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Abstract

1 Physical balance principles

When not considering electrical or electromagnetic effects there are only five balance principles to be considered in physics, namely: the balance of mass, balance of momentum, balance of angular momentum, balance of energy (i.e. the first axiom of thermodynamics) and the entropy inequality (i.e. the second axiom of thermodynamics). In every physical model these principles must be considered. Of course, approximations to the general balance principles can be adopted but only if these approximations can be justified by experiments together with physical stringent modelling.

1.1 Introduction to the balance principles

In continuum mechanics there are only five postulated balance equations where one of them being an inequality.. These equations are independent of the actual material studied.

Balance of *mass* can be expressed as

$$\dot{\rho} + \rho \mathrm{div}\,(\dot{\mathbf{x}}) = 0 \tag{1}$$

where $\dot{\rho}$ [kg/m^3/s] is the rate of change of the *density* where the dot denotes a time derivative following the motion (*material time derivative*) defined by the *velocity* $\dot{\mathbf{x}}$ [m/s].

Balance of *linear momentum* is

$$\rho\ddot{\mathbf{x}} = \mathrm{div}\,(\mathbf{T}) + \rho\mathbf{b} \tag{2}$$

where $\ddot{\mathbf{x}}$ [m/s^2] is the acceleration recorded by an observer following the motion of the actual body of interest (material time derivative). The *stress (tensor)* is denoted $\mathbf{T}$ [N/m^2] and $\mathbf{b}$ [kgm^2/s], or [N/m^3] is the so-called *body force density* due to gravity.

The result from the *angular momentum* that will be used here is that the stress tensor is symmetric, i.e.

$$\mathbf{T} = \mathbf{T}^{\mathrm{T}} \tag{3}$$

The balance of *energy* is

$$\rho\dot{\varepsilon} = \mathrm{tr}\,(\mathbf{TD}) - \mathrm{div}\,(\mathbf{q}) + \rho r \tag{4}$$

where $\dot{\varepsilon}$ [J/kg/s] is the rate of change of the *internal energy* where the dot denotes a time derivative following the actual motion of the body (material time derivative). The symmetric part of the *velocity gradient* is denoted $\mathbf{D}$ [1/s]. And the *heat flux vector* is denoted $\mathbf{q}$ [J/m^2/s/K], or [W/m^2/K]. The external heat source is denoted r [J/kg/s], or [W/kg] (*radiation*)

The *second axiom of thermodynamics* can be expressed as

$$\theta\rho\dot{\eta} - \mathrm{grad}\,(\theta)\cdot\mathbf{q}/\theta - \rho\dot{\varepsilon} + \mathrm{tr}\,(\mathbf{TD}) \geq 0 \tag{5}$$

where θ [K] is the *temperature* (necessarily positive) and $\dot{\eta}$ [J/kg/K/s], or [W/kg/K] is the rate of change of the *entropy.*

In the following sections it will be shown how the above balance principles can be obtained.

1.2 Global balance principles

The fact that mass cannot be 'destroyed', i.e. the mass cannot change, is within the continuum concept expressed as

$$\frac{D}{Dt}m\,(\Re) = 0 \tag{6}$$

where $m\,(\Re)$, e.g. [kg], denotes the mass of a particle denoted $\Re$. The time derivative $D/Dt\,[1/\mathrm{s}]$ follows the motion of the particle $\Re$.

The global rate of change of *momentum* $D/Dt\left(\bar{P}(\Re)\right)$ [kgm/s^2], or [N] is due to a force resultant vector $\bar{R}^{input}$ [N] acting on the body or particle $\Re$.

$$\frac{D}{Dt}\bar{P}(\Re) = \bar{R}^{input}(\Re) \tag{7}$$

The rate of change of the *angular momentum* $D/Dt\left(\bar{L}_o(\Re)\right)$ [kgm^2/s^2], or [Nm, or J] is due the angular force resultant vector $\bar{M}_o^{input}(\Re)$ [Nm] contributing to momentum taken for example at the origo of an introduced coordinate system.

$$\frac{D}{Dt}\bar{L}_o(\Re) = \bar{M}_o^{input}(\Re) \tag{8}$$

The rate of change of the *total energy* $D/Dt\,(E(\Re))$ [Nm/s], or [J/s, or W] is due to an input of power (energy) $P^{input}(\Re)$ [W] to the body $\Re$.

$$\frac{D}{Dt}E(\Re) = P^{input}(\Re) \tag{9}$$

The total energy is divided into a kinetic energy part $E_k(\Re)$ and a internal energy part $E_i(\Re)$, i.e.

$$E(\Re) = E_k(\Re) + E_i(\Re) \tag{10}$$

where the energy term $E_k(\Re)$ steams from a global acceleration force (i.e. energy) of the body $\Re$, and the internal energy is due to *internal thermal-motion* within the body $\Re$. From (9) and (10) one obtain

$$\frac{D}{Dt}E_k(\Re) + \frac{D}{Dt}E_i(\Re) = P^{input}(\Re) \tag{11}$$

Furthermore, the power input $P^{input}(\Re)$ is divided into a mechanical input part $P_{mek}^{input}(\Re)$ and a part associated with input as heat denoted $Q^{input}(\Re)$.

$$P^{input}(\Re) = P_{mek}^{input}(\Re) + Q^{input}(\Re) \tag{12}$$

From (12) it is concluded that (11) takes the form

$$\frac{D}{Dt}E_k(\Re) + \frac{D}{Dt}E_i(\Re) = P_{mek}^{input}(\Re) + Q^{input}(\Re) \tag{13}$$

The net power $N(\Re)$ [W] to the body $\Re$ is defined as the sum of the mechanical input $P_{mek}^{input}(\Re)$ and the rate of change of the kinetic energy of the body $\Re$, i.e.

$$N(\Re) = P_{mek}^{input}(\Re) + \frac{D}{Dt}E_k(\Re) \tag{14}$$

For a body having no acceleration and which do not deform from its original shape both $P_{mek}^{input}(\Re)$ and $D/Dt\,(E_k(\Re))$ is zero, i.e. $N(\Re)$ is zero for rigid body motion.

The rate of change of the internal energy, i.e. $D/Dt\,(E_i(\Re))$, can now be expressed as

$$\frac{D}{Dt}E_i(\Re) = N(\Re) + Q^{input}(\Re) \tag{15}$$

where (13) and (14) is used.

At last, the general experimental observation is that the rate of change of the *entropy* $D/Dt\,(\mathcal{H}(\Re))$ [J/s/K] is always greater (irreversible process) or equal (reversible process) to the input of the entropy $S^{input}(\Re)$ [J/s/K] (entropy influx) to the body $\Re$.

$$\frac{D}{Dt}\mathcal{H}(\Re) \geq S^{input}(\Re) \tag{16}$$

The nature of the inequality steams from the so-called thermal or mechanical *dissipation* energy productions.

1.3 Summery of the global 'balance' postulates

The global balance relations are

$$\frac{D}{Dt}m(\Re) = 0 \tag{17}$$

for the mass conservation. The momentum balance is in global form

$$\frac{D}{Dt}\bar{P}(\Re) = \bar{R}^{input}(\Re) \tag{18}$$

The angular momentum described in global form is

$$\frac{D}{Dt}\bar{L}_o(\Re) = \bar{M}_o^{input}(\Re) \tag{19}$$

and the global energy balance is

$$\frac{D}{Dt}E_k(\Re) + \frac{D}{Dt}E_i(\Re) = P_{mek}^{input}(\Re) + Q^{input}(\Re) \tag{20}$$

and the relation for the rate of change of entropy is

$$\frac{D}{Dt}\mathcal{H}(\Re) \geq S^{input}(\Re) \tag{21}$$

which is the second axiom of thermodynamics expressed in global form of a body $\Re$.

1.4 Towards a balance description in terms of densities

The mass $m(\Re)$ of a body $\Re$ can be regarded as a *mass density* ρ [kg/m^3]. The relation between the two can be obtained by integrate over a small representative volume v in which the density can be properly defined, i.e.

$$m(\Re) = \int_{\Re} \rho dv \tag{22}$$

The global *momentum* $\bar{P}(\Re)$ [Ns], or [kgm/s] of the body $\Re$ can be regarded as an integral of the momentum vectors, i.e. $\rho\dot{\mathbf{x}}$ [kg/s/m^2] acting within the body as

$$\bar{P}(\Re) = \int_{\Re} \rho\dot{\mathbf{x}} dv \tag{23}$$

where $\dot{\mathbf{x}}$ is the velocity [m/s].

The *angular momentum* $\bar{L}_o(\Re)$ [Nms], or [kgm^2/s] of the body $\Re$ can be regarded as an integral of the angular momentum vectors, i.e. $\mathbf{x}\times\rho\dot{\mathbf{x}}$ [kgm^2/s] within the body as

$$\bar{L}_o(\Re) = \int_{\Re} \mathbf{x}\times\rho\dot{\mathbf{x}} dv \tag{24}$$

The *kinetic energy* $E_k(\Re)$ [J], or [Nm, or kgm^2/s^2]of the body $\Re$ is

$$E_k(\Re) = \int_{\Re} \tfrac{1}{2}\dot{\mathbf{x}}\cdot\rho\dot{\mathbf{x}} dv \tag{25}$$

and the *internal energy* $E_i(\Re)$ [J], or [Nm, or kgm^2/s^2] of the body $\Re$ is

$$E_i(\Re) = \int_{\Re} \rho\varepsilon dv \tag{26}$$

where ε [J/kg] denotes the *internal energy density*.

