An Algebraic Test for Positive Realness

Molander, Per

1976

JFRT-3140

AN ALGEBRAIC TEST FOR POSITIVE REALNESS

P. MOLANDER

Report 7644 (C) September 1976
Department of Automatic Control
Lund Institute of Technology

[redacted]

AN ALGEBRAIC TEST FOR POSITIVE REALNESS

Per Molander[†]

Abstract

This study presents an algebraic test for positive realness of transfer functions and transfer matrices. A simple application to a problem that arises in identification is given.

[†] Department of Automatic Control, Lund Institute of Technology, Box 725, S-220 07 Lund 7, Sweden

1. Introduction

Def ([1], pp 57,59) A rational function $P(s)/Q(s)$ is termed positive real (PR) if

i) $P(s)/Q(s)$ is real when s is real

ii) $\mathrm{Re}\, P(s)/Q(s) \geq 0$ for all s with $\mathrm{Re}\, s \geq 0$

It is strictly positive real (SPR) if $P(s-\varepsilon)/Q(s-\varepsilon)$ is PR for some $\varepsilon > 0$. □

An equivalent definition for functions with real coefficients, which is easier to handle, is

Def ([1], p 59) $P(s)/Q(s)$ is PR if

i) it is analytic in the open right half-plane

ii) $\mathrm{Re}\, P(i\omega)/Q(i\omega) \geq 0$ for all real ω

It is SPR if the inequality in ii) is strict. □

Since condition i) may be verified by means of an ordinary Routh or Hurwitz test, we will only treat ii).

2. The Algorithm

The simpler case of strict positive realness will be treated first. Since $\mathrm{Re}\, P(i\omega)/Q(i\omega)$ is proportional to $\mathrm{Re}\, P(i\omega)Q(-i\omega)$, testing if a given transfer function is SPR is equivalent to verifying if the polynomial $\mathrm{Re}\, P(i\omega)Q(-i\omega)$ has no zeros on the real axis (and, of course, that it is > 0 for some ω). This is an old problem, first solved by Sturm. Here, only an outline of the proof will be given; for a more detailed account, see [2], pp 172-176.

Def The Cauchy index of a rational function $R(x)$ on the interval $[a,b]$, denoted $I_a^b R(x)$, is the difference between the number of jumps from $-\infty$ to $+\infty$ and that of the jumps from $+\infty$ to $-\infty$ between a and b. □

In particular, if $\varphi(x)$ is a polynomial with the real roots α_i, $i = 1,\ldots, m$, of multiplicity n_i, that is,

$$\varphi(x) = \tilde{\varphi}(x) \prod_{i=1}^{m} (x-\alpha_i)^{n_i}; \quad \tilde{\varphi}(x) \neq 0, \quad x \in R$$

then

$$\varphi'(x)/\varphi(x) = \sum_{i=1}^{m} \frac{n_i}{x-\alpha_i} + \frac{\tilde{\varphi}'(x)}{\tilde{\varphi}(x)},$$

so that $I_a^b \dfrac{\varphi'(x)}{\varphi(x)}$ equals the number of distinct real zeros of $\varphi(x)$ between a and b.

<u>Def</u> A sequence of polynomials $\psi_i(x)$, $i=1,\ldots,n$, is called a <u>Sturm chain</u> on $[a,b]$ if it satisfies

i) if $\psi_i(x) = 0$, $x \in [a,b]$, then $\psi_{i-1}(x)\psi_{i+1}(x) < 0$, $i = 2,\ldots,n-1$.

ii) $\psi_n(x) \neq 0$, $x \in [a,b]$. □

An easy argument shows that

$$I_a^b \frac{\psi_2(x)}{\psi_1(x)} = V(a) - V(b),$$

where $V(x)$ is the number of sign changes along the Sturm chain at x. Our problem will thus be solved if we can generate a Sturm chain starting from $\varphi(x)$ and $\varphi'(x)$. This may be achieved by means of the Euclidean algorithm, if the remainder in each division is given a minus sign:

$$\varphi_0(x) = \varphi(x), \quad \varphi_1(x) = \varphi'(x)$$

$$\varphi_0(x) = q_1(x)\varphi_1(x) - \varphi_2(x)$$

$$\ldots$$

$$\varphi_{k-1}(x) = q_k(x)\varphi_k(x) - \varphi_{k+1}(x)$$

$$\ldots$$

$$\varphi_{m-1}(x) = q_{m-1}(x)\varphi_m(x)$$

One special feature about Re $P(i\omega)Q(-i\omega) = S(\omega)$ is that it is an even polynomial in ω. As a consequence, the division above may be performed using a Routh scheme. To fit the problem into an ordinary Routh procedure, it is only necessary to change the sign of the coefficients in the columns of even index.

