# ON THE TURN OFF PHENOMENON IN ADAPTIVE CONTROL.

BJÖRN WITTENMARK

REPORT 7105 SEPTEMBER 1971
LUND INSTITUTE OF TECHNOLOGY
DIVISION OF AUTOMATIC CONTROL

# ON THE TURN OFF PHENOMENON IN ADAPTIVE CONTROL.[†]

B. Wittenmark

ABSTRACT.

"Turn off" is a phenomenon that can occur when using adaptive controllers. The phenomenon means that the control unintentionally can be turned off for longer or shorter periods of time. A zero order system with an unknown gain is studied and it is discussed why the control is turned off and then is turned on again. The phenomenon can be explained through analysing the stability properties of a special non-linear stochastic process. Further is discussed how the turn off phenomenon can be eliminated.

[†] This work has been supported by the Swedish Board of Technical Development under Contract 69-631/U489.

TABLE OF CONTENTS Page

# 1. INTRODUCTION.

When using adaptive controllers a couple of strange phenomena may occur [10]. The phenomena are difficult to analyze because of the complexity of the adaptive systems. But they can often be intuitively explained by the conflict between the identification part and the control part of the adaptive controller. To make a good identification of the unknown process parameters it is desirable to have as large input signals as possible, this in order to get a large signal to noise ratio. But if the purpose with the control is to have minimum variance on the output then one has to compromize in order to get a good overall performance of the system. These thoughts were introduced by Feldbaum [5] as dual control.

One of the phenomena is "burst". By this is meant that the system makes good control over long periods of time, but sometimes the output will be large and oscillating. After this burst the system will again return to normal conditions. The burst phenomenon can be explained by the conflict between identification and control.

This report will discuss and explain another phenomenon. This phenomenon is called "turn off" and is reported when using minimum variance control strategies in adaptive systems [4], [9], [10]. By this is meant that the identification and the control is turned off for long periods of time, but when the controller is not turned off the performance of the system is good.

The purpose with this report is to give an explanation to this phenomenon for a simple system. The turn off phenomenon is further illustrated and a more precise formulation of the problem is given in Section 2. The solution to the problem is obtained through studying the stability properties of a certain nonlinear stochastic process. This is done in Section 3 where also conditions are given when the system can switch between control and no control. In Section 4 an example is given to illustrate the results from Section 3. A discussion of how to avoid the turn off phenomenon is given in Section 5. Finally, in Section 6 is given references.

## 2. FORMULATION OF THE PROBLEM.

The turn off phenomenon has occured when using minimum variance control laws. The general problem is very difficult to analyse because of the nonlinearity of the adaptive system. In order to get some insight into the problem with turn off a simplified case will be discussed. Let the system be

$$y(t) = x(t)u(t-1) + e(t) \tag{2.1}$$

where

$y(t)$ - output signal
$u(t)$ - control signal
$x(t)$ - unknown gain
$e(t)$ - white noise normal distributed $N(0,\lambda)$

The system has no dynamics, it is only characterized by the unknown gain which is a scalar valued stochastic process

$$x(t+1) = ax(t) + v(t) \tag{2.2}$$

where $v(t)$ is white noise normal distributed $N(0,1)$ and $a$ is a known constant greater than zero.

The gain is estimated from the measurements $y(t)$ in real time using an estimator based upon Kalman theory (see e.g. [2]).

$$\begin{cases} \hat{x}(t+1) = a\hat{x}(t) + K(t)\big(y(t) - u(t-1)\hat{x}(t)\big) \\ K(t) = \dfrac{aP(t)u(t-1)}{\lambda^2+P(t)u(t-1)^2} \\ P(t+1) = a^2P(t) + 1 - \dfrac{a^2P(t)^2u(t-1)^2}{\lambda^2+P(t)u(t-1)^2} \end{cases} \tag{2.3}$$

$\hat{x}(t)$ is the estimate of $x(t)$ based upon old inputs and outputs up to and including time $t-1$. $P(t)$ is the variance of the estimation error and is a measure of the accuracy of the estimate.

The purpose with the control is to minimize

$$E\left(y(t) - 1\right)^2 \tag{2.4}$$

with respect to $u(t-1)$, when $u(t-1)$ is allowed to be a function of old inputs $u(t-2)$, $u(t-3)$, ... and outputs $y(t-1)$, $y(t-2)$, ...

The optimal control law is given by [3]:

$$u(t-1) = \frac{\hat{x}(t)}{\hat{x}^2(t) + P(t)} = \frac{1}{\hat{x}(t)} \cdot \frac{\hat{x}(t)^2}{\hat{x}(t)^2 + P(t)} \tag{2.5}$$

Result from a simulation of the process (2.1) when using the control law (2.5) is given in Figure 2.1.

We see that the estimate and control signal for long periods of time are almost zero. During the turn off periods the estimate is of order $10^{-4} - 10^{-7}$. We also see that under these periods the variance is close to its steady state value

$$P_{\infty} = \frac{1}{1 - a^2} = 5.26$$

Notice the rapid decrease in the variance when the control turns on again.

There are now several questions which are of great interest to answer:

- Why does the control switch between the two different behaviours?

