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1973

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Bengtsson, G. (1973). *Minimal System Inverses for Linear Multivariable Systems*. (Research Reports TFRT-3051). Department of Automatic Control, Lund Institute of Technology (LTH).

Total number of authors:
1

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Report 7305 March 1973
Lund Institute of Technology
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Gunnar Bengtsson

ABSTRACT.

The problem of minimal inverses for linear timeinvariant multivariable systems is formulated and constructively solved in a state space setting. Unknown initial states as well as zero initial states are considered. The spectrum of the minimal inverse is shown to be unique and constructable from the original system without first calculating the whole inverse. This leads to a simple way of introducing the equivalence of "zeroes" in state space terminology.

This paper has been submitted for outside publication. As a courtesy to the intended publisher it should not be widely distributed until after the date of outside publication.

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1. INTRODUCTION.

The concept of system inverses plays an important role in linear system theory. The reason for this is that the inverse system contains much information about the original system such as tracking ability and stabilizability. Many fundamental control and estimation problems are consequently closely related to system inversion. A few examples are decoupling [2, 4, 6, 8, 10], model matching [9, 11] and feedforward control. It has also been shown that systems with unstable inverses can be very difficult to control [12].

For linear time invariant systems with zero initial states the inversion problem can be treated in a completely algebraic fashion using the transfer function description of the system. The inversion problem then becomes a problem of inverting a matrix of rational functions. This can be done [5] for instance using the invariant factor theorem. However, this approach seems to be more or less technical and suffers also from computational difficulties.

Silverman [1] and Silverman and Payne [2] have developed a quite different inversion theory using state space terminology. The inverse system is constructed by means of a certain algorithm, called the structure algorithm, avoiding some of the computational difficulties in the transfer function approach. Moreover, some properties of the inverse system can be extracted from this algorithm [2]. Related work has also been done by Sain and Massey [3]. Quite recently Wonham and Morse [4] gave some necessary and sufficient conditions for left invertibility in terms of a certain invariant subspace.

In this paper the concept of minimal system inverse will be introduced as the inverse dynamical system having the lowest possible order. Minimal system inverses are constructed for systems with arbitrary unknown initial states and zero initial states. Even more interesting is that the order and the spectrum of the minimal inverse can be characterized using properties of the original system without first calculating the whole inverse.

The spectrum of the minimal inverse is shown to be unique in the sense that all minimal inverses have the same spectrum. This is an interesting fact since it leads to a proper definition of the zeroes in the multivariable case. The "zeroes" have a simple and straightforward interpretation in state space terminology.

The paper is organized as follows:

The concept of minimal system inverse is defined in Section 2. Some geometric concepts introduced by Wonham and Morse [8] are the basic mathematical tools. These concepts are introduced in Section 2.
In Section 3 the problem of minimal system inverses will be solved first for arbitrary unknown initial states and later for zero initial states. The concept of inverse spectrum as a state space equivalence to the concept of zeroes is also discussed.

2. PRELIMINARIES.

The notations of [4] are adopted with only smaller modifications. Basic knowledge of linear algebra and linear system theory is assumed, but for completeness some basic concepts and theorems are summarized below. For a more detailed account the reader is referred to [4, 8].

Algebraic Background.

Let $\mathcal{X}$ be a finite dimensional vector space and $A:\mathcal{X} \to \mathcal{X}$ a linear map. If $\mathcal{V} \subset \mathcal{X}$ is a linear subspace then $A\mathcal{V} = \{x \in \mathcal{X} | x = Az, z \in \mathcal{V}\}$ is the image of $\mathcal{V}$ under A and $A^{-1}\mathcal{V} = \{x \in \mathcal{X} | Ax \in \mathcal{V}\}$ is the inverse image of $\mathcal{V}$ under A. A subspace $\mathcal{V} \subset \mathcal{X}$ is said to be A-invariant if $A\mathcal{V} \subset \mathcal{V}$. If $\mathcal{V}$ is A-invariant and V is a basis matrix for $\mathcal{V}$, the restriction of A to $\mathcal{V}$, $A|\mathcal{V}$, is defined by $AV = V\tilde{A}$ and $\tilde{A} = A|\mathcal{V}$ cf.[15].

