$(56) \qquad \left(Id_{L^2(\Gamma_c, \nu)} + M_1^* \circ (M_4^0 - M_2)\right) \omega(z) = -M_1^*(\alpha z + \bar{\beta}\bar{z}), \qquad z \in \Gamma_c, ,$$

$$(57) \qquad \left(Id_{L^2(\Gamma_d)} + d_1 M_4 + d_2 M_2\right) \omega(z) = -d_1 \alpha z + d_2 \bar{\beta}\bar{z}, \qquad z \in \Gamma_d.$$

For pure crack problems the performance of equations (48) and (56) is almost identical. For crack and inclusion problems equations (56–57) outperform equations (48–49) as is shown by the example in §10.5.

**7. Stress intensity factors.** Stress intensity factors are quantities of great importance. These factors, which are assumed to control crack growth, can be defined as certain limits of the displacement at the crack tips of a cracked material subjected to loads. Below we will present simple expressions for stress intensity factors in terms of the densities $\omega(z)$ and $\Omega(z)$.

The opening displacement $(\delta u, \delta v)$ on a crack $\Gamma^j$ is the limit value of the displacement from the right minus the limit value of the displacement from the left. Using (2) it is easy to derive the following relation between the crack opening displacement and the density $\omega(z)$

$$(58) \qquad \delta u + i\delta v = -\left(\frac{1}{\kappa_1} + \frac{1}{\mu_1}\right)\frac{\omega(z)}{\rho(z)}.$$

For a crack that is open, $\omega(z)/\rho(z)$ approaches zero as the square root of the distance to the crack tips. The complex valued stress intensity factor $K_{\mathrm{I}} + iK_{\mathrm{II}}$ at the crack

tips $\gamma_\mathrm{s}^j$ and $\gamma_\mathrm{e}^j$ can be defined as

$$K_\mathrm{I} + iK_\mathrm{II} = -\frac{\sqrt{2\pi}}{g} \lim_{z \to \gamma_\mathrm{s}^j} \frac{n(z)}{2\sqrt{\delta s(z)}} \frac{\overline{\omega(z)}}{\overline{\rho(z)}},$$

and

$$(59) \qquad K_\mathrm{I} + iK_\mathrm{II} = -\frac{\sqrt{2\pi}}{g} \lim_{z \to \gamma_\mathrm{e}^j} \frac{n(z)}{2\sqrt{\delta s(z)}} \frac{\overline{\omega(z)}}{\overline{\rho(z)}},$$

where $\delta s(z)$ is arclength measured from the closest crack tip and $g$ is a normalization factor which varies with different authors.

The derivative of the crack opening displacement can be expressed in terms of $\Omega(z)$ via (4) as

$$(60) \qquad \frac{\mathrm{d}}{\mathrm{d}z}(\delta u + i\delta v) = -\left(\frac{1}{\kappa_1} + \frac{1}{\mu_1}\right)\Omega(z)\rho(z).$$

For a crack that is open, $\Omega(z)\rho(z)$ has a one over square root singularity at the crack tips. The complex valued stress intensity factor $K_\mathrm{I} + iK_\mathrm{II}$ at the crack tips $\gamma_\mathrm{s}^j$ and $\gamma_\mathrm{e}^j$ can also be defined as

$$K_\mathrm{I} + iK_\mathrm{II} = \frac{i\sqrt{2\pi}}{g} \lim_{z \to \gamma_\mathrm{s}^j} \overline{\Omega(z)\rho(z)}\sqrt{\delta s(z)},$$

and

$$(61) \qquad K_\mathrm{I} + iK_\mathrm{II} = -\frac{i\sqrt{2\pi}}{g} \lim_{z \to \gamma_\mathrm{e}^j} \overline{\Omega(z)\rho(z)}\sqrt{\delta s(z)}.$$