- Can the control be turned off for all future?
- Which conditions are sufficient to ensure that the control will be turned on?

The purpose with the report is to answer these questions for the simple system (2.1).

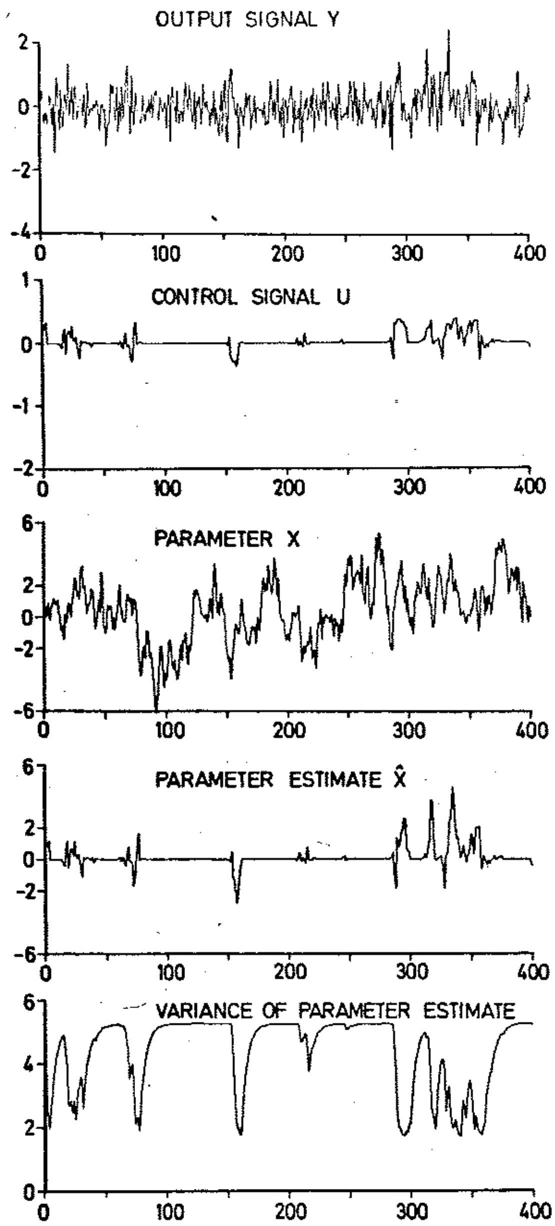


Fig. 2.1 - Result from a simulation of the system described by the equations (2.1), (2.2), (2.3) and (2.5) when $a = 0.9$ and $\lambda = 0.5$. (From [3])

## 3. SOLUTION OF THE PROBLEM.

The equations (2.1) - (2.3) and (2.5) describe a system which is very difficult to analyse in detail. This because of the strong nonlinearity of the equations. We will first give some heuristic arguments which will give some insight into the behaviour of the system.

From Figure 2.1 it seems as if the system jumps between two modes. First when there is a reasonable good tracking of the unknown state variable, $x(t)$, and second when the estimates of $x(t)$ is almost equal to zero. When $\hat{x}(t)$ is small $u(t)$ also will be small (see equation (2.5)) and the control is turned off. How can these two modes be explained? When using Kalman theory the estimation of the state variables is done in such a way that the variance of the estimation error, $P(t)$, is minimized [2]. But we can also interpret the equations in the following way: From all old data is formed an estimate in such a way that the prediction of the output at the next sampling point, $\hat{y}(t)$, is as good as possible. This means that the estimator tries to make the residuals equal to the measurement noise, i.e. to make

$$\varepsilon(t) = y(t) - \hat{y}(t) = y(t) - \hat{x}(t)u(t-1)$$

equal to white noise. Using (2.5) we get

$$\varepsilon(t) = y(t) - \hat{x}(t)u(t-1) =$$

$$= \frac{\hat{x}(t)}{\hat{x}(t)^2 + P(t)}\left(x(t) - \hat{x}(t)\right) + e(t)$$

The residuals can now be "whitened" in two ways. First make a good estimate of $x(t)$, i.e. $\hat{x}(t) \approx x(t)$, then the residuals almost become equal to the measurement noise, $e(t)$. Second put $\hat{x}(t) \approx 0$ then again the residuals become almost equal to $e(t)$. It is these two preferable states that correspond to the turned on respectively the turned off situations. In normal cases when using Kalman filter the turned off situation does not exist. But with the special control law used in this report it is possible to make the control signal almost equal to zero through making the estimate almost equal to zero.

How a turn off may start can be explained by looking at the following situation:

Assume that the estimator makes a good tracking and that the true state, $x(t)$, and then also $\hat{x}(t)$, become small over a longer or shorter interval of time. But if $\hat{x}(t)$ is small then $u(t)$ and $K(t)$ will be small too and the next estimate may be smaller. Then $P(t+1)$ will increase and $u(t+1)$ and $K(t+1)$ will be still smaller and so on. Using this type of arguments we heuristically can explain why a turn off may occur. But why is the control turned on again? The answer to this must be that the turn off is an unstable mode of the Kalman equations. But because of the complexity of the system it is difficult to directly analyse its stability properties.