(A,B)-Invariant Subspaces.

The concept of (A,B)-invariant subspaces was originally introduced by Bassile and Marro [7] and Wonham and Morse [8] in connection with the decoupling problem. A subspace $\mathcal{V} \subset R^n$ is said to be (A,B)-invariant if for some L

$$(A+BL)\mathcal{V} \subset \mathcal{V}$$

It is shown in [7, 8] that a necessary and sufficient condition for $\mathcal{V}$ to be (A,B)-invariant is

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B} \qquad (2.1)$$

where $\mathcal{B}$ denotes the range space of B. Let $\mathcal{D}$ be an arbitrary subspace of R^n. One can show [7, 8], that there exists a unique maximal subspace $\mathcal{V}^M$ contained in a given subspace $\mathcal{D}$, i.e. $\mathcal{V}^M \supset \mathcal{V}$ where $\mathcal{V}$ is any subspace satisfying $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$ and $\mathcal{V} \subset \mathcal{D}$. The subspace $\mathcal{V}^M$ can be constructed according to the following algorithm [7, 8]

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{D} \\ \mathcal{V}_1 &= \mathcal{D} \cap A^{-1}(\mathcal{V}_0 + \mathcal{B}) \\ \mathcal{V}_2 &= \mathcal{D} \cap A^{-1}(\mathcal{V}_1 + \mathcal{B}) \\ &\vdots \end{aligned} \tag{2.2}$$

Let i be the first integer such that $\mathcal{V}_i = \mathcal{V}_{i+1}$, then $\mathcal{V}^M = \mathcal{V}_i$. It can be shown that this algorithm converges after at most ν steps where $\nu = \dim(\mathcal{D})$.

Statement of the Problem.

The class of systems considered consists of all systems S(A,B,C) described by the differential equation

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0 \tag{2.3}$$

$$y = Cx$$

where $x(t) \in R^n$ is the vector of states, $u(t) \in R^m$ is the vector of inputs and $y(t) \in R^p$ is the vector of outputs. A, B and C are linear time invariant maps (matrices). It will be assumed that there are no redundant inputs or outputs in (2.3), i.e. the matrices B and C have full rank. The system S(A,B,C) is assumed to be completely observable. This is no restriction since the system can be reduced modulus the unobservable subspace.

The solution of (2.3) is

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-s)}Bu(s)ds$$

which can be regarded as an input-output map parameterized by the initial state x_0, i.e.

$$y = \theta(x_0,u) = \theta_1(x_0) + \theta_2(u)$$

The input space U consists of all piecewise continuous realvalued m-vector functions on (t_0,∞). The output space Y is defined as the image of $R^n \times U$ under θ, i.e. $Y = \theta(R^n \times U)$. All $y \in Y$ will then be continuous and continuous differentiable up to some finite order σ.

A left inverse to the system (2.3) is any operator $\hat{\theta}$: $Y \to U$ such that

$$\hat{\theta}y = \hat{\theta}\theta(x_0,u) = \hat{\theta}\theta_1(x_0) + \hat{\theta}\theta_2(u) = u \qquad (2.4)$$

for all input-output pairs $(u,y) \in U \times Y$ of (2.3). The inverse is linear and can be described by a dynamical system of order at most n (cf. Remark 2), where n is the dynamical order of S(A,B,C) [1]. The inverse system $\hat{\theta}$ is represented by a dynamical system of the form

$$\begin{aligned} \dot{w} &= \hat{A}w + N_1(p)y \\ u &= \hat{C}w + N_2(p)y \end{aligned} \qquad (2.5)$$

where $N_1(s)$ and $N_2(s)$ are polynomial matrices and $p = \frac{d}{dt}$. This representation of the inverse is assumed in the sequel. The concept of minimal system inverse can now be concisely defined.