**8. Effective moduli.** Effective elastic moduli are particularly simple to define and compute in the setting of a doubly periodic material with a square unit cell of unit area. The effective moduli of a material can be defined through the following relations between average stress and average strain

$$(62) \qquad \begin{pmatrix} \bar{\sigma}_{xx} \\ \bar{\sigma}_{yy} \\ \sqrt{2}\bar{\sigma}_{xy} \end{pmatrix} = \begin{pmatrix} c_{*1} & c_{*2} & c_{*3} \\ c_{*2} & c_{*4} & c_{*5} \\ c_{*3} & c_{*5} & c_{*6} \end{pmatrix} \begin{pmatrix} \bar{\epsilon}_{xx} \\ \bar{\epsilon}_{yy} \\ \sqrt{2}\bar{\epsilon}_{xy} \end{pmatrix}.$$

For the density $\Omega(z)$, let

$$(63) \qquad a = -\left(1 + \frac{\mu_1}{\kappa_1}\right)\int_{\Gamma_\mathrm{u}} z\Omega(z)\rho(z)\mathrm{d}z,$$

and

$$(64) \qquad b = -\left(1 + \frac{\kappa_1}{\mu_1}\right)\int_{\Gamma_\mathrm{cu}} \bar{z}\Omega(z)\rho(z)\mathrm{d}z - \left(1 + \frac{\kappa_1}{2\mu_1} + \frac{\kappa_2}{2\mu_2}\right)\int_{\Gamma_\mathrm{du}} \bar{z}\Omega(z)\mathrm{d}z$$

$$- \left(\frac{\kappa_1}{2\mu_1} - \frac{\kappa_2}{2\mu_2}\right)\int_{\Gamma_\mathrm{du}} \bar{z}M_3\Omega(z)\mathrm{d}z,$$

where $\Gamma_{\mathrm{cu}}$ denotes the cracks in the unit cell, $\Gamma_{\mathrm{du}}$ denotes the interfaces in the unit cell, and $\Gamma_{\mathrm{u}} = \Gamma_{\mathrm{cu}} + \Gamma_{\mathrm{du}}$. For the density $\omega(z)$, let

$$(65) \qquad a = \left(1 + \frac{\mu_1}{\kappa_1}\right) \int_{\Gamma_{\mathrm{u}}} \frac{\omega(z)}{\rho(z)} \mathrm{d}z,$$

and

$$(66) \qquad b = \left(1 + \frac{\kappa_1}{\mu_1}\right) \int_{\Gamma_{\mathrm{cu}}} \frac{\omega(z)}{\rho(z)} \mathrm{d}\bar{z} + \left(1 + \frac{\kappa_1}{2\mu_1} + \frac{\kappa_2}{2\mu_2}\right) \int_{\Gamma_{\mathrm{du}}} \omega(z) \mathrm{d}\bar{z}$$

$$+ \left(\frac{\kappa_2}{2\mu_2} - \frac{\kappa_1}{2\mu_1}\right) \int_{\Gamma_{\mathrm{du}}} M_2 \omega(z) \mathrm{d}\bar{z}.$$

If one imposes the average stress $\bar{\epsilon}_{\mathrm{I}}$, the effective moduli $c_{*1}$, $c_{*2}$, $c_{*3}$ can be computed as follows

$$c_{*1} = \kappa_1 + \mu_1 + \Im\mathrm{m}\{a - b\},$$
$$c_{*2} = \kappa_1 - \mu_1 - \Im\mathrm{m}\{a + b\},$$
$$(67) \qquad c_{*3} = -\sqrt{2}\Re\mathrm{e}\{a\}.$$

If one imposes the average stress $\bar{\epsilon}_{\mathrm{II}}$, then the effective moduli $c_{*2}$, $c_{*4}$, $c_{*5}$ can be computed by

$$c_{*2} = \kappa_1 - \mu_1 + \Im\mathrm{m}\{a - b\},$$
$$c_{*4} = \kappa_1 + \mu_1 - \Im\mathrm{m}\{a + b\},$$
$$(68) \qquad c_{*5} = -\sqrt{2}\Re\mathrm{e}\{a\}.$$

If one imposes the average stress $\bar{\epsilon}_{\mathrm{III}}$, then the effective moduli $c_{*3}$, $c_{*5}$, $c_{*6}$ can be computed by

$$c_{*3} = \Im\mathrm{m}\{a - b\}/\sqrt{2},$$
$$c_{*5} = -\Im\mathrm{m}\{a + b\}/\sqrt{2},$$
$$(69) \qquad c_{*6} = 2\mu_1 - \Re\mathrm{e}\{a\}.$$