To simplify the problem we use that during the turn off the estimates are very small, therefore expand the Kalman equations (2.3) around $\hat{x} = 0$. When omitting terms of magnitude $|\hat{x}|^2$ (2.3) is reduced to

$$\hat{x}(t+1) = a\left(1 + e(t)/\lambda^2\right)\hat{x}(t) \qquad (3.1)$$

This is a simpler, but still nonlinear stochastic process. The problem to explain why the control is turned on again is thus reduced to analyse the stability properties of the special stochastic process (3.1).

How to define the stability of stochastic processes? There are several ways to do this. The processes may be stable in probability, in mean square, in norm or with probability one (w p 1). In this case we have to use stability with probability one which is the strongest conception and the most difficult one to handle. We will use the following definitions:

Definition 1 [7]:

The origin is stable with probability one if and only if for any $\rho > 0$ and $\varepsilon > 0$ there is a $\delta(\rho,\varepsilon) > 0$ such that if $||x(0)|| < \delta(\rho,\varepsilon)$ then

$$P(\sup_{0 \leq t < \infty} ||x(t)|| \geq \varepsilon) \leq \rho$$

Definition 2:

The origin is unstable with probability one if and only if for any $\varepsilon > 0$ and $||x(0)|| > 0$

$$P(\sup_{0 \leq t < \infty} ||x(t)|| \geq \varepsilon) = 1$$

These definitions suit our purposes. We want to determine whether a realization of (3.1) can give large, positive or negative, values, because in that case the control will be turned on again. More explicitly we are interested in probabilites such as

$$P_N(t) = P(|\hat{x}(t)| > N) \tag{3.2}$$

and

$$M_N(t) = P(\sup_{0 \leq s \leq t} |\hat{x}(s)| > N) \tag{3.3}$$

We will now state

Theorem 1:

The solution $\hat{x}(t) = 0$ to the stochastic process

$$\hat{x}(t+1) = a\left(1 + e(t)/\lambda^2\right)\hat{x}(t), \quad e \in N(0,\lambda) \tag{3.1}$$

is stable with probability one if and only if

$$m = E \log \left|a\left(1 + e(t)/\lambda^2\right)\right| < 0 \tag{3.4}$$

To prove the theorem we will use

Lemma:

The process $z(t)$ defined by

$$z(t+1) = z(t) + \log \left|a\left(1 + e(t)/\lambda^2\right)\right| \tag{3.5}$$

is asymptotically normal with mean equal to mt and variance $\sigma^2 t$, where m and $\sigma^2$ are defined by (3.4) and

$$\sigma^2 = \mathrm{Var}\left(\log \left|a\left(1 + e(t)/\lambda^2\right)\right|\right) \tag{3.6}$$

Proof of Lemma:

If m and $\sigma$ exist the central limit theorem will prove the lemma as z(t) is a sum of independent equally distributed random variables. It is thus sufficient to show that $|m| < \infty$ and $E\left(\log|a(1 + e(t)/\lambda^2)|\right)^2 = \sigma^2 + m^2 < \infty$.

$$m = E \log|a(1 + e(t)/\lambda^2)| =$$

$$= \frac{1}{\sqrt{2\pi} \cdot \lambda} \cdot \int_{-\infty}^{\infty} \log\left\{|a(1 + s/\lambda^2)|\right\} e^{-s^2/2\lambda^2} ds =$$

$$= \log|a| + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \log\left\{|1 + x/\lambda|\right\} e^{-x^2/2} dx =$$

$$= \log|a| + \frac{\lambda}{\sqrt{2\pi}} \int_{0}^{\infty} \log x \cdot \left[e^{-\lambda^2(x+1)^2/2} + e^{-\lambda^2(x-1)^2/2}\right] dx$$

The only crucial part of the integral is when x = 0, but the integral

$$\int_0^{\varepsilon} \log x \cdot e^{-x^2(x+a)^2} dx \leqslant \int_0^{\varepsilon} \log x dx$$

is bounded and thus it is clear that $|m| < \infty$.

In the same way it can be shown that $\sigma^2 + m^2 < \infty$.

Proof of Theorem 1:

Define the new stochastic process

$$z(t) = \log|\hat{x}(t)|$$

then from (3.1)

$$z(t+1) = z(t) + v(t)$$

where

$$v(t) = \log|a(1 + e(t)/\lambda^2)|$$

is a sequence of independent equally distributed random variables. The process z(t) is thus a random walk. As, from the lemma, z(t) is asymptotically normal then

$$P_N(t) = P(|\hat{x}(t)| > N) = P(\log|\hat{x}(t)| > \log N)$$

$$= P(z(t) > \log N) \approx 1 - \Phi\left(\frac{\log N - mt}{\sigma\sqrt{t}}\right)$$

where $\Phi(x)$ is the normal distribution function. This means that if $m > 0$ then $P_N(t) \to 1$ when $t \to \infty$. But as $M_N(t) \geq P_N(t)$ this implies that $M_N(t) \to 1$ as $t \to \infty$. Thus (3.1) is unstable w p 1 if $m > 0$.