Definition. A minimal (left) inverse of S(A,B,C) is any operator $\hat{\theta}$ with representation (2.5) such that the condition (2.4) is satisfied and w in (2.5) is of minimal dimension.

Remark 1. Only the case of left inverses has been considered, i.e. the problem of finding an operator that with y as input produces u as output. The corresponding right inversion problem, i.e. to find some input u to the system to produce a predefined output y, may be more interesting in some control problems. Such an input is produced by the right inverse with the desired output as forcing function. The results of this paper can be extended to the case of right inverses by considering the adjoint system

$$\dot{z} = A^T z + C^T v$$

$$w = B^T z$$

It can be shown [3] that, for zero initial state, the original system is right invertible if and only its adjoint is left invertible. As an illustration consider the transfer-functions $G(s) = C(sI-A)^{-1}B$ and $G^T(s) = B^T(sI-A^T)^{-1}C^T$ and their left and right inverses. Some care must however be taken in defining the appropriate input and output spaces.

Remark 2. By dynamical order we here mean the dimension of the state vector.

3. MINIMAL SYSTEM INVERSES.

In this section the properties of minimal system inverses for left invertible systems with arbitrary unknown initial states and zero initial states will be investigated. In the former case existence conditions are provided, since such conditions do not seem to be known previously. Naturally, the class of invertible systems is much broader in the latter case, which will also be clear from the invertibility conditions.

Systems with Unknown Initial States.

Consider the system S(A,B,C) and assume the initial state is arbitrary and unknown. The inverse shall reproduce the input irrespective of what the initial state is. We can express this in terms of the following conditions on the inverse $\hat{\theta}$ using (2.4)

$$\hat{\theta}\theta_1 = 0 \quad ; \qquad \hat{\theta}\theta_2 = I \tag{3.1}$$

Introduce V^M as the maximal (A,B)-invariant subspace contained in ker(C) (the null space of C). The following lemma will be needed in the proof of the theorem below

__Lemma 1.__ If $V^M = 0$ there are maps $N_i: R^p \rightarrow R^n$, $i = 0,1,..,n$, such that

$$\sum_{i=0}^{n} N_i CA^i = I_n$$

$$\sum_{i=k}^{n} N_i CA^{i-k}B = 0 \quad ; \qquad k = 1,2,..,n$$

__Proof.__ Show first that $V^M = 0$ implies that $\{Q\} \cap \{R\} = 0$

where {˙} denotes the range space and R and Q are the the following block matrices

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^n \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \cdots & & & \\ CA^{n-1}B & \cdots & CAB & CB \end{bmatrix} \tag{3.2}$$

Assume that $\{Q\} \cap \{R\} \neq 0$. There exist vectors x and $r^T = [r_1^T; r_2^T; \ldots; r_n^T]$ such that $Qx = Rr$, i.e. using (3.2)

$$\begin{aligned} &Cx = 0 \\ &CAx - CBr_1 = 0 \\ &CA^2x - CABr_1 - CBr_2 = 0 \\ &\vdots \end{aligned} \tag{3.3}$$

Introduce $v_1 = x$ and $v_{i+1} = Av_i - Br_i$, $i = 1,2,\ldots,n$. The sequence (3.3) then becomes $Cv_i = 0$, $i = 1,2,\ldots, n+1$. Since $Av_i = v_{i+1} + Br_i$ we can write in subspace notations

$$A\{v_i\} \subset \{v_{i+1}\} + \mathcal{B} \; ; \; \{v_i\} \subset \ker(C) \tag{3.4}$$

where $\mathcal{B}$ denotes the range space of B. Now define a sequence of subspaces $\mathcal{V}_k$, $k = 0,1,2,\ldots,n$, by

$$\mathcal{V}_k = \sum_{i=1}^{n-k+1} \{v_i\}$$

Using (3.4):