**9. Branch chasing and evaluation of integral operators.** The smooth integral kernels of $M_1^0$, $M_2$, $M_3$, and $M_4^0$ are simple to evaluate with composite Gauss-Legendre and Gauss-Jacobi quadrature. In this section we will discuss technical details of the more complicated evaluations of the singular kernels of $M_1^*$ and $M_4^*$, and the evaluation of the desired branch of the weight function $\rho(z)$,

**9.1. Evaluation of $\rho(z)$.** The value of the weight function $\rho(z)$ in (8) on a curve $\Gamma^j$ is defined as the limit from the right (relative to the orientation of $\Gamma^j$) of the branch of $\rho(z)$ given by a cut in the plane along $\Gamma^j$ and by condition (10). To calculate numerical values of $\rho(z)$ we note that

$$(70) \qquad \frac{1}{\rho(z)} = \nu(z) \cdot \mathrm{Sqrt}\,(z - \gamma_{\mathrm{s}}^j)\mathrm{Sqrt}\,(z - \gamma_{\mathrm{e}}^j)$$

where $\mathrm{Sqrt}\,()$ is the principal branch of the square root given by a cut along the negative real axis and $\mathrm{Sqrt}\,(1) = 1$, and $\nu(z)$ is the relative sign of our branch of $\rho(z)$

as compared to the principal branches in (70). Since the principal branch is easy to calculate and in fact is implemented in most programming languages it remains to calculate the sign function $\nu(z)$ along the curve $\Gamma$. To this end we define the angles $\theta_1(t)$ and $\theta_2(t)$ as

(71) $\qquad \theta_1(t) = arg(z(t) - \gamma_s), \quad 0 < t \le 1, \qquad \theta_1(0) = arg(z'(0)),$

(72) $\qquad \theta_2(t) = arg(z(t) - \gamma_e), \quad 0 \le t < 1, \qquad \theta_2(1) = arg(-z'(1)).$

The function $\nu(z)$ takes values $\pm 1$ and changes sign exactly when either $\theta_1$ or $\theta_2$ passes through an odd multiple of $\pi$. This gives $\nu(z)$ up to a global sign, which is obtained as follows. Let $m$ be the total number of times $\theta_1$ passes through an odd multiple of $\pi$ along $\Gamma$. Then the initial value of $\nu(z)$ as $z$ starts out from $\gamma_s$ is

(73) $\qquad\qquad\qquad (-1)^m, \qquad 0 < arg(\gamma_e - \gamma_s) \le \pi,$

(74) $\qquad\qquad\qquad -(-1)^m, \qquad -\pi < arg(\gamma_e - \gamma_s) \le 0.$

We discuss in more detail the calculation of $\theta_1$. The angle $\theta_2$ is handled analogously. Let $t_i$ and $t_{i+1}$ be two consecutive points in the discretization of $\Gamma^j$, and let $\Delta\theta_1$ be the increment of $\theta_1$ as we go along the curve segment $[z(t_i), z(t_{i+1})]$. Due to the multivaluedness of $\theta_1$ this change in the angle can not be calculated from $z(t_i)$ and $z(t_{i+1})$ alone but depends on the number of turns the curve segment $[z(t_i), z(t_{i+1})]$ winds around $\gamma_s$. To avoid this problem we are going to use a geometrical condition on the curve segment which implies that we are not winding more than half a turn around $\gamma_s$. Given $z(t_i)$, $z(t_{i+1})$ and $\gamma_s$ there exists a unique $-\pi < \alpha < \pi$ such that

$$\Delta\theta_1 = \alpha + k \cdot 2\pi, \text{ for some } k \in \mathbb{Z}.$$

Here a $k \ne 0$ indicates that the curve segment winds more than half a turn around $\gamma_s$. Let $T_i$ be the total curvature, see [11], of the curve segment $[z(t_i), z(t_{i+1})]$ and let $\alpha$ be as above. Then $\Delta\theta_1 = \alpha$ if and only if

(75) $$\pi - \alpha > T_i.$$

A similar problem arises at the endpoints. The geometry is different but when going from $z(0)$ to $z(t_1)$ the condition in (75) ensures that the curve does not wind more then half a turn around $\gamma_s$. The total curvature $T_i$ can easily be estimated by a simple quadrature rule.