To fully prove the theorem we also have to investigate the two cases $m = 0$ and $m < 0$. These two cases can be treated using a theorem from [6 , theorem 1, p 379]. This theorem states that if $m = 0$ then z(t) will oscillate between $+\infty$ and $-\infty$ with probability one and thus the process is unstable. The theorem also states that if $m < 0$ then

$$\sup_t \sum_{s=0}^{t} v(s) = M < \infty$$

exists w p 1. It is then always possible to choose a starting value, $\hat{x}(0)$, such that

$$P\left(\sup_t |\hat{x}(t)| \geq \varepsilon\right) = P\left\{\sup_t \sum^{t} v \geq \log \varepsilon - \log|\hat{x}(0)|\right\} \leq \rho$$

for any $\varepsilon$ and $\rho$. E.g. choose $|\hat{x}(0)| < \delta = \varepsilon \cdot e^{-M}$. Comparing with Definition 1 we thus find that if $m < 0$ then the process is stable with probability one. Thus $m < 0$ is both a necessary and sufficient condition for stability w p 1 for the process (3.1).

Theorem 1 can now be used to understand when and why the control will be turned on again after a turn off. When the control is turned off the estimator can be approximated by the process (3.1) which is independent of the actual value of the gain, $x(t)$. From Theorem 1 we have that if $m \geq 0$ then the process (3.1) is unstable and the magnitude of any realization will be large with probability one. This means that $\hat{x}(t)=0$ is an unstable solution to the estimator if $m \geq 0$. Thus $|\hat{x}|$ will increase and the control will be turned on again. As soon as $u(t)$ is not too small then very quick the Kalman filter will improve the estimate of $x(t)$ and the accuracy will increase rapidly, i.e. that $P(t)$ will decrease rapidly (see Figure 2.1).

Theorem 1 gives a necessary and sufficient condition for stability of the special stochastic process (3.1). It is also possible to derive only necessary conditions for stability w p 1. Using the theory of supermartingales it is possible to derive conditions which are very similar to the conditions in the Liapunov theory for deterministic systems. This approach is therefore often called the stochastic Liapunov function approach [7].

For discrete time processes we have

Theorem 2 (Kushner [8]):

Let $V(x)$ and $h(x)$ be nonnegative scalar-valued functions. Let $V(0) = 0$, $h(0) = 0$, $h(x) > 0$ if $x \neq 0$ and $h(x)$ continuous and nondecreasing. Let $Q$ be the set in which $V(x) \leq q < \infty$ and $V(x) = q$ on the boundary of $Q$. Let in $Q$, with probability one

$$E[V(x(n))|x(n-1)] - V(x(n-1)) = - k(x(n-1)) \leq 0$$

Then for $\lambda' \leq q$

$$P[\sup_n V(x(n)) \geq \lambda'] \leq EV(x(0))/\lambda'$$

For $0 \leq ||x|| \leq r$ let $V(x) \geq h(||x||)$.

Then for $\lambda' \leq r$

$$P[\sup_n h(||x(n)||) \geq \lambda'] \leq P[\sup_n V(x(n)) \geq \lambda']$$

For any $\rho, \lambda' > 0$ let there be a $\delta(\rho,\lambda') > 0$ such that $EV(x_0) < \rho\lambda'$ if $||x_0|| \leq \delta(\rho,\lambda')$. Then the origin is stable w p 1.

Notice that the stochastic Liapunov theory has the same disadvantage as ordinary Liapunov theory in the sense that the choice of Liapunov function is crucial.

It can be interesting to compare the necessary condition for stability of the process (3.1) given by Theorem 2 with the results given by Theorem 1.

To use Theorem 2 the Liapunov function is chosen to

$$V(x) = |x|^r \qquad r > 0$$

Then the necessary condition for stability w p 1 is

$$E(V[x(t+1)]|x(t)) - V(x(t)) =$$

$$= [E|a(1 + e(t)/\lambda^2)|^r - 1]|x(t)|^r \leq 0$$

or

$$E|a(1 + e(t)/\lambda^2)|^r \leq 1$$

It is possible to get an analytic solution for the left hand side

$$E|a(1 + e(t)/\lambda^2)|^r =$$

$$= \left(\frac{a}{\lambda}\right)^r \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\lambda^2/4}\{U(r+0.5,\lambda) \cdot$$

$$\cdot (1 + \sin \pi(r+0.5)) \cdot \Gamma(r+1) + \pi \cdot V(r+0.5,\lambda)\} \tag{3.7}$$

where U and V are parabolic cylinder functions and $\Gamma$ is the gamma function. Values of U, V and $\Gamma$ are found e.g. in [1]. For certain values of r the expression (3.7) is reduced to a much simpler form, e.g.:

$r = 2$

$$E|a(1 + e(t)/\lambda^2)|^2 = a^2(1+1/\lambda^2)$$

$r = 1$

$$E|a(1 + e(t)/\lambda^2)| = a\left(2\Phi(\lambda) - 1 + \frac{1}{\lambda}\sqrt{\frac{2}{\pi}}\, e^{-\lambda^2/2}\right)$$

A smaller value on r will for fixed a ensure asymptotic stability for processes with a smaller value on $\lambda$. But when using the stochastic Liapunov functions it is not possible to say when the process (3.1) becomes unstable.