The entropy within the body $\Re$ can be expressed by the volume integral

$$\mathcal{H}(\Re) = \int_{\Re} \rho\eta dv \tag{27}$$

where η is the *entropy density* [J/kg/K].

The rate of change of the mass can be written in terms of the rate of change of the mass density by differentiate both sides of (22) with respect to time, as

$$\frac{D}{Dt} m(\Re) = \frac{D}{Dt} \int_{\Re} \rho dv \tag{28}$$

The rate of change of the linear momentum in the body $\Re$ is obtained from (23), as

$$\frac{D}{Dt}\bar{P}(\Re) = \frac{D}{Dt}\int_{\Re} \rho\dot{\mathbf{x}}dv \tag{29}$$

This equation is the *Euler's first law of motion.*

The rate of change of the angular momentum in the body $\Re$ is

$$\frac{D}{Dt}\bar{L}_o(\Re) = \frac{D}{Dt}\int_{\Re} \mathbf{x}\times\rho\dot{\mathbf{x}}dv \tag{30}$$

which is the *Euler's second law of motion,* where (24) is differentiated. In words the Euler's laws of motion are: *(i) The total force acting on a body is equall to the rate of change of the linear momentum of the body, (ii) The total torque acting on a body is equal to the rate of change of the moment of momentum of the body.*

The rate of change of the kinetic energy is obtained from (25), as

$$\frac{D}{Dt}E_k(\Re) = \frac{D}{Dt}\int_{\Re} \tfrac{1}{2}\rho\dot{x}^2 dv \tag{31}$$

where $\dot{x}^2 = \dot{\mathbf{x}}\cdot\dot{\mathbf{x}}$. And the rate of change of the internal energy can be obtained from (26) as

$$\frac{D}{Dt}E_i(\Re) = \frac{D}{Dt}\int_{\Re} \rho\varepsilon dv \tag{32}$$

At last the rate of change of the entropy can be written in term of a volume integral as

$$\frac{D}{Dt}\mathcal{H}(\Re) = \frac{D}{Dt}\int_{\Re} \rho\eta dv \tag{33}$$

where (27) is used.

It should be carefully noted that all the volume integrals $\int_{\Re} dv$ above is following the motion of the body $\Re$, and this means, further, that the time derivative D/Dt should be interpreted as a material time derivative. That is a derivative which is referred to an observer following the motion of the body in question.

1.5 Reynolds transport theorem, material time derivative of volume integrals

In the above sections volume integrals where discussed. These integrals was described as a property following the motion of the body. This type of description of volume integrals is not very useful.. Instead one often want to use fixed integrals in space. The Reynolds transport theorem is a mathematical relation which transforms the volume integrals following the motion to a description where the volume integrals are fixed in space.

For a general physical property Γ, which may be a scalar or a vector, is transformed from a state where the volume integral follows the motion, denoted with the material time derivative D/Dt in front of the volume integral, to a state where the volume integrals are fixed in space, as

$$\frac{D}{Dt}\int_{\Re}\Gamma\rho dv=\frac{\partial}{\partial t}\int_{\Re}\Gamma\rho dv+\oint_{\partial\Re}\Gamma\rho\left(\dot{\mathbf{x}}\cdot d\mathbf{s}\right) \tag{34}$$

where $\partial/\partial t$ is a spatial derivative of the fixed volume integral $\int_{\Re}\Gamma\rho dv$ it should also be understood that the surface integral $\oint_{\partial\Re}\Gamma\rho\left(\dot{\mathbf{x}}\cdot d\mathbf{s}\right)$ is fixed in space. The velocity is denoted $\dot{\mathbf{x}}$ and the normalized vector (directed outwards from the boundary surface in the point considered) is denoted $d\mathbf{s}$.

First, it is noted that by setting $\Gamma\equiv 1$ one obtain

$$\frac{D}{Dt}m\left(\Re\right)=\frac{D}{Dt}\int_{\Re}\rho dv=\frac{\partial}{\partial t}\int_{\Re}\rho dv+\oint_{\partial\Re}\rho\dot{\mathbf{x}}\cdot d\mathbf{s} \tag{35}$$

which is a transformation from a material description of the volume integral to a spatial description in terms of a fixed volume integral and a fixed surface integral in space. The right hand-side of (35) represents the rate of change of the mass density from a fixed observers point of view (i.e. the spatial description).

By setting $\Gamma\equiv\dot{\mathbf{x}}$, one obtain

$$\frac{D}{Dt}\bar{P}\left(\Re\right)=\frac{D}{Dt}\int_{\Re}\rho\dot{\mathbf{x}}dv=\frac{\partial}{\partial t}\int_{\Re}\rho\dot{\mathbf{x}}dv+\oint_{\partial\Re}\rho\dot{\mathbf{x}}\left(\dot{\mathbf{x}}\cdot d\mathbf{s}\right) \tag{36}$$

this means that the right hand-side of this equation represents the spatial description (i.e. the fixed description in space) of the rate of change of momentum.

The rate of change of the spatial description of the angular momentum is obtained by setting $\Gamma\equiv\mathbf{x}\times\dot{\mathbf{x}}$, i.e. (34) results in that (30) can be expressed as

$$\frac{D}{Dt}\bar{L}_o\left(\Re\right)=\frac{D}{Dt}\int_{\Re}\mathbf{x}\times\rho\dot{\mathbf{x}}dv=\frac{\partial}{\partial t}\int_{\Re}\mathbf{x}\times\rho\dot{\mathbf{x}}dv+\oint_{\partial\Re}\mathbf{x}\times\rho\dot{\mathbf{x}}\left(\dot{\mathbf{x}}\cdot d\mathbf{s}\right) \tag{37}$$

By setting $\Gamma\equiv\frac{1}{2}\dot{x}^2$

$$\frac{D}{Dt}E_k\left(\Re\right)=\frac{D}{Dt}\int_{\Re}\tfrac{1}{2}\rho\dot{x}^2dv=\frac{\partial}{\partial t}\int_{\Re}\tfrac{1}{2}\rho\dot{x}^2dv+\oint_{\partial\Re}\tfrac{1}{2}\rho\dot{x}^2\left(\dot{\mathbf{x}}\cdot d\mathbf{s}\right) \tag{38}$$

the spatial description of the rate of change of the kinetic energy is obtained for the body $\Re$.

Equation (34) with $\Gamma \equiv \varepsilon$, yields the spatial description of the internal energy, i.e.

$$\frac{D}{Dt}E_i(\Re) = \frac{D}{Dt}\int_{\Re} \rho\varepsilon dv = \frac{\partial}{\partial t}\int_{\Re} \rho\varepsilon dv + \oint_{\partial\Re} \rho\varepsilon\dot{\mathbf{x}} \cdot ds \tag{39}$$

And the rate of change of the entropy in terms of spatially fixed volume and surface integrals is obtained by setting $\Gamma \equiv \eta$, i.e.

$$\frac{D}{Dt}\mathcal{H}(\Re) = \frac{D}{Dt}\int_{\Re} \rho\eta dv = \frac{\partial}{\partial t}\int_{\Re} \rho\eta dv + \oint_{\partial\Re} \rho\eta\dot{\mathbf{x}} \cdot ds \tag{40}$$

Yet, only the internal acting forces and energies has been discussed. In the next section the external acting forces and energies will be specified.

1.6 Input of energy and momentum to a body

A material or a specified body within a material of any kind can of coarse be subjected to forces and energies from its surrounding e.g. *external* forces and energies can act on the boundary surface of the body $\Re$ of interest. The surrounding is here referred to as the neighboring material points located near the body $\Re$, i.e.e the surrounding is expresses as a surface integral and this defined surface is the boundary to the surrounding bodies.

The effect of the input of forces or energies to the system is solely determined by the stress $\mathbf{T}$, the heat flux $\mathbf{q}$ and the radiation r and also due to the body forces $\mathbf{b}$.

Indeed, it was seen from the Reynolds transport theorem that the surface integrals appeared also when dealing with *internal* physical properties. But these surface integrals did not appear due to external acting forces or energies on the body, but rather due to the fact that the global internal physical properties, such as mass and total energy, was initially defined as forces and energies following the motion of the body.

The external input of 'force' in equation (18) (or input of momentum $\bar{R}^{input}(\Re)$) at the external surface boundary to the body $\Re$ and within the body, is

$$\bar{R}^{input}(\Re) = \oint_{\partial\Re} \mathbf{T}ds + \int_{\Re} \rho\mathbf{b}dv \tag{41}$$

where is should be noted that $\mathbf{T}ds$ is the so-called traction $\mathbf{t} = \mathbf{T}ds$, i.e. the force acting on the surface per square length or simply the stress vector. The relation $\mathbf{t} = \mathbf{T}ds$ is the so-called *Cauchy's fundamental theorem for the stress*, see any text book in mechanics. The body force $\rho\mathbf{b}$ is also interpreted as an external input of momentum, but this momentum 'force' acts within the whole volume.