$$A\mathcal{V}_k = \sum_{i=1}^{n-k+1} A\{v_i\} \subset \sum_{i=1}^{n-k+1} \{v_{i+1}\} + \mathcal{B} \subset \mathcal{V}_{k+1} + \mathcal{B} \tag{3.5}$$

$$\mathcal{V}_k \subset \ker(C)$$

We then have a sequence of subspaces satisfying $0 \neq V_n \subset \subset V_{n-1} \ldots \subset V_0 \subset \ker(C)$. Since V_n is nonzero and ker(C) has dimension at most n-1 it follows that $V_j = V_{j+1}$ for some j. Then from (3.5)

$$AV_{j+1} \subset V_{j+1} + B \quad ; \quad V_{j+1} \subset \ker(C)$$

Thus $V^M \neq 0$ and $\{Q\} \cap \{R\} = 0$ by contradiction. However, by the observability assumption, $\dim(\{Q\}) = n$ and the columns $q_1, q_2, \ldots, q_n$ of Q is a basis for {Q}. Moreover let $r_1, r_2, \ldots, r_s$ be a basis for {R}. Since $\{Q\} \cap \{R\} = 0$ from above, the vectors $q_1, q_2, \ldots, q_n, r_1, r_2, \ldots, r_s$ are linearly independent. This implies that there is a map $N: R^{p(n+1)} \to R^n$ such that $Nq_i = e_i$, where e_i is the i:th unit vector, and $Nr_j = 0$. For this map we have $NQ = I_n$ and $NR = 0$. Partion $N = [N_0\ N_1\ \ldots\ N_n]$ compatibly with the blocks in Q and R. An evaluation of the matrix products $NQ = I_n$ and $NR = 0$ give the sums in the lemma □

The minimal inverse in the case of arbitrary unknown initial state is then charactarized in the following theorem.

Theorem 1. Assume the system S(A,B,C) is completely observable. There exists a left inverse with the property (3.1) if and only if $V^M = 0$. Moreover, if $V^M = 0$ there is a polynomial matrix N(s) such that for all $(u,y) \in U*Y$ and all x_0

$$x = N(p)y \qquad\qquad t \in (t_0, \infty)$$
$$u = \hat{B}(pI-A)N(p)y \qquad\qquad (3.6)$$

where $p = \frac{d}{dt}$ and $\hat{B}$ is a left inverse of B.

Proof. Assume first there is an operator $\hat{\theta}$ with the property (3.1) and $V^M = 0$. Let L_M be a map such that $(A+BL_M)V^M \subset V^M$ and consider the input $u_1 = L_M x$ with $x_0 \in V^M$. Then

$$\dot{x} = (A+BL_M)x \qquad x(t_0) = x_0$$

$$y = Cx \tag{3.7}$$

$$u_1 = L_M x$$

Since $x_0 \in V^M$ and $(A+BL_M)V^M \subset V^M$ it follows that $x(t) \in V^M$ and thus $y(t) = 0$ for $t \geq t_0$. The input u_1 is not identically zero for all $x_0 \in V^M$, since this would imply that $L_M V^M = 0$ and $\ker(C) \supset V^M \supset (A+BL)V^M = AV^M$ and the observability assumption is contradicted. The same output is however produced by $x_0 = 0$ and $u_2 = 0$, and it will be impossible to distinguish between the inputs u_1 and u_2 by observing the output and left invertibility in the sense of (3.1) fails.

Conversely, assume $V^M = 0$. By successive differentiation of y in (2.3) we have using Lemma 1 and the substitution $x(t) = A \int^t x(s)ds + B \int^t u(s)ds + x_0$

$$N_n y = N_n Cx = N_n CA \int x(s)ds + N_n CB \int u(s)ds + N_n Cx_0 =$$

$$= N_n CA \int x(s)ds + N_n Cx_0$$

$$N_n y^{(1)} = N_n CAx$$

$$N_n y^{(1)} + N_{n-1} y = N_n CAx + N_{n-1} Cx = (N_n CA^2 + N_{n-1} CA) \int x(s)ds$$