That $m < 0$ is a stronger condition than (3.7) can be shown using Jensen's inequality.

$$rE \log|x| = E \log|x|^r \leq \log E|x|^r$$

The condition given by Theorem 2 is equivalent to

$$\log E|a(1 + e(t)/\lambda^2)|^r \leq 0$$

and we find that

$$rm \leq \log E|a(1 + e(t)/\lambda^2)|^r$$

Thus for any r, a and $\lambda$ which makes

$$\log E|a(1 + e(t)/\lambda^2)|^r$$

less than zero we also have that $m < 0$.

Using stochastic Liapunov theory it is possible to get an upper bound on $M_N(t)$. From [7 , theorem 3, p 86] we have that

$$M_N(t) \leq \frac{R^t}{N^r} \tag{3.8}$$

when $R = E(|x(t+1)|^r | x(t)) > 1$

The probability $M_N(t)$ is thus limited by

$$P_N(t) \leq M_N(t) \leq \frac{R^t}{N^r} \tag{3.9}$$

The turn off phenomenon for the simple process (2.1) - - (2.3) and (2.5) can thus be explained through analysing an approximation of the estimator when the magnitude of the estimate is small. Depending on the parameters a and $\lambda$ this simplified process is stable or unstable with probability one. If the estimate, $\hat{x}(t)$, once has become small then it can be approximated by (3.1). If m defined by (3.4) is greater than or equal to zero then we know that the system is forced away from $\hat{x} = 0$ and the control will be turned on again. If $m < 0$ the solution $\hat{x} = 0$ is stable solution and the control may be turned off for ever. But even if $m < 0$ there is a possibility that $\hat{x}(t)$ may be large and that the control is turned on. An upper bound for this probability is given by Theorem 2.

## 4. EXAMPLE.

To illustrate the results from the previous section we will use the system

$$\begin{cases} x(t+1) = 0.9\, x(t) + v(t) & v(t) \in N(0,1) \\ y(t) = x(t)u(t-1) + e(t) & e(t) \in N(0,\lambda) \\ u(t-1) = \dfrac{\hat{x}(t)}{\hat{x}(t)^2+P(t)} & \end{cases}$$

where $\hat{x}(t)$ and $P(t)$ are given by equation (2.3).

A simulation of this system was shown in Section 2. In that case $\lambda$ was equal to 0.5 and the control was turned off over rather long intervals of time. During the turn off periods the estimate was of order $10^{-4}$ - $10^{-7}$. Simulations have verified that for small $|\hat{x}|$ the process (3.1) is a good approximation of the estimate $\hat{x}$. (By small $|\hat{x}|$ is in this case meant that $|\hat{x}|$ is less than 0.1 - 0.05.) It can thus be reasonable to investigate the probability that the process (3.1) will increase by a factor $10^4$. If this happens the estimate and the control signal will no longer be small and the control is turned on.

The parameters $m$ and $\sigma$, see equations (3.4) and (3.6), have been obtained through numerical integration over the normal density function. The values of $m$ and $\sigma$ are given in Table 4.1 for different values of $\lambda$.

From Table 4.1 it is seen that the process (3.1) is stable with probability one for values of $\lambda$ greater than about 0.55. This limit is also verified by simulations of the process.

| $\lambda$ | m | $\sigma$ |
|---|---|---|
| 0.50 | 0.0726 | 1.1064 |
| 0.52 | 0.0427 | 1.1057 |
| 0.54 | 0.0146 | 1.1050 |
| 0.56 | -0.0118 | 1.1041 |
| 0.58 | -0.0366 | 1.1033 |
| 0.60 | -0.0600 | 1.1022 |

Table 4.1.

| $\lambda$ | r | | | | |
|---|---|---|---|---|---|
| | 1.00 | 0.50 | 0.25 | 0.10 | 0.05 |
| 0.50 | 1.6120 | 1.1699 | 1.0531 | 1.0131 | 1.0051 |
| 0.52 | 1.5636 | 1.1524 | 1.0452 | 1.0101 | 1.0036 |
| 0.54 | 1.5191 | 1.1361 | 1.0379 | 1.0073 | 1.0022 |
| 0.56 | 1.4783 | 1.1209 | 1.0310 | 1.0046 | 1.0009 |
| 0.58 | 1.4407 | 1.1068 | 1.0245 | 1.0021 | 0.9996 |
| 0.60 | 1.4060 | 1.0937 | 1.0185 | 0.9997 | 0.9985 |

Table 4.2 - $E\{|\hat{x}(t+1)|^r|\hat{x}(t)\}$ for the process (3.1) when a = 0.9.

For the process (3.1) with a = 0.9 values of

$$E\{|\hat{x}(t+1)|^r|\hat{x}(t)\}$$

are given for different values of r and $\lambda$ in Table 4.2. Using r = 0.10 Theorem 2 states that the process is stable w p 1 if $\lambda \geq 0.60$. But using r = 0.05 the same theorem states that $\lambda \geq 0.58$ gives a stable process. It would be possible to use a still smaller r in order to move the stability boundary further against the critical value $\lambda \approx 0.55$. But using stochastic Liapunov theory we do not know if any other Liapunov function would give a still larger stability area.