The external input of angular momentum $\bar{M}_o^{input}(\Re)$ in equation (19) is the moment of momentum i.e.

$$\bar{M}_o^{input}(\Re) = \oint_{\partial\Re} \mathbf{x} \times \mathbf{T}ds + \int_{\Re} \mathbf{x} \times \rho\mathbf{b}dv \tag{42}$$

The power input $P_{mek}^{input}(\Re)$ in equation (20) is due to mechanical forces such as

$$P_{mek}^{input}(\Re) = \oint_{\partial\Re} \mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}} \cdot d\mathbf{s} \tag{43}$$

And the heat input $Q^{input}(\Re)$ in equation (20) is due to influx of heat $\mathbf{q}$ through the surface boundary and due to work done by the body forces $\rho\dot{\mathbf{x}} \cdot \mathbf{b}$ within the body and also due to local energy supply r within the material due to, for example, radiation.

$$Q^{input}(\Re) = \oint_{\partial\Re} \mathbf{q} \cdot d\mathbf{s} + \int_{\Re} \rho\dot{\mathbf{x}} \cdot \mathbf{b}dv + \int_{\Re} \rho r dv \tag{44}$$

The input of entropy $S^{input}(\Re)$ in equation (21) is defined to be caused only by heat fluxes entering the body and due to local energy supply r within the body and these properties are also divided by the temperature θ, i.e.

$$S^{input}(\Re) = -\oint_{\partial\Re} \mathbf{q}/\theta \cdot d\mathbf{s} + \int_{\Re} \rho r/\theta dv \tag{45}$$

1.7 Balance principles described with fixed volume and surface integrals in space

By using the mass balance described in terms of mass densities, i.e. equation (22) together with the Reynolds transport theorem (35) the mass balance equation described with fixed volume and surface integrals in space is obtained as

$$\frac{\partial}{\partial t}\int_{\Re} \rho(\mathbf{x},t)\, dv = -\oint_{\partial\Re} \rho\dot{\mathbf{x}} \cdot d\mathbf{s} \tag{46}$$

The (linear) momentum for fixed volume and surface integrals becomes

$$\frac{\partial}{\partial t}\int_{\Re}\rho\dot{\mathbf{x}}dv = -\oint_{\partial\Re}\rho\dot{\mathbf{x}}\left(\dot{\mathbf{x}}\cdot d\mathbf{s}\right)+\oint_{\partial\Re}\mathbf{T}d\mathbf{s}+\int_{\Re}\rho\mathbf{b}dv \tag{47}$$

where (18), (23), (29), (36) and (41) are used.

The angular momentum is obtained in the same manner

$$\frac{\partial}{\partial t}\int_{\Re}\mathbf{x}\times\rho\dot{\mathbf{x}}dv = -\oint_{\partial\Re}\mathbf{x}\times\rho\dot{\mathbf{x}}\left(\dot{\mathbf{x}}\cdot d\mathbf{s}\right)+\oint_{\partial\Re}\mathbf{x}\times\mathbf{T}d\mathbf{s}+\int_{\Re}\mathbf{x}\times\rho\mathbf{b}dv \tag{48}$$

where (19), (24), (30), (37) and (42) are used and where the volume and surface integrals are fixed in space.

The energy balance with fixed volume and surface integrals becomes

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\Re}\rho\left(\varepsilon+\tfrac{1}{2}\dot{x}^2\right)dv \;=\; & -\oint_{\partial\Re}\rho\left(\varepsilon+\tfrac{1}{2}\dot{x}^2\right)\dot{\mathbf{x}}\cdot d\mathbf{s} \\ & +\oint_{\partial\Re}\left(\mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}}+\mathbf{q}\right)\cdot d\mathbf{s} \\ & +\int_{\Re}\rho\dot{\mathbf{x}}\cdot\mathbf{b}dv+\int_{\Re}\rho r dv\end{aligned} \tag{49}$$

where (20), (25), (26), (31), (32), (38), (39), (43) and (44) are used.

And the second axiom of thermodynamics expressed with fixed volume and surface integrals is

$$\frac{\partial}{\partial t}\int_{\Re}\rho\eta dv = -\oint_{\partial\Re}\rho\eta\dot{\mathbf{x}}\cdot d\mathbf{s}-\oint_{\partial\Re}\mathbf{q}/\theta\cdot d\mathbf{s}+\int_{\Re}\rho r/\theta dv \tag{50}$$

where (21), (27), (33), (40) and (45) are used.

1.8 Divergence theorem, fixed surface integrals to fixed volume integrals

A physical phenomenon described with a traction on a fixed boundary surface $\partial\Re$ or a flux through a fixed boundary surface $\partial\Re$ can be converted to a volume integral with the help of the divergence theorem. Loosely speaking, the divergence theorem states that it is immaterial if a physical 'change' is recorded within the material or if the same 'change' is recorded as a 'flux' of the same physical property in or out throw the boundary surface of a body $\Re$ having a boundary surface denoted $\partial\Re$.

Consider an arbitrary vector Γ, this vector is acting on a boundary surface $\partial\Re$, i.e. $\oint_{\partial\Re}\Gamma\cdot ds$ the effect of the same physical phenomenon can be recorded within the material and becomes $\int_{\Re}\mathrm{div}(\Gamma)\,dv$. For this case the divergence theorem can be written as

$$\oint_{\partial\Re}\Gamma\cdot ds=\int_{\Re}\mathrm{div}\,(\Gamma)\,dv;\qquad \Gamma\Rightarrow \text{ is a vector} \tag{51}$$

where, again, ds is an out-ward drawn normalized vector (normal) to the point at the boundary surface considered. The divergence operator is denoted div.

By identifying the arbitrary vector Γ as the mass density flux (or the momentum flux) $\Gamma\equiv\rho\dot{\mathbf{x}}$ the divergence theorem (51), gives the transformation of a fixed surface integral to a fixed volume integral as

$$\oint_{\partial\Re}\rho\dot{\mathbf{x}}\cdot ds=\int_{\Re}\mathrm{div}\,(\rho\dot{\mathbf{x}})\,dv \tag{52}$$

By setting $\Gamma\equiv\frac{1}{2}\dot{x}^2\dot{\mathbf{x}}$ in (51) one obtain

$$\oint_{\partial\Re}\tfrac{1}{2}\dot{x}^2\rho\,(\dot{\mathbf{x}}\cdot d\mathbf{s})=\int_{\Re}\mathrm{div}\left(\tfrac{1}{2}\dot{x}^2\dot{\mathbf{x}}\right)dv \tag{53}$$

By identifying the arbitrary vector as $\Gamma\equiv\varepsilon\rho\dot{\mathbf{x}}$, the result is

$$\oint_{\partial\Re}\varepsilon\rho\dot{\mathbf{x}}\cdot ds=\int_{\Re}\mathrm{div}\,(\varepsilon\rho\dot{\mathbf{x}})\,dv \tag{54}$$

and setting $\Gamma\equiv\eta\rho\dot{\mathbf{x}}$, gives

$$\oint_{\partial\Re}\eta\rho\dot{\mathbf{x}}\cdot ds=\int_{\Re}\mathrm{div}\,(\eta\rho\dot{\mathbf{x}})\,dv \tag{55}$$

The heat flux entering (or leaving) through the boundary surface $\partial\Re$ is transformed to a volume integral by using $\Gamma\equiv\mathbf{q}$ in (51) to obtain

$$\oint_{\partial\Re}\mathbf{q}\cdot ds=\int_{\Re}\mathrm{div}\,(\mathbf{q})\,dv \tag{56}$$

By setting $\Gamma\equiv\mathbf{q}/\theta$

$$\oint_{\partial\Re}\mathbf{q}/\theta\cdot ds=\int_{\Re}\mathrm{div}\,(\mathbf{q}/\theta)\,dv \tag{57}$$

is obtained, and by $\Gamma\equiv\mathbf{T}\dot{\mathbf{x}}$, one get

$$\oint_{\partial\Re}\mathbf{T}\dot{\mathbf{x}}\cdot ds=\int_{\Re}\mathrm{div}\,(\mathbf{T}\dot{\mathbf{x}})\,dv \tag{58}$$

If the arbitrary property Γ is a second order tensor the divergence theorem can be written as

$$\oint_{\partial\Re} \Gamma ds = \int_{\Re} \operatorname{div}(\Gamma)\, dv; \quad \Gamma \Rightarrow \text{ is a second order tensor} \tag{59}$$

and by using this equation with $\Gamma \equiv \mathbf{T}$, one obtain

$$\oint_{\partial\Re} \mathbf{T} ds = \int_{\Re} \operatorname{div}(\mathbf{T})\, dv \tag{60}$$

In the same way surface integrals which is subjected to terms including cross-products can be transformed to volume integrals, e.g. with $\Gamma \equiv \mathbf{x} \times \mathbf{T}$ the divergence theorem (59) gives

$$\oint_{\partial\Re} \mathbf{x} \times \mathbf{T} ds = \int_{\Re} \operatorname{div}(\mathbf{x} \times \mathbf{T})\, dv \tag{61}$$

Another case of transformations is when quadratic terms of the velocity is included. For this case the divergence theorem can be written

$$\oint_{\partial\Re} \Gamma\left(\dot{\mathbf{x}} \cdot ds\right) = \int_{\Re} \operatorname{div}(\Gamma \otimes \dot{\mathbf{x}})\, dv; \quad \Gamma \Rightarrow \text{ is a vector including the velocity} \tag{62}$$

where Γ is a vector including the velocity $\dot{\mathbf{x}}$, and where $\otimes$ is the dyad product or equally the tensor product. With $\Gamma \equiv \rho\dot{\mathbf{x}}$, (62) gives

$$\oint_{\partial\Re} \rho\dot{\mathbf{x}}\left(\dot{\mathbf{x}} \cdot ds\right) = \int_{\Re} \operatorname{div}(\rho\dot{\mathbf{x}} \otimes \dot{\mathbf{x}})\, dv \tag{63}$$

and with $\Gamma \equiv \mathbf{x} \times \dot{\mathbf{x}}\rho$

$$\oint_{\partial\Re} \mathbf{x} \times \dot{\mathbf{x}}\rho\left(\dot{\mathbf{x}} \cdot ds\right) = \int_{\Re} \operatorname{div}(\rho\left(\mathbf{x} \times \dot{\mathbf{x}}\right) \otimes \dot{\mathbf{x}})\, dv \tag{64}$$

is obtained.