This leads to the following recursive algorithm for the calculation of $\Delta\theta_1$ going from parameter value $t$ to $s$ along the curve (and similarly for $\Delta\theta_2$).

```
function Delta(t, s)
calculate θ₁(s) and α
if  (π − α > T(s, t)) then
return  α
else
t1 := (s+t)/2
return  Delta(s, t1) + Delta(t1, s)
endif
```

Simple analytical arguments for curves of finite total curvature show that this algorithm stops.

**9.2. Evaluation of $M_1^*$ and $M_4^*$.** We now turn to the numerical evaluation of the Cauchy type singular kernels of $M_1^*$ and $M_4^*$. The framework is composite Gaussian quadrature. Each crack $\Gamma^j$ is divided into $m^j$ Gaussian segments. The first segment, $\Gamma_s^j$, starts at $\gamma_s^j$ and end at $\gamma_2^j$. The second segment starts at $\gamma_2^j$ and ends at $\gamma_3^j$, and so on. The last segment, $\Gamma_e^j$, starts at $\gamma_{m^j}^j$ and ends at $\gamma_e^j$.

To facilitate numerical integration it will be convenient to introduce a piece-wise smooth real valued weight function $h(z)$ on $\Gamma^j$ given by

$$h(z) = \sqrt{\frac{s(\gamma_2^j) - s(\gamma_s^j)}{s(z) - s(\gamma_s^j)}}, \qquad z \in \Gamma_s^j.$$

$$h(z) = \sqrt{\frac{s(\gamma_e^j) - s(\gamma_{m^j}^j)}{s(\gamma_e^j) - s(z)}}, \qquad z \in \Gamma_e^j,$$

and

(76) $$h(z) = 1, \qquad z \in \Gamma^j - \Gamma_s^j - \Gamma_e^j.$$

The weight function $h(z)$ is constructed so that the ratio $\rho(z)/h(z)$ is bounded, non-zero, and continuous.

Lemma 4 and equation (27) of Lemma 1 allow us to rewrite the integral operators $M_1^*$ and $M_4^*$ in ways that make it simple to compute their Cauchy principal values. On $\Gamma^j$ and operating on a function $f(z)$ we get for $M_1^*$

(77) $$\frac{1}{\pi i} \int_{\Gamma^j} \frac{f(\tau)\rho(\tau)\mathrm{d}\tau}{\tau - z} = \frac{1}{\pi i} \int_{\Gamma^j} \frac{(f(\tau) - f(z))}{(\tau - z)} \frac{\rho(\tau)}{h(\tau)} h(\tau)\mathrm{d}\tau,$$

and for $M_4^*$

(78) $$\frac{1}{\pi i} \int_{\Gamma^j} \frac{f(\tau)\mathrm{d}\tau}{\rho(\tau)(\tau - z)} = \frac{1}{\pi i} \int_{\Gamma^j} \frac{(f(\tau) - f(z))}{(\tau - z)} \frac{h(\tau)}{\rho(\tau)} \frac{\mathrm{d}\tau}{h(\tau)} + \left(z - \frac{\gamma_s^j + \gamma_e^j}{2}\right) f(z).$$

The integrals on the right hand sides of (77) and (78) have smooth integrands if $f(z)$ is a smooth function. Given a parameterization $z(s)$ of $\Gamma^j$ the integrals can be evaluated using Gauss-Jacobi quadrature on the intervals containing the crack tips, and using Gauss-Legendre quadrature on all other intervals. If $f(z)$ is only known at $n$ discrete points this evaluation will include taking the limit

$$\lim_{t \to s} \frac{f(z(t)) - f(z(s))}{t - s}$$

numerically, which can be achieved by analytic differentiation of an interpolating polynomial of degree $n - 1$. As a result, one may compute the principal value with at worst $(n-2)$th order accuracy and not with $(2n-1)$th order accuracy which is normally the case for Gaussian quadrature.