The values in Table 4.2 can together with (3.8) be used to get an upper bound of $M_N(t)$. Further $P_N(t)$ gives a lower bound of $M_N(t)$. These two bounds of $M_N(t)$ are shown in Figure 4.1. In the figure $P_N(t)$ is approximated with

$$1 - \phi\left(\frac{\log N - mt}{\sigma\pi}\right)$$

for all values of t. It is possible to recursively compute the exact value of $P_N(1)$, $P_N(2)$, ..., but after a few steps there is no greater difference between the true and the approximated values of $P_N(t)$.

In Figure 4.1 is also shown an estimate of $M_N(t)$. This estimate is obtained through repeated simulation of the process (3.1). The estimated $M_N(t)$ gives that for $t \approx 150$ the probability is 0.75 that the process has increased its value by a factor $10^4$.

A couple of simulations of the system have also been done with $\lambda = 0.6$. The simulations were done over 2500 steps. After a short period of time the control was turned off, but it could be turned on again once or twice for a short while whereafter it was turned off definitely. This result is expected since for $\lambda = 0.6$ the process (3.1) is stable w p 1. Theorem 2 then gives an upper bound of $M_N(t)$. Using $\lambda = 0.6$ and $E\{|\hat{x}(t+1)|^r|\hat{x}(t)\}$ with $r = 0.1$ from Table 4.2 we get

$$P\left(\sup_t|\hat{x}(t)| \geq 10^4|\hat{x}(0)|\right) =$$

$$= P\left(\sup|\hat{x}(t)|^{0.1} \geq 10^{0.4}|\hat{x}(0)|^{0.1}\right) \leq \frac{1}{10^{0.4}} = 0.40$$