1.9 Balance principles described only with volume integrals, and the local forms

It is an advantage to consider only fixed volume integral when establishing the physical balance principles and the second axiom of thermodynamics. The divergence theorem discussed earlier was used to transform surface integrals to volume integrals. All surface integrals are now replaced by its corresponding volume integrals to obtain balance equations described only with volume integrals. It should be noted that all equations are now described with the spatial time derivatives $\partial/\partial t$, i.e. a change of a physical property as observed by a fixed observer having no motion.

The balance of mass can be brought to the form by using (46) and (52) to yield.

$$\frac{\partial}{\partial t}\int_{\Re}\rho dv+\int_{\Re}\operatorname{div}(\rho\dot{\mathbf{x}})\,dv=0 \tag{65}$$

Furthermore, it is realized that this equation holds for arbitrary volumes v, this results in that (65) can be written as

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\dot{\mathbf{x}})=0 \tag{66}$$

this form is referred to as the *local form* of the balance of mass.

The linear momentum equation (47) is reformulated by transforming the surface integrals to volume integrals by using the divergence theorem, i.e. (63) and (60) gives

$$\frac{\partial}{\partial t}\int_{\Re}\rho\dot{\mathbf{x}}dv=-\int_{\Re}\operatorname{div}(\rho\dot{\mathbf{x}}\otimes\dot{\mathbf{x}})\,dv+\int_{\Re}\operatorname{div}(\mathbf{T})\,dv+\int_{\Re}\rho\mathbf{b}dv \tag{67}$$

The local form of the linear momentum is therefore

$$\frac{\partial(\rho\dot{\mathbf{x}})}{\partial t}=-\operatorname{div}(\rho\dot{\mathbf{x}}\otimes\dot{\mathbf{x}})+\operatorname{div}(\mathbf{T})+\rho\mathbf{b} \tag{68}$$

The angular momentum is obtained by the same procedure, i.e. the angular momentum equation (48) together with the divergence theorem, i.e. equations (64) and (61), gives

$$\frac{\partial}{\partial t}\int_{\Re}\mathbf{x}\times\rho\dot{\mathbf{x}}dv = -\int_{\Re}\operatorname{div}(\rho(\mathbf{x}\times\dot{\mathbf{x}})\otimes\dot{\mathbf{x}})\,dv \tag{69}$$

$$+\int_{\Re}\operatorname{div}(\mathbf{x}\times\mathbf{T})\,dv+\int_{\Re}\mathbf{x}\times\rho\mathbf{b}dv \tag{70}$$

and its corresponding local form is

$$\frac{\partial(\mathbf{x}\times\dot{\mathbf{x}}\rho)}{\partial t}=-\operatorname{div}(\rho(\mathbf{x}\times\dot{\mathbf{x}})\otimes\dot{\mathbf{x}})+\operatorname{div}(\mathbf{x}\times\mathbf{T})+\mathbf{x}\times\rho\mathbf{b} \tag{71}$$

The energy balance equation (49) is in terms of volume integrals written as

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{\Re}\rho\left(\varepsilon+\tfrac{1}{2}\dot{x}^2\right)dv &= -\int_{\Re}\operatorname{div}(\varepsilon\rho\dot{\mathbf{x}})\,dv-\int_{\Re}\operatorname{div}\left(\tfrac{1}{2}\dot{x}^2\dot{\mathbf{x}}\right)dv \\
&\quad+\int_{\Re}\operatorname{div}(\mathbf{T}\dot{\mathbf{x}})\,dv+\int_{\Re}\operatorname{div}(\mathbf{q})\,dv \\
&\quad+\int_{\Re}\rho\dot{\mathbf{x}}\cdot\mathbf{b}dv+\int_{\Re}\rho r dv
\end{aligned} \tag{72}
$$

where the divergence theorem was used, i.e. the equations (54), (53), (58) and (56). The local form of the balance of energy, therefore, is

$$
\begin{aligned}
\frac{\partial\left(\rho\left(\varepsilon+\frac{1}{2}\dot{x}^2\right)\right)}{\partial t} &= -\operatorname{div}(\varepsilon\rho\dot{\mathbf{x}})-\operatorname{div}\left(\tfrac{1}{2}\dot{x}^2\dot{\mathbf{x}}\right) \\
&\quad+\operatorname{div}(\mathbf{T}\dot{\mathbf{x}})+\operatorname{div}(\mathbf{q}) \\
&\quad+\rho\dot{\mathbf{x}}\cdot\mathbf{b}+\rho r
\end{aligned} \tag{73}
$$

For entropy, i.e. the inequality (50), one obtain

$$
\frac{\partial}{\partial t}\int_{\Re}\rho\eta dv \geq -\int_{\Re}\operatorname{div}(\eta\rho\dot{\mathbf{x}})\,dv-\int_{\Re}\operatorname{div}(\mathbf{q}/\theta)\,dv+\int_{\Re}\rho r/\theta dv \tag{74}
$$

by using (55) and (57), and its local form therefore is

$$
\frac{\partial(\rho\eta)}{\partial t}\geq-\operatorname{div}(\eta\rho\dot{\mathbf{x}})-\operatorname{div}(\mathbf{q}/\theta)+\rho r/\theta \tag{75}
$$

1.10 Summery of the local 'balance' postulates described with spatial time derivatives

The local forms of the balance principles described with the spatial time derivatives $\partial/\partial t$ is summarized below.

The global postulate of balance of mass (17) takes the local form

$$
\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\dot{\mathbf{x}})=0 \tag{76}
$$

where, again, ρ is the mass density, and $\dot{\mathbf{x}}$ is the velocity. The divergence operator is denoted div.

The global postulate of balance of linear momentum (18) takes the local form

$$
\frac{\partial(\rho\dot{\mathbf{x}})}{\partial t}=-\operatorname{div}(\rho\dot{\mathbf{x}}\otimes\dot{\mathbf{x}})+\operatorname{div}(\mathbf{T})+\rho\mathbf{b} \tag{77}
$$

where, again, $\mathbf{T}$ is the stress tensor and $\mathbf{b}$ is the body force.

The global postulate of balance of angular momentum (19) takes the local form

$$\frac{\partial\left(\mathbf{x}\times\dot{\mathbf{x}}\rho\right)}{\partial t}=-\mathrm{div}\left(\rho\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\otimes\dot{\mathbf{x}}\right)+\mathrm{div}\left(\mathbf{x}\times\mathbf{T}\right)+\mathbf{x}\times\rho\mathbf{b} \tag{78}$$

The balance of energy is the local postulate

$$\begin{aligned}\frac{\partial\left(\rho\left(\varepsilon+\frac{1}{2}\dot{x}^2\right)\right)}{\partial t} &= -\mathrm{div}\left(\varepsilon\rho\dot{\mathbf{x}}\right)-\mathrm{div}\left(\tfrac{1}{2}\dot{x}^2\dot{\mathbf{x}}\right) \\ &\quad +\mathrm{div}\left(\mathbf{T}\dot{\mathbf{x}}\right)+\mathrm{div}\left(\mathbf{q}\right) \\ &\quad +\rho\dot{\mathbf{x}}\cdot\mathbf{b}+\rho r\end{aligned} \tag{79}$$

which was obtained from the global postulate of balance of energy (19). Again, ε is the internal energy, $\mathbf{q}$ is the heat flux vector and r is an external heat source, e.g. radiation.

And at last, the second axiom of thermodynamics is the local postulate

$$\frac{\partial\left(\rho\eta\right)}{\partial t}\geq-\mathrm{div}\left(\eta\rho\dot{\mathbf{x}}\right)-\mathrm{div}\left(\mathbf{q}/\theta\right)+\rho r/\theta \tag{80}$$

where η is the entropy density. This form was obtained from the global postulate for entropy, i.e. the inequality (21).

It turns out that yet another version of the balance principles can be obtained, namely a local version similar to the above equations but in terms of material time derivatives. The material time derivative is most often used due to the balance principles taking a more compact and simple form using this type of description.

1.11 Different forms of balance of mass

If $\Re$ is a fixed spatial volume and $\partial\Re$ is the boundary area of the volume $\Re$, the axiom of balance is

$$\frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\dot{\mathbf{x}})=0 \tag{81}$$

compare the derivation in previous sections.