Since we are only interested in the number of sign changes at $\pm\infty$, it is sufficient to determine the sign of the leading term in each member of the Sturm chaing, i.e. the signs of the coefficients in the first column.

The algorithm may thus be summarized as follows:

Algorithm

i) Given $P(s)/Q(s)$, check that it is analytical in the ORHP and that $P(0)/Q(0) > 0$.

ii) Form $S(\omega) = \text{Re } P(i\omega)Q(-i\omega)$ and $S'(\omega)$.

iii) Use the coefficients of $S(\omega)$ and $S'(\omega)$ to build up the first two rows of the Routh scheme, remembering that the sign of the coefficients in columns of even index should be changed. Complete the Routh scheme as usual.

iv) Denote the coefficients of the first column by c_i, $i = m, m-1, \ldots, 0$. Then $V(-\infty)$, the number of sign changes at $-\infty$, equals the number of sign changes in the sequence $\left((-1)^m c_m, (-1)^{m-1} c_{m-1}, \ldots, (-1)' c_1, c_0\right)$, and $V(\infty)$ equals the number of sign changes in $(c_m, c_{m-1}, \ldots, c_1, c_0)$.

v) If $V(-\infty) = V(\infty)$, $P(s)/Q(s)$ is strictly positive real; if $V(-\infty) > V(\infty)$ it is not even positive real.

If $P(s)/Q(s)$ is PR but not SPR, $S(\omega)$ will have multiple real zeros, which are also zeros of $S'(\omega)$, and the Routh scheme collapses. This case may be taken care of by differentiation with respect to ω; for the details, see [2], pp 181-185.

Example $P(s)/Q(s) = (s^2 + 6s + 8) / (s^2 + 4s + 3)$

i) $Q(s)$ is strictly Hurwitz, and $P(0)/Q(0) = 8/3 > 0$.

ii) $P(i\omega)Q(-i\omega) = S(\omega) = \omega^4 + 13\omega^2 + 24$

$$S'(\omega) = 4\omega^3 + 26$$

iii) The Routh scheme is formed (remember the sign changes):

$i = 4$	1	-13	24
3	4	-26	
2	$\frac{-13}{2}$	24	
1	$\frac{-146}{13}$		
0	24		

iv) $V(-\infty)$ equals the number of sign changes in the sequence $\left((-1)^4 \cdot 1,\ (-1)^3 \cdot 4,\ (-1)^2 (-\frac{13}{2}),\ (-1)^1 (-\frac{146}{13}),\ 24\right)$, i.e. 2.

$V(\infty)$: $(1,\ 4,\ -\frac{13}{2},\ -\frac{146}{13},\ 24)$ gives 2 sign changes.

v) $V(-\infty) = V(\infty)$, i.e. $P(s)/Q(s)$ is SPR.

3. The Multivariable Case

In the multivariable case, the inequality to be verified (apart from the analyticity) is

$$\text{He } G(i\omega) \triangleq \frac{1}{2} \left(G(i\omega) + G^T(-i\omega)\right) \geq 0, \quad \omega \in R$$

This may be done by testing the signs of all principal minors; if these are all non-negative, He $G(i\omega)$ is positive semidefinite. If strict positive realness is required, it is sufficient to check that all the leading principal minors are positive. Since all this amounts to verifying that certain polynomials in ω have no real zeros, the algorithm developed for the SISO case may be applied in this case as well.

4. The Application

In the identification of the coefficients of a polynomial $P(s)$, it is necessary in certain algorithms [3] to require that the initial estimate $\hat{P}(s)$ satisfies $\hat{P}(s)/P(s)$ PR in order to guarantee that the algorithm converges. Since $P(s)$ is not known, this cannot be guaranteed for any choice of $\hat{P}(s)$. However, certain general indications may be formulated regarding an optimal choice of the zeros of $\hat{P}(s)$. The analysis will be carried through in the second order case.