$$+ (N_n CAB + N_{n-1} CB) \int u(s)ds + (N_n CA + N_{n-1} C)x_0$$

$$= (N_n CA^2 + N_{n-1} CA) \int x(s)ds + (N_n CA + N_{n-1} C)x_0$$

$$N_n y^{(2)} + N_{n-1} y^{(1)} = (N_n CA^2 + N_{n-1} CA)x$$

Proceeding recursively in this way we have after n steps

$$\sum_{i=1}^{n} N_i y^{(i)} = \sum_{i=1}^{n} N_i CA^i x$$

and adding $N_0 y = N_0 Cx$ to either side

$$\sum_{i=0}^{n} N_i y^{(i)} = \sum_{i=0}^{n} N_i CA^i x = x \qquad (3.8)$$

where Lemma 1 has been used once more. The last expression can be written in operator form as $N(p)y = x$ with $N(p) = N_0 + N_1 p + \ldots + N_n p^n$. From (2.3) we have $(pI-A)x = Bu$. Substitute $x = N(p)y$ and multiply from left by $\hat{B}$, where $\hat{B}$ is a left inverse of B. We obtain

$$\hat{B}(pI-A)N(p)y = u \qquad (3.9)$$

The last relation holds for all x_0 and a left inverse in the sense of (3.1) exists. The second statement in the theorem is also proven by (3.8) and (3.9) □

Remark 1. The inverse operator $\hat{\theta} = \hat{B}(pI-A)N(p)$ is obviously in the required minimal form since w in the representation (2.5) has zero dimension. The construction of the operator N(p) can be done as outlined in the proof above.

Remark 2. For systems with one input and output, the condition $V^M = \emptyset$ is equivalent to the condition that the transfer function has no zeroes.

Systems with Zero Initial States.

For zero initial states, the input-output operators of S(A,B,C) and its controllable and observable subsystem are the same. Therefore it is no restriction to assume the system is completely controllable and observable. This property is assumed in the sequel. In this case the inverse shall satisfy $\hat{\theta}\theta_2 = I$, which can be compared with (3.1).

If $\mathcal{V}^M$ is the maximal (A,B)-invariant subspace contained in ker(C), a necessary and sufficient condition for the system to be left invertible in the case of zero initial state is given by [4]

$$\text{i)}\quad \mathcal{V}^M \cap \mathcal{B} = 0 \qquad\qquad \text{ii)}\quad \ker(B) = 0 \tag{3.10}$$

where $\mathcal{B}$ denotes the range space of B. The second condition is here satisfied by assumption.

To construct the minimal inverse it will be convenient to first make the transformation

$$S(A,B,C) \overset{(T,L)}{\rightarrow} S(T^{-1}(A+BL)T,\ T^{-1}B,\ CT) \tag{3.11}$$

with suitable T and L. This transformation is achieved by a state feedback $u = Lx + u_0$ and a coordinate transformation $z = T^{-1}x$.

Let L_M be a map such that $(A+BL_M)\mathcal{V}^M \subset \mathcal{V}^M$. From the invertibility condition (3.10) it can be seen that the whole space can be factorized into independent subspaces as $R^n = \hat{\mathcal{X}} \oplus \mathcal{B} \oplus \mathcal{V}^M$, where $\hat{\mathcal{X}}$ is any extension space. Introduce

$$T_M = [\hat{X}\ B\ V_M] \tag{3.12}$$

where $\hat{X}$, B and V_M are basis matrices for $\hat{\mathcal{X}}$, $\mathcal{B}$ and $\mathcal{V}^M$ respectively. Consider now the transformation (3.11) with (T_M, L_M). Since $\mathcal{V}^M$ is $(A+BL_M)$-invariant and contained in ker(C) the transformed system must take the form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u_0 \tag{3.13a}$$

$$y = [C_1 \quad 0]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

(3.13b)

$$L_M x = L_M T_M z = L_{M1} z_1 + L_{M2} z_2$$

where z_1 and z_2 are given by $x = (\hat{X}B)z_1 + V_M z_2$. Some properties of the system (3.13) are given below

Lemma 2. Consider the system (3.13). There exists a polynomial matrix N(s) such that

$$z_1 = N(p)y$$

(3.14)

$$u = \hat{B}_1(pI-A_{11})N(p)y$$

where $\hat{B}_1$ is a left inverse of B_1 and $p = \frac{d}{dt}$.