In (56) the operator $M_1^*$ operates on $M_4^0 - M_2$, which in turn operates on $\omega(z)$. Due to different weights in the integrals, $M_1^*$ and $M_4^0 - M_2$ will need different sets of source points in the quadrature on segments containing crack tips if the order of the quadrature is to be the best possible. Let these sets be denoted $\Lambda_I$ and $\Lambda_{II}$. The target points for $M_4^0 - M_2$ are in $\Lambda_I$ and the target points for $M_1^*$ are in $\Lambda_{II}$.

INTEGRAL EQUATIONS FOR CRACKS

The situation in (48) for $\Omega(z)$ is similar. We need $\Lambda_{\mathrm{II}}$ and $\Lambda_{\mathrm{I}}$ as source points for $M_4^*$ and $M_1^0 - M_3$ to get best possible order in the quadrature. The target points for $M_4^*$ and $M_1^0 - M_3$ are taken as $\Lambda_{\mathrm{I}}$ and $\Lambda_{\mathrm{II}}$, respectively. A complication is that, because of the second term on the right hand side of (78), the operator $M_4^*$ also needs the values of $(M_1^0 - M_3)\Omega$ at $\Lambda_{\mathrm{I}}$ as source points. In our implementation these values are computed by interpolation.

**10. Numerical results.** In this section we solve (48–49) and (56–57) using composite 16-point Gaussian quadrature for a few free-space and periodic configurations of cracks and inclusions. To achieve high accuracy for difficult geometries we use *a posteriori* refinement. Initially we place a certain number of Gaussian segments on each crack and interface and solve for the density $\Omega(z)$ or $\omega(z)$. We determine whether the density is well resolved by looking at its Legendre expansion. Those segments on which the coefficients in the Legendre expansion are slowly decaying will be subdivided, and the integral equations are solved again. The final accuracy is determined by using convergence studies explained in [5] and [4]. The linear system of equations in each refinement step is solved with the GMRES iterative solver [17] and the iterations are terminated when the residual is less than $10^{-13}$. For simplicity we allow the possibility of negative crack opening displacement. The computed effective moduli then become those of a material where the cracks are not closed prior to loading, but have a small initial opening.

**10.1. A branched crack.** Let us start with a seemingly difficult geometry containing just one crack and no inclusion: The asymmetrically branched crack in freespace [9]. This geometry has been treated by several previous investigators. Here we choose a crack of length $2a$ with a branch of length $a$. The angle between the main crack and the branch is $\pi/4$. A uniform strain is applied at infinity perpendicular to the main crack. Lo [9] obtained normalized stress intensity factors for the branched tip $K_{\mathrm{I}} = 0.66$ and $K_{\mathrm{II}} = 0.72$. Zang and Gudmundson [19] reported $K_{\mathrm{I}} = 0.664$ and $K_{\mathrm{II}} = 0.722$, and Helsing [6] got $K_{\mathrm{I}} = 0.66278$ and $K_{\mathrm{II}} = 0.72093$ with an algorithm involving numerical preconditioning of a first kind Fredholm integral equation. With 1344 discretization points and using either (48) or (56) we get convergence to $K_{\mathrm{I}} = 0.662782122673$ and $K_{\mathrm{II}} = 0.720931095103$. With the angle reduced to $\pi/12$ and with 1344 discretization points we get convergence to $K_{\mathrm{I}} = 1.1519846410388$ and $K_{\mathrm{I}} = 0.31778821583537$, values which should be compared with $K_{\mathrm{I}} = 1.15$ and $K_{\mathrm{II}} = 0.32$ of Lo [9], $K_{\mathrm{I}} = 1.154$ and $K_{\mathrm{II}} = 0.318$ of Zang and Gudmundson [19], and $K_{\mathrm{I}} = 1.1519846$ and $K_{\mathrm{II}} = 0.3177882$ of Helsing [6].