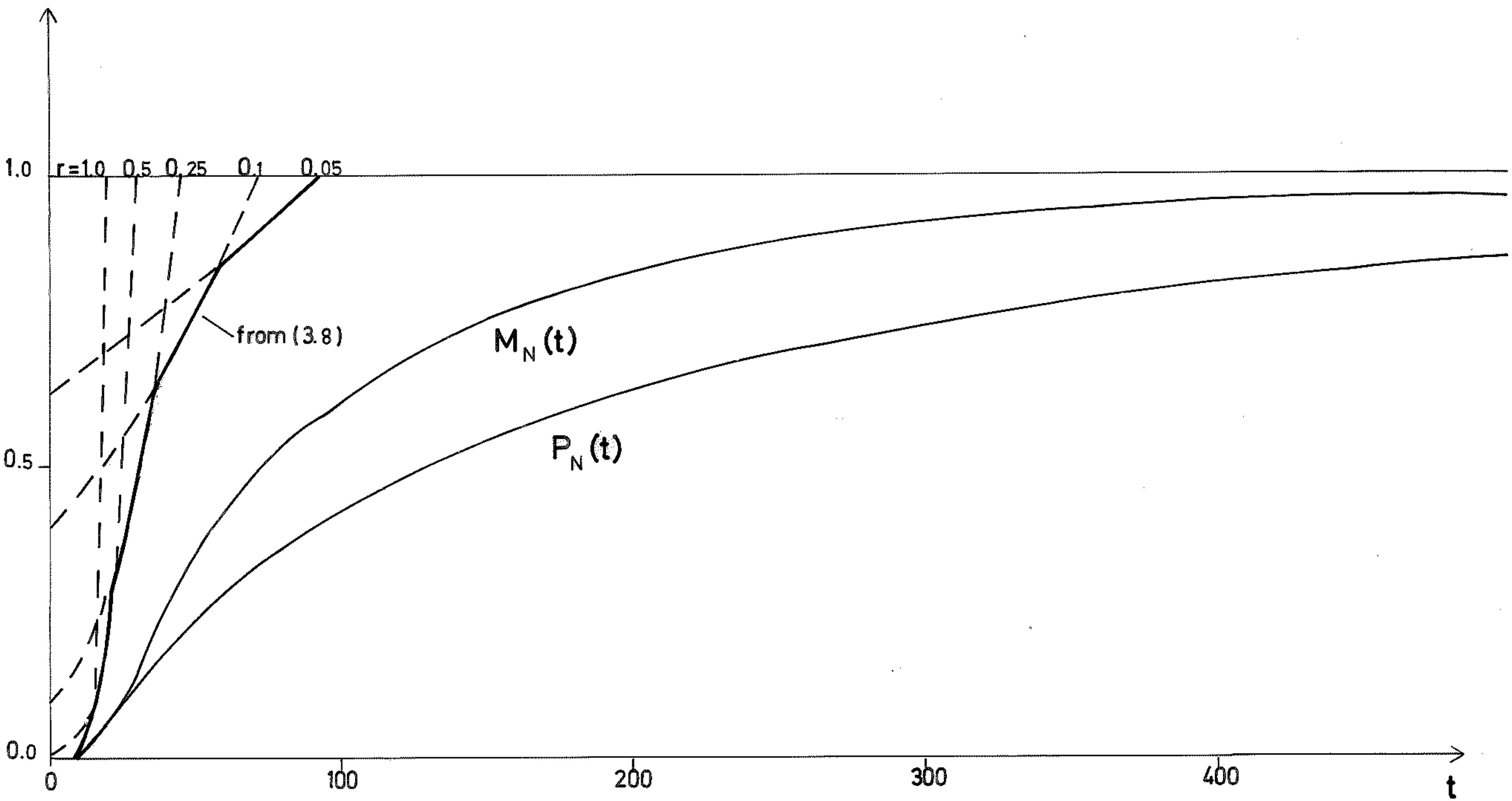


Figure 4.1 - An estimation of $M_N(t)$ obtained by repeated simulation of (3.1) is shown together with the bounds of $M_N(t)$ given by (3.9) when $a = 0.9$, $\hat{x}(0) = 1$, $\lambda = 0.5$ and $N = 10^4$. $P_N(t)$ is approximated by $1 - \Phi\left[\frac{\log N - mt}{\sigma\pi}\right]$ for all t.

## 5. DISCUSSION.

The report has shown that for a simple system the turn off phenomenon can be explained through investigation of a certain nonlinear stochastic process. Necessary and sufficient conditions for this process to be stable with probability one are given. If the process is unstable the control will be turned on again with probability one. But also if the process is stable there is a probability greater than zero that the control may be turned on again. Even if the control will be turned on again with probability one it can be an unsatisfactory behaviour of the system that the control is switched between on and off. We will therefore give a few comments on how to avoid the turn off phenomenon.

In Section 3 it was mentioned that it is the special type of control law that can give rise to the two preferable modes of the Kalman equations. The control law was derived through minimization of

$$E\big(y(t) - 1\big)^2 \tag{2.4}$$

With this loss function the control law only acts to reduce the predicted error one step ahead. But it does not make any attempt to get better estimates in order to be able to make a better control in future steps. This type of control is called non-dual. To get a dual control law 5 the loss function must be chosen in such a way that the minimization is done over several steps. Such a loss function is

$$\frac{1}{N} E \sum_{t=1}^{N} \big(y(t) - 1\big)^2 \tag{5.1}$$

By dual control is meant that the controller acts in a two-fold way. First it makes the control action which shall minimize the loss. Second it also makes the control in such a way that the estimation of the process parameters is as good as possible.

When using the loss function (5.1) the solution leads to a functional equation which has to be solved using dynamic programming. This reduces the number of systems for which the optimal controller can be derived. The dimension of the dynamic programming problem is increasing with $n^2$, where n is the order of the system [3].

In [10] the functional equation is solved for the simple process described by (2.1) and (2.2). Simulations show that the behaviour of the system is widely different when using the dual controller than when using the non-dual controller (2.5) (see [3] or [10] ). Especially there is no turn off of the control and the estimate of the gain is good all the time. The dual controller thus ensures that the estimate is good and then it is easy to make a good control action.

The disadvantage with the optimal dual controller is the time consuming solution of the functional equation. But it is possible to make a suboptimal dual controller. This can be done using the following arguments: if the estimate of the unknown process parameters is accurate then a good control law is obtained through minimization of (2.4). A way to ensure good estimates is to superimpose a small perturbation signal. This perturbation may be a square wave or a random signal, e.g. a pseudo random binary sequence. The perturbation can be used all the time or only when the estimate is poor. In the latter case the variance, P(t), can be used to determine when the perturbation signal shall be switched on. This simple type of suboptimal dual controllers have a good effect on the discussed system. Let the control law be

$$u(t-1) = \frac{\hat{x}(t)}{\hat{x}(t)^2+P(t)} + (-1)^t \cdot \delta \qquad (5.2)$$

then the effect of the perturbation signal can be seen in Figure 5.1, where the accumulated loss is shown for the three discussed control laws. When using (2.5) the control becomes only slightly better than making no control at all. But when using the control law (5.2) the loss is reduced almost as much as can be done with the optimal dual control law. The control law (5.2) is thus a very simple way to eliminate the turn off of the control.

In this report the turn off has only been discussed for a very simple system. But the turn off phenomenon can also occur when controlling more complex systems. It is then more difficult to analyse what happens when the control is turned off, but the principle behaviour seems to be the same. If a higher order system has a tendency to turn off the control it is impossible, because of the immense computations to avoid the turn off by deriving an optimal dual control law. A more practical way is instead to use a perturbation signal of small amplitude. The perturbation signal will ensure that the process parameters are satisfactorily estimated all the time. Thus in order to avoid the turn off phenomenon it can be justified to introduce extra signals into the system. But the extra loss introduced by the perturbation signal is compensated by the better control that is possible to do when the parameter estimates are good.