Assuming that $P(s) = s^2 + as + b$, $\hat{P}(s) = s^2 + \hat{a}s + \hat{b}$, the ratio $\hat{P}(s)/P(s)$ is PR if the polynomial $s(\omega) = \omega^4 + \omega^2(a\hat{a} - b - \hat{b}) + b\hat{b}$ has no real zeros. This is equivalent to requiring that not both of the following be satisfied:

i) $a\hat{a} - \hat{b} - b < 0$

ii) $(a\hat{a} - \hat{b} - b)^2 > 4b\hat{b}$

The prohibited area is shown in the figure for a typical case.

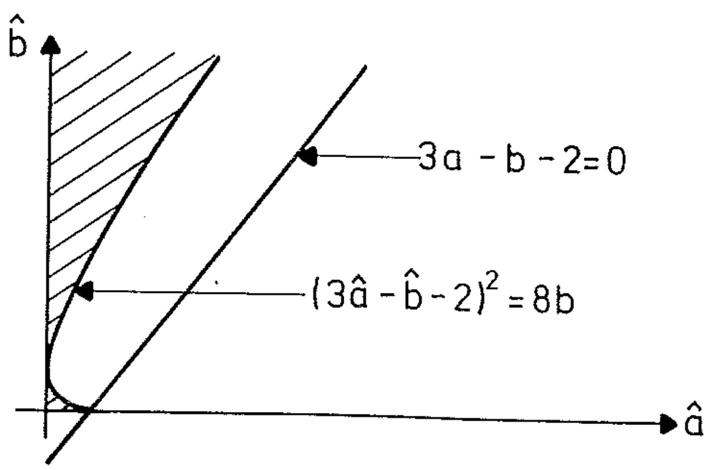

Fig 1. The prohibited area in the parameter space ($a = 3$, $b = 2$).

It can be inferred that $\hat{a}$ and $\hat{b}$ should be chosen large and small respectively, which implies that the zeros of $\hat{P}(s)$ are located far apart.

It is a well-known fact that a rational function $P(s)/Q(s)$ is PR if its poles and zeros are interlacing (in fact, such a transfer function may be realized by an RL-net). By spreading the zeros of $\hat{P}(s)$ instead of clustering them, one increases the probability that such a configuration, or one close to it, crises (loosely speaking). See the figure below.

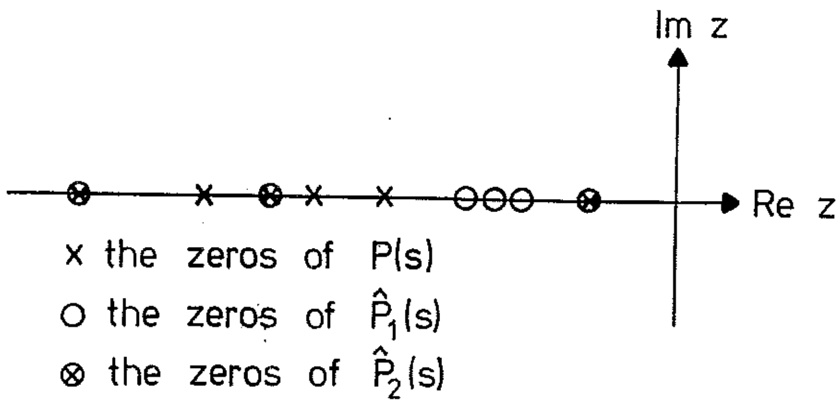

Fig 2. $\hat{P}_2(s)/P(s)$ is closer to an "interlacing configuration" than $\hat{P}_1(s)/P(s)$ and is more likely to be positive real.

5. Conclusion

It is possible to determine whether a given transfer function is PR or not using the Kalman-Yakubovich lemma [4] and, for the multivariable case, its matrix version [5]. This implies, among other things, solving a Lyapunov equation. The fact that the PR test may be carried through without recourse to the state space representation means that computer time is saved, apart from aesthetic reasons that motivate such an approach.

Bibliography

[1] Narendra, K, Taylor, J H: Frequency Domain Criteria for Absolute Stability (AP 1973).

[2] Gantmacher, F R: Matrix Theory, vol II (Chelsea 1960).

[3] Landau, I: Unbiased Recursive Identification Using Model Reference Adaptive Techniques. IEEE Trans Aut Control AC-21 (April 1976).

[4] Yakubovich, V A: Solution of Certain Matrix Inequalities Occurring in the Theory of Automatic Controls. Dokl Akad Nauk USSR 143, No 1 (1962).

[5] Anderson, B D O: A System Theory Criterion for Positive Real Matrices. SIAM J Control 5, No 2 (1967).