Proof. Let V_1^M be the maximal (A_{11},B_1)-invariant contained in $\ker(C_1)$. By the maximal property of V^M it follows that $V_1^M = 0$.

Consider then the system (3.13). Since the initial state $z_0 = 0$, the input-output operator of (3.13) becomes equal to the input-output operator for the subsystem $S(A_{11},B_1,C_1)$ by neglecting the unobservable part, i.e.

$$\dot{z}_1 = A_{11}z_1 + B_1 u_0$$

$$y = C_1 z_1$$

The lemma then follows directly from Theorem 1.

Lemma 3. The pair (L_{M2}, A_{22}) is completely observable.

Proof. If the pair (L_{M2}, A_{22}) is not completely observable there is an A_{22}-invariant subspace $\mathcal{W}$ contained in $\ker(L_{M2})$. If W is a basis matrix for $\mathcal{W}$ this implies that $A_{22}W = WQ$ for some matrix Q and $L_{M2}W = 0$. Introduce $\tilde{A} = T_M^{-1}(A+BL_M)T_M$ and $\tilde{W}^T = [0; W^T]$. From the special block form of $\tilde{A}$ shown in (3.13) it immediately follows that $\tilde{A}\tilde{W} = \tilde{W}Q$. Consider then $V = T_M\tilde{W}$. By some simple manipulations

$$(A+BL_M)V = AV + BL_M T_M \tilde{W} = AV + BL_{M2}W = AV$$

$$(A+BL_M)V = T_M\tilde{A}\tilde{W} = T_M\tilde{W}Q = VQ$$

Thus $AV = VQ$ and $\mathcal{V} = \{V\}$ is A-invariant. Moreover by the form of T_M (3.12) and $\tilde{W}$, $V = T_M\tilde{W} = V_M W$ and thus $\mathcal{V} \subset \mathcal{V}^M$. Since $\mathcal{V}^M \subset \ker(C)$ this implies that $\mathcal{V}$ is an A-invariant contained in ker(C) and the observability assumption is contradicted. □

With these notations the following theorems may now be stated characterizing the minimal inverse for systems with zero initial states.

Theorem 2. Denote the characteristic polynomial of A_{22} by $\alpha_M(s)$. Let $\hat{\theta}$ be an arbitrary left inverse of S(A,B,C) with $x_0 = 0$ and let $\hat{\alpha}(s)$ be the characteristic polynomial of $\hat{A}$ in the representation (2.5). Then $\alpha_M(s)$ divides $\hat{\alpha}(s)$.

Proof. Let $x_1 \in \mathcal{V}^M$. Since the system S(A,B,C) is completely controllable, there exists an input $u_0 \in U$ such that $x(t_1) = x_1$ for some fixed point of time $t_1 > t_0$. Consider then the input

$$u(t) = \begin{cases} u_0(t) & t_0 \leq t < t_1 \\ L_M x(t) & t \geq t_1 \end{cases}$$

where L_M is given by (3.11). Obviously $u \in U$. For $t \geq t_1$ the solution of $S(A,B,C)$ becomes

$$\dot{x} = (A+BL_M)x \qquad x(t_1) = x_1$$

$$y = Cx$$

$$u = L_M x$$

Consider now the transformation $z = T_M^{-1}x$ with T_M as in (3.12). The transformed system is described by (3.13) with $u_0 = 0$ and subject to the initial condition $z(t_1)^T = [0; z_{21}^T]$ since $x(t_1) \in V^M$. Thus for $t \geq t_1$

$$u(t) = L_{M2} e^{A_{22}(t-t_1)} z_{21}; \qquad y(t) \equiv 0$$

However, u is also produced as the output of any left inverse $\hat{\theta}$ with y as input.