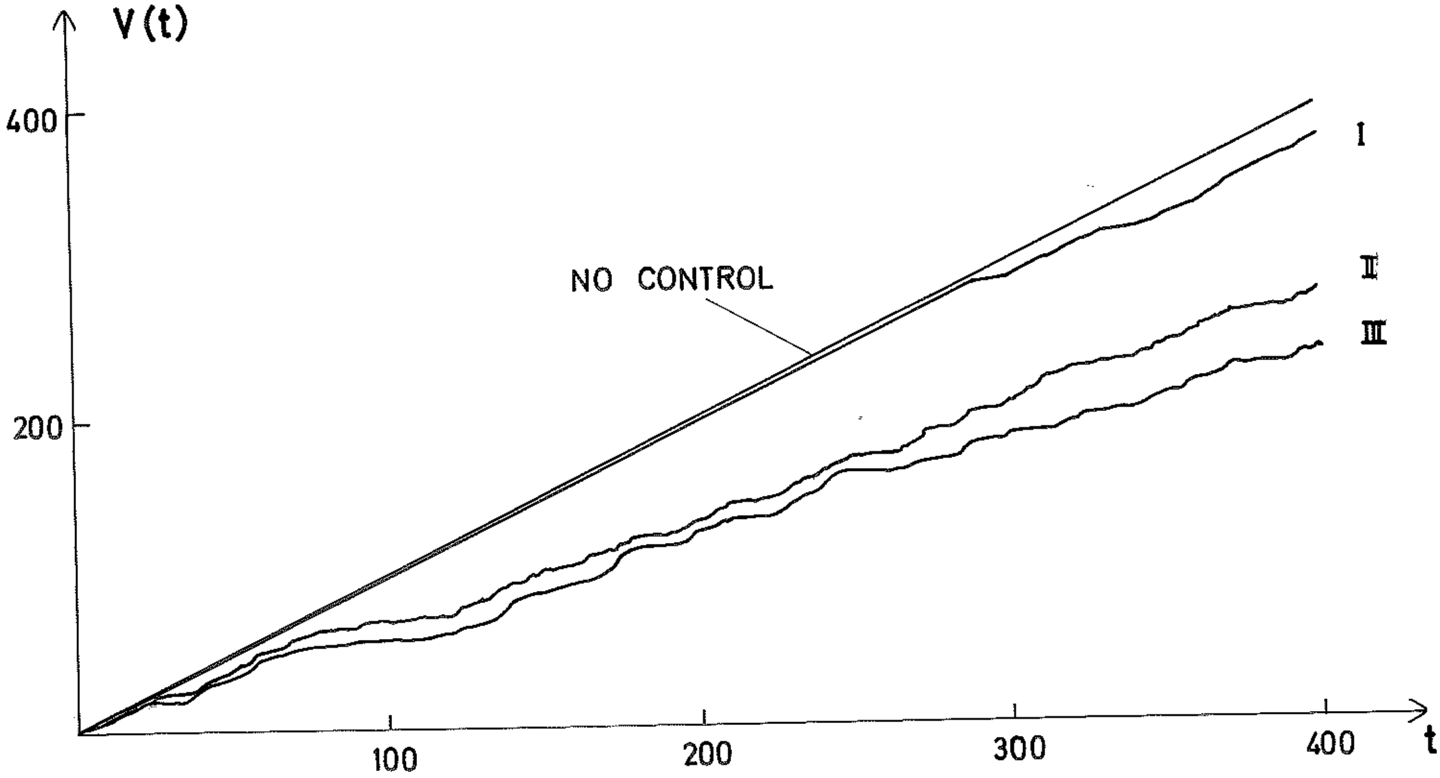


Figure 5.1 - Accumulated loss

$$V(t) = \sum_{s=1}^{t} \left(u(s-1)x(s) - 1\right)^2$$

for

I $\quad u(t-1) = \dfrac{\hat{x}(t)}{\hat{x}(t)^2+P(t)}$

II $\quad u(t-1) = \dfrac{\hat{x}(t)}{\hat{x}(t)^2+P(t)} + (-1)^t \cdot 0.15$

III optimal dual control law (see [9])

## 6. REFERENCES.

[1] Abramowitz M, Stegun I (Editors): Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series 55, 1968.

[2] Åström K J: Introduction to Stochastic Control Theory, Academic Press, 1970.

[3] Åström K J, Wittenmark B: Problems of Identification and Control, Journal of Mathematical Analysis and Applications, Vol 34, No 1, April 1971, pp 90-113.

[4] Bohlin T: Optimal Dual Control of a Simple Process with Unknown Gain, IBM Nordic Laboratory, TP 18-196, 1969.

[5] Feldbaum A A: Dual Control Theory, I-IV, Automation and Remote Control, Vol. 21, No. 9, April 1961; Vol 21, No. 11, May, 1961; Vol 22, No. 1, Jan.,1962; Vol 22. No. 2, Febr., 1962.

[6] Feller W: An Introduction to Probability Theory and Its Applications, Vol II, Wiley, 1966.

[7] Kushner H J: Stochastic Stability and Control, Academic Press, 1967.

[8] Kushner H J: On the Stability of Stochastic Dynamical Systems, Proc. Natl. Acad. Sci. 53, p 8-12, 1965.

[9] Wieslander J, Wittenmark B: An Approach to Adaptive Control Using Real-Time Identification, IFAC Symposium on Identification and Process Parameter Estimation, Prague, 1970.

[10] Wittenmark B: On Adaptive Control of Low Order Systems, Report 6918, Lund Institute of Technology, Division of Automatic Control, 1969.