**10.2. A spiral crack.** In Figure 1 we see a crack in the shape of a spiral parameterized by

$$(79) \qquad\qquad z(t) = te^{i8.3\pi t}, \qquad 0 \leq t \leq 1.$$

In the definition of the stress intensity factors of (59) we use a normalization factor $g = 1$. The crack is subjected to a uniform remote biaxial unit load. After one stage of refinement, using a total of 880 discretization points in equation (48), we get for the outer crack tip $K_{\mathrm{I}} = 0.43420263702$ and $K_{\mathrm{II}} = 0.48761577661$. The stress intensity factors for the crack tip in the center have magnitudes smaller than $10^{-12}$.

**10.3. A periodic configuration of aligned cracks.** We take a doubly periodic square array of aligned straight cracks. The crack in the unit cell has length $2a$ and is directed parallel to the $x$-axis. Average displacements $\bar{\epsilon}_{\mathrm{II}}$ and $\bar{\epsilon}_{\mathrm{III}}$ are applied.
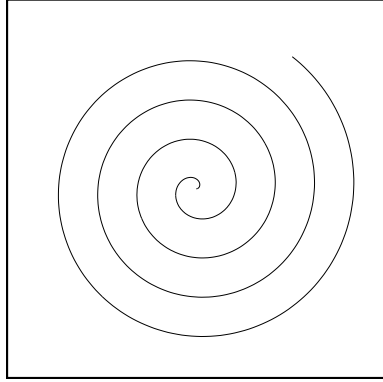
FIG. 1. *A crack in the shape of a spiral.*

TABLE 1

*Stress intensity factor $K_\mathrm{I}$ normalized with a factor $\sigma_\mathrm{yy}\sqrt{a\pi}$, stress intensity factor $K_\mathrm{II}$ normalized with a factor $\sigma_\mathrm{xy}\sqrt{a\pi}$, and effective moduli $c_{*4}$ and $c_{*6}$ for a square array of cracks of length $2a$. The number of discretization points per crack in the final estimate is 'np'. The two dimensional elastic moduli of the material are $\kappa = 0.5$ and $\mu = 0.5$.*

| $a$ | $c_{*4}$ | $c_{*6}$ | $K_\mathrm{I}$ | $K_\mathrm{II}$ | np |
|---|---|---|---|---|---|
| 0.1 | 0.9402238722753 | 0.9689332424173 | 1.012189656971 | 1.020492383609 | 32 |
| 0.2 | 0.7900434229480 | 0.8799346852987 | 1.062257994344 | 1.085071774160 | 32 |
| 0.3 | 0.6016657820793 | 0.7441611351022 | 1.193897589344 | 1.219483489482 | 32 |
| 0.4 | 0.4052636517192 | 0.5687774436483 | 1.557757225697 | 1.570463954286 | 64 |
| 0.49 | 0.1861381455207 | 0.3109493450269 | 4.54629613227 | 4.54671859270 | 190 |
| 0.499 | 0.120426466863 | 0.213595309831 | 14.2494765410 | 14.2494897992 | 290 |
| 0.4999 | 0.089003112607 | 0.162664545319 | 45.02031709 | 45.02031750 | 420 |
| 0.49999 | 0.07058505740 | 0.13134576810 | 142.353932 | 142.353932 | 540 |
| 0.499999 | 0.0584827919 | 0.1101398788 | 450.1586 | 450.1586 | 600 |
| 0.4999999 | 0.049923145 | 0.094829653 | 1423.525 | 1423.525 | 700 |
| 0.49999999 | 0.0435492 | 0.0832563 | 4501.58 | 4501.58 | 800 |

The two dimensional bulk and shear moduli are chosen as $\kappa = 0.5$ and $\mu = 0.5$. Results for the effective moduli $c_{*4}$ and $c_{*6}$ and for the stress intensity factors $K_\mathrm{I}$ and $K_\mathrm{II}$, normalized with factors $g = \sigma_\mathrm{yy}\sqrt{a\pi}$ and $g = \sigma_\mathrm{xy}\sqrt{a\pi}$, are presented in Table 1. Equations (48) and (56) were used with virtually identical performance. GMRES typically converged to the desired accuracy in less than ten iterations.