The formula for transforming spatial time derivatives, i.e. $\partial\Gamma/\partial t$, to material time derivatives, i.e. $\dot{\Gamma}$, for an arbitrary, scalar, vector or tensor property Γ, is

$$\dot{\Gamma}\left(\mathbf{x},t\right)=\frac{\partial\Gamma}{\partial t}\left(\mathbf{x},t\right)+\left[\mathrm{grad}\,\Gamma\left(\mathbf{x},t\right)\right]\dot{\mathbf{x}}\left(\mathbf{x},t\right) \tag{82}$$

By setting $\Gamma \equiv \rho$,

$$\dot{\rho} = \frac{\partial \rho}{\partial t} + \mathrm{grad}\, \rho \cdot \dot{\mathbf{x}} \tag{83}$$

The balance of mass (81), described with a spatial time derivative of the mass density, can therefore be written

$$\dot{\rho} - \mathrm{grad}\, \rho \cdot \dot{\mathbf{x}} + \mathrm{div}(\rho \dot{\mathbf{x}}) = 0 \tag{84}$$

by combining (81) and (83). Noting that, by partial derivation the term $\mathrm{div}(\rho \dot{\mathbf{x}})$ can be written with the identity

$$\mathrm{div}(\rho \dot{\mathbf{x}}) = \rho \mathrm{div}(\dot{\mathbf{x}}) + \mathrm{grad}\, \rho \cdot \dot{\mathbf{x}} \tag{85}$$

Therefore, (84) and (85) combines to yield

$$\dot{\rho} + \rho \mathrm{div}(\dot{\mathbf{x}}) = 0 \tag{86}$$

Yet, another useful form expressing balance of mass can be obtained by considering the identities

$$\mathrm{div}\, \dot{\mathbf{x}} = \mathrm{tr}\, (\mathrm{grad}\, \dot{\mathbf{x}}) = \mathrm{tr}\mathbf{L} = \mathrm{tr}\mathbf{D} \tag{87}$$

where $\mathbf{D}$ is the symmetric part of the velocity gradient, i.e.

$$\mathbf{L} = \mathrm{grad}\, \dot{\mathbf{x}}; \quad \mathbf{L} = \mathbf{D} + \mathbf{W} \tag{88}$$

where

$$\mathbf{D} = \tfrac{1}{2} \left(\mathbf{L} + \mathbf{L}^{\mathrm{T}} \right); \quad \text{and} \quad \mathbf{W} = \tfrac{1}{2} \left(\mathbf{L} - \mathbf{L}^{\mathrm{T}} \right) \tag{89}$$

The term $\mathbf{W}$ is the skew-symmetric part of the velocity gradient $\mathbf{L}$.

The expressions (86) and (87) gives

$$\dot{\rho} = -\rho\, \mathrm{tr}\mathbf{D} \tag{90}$$

1.12 Material time derivative description of the Balance of momentum

The local form of balance of linear momentum is

$$\frac{\partial\, (\rho \dot{\mathbf{x}})}{\partial t} = -\mathrm{div}\, (\rho \dot{\mathbf{x}} \otimes \dot{\mathbf{x}}) + \mathrm{div}\, \mathbf{T} + \rho \mathbf{b} \tag{91}$$

compare previous sections, i.e. equation (77). If the rule of differentiating a product the term on the left-hand side of (91) can be rewritten as

$$\frac{\partial\left(\rho\dot{\mathbf{x}}\right)}{\partial t}=\dot{\mathbf{x}}\frac{\partial\rho_a}{\partial t}+\rho\frac{\partial\dot{\mathbf{x}}}{\partial t}=\dot{\mathbf{x}}\frac{\partial\rho}{\partial t}+\rho\ddot{\mathbf{x}}-\rho\left[\mathrm{grad}\ \dot{\mathbf{x}}\right]\dot{\mathbf{x}} \tag{92}$$

where

$$\dot{\Gamma}\left(\mathbf{x},t\right)=\frac{\partial\Gamma}{\partial t}\left(\mathbf{x},t\right)+\left[\mathrm{grad}\,\Gamma\left(\mathbf{x},t\right)\right]\dot{\mathbf{x}}\left(\mathbf{x},t\right) \tag{93}$$

is used with $\Gamma\equiv\dot{\mathbf{x}}$, i.e.

$$\ddot{\mathbf{x}}=\frac{\partial\dot{\mathbf{x}}}{\partial t}+\left[\mathrm{grad}\,\dot{\mathbf{x}}\right]\dot{\mathbf{x}} \tag{94}$$

The term $\mathrm{div}\left(\rho\dot{\mathbf{x}}\otimes\dot{\mathbf{x}}\right)$ in (91) can, further, be rewritten with the identity

$$\mathrm{div}\left(\rho\dot{\mathbf{x}}\otimes\dot{\mathbf{x}}\right)=\dot{\mathbf{x}}\,\mathrm{div}\ \left(\rho\dot{\mathbf{x}}\right)+\rho\left[\mathrm{grad}\ \dot{\mathbf{x}}\right]\dot{\mathbf{x}} \tag{95}$$

If (92) and (95) are used, the local version of the linear momentum for the a:th constituent, i.e. equation (91) becomes

$$\rho\ddot{\mathbf{x}}+\dot{\mathbf{x}}\left[\frac{\partial\rho}{\partial t}+\mathrm{div}\ \left(\rho\dot{\mathbf{x}}\right)\right]=\mathrm{div}\,\mathbf{T}+\rho\mathbf{b} \tag{96}$$

From the balance of mass (76), it is concluded that

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\ \left(\rho\dot{\mathbf{x}}\right)=0 \tag{97}$$

That is, the terms in brackets in (96) cancels due to (97), i.e. the linear momentum in local form can be written as

$$\rho\ddot{\mathbf{x}}=\mathrm{div}\,\mathbf{T}+\rho\mathbf{b} \tag{98}$$

or equally

$$\rho\frac{\partial\dot{\mathbf{x}}}{\partial t}+\rho\left[\mathrm{grad}\,\dot{\mathbf{x}}\right]\dot{\mathbf{x}}=\mathrm{div}\,\mathbf{T}+\rho\mathbf{b} \tag{99}$$

which is the corresponding version of the balance of momentum using the spatial time derivative of the velocity $\dot{\mathbf{x}}$. This expression was obtained by using equation (94).

1.13 Balance of angular momentum gives that the stress tensor is symmetric

The angular momentum is usually used only to show that the stress tensor must be symmetric. This derivation will be performed in this section.

From the previous sections it was shown that the balance of angular momentum could be brought to the form

$$\frac{\partial\left(\mathbf{x}\times\dot{\mathbf{x}}\rho\right)}{\partial t}=-\operatorname{div}\left(\rho\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\otimes\dot{\mathbf{x}}\right)+\operatorname{div}\left(\mathbf{x}\times\mathbf{T}\right)+\mathbf{x}\times\rho\mathbf{b} \tag{100}$$

This equation will now be simplified further.. Consider the identity

$$\begin{aligned}\frac{\partial}{\partial t}\left(\mathbf{x}\times\dot{\mathbf{x}}\rho\right) &= \left(\mathbf{x}\times\dot{\mathbf{x}}\right)\frac{\partial\rho}{\partial t}+\rho\frac{\partial\left(\mathbf{x}\times\dot{\mathbf{x}}\right)}{\partial t} \\ &= \left(\mathbf{x}\times\dot{\mathbf{x}}\right)\frac{\partial\rho}{\partial t}+\rho\,\dot{\overline{\mathbf{x}\times\dot{\mathbf{x}}}}-\rho\left[\operatorname{grad}\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\right]\dot{\mathbf{x}}\end{aligned} \tag{101}$$

where

$$\dot{\Gamma}\left(\mathbf{x},t\right)=\frac{\partial\Gamma}{\partial t}\left(\mathbf{x},t\right)+\left[\operatorname{grad}\Gamma\left(\mathbf{x},t\right)\right]\dot{\mathbf{x}}\left(\mathbf{x},t\right) \tag{102}$$

is used with $\Gamma\equiv\mathbf{x}\times\dot{\mathbf{x}}$, i.e.

$$\dot{\overline{\mathbf{x}\times\dot{\mathbf{x}}}}=\frac{\partial\left(\mathbf{x}\times\dot{\mathbf{x}}\right)}{\partial t}+\left[\operatorname{grad}\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\right]\dot{\mathbf{x}} \tag{103}$$

Equations (100) and (101) combines to yield

$$\begin{aligned}\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\frac{\partial\rho}{\partial t}+\rho\,\dot{\overline{\mathbf{x}\times\dot{\mathbf{x}}}} &= \rho\left[\operatorname{grad}\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\right]\dot{\mathbf{x}} \\ &\quad -\operatorname{div}\left(\rho\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\otimes\dot{\mathbf{x}}\right) \\ &\quad +\operatorname{div}\left(\mathbf{x}\times\mathbf{T}\right)+\mathbf{x}\times\rho\mathbf{b}\end{aligned} \tag{104}$$

The first term on the right hand side of (100) is rewritten with the identity

$$\operatorname{div}\left(\rho\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\otimes\dot{\mathbf{x}}\right)=\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\operatorname{div}\left(\rho\dot{\mathbf{x}}\right)+\rho\left[\operatorname{grad}\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\right]\dot{\mathbf{x}} \tag{105}$$

Combining (104) and (105) to yield

$$\begin{aligned}\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\frac{\partial\rho}{\partial t}+\rho\,\dot{\overline{\mathbf{x}\times\dot{\mathbf{x}}}} &= \rho\left[\operatorname{grad}\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\right]\dot{\mathbf{x}}-\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\operatorname{div}\left(\rho\dot{\mathbf{x}}\right) \\ &\quad -\rho\left[\operatorname{grad}\left(\mathbf{x}\times\dot{\mathbf{x}}\right)\right]\dot{\mathbf{x}} \\ &\quad +\operatorname{div}\left(\mathbf{x}\times\mathbf{T}\right)+\mathbf{x}\times\rho\mathbf{b}\end{aligned} \tag{106}$$

i.e.

$$(\mathbf{x}\times\dot{\mathbf{x}})\,\frac{\partial\rho}{\partial t}+\rho\,\overline{\mathbf{x}\times\dot{\mathbf{x}}}^{\boldsymbol{\cdot}}=-\;(\mathbf{x}\times\dot{\mathbf{x}})\;\mathrm{div}\,(\rho\dot{\mathbf{x}})+\mathrm{div}\,(\mathbf{x}\times\mathbf{T})+\mathbf{x}\times\rho\mathbf{b} \tag{107}$$

This equation is further rearranged to yield

$$(\mathbf{x}\times\dot{\mathbf{x}})\left(\frac{\partial\rho}{\partial t}+\mathrm{div}\,(\rho\dot{\mathbf{x}})\right)+\rho\,\overline{\mathbf{x}\times\dot{\mathbf{x}}}^{\boldsymbol{\cdot}}=\mathrm{div}\;(\mathbf{x}\times\mathbf{T})+\mathbf{x}\times\rho\mathbf{b} \tag{108}$$

The balance of mass is

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\,(\rho\dot{\mathbf{x}})=0 \tag{109}$$

And due to this the balance of angular momentum reduces to

$$\rho\,\overline{\mathbf{x}\times\dot{\mathbf{x}}}^{\boldsymbol{\cdot}}=\mathrm{div}\;(\mathbf{x}\times\mathbf{T})+\mathbf{x}\times\rho\mathbf{b} \tag{110}$$

This equation will be analyzed further to show that the stress tensor $\mathbf{T}$ is symmetric.

