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AN APPROXIMATE ANALYTIC CONTROL LAW FOR AN ACTIVE SUBOPTIMAL DUAL CONTROLLER

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Abstract

Implementation for an active suboptimal dual controller is simplified. The controller achieves dual qualities through combining a term which corresponds to output loss and a term which reflects parameter uncertainty. Thus the dual controller compromises between good control and good parameter estimation. The result is a fifth-order loss function. Approximation of the resulting loss function furnishes a control law which can be determined analytically and which gives an approximate minimum for the function. From an implementation standpoint, the analytic solution is superior to previous iterative minimization algorithms. Two simple systems illustrate the performance of the approximate analytic control law, and comparisons are made with cautious control and a near-optimal control law for a special case.

Sammanfattning

Implementering av en aktiv suboptimal dual regulator förenklas. Regulatorn får duala egenskaper genom en sammansättning utav en term som svarar mot utsignal förlust, och en term som avspeglar osäkerhet hos parametrarna. Approximation av den erhållna förlustfunktionen leder till en styrlag som approximativt minimerar förlustfunktionen och kan beräknas analytiskt. Från implementeringssynpunkt utgör den analytiska lösningen en förbättring över tidigare iterativa minimeringsalgorithmer. Två enkla system illustrerar den approximativa styrlagens uppförande och styrlagen jämförs med cautious reglering och en approximativt optimal dual styrlag som gäller i särskilt fall.

An Approximate Analytic Control Law for an Active Suboptimal Dual Controller

I. Introduction

This report summarizes the results of project work in the area of dual adaptive control. The project aims toward simplifying implementation of an active suboptimal dual (ASOD) regulator first proposed by Björn Wittenmark in "An Active Suboptimal Dual Controller For Systems With Stochastic Parameters" [1]. By adaptive control is meant the control of systems with unknown or poorly measured, and therefore, stochastic, parameters. Dual control implies that the controller both controls well and helps provide useful information about the system. The latter goal implies that the controller disturb the system when more information about system parameters is required. For the ASOD controller, the dual strategy leads to a loss function composed of an output loss term and a term reflecting parameter uncertainty two steps ahead. This loss function is fifth order with respect to the variable of minimization.

The goal of the project is to improve the feasibility of computer implementation of the ASOD controller by decreasing the computing time. The results presented in this paper are based on series expansion of the ASOD controller's fifth order loss function, in order to analytically solve for a control signal which approximately yields minimum loss. The concept of expanding the expected loss function in a series is found for instance in "A Linear Feedback Dual Controller for a Class of Stochastic Systems," by Bar-Shalom, Mookerjee, and Molusis [6]. The mathematics involved in expanding the loss function in a second order Taylor series are not difficult. A greater portion of the calculations takes place in determining an appropriate point to expand about. The thinking behind the ASOD controller will first be covered, and an approximate analytic solution to the ASOD control law will then be developed in detail.

Following the development of the control algorithm, simulations performed using two simple systems and the cautious controller and a second dual control law will be presented. Statistical comparisons of Monte Carlo simulations indicate that the new ASOD control law performs at least as satisfactorily as the ASOD control law solved exactly through numerical algorithms. The ASOD controller also is shown to well outperform cautious control, as well as under certain circumstances matching the performance of dual controllers which are expected to be near optimal.

II. Problem Formulation

The system to be controlled is described as

$$y(t) + a_1(t)y(t-1) + \ldots + a_n(t)y(t-n) = b_1(t)u(t-1) + \ldots + b_n(t)u(t-n) + e(t)$$

where $e(t)$ is discrete white noise with mean 0 and variance σ^2. The parameters a_i and b_i $(i=1,2,\ldots n)$ are themselves stochastic variables.

Introducing the vector

$$x(t)=[a_1(t),a_2(t),\ldots a_n(t),b_1(t),b_2(t),\ldots b_n(t)]^T$$

gives the more convenient parameter representation

$$x(t+1) = \Phi x(t) + v(t)$$

It is assumed that this stochastic model suffices for the most general parameter variations. The known matrix Φ is assumed for the purposes of this paper to be a diagonal matrix of full rank. Φ is often refered to as the "forgetting matrix" for $x(t)$ since Φ determines the rate at which old values of $x(t)$ are reduced. The discrete noise vector $v(t)$ has zero mean and variance matrix R_1. Additionally, the processes $e(t)$ and $v(t)$ are assumed to be independent.

For further simplification of the system representation the vector

$$\Theta(t) = [-y(t-1),-y(t-2),\ldots -y(t-n),u(t-1),u(t-2),\ldots u(t-n)]$$

is used and the system is then given by

$$x(t+1)= \Phi x(t) + v(t)$$
$$y(t)=\Theta(t)x(t) + e(t) .$$

The loss function which ultimately gauges regulator performance is

$$V_1 = E \frac{1}{N} \sum_{n=1}^{N} (y(t+n)-y_r)^2 \qquad (1)$$

which is to be minimized for all $u(t)$ $t=1,2,\ldots(N-1)$. Since no general solution to this minimization problem exists except in special cases, a simplified version of V_1 may be considered:

$$V_1' = E[(y(t+1)-y_r)^2 | Y_t] \qquad (2)$$

where

$$Y_t=[y(t), y(t-1),\ldots,y(0),u(t-1),u(t-2),\ldots,u(0)]$$

V_1' is called the "one-stage" or "cautious" control law, which seeks to minimize the output loss one step ahead. The expression for V_1' can be expanded through use of standard formulas to give

$$V_1' = (\Theta(t+1)\hat{x}(t+1)-y_r)^2 + \Theta(t+1)P(t+1)\Theta(t+1)^T + \sigma^2,$$

which is quadratic in the variable of minimization, and thus analytically solvable.