Since $y(t) \equiv 0$ for $t \geq t_1$ we have from (2.5)

$$u(t) = \hat{C} e^{\hat{A}(t-t_1)} w(t_1) = \bar{C} e^{\bar{A}(t-t_1)} \bar{w}(t_1)$$

where $(\bar{C},\bar{A})$ denotes the observable subsystem of $(\hat{C},\hat{A})$. It is easy to show that the characteristic polynomial $\bar{\alpha}(s)$ of $\bar{A}$ divides $\hat{\alpha}(s)$. Since z_{21} is an arbitrary ν-vector where $\nu = \dim V^M$ $(x(t_1)$ is arbitrary in $V^M)$ we have derived the following relation between A_{22} and $\bar{A}$

$$L_{M2}e^{A_{22}(t-t_1)} = \bar{C}e^{\bar{A}(t-t_1)}\bar{W} \tag{3.15}$$

for some matrix $\bar{W}$. Introduce the observability matrices

$$Q_1 = \begin{pmatrix} L_{M2} \\ L_{M2}A_{22} \\ \vdots \\ L_{M2}A_{22}^{n-1} \end{pmatrix} \qquad Q_2 = \begin{pmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{pmatrix}$$

where n equals the dimension of $\bar{A}$. A successive differentiation of (3.15) gives

$$Q_1e^{A_{22}(t-t_1)} = Q_2e^{\bar{A}(t-t_1)}\bar{W} \tag{3.16}$$

According to Lemma 2, the pair (L_{M2},A_{22}) is completely observable, i.e. rank$(Q_1) = \nu$. Setting $t = t_1$ in (3.16) we have $Q_1 = Q_2\bar{W}$ and it follows that rank $(\bar{W}) = \nu$. Since the pair $(\bar{C},\bar{A})$ is completely observable Q_2 has a left inverse $\hat{Q}_2$ and $\hat{Q}_2Q_1 = \bar{W}$. Another differentiation of (3.16) gives with $t = t_1$

$$Q_1A_{22} = Q_2\bar{A}\bar{W}$$

Multiply from left by $\hat{Q}_2$

$$\bar{W}A_{22} = \bar{A}\bar{W}$$

From the last expression we conclude that $\mathcal{W} = \{\bar{W}\}$ is $\bar{A}$-invariant and $A_{22} = \bar{A}|\mathcal{W}$. Thus $\alpha_M(s)$ divides $\bar{\alpha}(s)$. Since $\bar{\alpha}(s)$ divides $\hat{\alpha}(s)$, the theorem follows trivially□

Remark. The theorem above sets a lower limit on the dynamical order of any inverse $\hat{\theta}$ of the form (2.5). This limit equals $\deg\left(\alpha_M(s)\right) = \dim(V^M)$.

Theorem 3. Assume the system S(A,B,C) is left invertible. With notations as above a minimal inverse of dynamical order $\nu = \dim(V^M)$ is given by

$$\dot{w} = A_{22}w + N_1(p)y \qquad w(t_0) = 0$$

$$u = L_{M2}w + N_2(p)y$$

where

$$N_1(p) = A_{21}N(p)$$

$$N_2(p) = \left(L_{M1} + \hat{B}_1(pI-A_{11})\right)N(p)$$

and N(p) is given by Lemma 2 and $\hat{B}_1$ is a left inverse of B_1.