Consider the identity

$$\rho\,\overline{\mathbf{x}\times\dot{\mathbf{x}}}^{\boldsymbol{\cdot}}=\rho\mathbf{x}\times\ddot{\mathbf{x}} \tag{111}$$

and also the identity

$$\begin{aligned}\mathrm{div}\;(\mathbf{x}\times\mathbf{T})\;&=\;\mathbf{x}\times\mathrm{div}\;(\mathbf{T})\\ &\quad+(T_{32}-T_{23})\,\mathbf{i}_1+(T_{13}-T_{31})\,\mathbf{i}_2+(T_{21}-T_{12})\,\mathbf{i}_3\end{aligned} \tag{112}$$

Combining (111) and (112) with (110) to yield

$$\begin{aligned}\rho\mathbf{x}\times\ddot{\mathbf{x}}\;&=\;\mathbf{x}\times\mathrm{div}\;(\mathbf{T})\\ &\quad+(T_{32}-T_{23})\,\mathbf{i}_1+(T_{13}-T_{31})\,\mathbf{i}_2+(T_{21}-T_{12})\,\mathbf{i}_3+\mathbf{x}\times\rho\mathbf{b}\end{aligned} \tag{113}$$

Rearrangement of this equation gives

$$\begin{aligned}\mathbf{x}\times(\rho\ddot{\mathbf{x}}-\mathrm{div}\;(\mathbf{T})-\rho\mathbf{b})\;&=\;(T_{32}-T_{23})\,\mathbf{i}_1\\ &\quad+(T_{13}-T_{31})\,\mathbf{i}_2+(T_{21}-T_{12})\,\mathbf{i}_3\end{aligned} \tag{114}$$

Due to the linear balance of momentum, i.e.

$$\rho\ddot{\mathbf{x}}-\mathrm{div}\;(\mathbf{T})-\rho\mathbf{b}=\mathbf{0} \tag{115}$$

Equation (114) simplifies to

$$\mathbf{0} = (T_{32} - T_{23})\,\mathbf{i}_1 + (T_{13} - T_{31})\,\mathbf{i}_2 + (T_{21} - T_{12})\,\mathbf{i}_3 \tag{116}$$

Using the rectangular base vectors $\mathbf{i}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{i}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{i}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$ one obtain

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{32} - T_{23} \\ T_{13} - T_{31} \\ T_{21} - T_{12} \end{bmatrix} \tag{117}$$

from equation (116). That is, the stress tensor

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{118}$$

is symmetric. This fact can be illustrated by writing $T_{32} = T_{23}$, $T_{13} = T_{31}$ and $T_{21} = T_{12}$ or equally

$$\mathbf{T} = \mathbf{T}^{\mathrm{T}} \tag{119}$$

1.14 Summery of the balance of momentum equations: Cauchy's laws of motion

Cauchy's laws of motion is simply the linear momentum equation (98) together with the fact that the stress tensor is symmetric which was shown by using the postulate of angular momentum (110), compare previous sections. That is

$$\rho\ddot{\mathbf{x}} = \operatorname{div}\mathbf{T} + \rho\mathbf{b}; \quad \text{and} \quad \mathbf{T} = \mathbf{T}^{\mathrm{T}} \tag{120}$$

1.15 Balance of energy

The balance of energy can be written as

$$\begin{aligned} \frac{\partial\left(\rho\left(\varepsilon + \frac{1}{2}\dot{x}^2\right)\right)}{\partial t} = & -\operatorname{div}(\varepsilon\rho\dot{\mathbf{x}}) - \operatorname{div}\left(\tfrac{1}{2}\dot{x}^2\dot{\mathbf{x}}\right) \\ & +\operatorname{div}(\mathbf{T}\dot{\mathbf{x}}) + \operatorname{div}(\mathbf{q}) \\ & +\rho\dot{\mathbf{x}}\cdot\mathbf{b} + \rho r \end{aligned} \tag{121}$$

compare previous sections. Again, ε is the internal energy, $\mathbf{q}$ is the heat flux vector and r is an external heat source, e.g. radiation.

The term on the left hand side of (121) can be rewritten as

$$\frac{\partial}{\partial t}\left(\rho\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\right) = \left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\frac{\partial \rho}{\partial t} + \rho\frac{\partial}{\partial t}\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right) \tag{122}$$

The two first terms on the right hand side of (121) can be rewritten, in the same manner, as

$$\begin{aligned} -\mathrm{div}\,\left(\rho\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\dot{\mathbf{x}}\right) &= -\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\mathrm{div}\,\left(\rho\dot{\mathbf{x}}\right) \\ &\quad -\mathrm{grad}\,\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\cdot\rho\dot{\mathbf{x}} \end{aligned} \tag{123}$$

Using

$$\dot{\Gamma}\left(\mathbf{x}, t\right) = \frac{\partial \Gamma}{\partial t}\left(\mathbf{x}, t\right) + \left[\mathrm{grad}\,\Gamma\left(\mathbf{x}, t\right)\right]\dot{\mathbf{x}}\left(\mathbf{x}, t\right) \tag{124}$$

with $\Gamma \equiv \left(\varepsilon + \frac{1}{2}\dot{x}^2\right)$, gives.

$$\rho\dot{\overline{\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)}} = \rho\frac{\partial}{\partial t}\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right) + \rho\,\mathrm{grad}\,\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\cdot\dot{\mathbf{x}} \tag{125}$$

Combining (123) and (125) gives further

$$\begin{aligned} \mathrm{div}\,\left(\rho\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\dot{\mathbf{x}}\right) &= \left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\mathrm{div}\,\left(\rho\dot{\mathbf{x}}\right) + \rho\dot{\overline{\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)}} \\ &\quad -\rho\frac{\partial}{\partial t}\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right) \end{aligned} \tag{126}$$

Consider, also, the mass balance equation, i.e.

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\left(\rho\dot{\mathbf{x}}\right) = 0 \tag{127}$$

this equation is multiplied with $\left(\varepsilon + \frac{1}{2}\dot{x}^2\right)$ to yield

$$\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\frac{\partial \rho}{\partial t} + \left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\mathrm{div}\,\left(\rho\dot{\mathbf{x}}\right) = 0 \tag{128}$$

Combining (126) and (128), yields

$$\mathrm{div}\,\left(\rho\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\dot{\mathbf{x}}\right) = -\frac{\partial}{\partial t}\left(\rho\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)\right) + \rho\dot{\overline{\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)}} \tag{129}$$

replace the two first terms on the right hand side in the energy balance equation (121) with the expression given by (129) to yield the balance of energy in the following form:

$$\rho\overline{\left(\varepsilon + \tfrac{1}{2}\dot{x}^2\right)}^{\boldsymbol{\cdot}} = \text{div}\left(\mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}} - \mathbf{q}\right) + \rho r + \rho\dot{\mathbf{x}} \cdot \mathbf{b} \tag{130}$$

The kinetic energy term $\overline{\left(\frac{1}{2}\dot{x}^2\right)}^{\boldsymbol{\cdot}}$ in (130) is now rewritten as

$$\overline{\left(\tfrac{1}{2}\dot{x}^2\right)}^{\boldsymbol{\cdot}} = \overline{\left(\tfrac{1}{2}\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}\right)}^{\boldsymbol{\cdot}} = \tfrac{1}{2}\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \tfrac{1}{2}\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} = \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \tag{131}$$

and the term div $\left(\mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}}\right)$ in (130) is rewritten with the identity

$$\text{div}\left(\mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}}\right) = \dot{\mathbf{x}}\cdot\text{div}\,(\mathbf{T}) + \mathbf{T}\cdot\text{grad}\,(\dot{\mathbf{x}}) \tag{132}$$

Noting that the velocity gradient is defined as

$$\mathbf{L} = \text{grad}\,(\dot{\mathbf{x}}) \tag{133}$$

This means that the last term in (132) alternatively can be written

$$\mathbf{T}\cdot\text{grad}\,(\dot{\mathbf{x}}) = \mathbf{T} \cdot \mathbf{L} = \text{tr}\mathbf{T}^{\mathrm{T}}\mathbf{L} \tag{134}$$

i.e. the identity (132) takes the form

$$\text{div}\left(\mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}}\right) = \dot{\mathbf{x}}\cdot\text{div}\,(\mathbf{T}) + \text{tr}\mathbf{T}^{\mathrm{T}}\mathbf{L} \tag{135}$$

Insertion of (131) and (135) into (130), yields

$$\rho\dot{\varepsilon} + \rho\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} = \dot{\mathbf{x}}\cdot\text{div}\,(\mathbf{T}) + \text{tr}\mathbf{T}^{\mathrm{T}}\mathbf{L} - \text{div}\,(\mathbf{q}) + \rho r + \rho\dot{\mathbf{x}} \cdot \mathbf{b} \tag{136}$$