**10.4. Five spiral shaped cracks in a square unit cell.** Figure 2 shows a square unit cell of unit side length with fives cracks in the shape of spirals. The cracks are parameterized

$$(80) \qquad\qquad z(t) = z_\mathrm{cent} + 0.23te^{\pm i5.3\pi t}, \qquad 0 \le t \le 1,$$

where $z_\mathrm{cent}$ are the five spiral centers $z_1 = (-0.42, 0.22)$, $z_2 = (0.22, 0.45)$, $z_3 = (0, 0.06)$, $z_4 = (0.45, -0.16)$ $z_5 = (-0.16, -0.38)$. Plus and minus indicate counterclockwise or clockwise orientation.

Thirty Gaussian segments were initially placed on each crack, corresponding to 2,400 discretization points. One stage of iterative refinement was then used in (48), adding another 112 discretization points. The effective elastic moduli converged to $c_{*1} = 0.17047722558$, $c_{*2} = 0.06588860514$, $c_{*3} = -0.04780552823$, $c_{*4} = 0.15851961817$, $c_{*5} = 0.05120153242$, and $c_{*6} = 0.12783097335$.
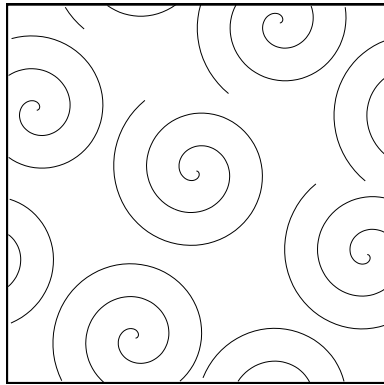
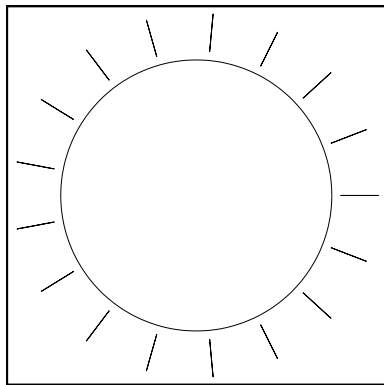FIG. 2. Five spiral shaped cracks in a square unit cell.



FIG. 3. A unit cell of unit side length consisting of a disk of area fraction 0.4 and 17 cracks of length 0.1.

**10.5. A Disk surrounded by cracks in the form of rays.** We solve (48–49) and (56–57) for the geometry in Figure 3. The unit cell, in a doubly periodic square array, has a disk of area fraction $p_2 = 0.4$ centered around its origin. The disk is surrounded by 17 equispaced cracks of length 0.1. The midpoints of the cracks are placed at a distance of 0.43 from the origin. The elastic moduli of the matrix material is $\kappa_1 = 0.5$ and $\mu_1 = 0.5$. The elastic moduli of the disk are $\mu_2 = 135$ and $\kappa_2 = 225$.

For (48–49), two Gaussian segments were initially placed on each crack and 50 segments were placed on the disk. Using two stages of iterative refinement with 27 new segments each, leading up to a total of 2208 points discretization points, we arrive at the effective elastic moduli: $c_{*1} = 2.086956695352$, $c_{*2} = 0.2073748112781$, $c_{*4} = 2.087501278993$, and $c_{*6} = 1.547493105594$. The same moduli were then computed using (56–57). For this calculation we only needed 40 initial segments on the disk and one stage of refinement, leading up to a total of 1616 discretization points. We conclude that for this crack and inclusion problem the formulation (56–57) is about 30 per cent more efficient in terms of discretization points and computing time.

**11. Conclusions.** We have derived two systems of second kind Fredholm integral equations for elastostatics of cracked composites in the absence of interface cracks. The first system, (48–49), is based on a density $\Omega$, closely related to the derivative of

the crack opening displacement. The second system, (56–57), is based on a density $\omega$, closely related to the crack opening displacement itself. The second system of integral equations has a smoother right hand side than the first system.

Based on these equations two robust and adaptive high-order solvers with flexible and user friendly interfaces were developed and implemented. Numerical calculations gave results with unprecedented accuracy and indicated that for pure crack problems the two systems of equations were comparable, while for crack and inclusion problems the system (56–57) was most efficient. We are currently considering an extension of this scheme to geometries with interface cracks.

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