Proof. Notice first that it from Theorem 2 follows that the dynamical order ν_0 of any inverse must satisfy $\nu_0 \geq \dim(V^M)$. Let $u \in U$ be an arbitrary input and define u_0 by $u_0 = u - L_M x$. Make the transformation (3.10) with (T_M, L_M). From (3.13)

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u_0 \qquad (3.19)$$

$$y = C_1 z_1$$

$$u = L_{M1}z_1 + L_{M2}z_2 + u_0$$

The input-output operator for this system equals the input-output operator for the subsystem $S(A_{11},B_1,C_1)$ by neglecting the nonobservable part. From Lemma 2

$$N(p)y = z_1$$

$$\hat{B}_1(pI-A_{11})N(p)y = u_0$$

where $N(s)$ is a polynomial matrix and $p = \frac{d}{dt}$. Using (3.19)

$$u = L_{M1}z_1 + L_{M2}z_2 + u_0$$

$$= L_{M2}z_2 + \left(L_{M1} + \hat{B}_1(pI-A_{11})\right)N(p)y \qquad (3.20)$$

where z_2 satisfies

$$\dot{z}_2 = A_{22}z_2 + A_{21}z_1 = A_{22}z_2 + A_{21}N(p)y \qquad (3.21)$$

$$z_2(t_0) = 0$$

Then (3.20) and (3.21) obviously constitutes a left inverse for $S(A,B,C)$▫

The spectrum of the minimal inverse, i.e. the eigenvalues of the matrix A_{22} will satisfy some uniqueness conditions:

<u>Corollary</u>. The spectrum of the minimal inverse is unique and is a subset of the spectrum of any other inverse.

<u>Proof</u>. Follows directly from Theorem 2 and Theorem 3 and the uniqueness of the subspace $\mathcal{V}^M$.

<u>Remark</u>. The minimal inverse may be constructed as outlined above. The computational steps involved consist of

- o calculating a maximal (A,B)-invariant $\mathcal{V}^M$ and an associated map L_M,
- o transforming the system by (3.10),
- o calculating the operator N(p) using the sufficiency part of Theorem 1 and Lemma 2.

The Inverse Spectrum.

The inverse spectrum, or the zeroes in the transfer function case, may now be characterized in simple terms from the system description S(A,B,C). According to Theorem 3, and its corollary, the inverse spectrum for left invertible systems is unique and equals the matrix A_{22} in (3.13) i.e. the spectrum of the map

$$(A+BL_M)|\mathcal{V}^M$$

where $\mathcal{V}^M$ is the maximal (A,B)-invariant subspace contained in ker(C) and L_M is such that $(A+BL_M)\mathcal{V}^M \subset \mathcal{V}^M$. $\mathcal{V}^M$ can be constructed according to the algorithm (2.2). The corresponding result for right invertible systems is obtained via the adjoint system $S(A^T,C^T,B^T)$. Let $\mathcal{V}^M_*$ be the maximal (A^T,C^T)-invariant subspace contained in $\ker(B^T)$ and let K_M be such that $(A+K_MC)^T\mathcal{V}^M_* \subset \mathcal{V}^M_*$. The inverse spectrum in this case is the spectrum of the map

$$(A+K_MC)^T|\mathcal{V}^M_*$$

Remark. Computationally the spectrum of $(A+BL)|\mathcal{V}^M$ can be obtained as the eigenvalues of $V_M^\dagger(A+BL_M)V_M$ where V_M is a basis matrix for $\mathcal{V}^M$ and $(\cdot)^\dagger$ denotes the pseudoinverse [13, 14].

4. CONCLUSIONS.

The problem of minimal system inverses for linear time invariant systems has been formulated and solved for systems with unknown initial states as well as for systems with zero initial states. The basic mathematical tool is some geometric concepts introduced by Wonham and Morse [8] which can be transformed to computational algorithms [4]. This will be devoted to a future paper.

The spectrum of the minimal inverse, or the zeroes in the transfer function case, has been characterized in a simple way from the state space description S(A,B,C) of the system. This means that the "zeroes" have been available as dynamical characteristics in state space synthezis.

5. ACKNOWLEDGEMENTS.

The author wishes to express his great gratitude to Professor Karl-Johan Åström and Tekn.lic. Per Hagander. Their helpful advices and suggestions have been of great value. He also thanks Mrs. Gudrun Christensen who typed the manuscript.

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