A pure rearrangement gives, further

$$\rho\dot{\varepsilon} + \dot{\mathbf{x}} \cdot (\rho\ddot{\mathbf{x}} - \text{div}\,(\mathbf{T}) - \rho\mathbf{b}) = \text{tr}\mathbf{T}^{\mathrm{T}}\mathbf{L} - \text{div}\,(\mathbf{q}) + \rho r \tag{137}$$

And it is recalled that the linear momentum is the expression

$$\rho\ddot{\mathbf{x}} - \text{div}\,(\mathbf{T}) - \rho\mathbf{b} = \mathbf{0} \tag{138}$$

Equation (137) and (138) gives an alternative form of the energy equation, i.e.

$$\rho\dot{\varepsilon} = \text{tr}\mathbf{T}^{\mathrm{T}}\mathbf{L} - \text{div}\,(\mathbf{q}) + \rho r \tag{139}$$

Note also that the symmetric part of the velocity gradient is defined by

$$\mathbf{D} = \tfrac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right) \tag{140}$$

and also that the stress tensor is symmetric, due to the result given from the balance of angular momentum, i.e. $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$. This gives that the trace of the symmetric part of the velocity gradient $\mathbf{D}$ and the trace of the velocity gradient $\mathbf{L}$ is identical, i.e. $\mathrm{tr}\mathbf{L} = \mathrm{tr}\mathbf{D}$. Therefore, the energy balance equation (121) takes the, more simple alternative form

$$\rho\dot{\varepsilon} = \mathrm{tr}\mathbf{TD} - \mathrm{div}\ (\mathbf{q}) + \rho r \tag{141}$$

This equation can also be written with a spatial time derivative of the internal energy ε by replacing $\dot{\varepsilon}$ with $\dot{\varepsilon} = \partial\varepsilon/\partial t + \mathrm{grad}\,\varepsilon \cdot \dot{\mathbf{x}}$.

1.16 Second axiom of thermodynamics

The second axiom of thermodynamics can be written as

$$\frac{\partial}{\partial t}(\rho\eta) \geq -\mathrm{div}\ (\rho\eta\dot{\mathbf{x}}) - \mathrm{div}\ (\mathbf{q}/\theta) + \rho r/\theta \tag{142}$$

compare previous sections.

The first term on the right hand-side of (142) can be rewritten by the identity

$$\mathrm{div}\ (\rho\eta\dot{\mathbf{x}}) = \eta\mathrm{div}\ (\rho\dot{\mathbf{x}}) + \rho\dot{\mathbf{x}}\cdot\mathrm{grad}\ (\eta) \tag{143}$$

and the term on the left hand side of (142) can be differentiated as

$$\frac{\partial}{\partial t}(\rho\eta) = \eta\frac{\partial\rho}{\partial t} + \rho\frac{\partial\eta}{\partial t} \tag{144}$$

That is, the second axiom of thermodynamics (142) can be written

$$\eta\frac{\partial\rho}{\partial t} + \rho\frac{\partial\eta}{\partial t} \geq -\eta\mathrm{div}\ (\dot{\mathbf{x}}) - \rho\dot{\mathbf{x}}\cdot\mathrm{grad}\ (\eta) - \mathrm{div}\ (\mathbf{q}/\theta) + \rho r/\theta \tag{145}$$

where (143) and (144) is used. A pure rearrangement of this equation yields

$$\eta\left(\frac{\partial\rho}{\partial t} + \mathrm{div}\ (\dot{\mathbf{x}})\right) + \rho\frac{\partial\eta}{\partial t} \geq -\rho\dot{\mathbf{x}}\cdot\mathrm{grad}\ (\eta) - \mathrm{div}\ (\mathbf{q}/\theta) + \rho r/\theta \tag{146}$$

And it is noted that the term in the brackets is zero due to the balance of mass, i.e.

$$\frac{\partial \rho}{\partial t} + \text{div}\ (\dot{\mathbf{x}}) = 0 \tag{147}$$

This means that (146) is reduced to

$$\rho \frac{\partial \eta}{\partial t} \geq -\rho \dot{\mathbf{x}} \cdot \text{grad}\ (\eta) - \text{div}\ (\mathbf{q}/\theta) + \rho r/\theta \tag{148}$$

Using

$$\dot{\Gamma}\,(\mathbf{x},t) = \frac{\partial \Gamma}{\partial t}\,(\mathbf{x},t) + [\text{grad}\,\Gamma\,(\mathbf{x},t)]\,\dot{\mathbf{x}}\,(\mathbf{x},t) \tag{149}$$

with $\Gamma \equiv \eta$ gives.

$$\dot{\eta} = \frac{\partial \eta}{\partial t} + \dot{\mathbf{x}} \cdot \text{grad}\ (\eta) \tag{150}$$

Therefore, equation (148) can also be written

$$\rho \dot{\eta} \geq -\text{div}\ (\mathbf{q}/\theta) + \rho r/\theta \tag{151}$$

which is an alternative form of the second axiom of thermodynamics, compare equation (142). Again it is noted that the material time derivative can be replaced by the spatial time derivative by using the transformation $\dot{\eta} = \partial \eta / \partial t + \text{grad}\,\eta \cdot \dot{\mathbf{x}}$.

1.17 Definitions of different thermodynamic properties

As will be shown later, it can be advantageous to use alternative thermodynamic properties than the internal energy ε and the entropy density η.

The *Helmholtz's free energy* ψ is defined by the internal energy ε, the entropy density η and by the temperature θ, as

$$\psi = \varepsilon - \eta \theta \tag{152}$$

The chemical potential tensor $\mathbf{K}$ is defined by the Helmholtz's free energy ψ, the stress tensor $\mathbf{T}$ and by the mass density ρ, as

$$\mathbf{K} = \psi \mathbf{I} - \mathbf{T}^{\mathrm{T}}/\rho \tag{153}$$

It should be carefully noted that the chemical potential tensor $\mathbf{K}$ only reduces to the chemical potential μ, used in classical thermochemistry, when assuming that the stress tensor $\mathbf{T}$ can sustain hydrostatic pressures only (or thermodynamical pressures), i.e.

$$\mathbf{T} = -p\mathbf{I}; \qquad \mathrm{K} = \mu\mathbf{I} \qquad \text{(special case!)} \tag{154}$$

and then the classical chemical potential μ can be expressed as

$$\mu = \psi + p/\rho \qquad \text{(special case!)} \tag{155}$$

1.18 Alternative versions of the Second axiom of thermodynamics

A special useful form of the second axiom of thermodynamics can be obtained by combining the energy equation, i.e. (141), and the second axiom of thermodynamics, written in the form illustrated in (151), by elimination of the external heat source r, which is present in both equations of interest.

The energy equation to be used is

$$\rho\dot{\varepsilon} = \mathrm{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) - \mathrm{div}\ (\mathbf{q}) + \rho r \tag{156}$$

and the second axiom of thermodynamics is written as

$$\rho\dot{\eta} + \mathrm{div}\ (\mathbf{q}/\theta) - \rho r/\theta \geq 0 \tag{157}$$

This equation is rewritten as

$$\frac{1}{\theta}\left(\theta\rho\dot{\eta} + \theta\,\mathrm{div}\ (\mathbf{q}/\theta) - \rho r\right) \geq 0 \tag{158}$$

Noting also that $\mathrm{div}\,(\mathbf{q})$ can be expressed with the identity

$$\mathrm{div}\ (\mathbf{q}) = \mathrm{div}\ (\mathbf{q}\theta/\theta) = \theta\,\mathrm{div}\ (\mathbf{q}/\theta) + \mathrm{grad}\ (\theta)\cdot\mathbf{q}/\theta \tag{159}$$

By insertion of this identity into (158) one obtain

$$\frac{1}{\theta}\left(\theta\rho\dot{\eta} + \mathrm{div}\ (\mathbf{q}) - \mathrm{grad}\ (\theta)\cdot\mathbf{q}/\theta - \rho r\right) \geq 0 \tag{160}$$

The term ρr is according to the balance of energy, i.e. equation (158), equal to

$$\rho r = \rho\dot{\varepsilon} - \mathrm{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) + \mathrm{div}\ (\mathbf{q}) \tag{161}$$

By elimination of ρr by use of (160) and (161), the following inequality is obtained

$$\theta\rho\dot{\eta} - \text{ grad } (\theta) \cdot \mathbf{q}/\theta - \rho\dot{\varepsilon} + \text{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) \geq 0 \tag{162}$$

which is the second axiom of thermodynamics expressed in terms of the rate of change of the internal energy ε and the entropy density η.

Naturally, one can also express the second axiom of thermodynamics with other thermodynamic properties, for example, the Helmholtz's free energy ψ, defined as

$$\psi = \varepsilon - \eta\theta \tag{163}$$

Differentiation gives

$$\dot{\psi} = \dot{\varepsilon} - \dot{\eta}\theta - \eta\dot{\theta} \tag{164}$$

That is, the rate of change of the entropy η, is

$$\dot{\eta} = \frac{\dot{\varepsilon}}{\theta} - \frac{\eta\dot{\theta}}{\theta} - \frac{\dot{\psi}}{\theta} \tag{165}$$

By replacing $\dot{\eta}$ in the inequality (162) the following equation is achieved

$$-\rho\eta\dot{\theta} - \rho\dot{\psi} - \text{grad } (\theta) \cdot \mathbf{q}/\theta + \text{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) \geq 0 \tag{166}$$

which is the in terms second axiom of thermodynamics expressed in terms of the rate of change of the temperature θ and the Helmholtz's free energy ψ.

Yet another form can be obtained if the definition of the chemical potential tensor $\mathbf{K}$, see equation (153), is used. In order to illustrate this form of the second axiom of thermodynamics consider, first, the balance of mass written as

$$\dot{\rho} + \rho\, \text{div}\, \dot{\mathbf{x}} = 0 \tag{167}$$

Noting, also, that $\text{div}\, \dot{\mathbf{x}}$ can be written as

$$\text{div}\, \dot{\mathbf{x}} = \text{tr}\,(\text{grad}\, \dot{\mathbf{x}}) = \text{tr}\mathbf{L} \tag{168}$$

where it should be recalled that the velocity gradient $\mathbf{L}$ is defined as $\mathbf{L} = \text{grad}\, \dot{\mathbf{x}}$. Hence the postulated mass balance (167) can also be written

$$\dot{\rho} = -\rho\, \text{tr}\mathbf{L} \tag{169}$$

Consider, further, the partial differential of $\overline{\rho\psi}^{\cdot}$, i.e.

$$\overline{\rho\psi}^{\cdot}= \rho\dot{\psi} + \dot{\rho}\psi = \rho\dot{\psi} - \rho\psi\,\mathrm{tr}\mathbf{L} \tag{170}$$

where, also, the mass balance (169) is used. The above expression can be written

$$\rho\dot{\psi} =\overline{\rho\psi}^{\cdot} +\rho\psi\,\mathrm{tr}\mathbf{L} \tag{171}$$

The expression for $\rho\dot{\psi}$ in (171) is inserted into the entropy inequality (166) to yield

$$-\rho\eta\dot{\theta}- \overline{\rho\psi}^{\cdot} -\rho\psi\,\mathrm{tr}\mathbf{L} - \mathrm{grad}\,(\theta)\cdot\mathbf{q}/\theta + \mathrm{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) \geq 0 \tag{172}$$

The chemical potential tensor $\mathbf{K}$ is defined as

$$\mathbf{K} =\psi\mathbf{I} - \mathbf{T}^{\mathrm{T}}/\rho \tag{173}$$

i.e.

$$\mathbf{T}^{\mathrm{T}}=\rho\psi\mathbf{I}-\rho\mathbf{K} \tag{174}$$

By replacing $\mathbf{T}^{\mathrm{T}}$ in (172) with the above expression one obtain

$$-\rho\eta\dot{\theta}- \overline{\rho\psi}^{\cdot} -\rho\psi\,\mathrm{tr}\mathbf{L} - \mathrm{grad}\,(\theta)\cdot\mathbf{q}/\theta + \mathrm{tr}\,((\rho\psi\mathbf{I}-\rho\mathbf{K})\,\mathbf{L}) \geq 0 \tag{175}$$

noting also that

$$\rho\psi\,\mathrm{tr}\mathbf{L} =\mathrm{tr}\,((\rho\psi\mathbf{I})\,\mathbf{L}) \tag{176}$$

That is, the expression (175) simplifies to

$$-\rho\eta\dot{\theta}- \overline{\rho\psi}^{\cdot} -\mathrm{grad}\,(\theta)\cdot\mathbf{q}/\theta - \mathrm{tr}\,(\rho\mathbf{K}\mathbf{L}) \geq 0 \tag{177}$$

which is the second axiom of thermodynamics including the thermodynamic property of $\mathbf{K}$. It should be observed that the stress tensor $\mathbf{T}$ is not described explicitly when using this version of the entropy inequality, but rather described with chemical potential tensor $\mathbf{K}$.

1.19 Summery of the Second axiom of thermodynamics, expressed with different thermodynamic properties

The four equivalent forms of the second axiom of thermodynamics discussed in the previous section was:

$$\rho\dot{\eta} + \text{div}\ (\mathbf{q}/\theta) - \rho r/\theta \geq 0 \tag{178}$$

It turned out that this also could be expressed in terms of both the rate of change of internal energy rate ε and η, as

$$\theta\rho\dot{\eta} - \text{ grad }(\theta) \cdot \mathbf{q}/\theta - \rho\dot{\varepsilon} + \text{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) \geq 0 \tag{179}$$

In terms of the rate of change of the Helmholtz's free energy ψ and entropy η, the second axiom of thermodynamics takes the form

$$-\rho\eta\dot{\theta} - \rho\dot{\psi} - \text{grad }(\theta) \cdot \mathbf{q}/\theta + \text{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) \geq 0 \tag{180}$$

The second axiom of thermodynamics including the chemical potential tensor $\mathbf{K}$, is

$$-\rho\eta\dot{\theta} - \dot{\overline{\rho\psi}} - \text{grad }(\theta) \cdot \mathbf{q}/\theta - \text{tr}\,(\rho\mathbf{K}\mathbf{L}) \geq 0 \tag{181}$$

1.20 Alternative versions of the balance of energy, using different thermodynamic properties

Some times it is advantageous to write also the energy equation in terms of different thermodynamical properties depending on which physical problem studied.

The energy equation already discussed in detail is

$$\rho\dot{\varepsilon} = \text{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) - \text{div }(\mathbf{q}) + \rho r \tag{182}$$

Consider, again, the Helmholtz's free energy ψ defined by

$$\psi = \varepsilon - \eta\theta \tag{183}$$

Differentiation gives

$$\dot{\psi} = \dot{\varepsilon} - \dot{n}\theta - \eta\dot{\theta} \tag{184}$$

That is, the rate of change of the internal energy is

$$\dot{\varepsilon} = \dot{\psi} + \dot{n}\theta + \eta\dot{\theta} \tag{185}$$

By insertion of this expression into the balance of energy (182) an alternative version expressing balance of energy is obtained as

$$\rho\dot{\psi} + \rho\dot{n}\theta + \rho\eta\dot{\theta} = \text{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) - \text{div }(\mathbf{q}) + \rho r \tag{186}$$

furthermore energy balance can be expressed with the chemical potential tensor $\mathbf{K}$ defined by

$$\mathbf{K} = \psi \mathbf{I} - \mathbf{T}^{\mathrm{T}}/\rho \tag{187}$$

i.e.

$$\mathbf{T}^{\mathrm{T}} = \rho\psi\mathbf{I} - \rho\mathbf{K} \tag{188}$$

Again, the relation

$$\rho\dot{\psi} = \overline{\rho\psi}^{\cdot} + \rho\psi \,\mathrm{tr}\mathbf{L} \tag{189}$$

is obtained, see previous section. insertion of $\rho\dot{\psi}$ into (186) gives yet another form expressing balance of energy, i.e.

$$\overline{\rho\psi}^{\cdot} + \rho\psi \,\mathrm{tr}\mathbf{L} + \rho\dot{\eta}\theta + \rho\eta\dot{\theta} = \mathrm{tr}\left(\left(\rho\psi\mathbf{I} - \rho\mathbf{K}\right)\mathbf{L}\right) - \mathrm{div}\,(\mathbf{q}) + \rho r \tag{190}$$

Noting also, see previous section, that

$$\rho\psi \,\mathrm{tr}\mathbf{L} = \mathrm{tr}\left(\left(\rho\psi\mathbf{I}\right)\mathbf{L}\right) \tag{191}$$

which means that (190) is simplified to

$$\overline{\rho\psi}^{\cdot} + \rho\dot{\eta}\theta + \rho\eta\dot{\theta} = -\mathrm{tr}\left(\rho\mathbf{K}\mathbf{L}\right) - \mathrm{div}\,(\mathbf{q}) + \rho r \tag{192}$$

1.21 Summery of the alternative versions of the balance of energy

The three equivalent forms of the balance of energy (i.e. the first axiom of thermodynamics) discussed in the previous section was:

$$\rho\dot{\varepsilon} = \mathrm{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) - \mathrm{div}\,(\mathbf{q}) + \rho r \tag{193}$$

and

$$\rho\dot{\psi} + \rho\dot{\eta}\theta + \rho\eta\dot{\theta} = \mathrm{tr}\left(\mathbf{T}^{\mathrm{T}}\mathbf{L}\right) - \mathrm{div}\,(\mathbf{q}) + \rho r \tag{194}$$

and, at last

$$\overline{\rho\psi}^{\cdot} + \rho\dot{\eta}\theta + \rho\eta\dot{\theta} = -\mathrm{tr}\left(\rho\mathbf{K}\mathbf{L}\right) - \mathrm{div}\,(\mathbf{q}) + \rho r \tag{195}$$

More, equivalent, forms can be formulated , for example, by introducing the definition of the thermodynamic property entalpy.

1.22 Summery

In this chapter the following five balance principles was derived

$$\dot{\rho} + \rho \mathrm{div}\,(\dot{\mathbf{x}}) = 0 \tag{196}$$

$$\rho \ddot{\mathbf{x}} = \mathrm{div}\,(\mathbf{T}) + \rho \mathbf{b}; \qquad \mathbf{T} = \mathbf{T}^{\mathrm{T}} \tag{197}$$

$$\rho \dot{\varepsilon} = \mathrm{tr}\,(\mathbf{TD}) - \mathrm{div}\,(\mathbf{q}) + \rho r \tag{198}$$

$$\theta \rho \dot{\eta} - \ \mathrm{grad}\ (\theta) \cdot \mathbf{q}/\theta - \rho \dot{\varepsilon} + \mathrm{tr}\,(\mathbf{TD}) \geq 0 \tag{199}$$

It is important to not that there is more introduced physical properties than introduced balance law's. This means that when a material is studied, supplementary so-called material functions, or equally, constitutive equations must be specified. These functions is associated with a certain material often referred to a class of a material. Of course, a material function describing for example a stress-strain relation (which is a constitutive assumption) for plastic materials and for concrete will be different. It is, however, very important to note that the balance principles for energies and forces discussed in this chapter is completely independent of the characteristics of the material itself.