The Kalman filter is used to give the conditional distribution of $x(t)$ given Y_{t-1}. The mean estimate $\hat{x}(t+1)$ and covariance matrix $P(t+1)$ are solved through the equations:

$$K(t) = \Phi P(t)\Theta(t)^T/(\Theta(t)P(t)\Theta(t)^T + \sigma^2)$$

$$\hat{x}(t+1) = \Phi\hat{x}(t) + K(t)[y(t) - \Theta(t)\hat{x}(t)]$$

$$P(t+1) = [\Phi - K(t)\Theta(t)]P(t)\Phi^T + R_1 \quad .$$

In order to simplify the algebra for this solution, as well as the algebra throughout this paper, introduce the vectors

$$\tilde{\Theta}(t)=[-y(t-1),-y(t-2),...,y(t-n),0,u(t-2),u(t-3),...,u(t-n)]$$

and

$$\ell=[0,0,...,1,...,0] \ .$$

The vector ℓ has 2n elements, all of which are zero except for a one in the $n+1^{\underline{th}}$ position, corresponding to the $u(t-1)$ element in $\Theta(t)$. Assume also that ℓ^T is an eigenvector of Φ^T, with eigenvalue Φ_b. It has already been stated that Φ^T is a diagonal matrix with full rank, and thus it is reasonable to assume that ℓ^T satisfies the relation $\Phi^T\ell^T=\Phi_b\ell^T$.

The $u(t)$ which then minimizes V_1' forms the control law

$$u_c(t) = - \frac{\tilde{\Theta}(t+1)\hat{x}(t+1)\hat{b}_1(t+1) - y_r\hat{b}_1(t+1) + \ell P(t+1)\tilde{\Theta}(t+1)^T}{\hat{b}_1(t+1)^2 + p_{b1}(t+1)} \ . \quad (3)$$

Implementation of this control law presents no difficulty. The cautious controller is, however, not a dual controller. Dual control consists of both minimizing output error and making accurate estimates of system parameters. The latter goal of accurate parameter estimation demands that the system be disturbed by the control signal. Since the controller is responsible for excitations of the system, the dual control law then must constitute a compromise between good control and good estimation. The loss function V_1' seeks only to control output error one step ahead, and contains no component which is rewarded for excitation of the system. The lack of excitation often leads to poor parameter estimation. In practice, the one-stage control law often leads to the phenomenon of "turn-off", where the control signal tapers off to zero, and parameter estimation is greatly retarded. Thus the controller looses its capacity to regulate the system. The turn-off phenomenon is illustrated in the second example presented in this paper.

A ready solution to increasing the cautious controller's ability to estimate parameters might be to add a disturbance to $u_c(t)$. This has given rise to regulators which are similar to the cautious controller, and which give improved results. (See Jan Sternby's "Topics in Dual Control," [2].)

The principle behind the active sub-optimal dual (ASOD) controller is to incorporate the dual control characteristics into the original loss function. The ASOD control law seeks to minimize two coupled loss terms: the square of the error in the output, and a term which is a function of the parameter uncertainties. The ASOD control law, then, actively works for better parameter estimation.

The ASOD loss function has the form

$$V_2 = E\left\{[y(t+1) - y_r]^2 + \lambda f[P(t+2)] \mid Y_t\right\} . \tag{4}$$

In general, the function $f(\cdot)$ must be positive, monotonically increasing, and twice continuously differentiable. The choice of $f(\cdot)$ will be treated shortly. The constant λ weights the identification term in V_2 against the output loss term.

Using the vectors $x(t)$ and $\Theta(t)$ the loss function V_2 can be written in the form

$$\begin{aligned} V_2 = {} & (\Theta(t+1)\hat{x}(t+1) - y_r)^2 + \Theta(t+1)P(t+1)\Theta(t+1)^T + \sigma^2 \\ & + \lambda f\left[\Phi P(t+1)\Phi^T + R_1 + \frac{\Phi P(t+1)\Theta(t+1)^T\Theta(t+1)P(t+1)\Phi^T}{(\Theta(t+1)P(t+1)\Theta(t+1)^T + \sigma^2)} \right] \end{aligned} \tag{5}$$

which is to be minimized with respect to $u(t)$.

It has been shown that $\hat{b}_1$ is the most critical coefficient to identify correctly (see Wittenmark (1975) [1]). A suitable choice for the function $f[P(t+2)]$ then is found to be $\ell P(t+2)\ell^T$ so that $f(\cdot)$ yields a result which is a function of the variance of the coefficient $\hat{b}_1(t+1)$, $p_{b1}(t+1)$.

The expression for V_2 is then fifth order in $u(t)$, and has no analytical solution. Figure 1 below shows the typical appearance for the function. Previously, numerical algorithms have been employed with help of knowledge about V_2 and search procedures to solve for a u(t) which exactly minimizes V_2. The following analysis leads to a second order approximation for V_2 which makes possible an analytical solution for a u(t). This $\hat{u}(t)$, denoted $u_{as}(t)$ closely minimizes the loss function V_2. The same knowledge about the behavior of V_2 used in numerical algorithms aids in the derivation of the approximate control law.

III. Approximation of V_2

Approximation of the loss function V_2 is carried out through expanding the function in a series, keeping only terms linear and quadratic in u(t). The approximated loss can then be minimized analytically.

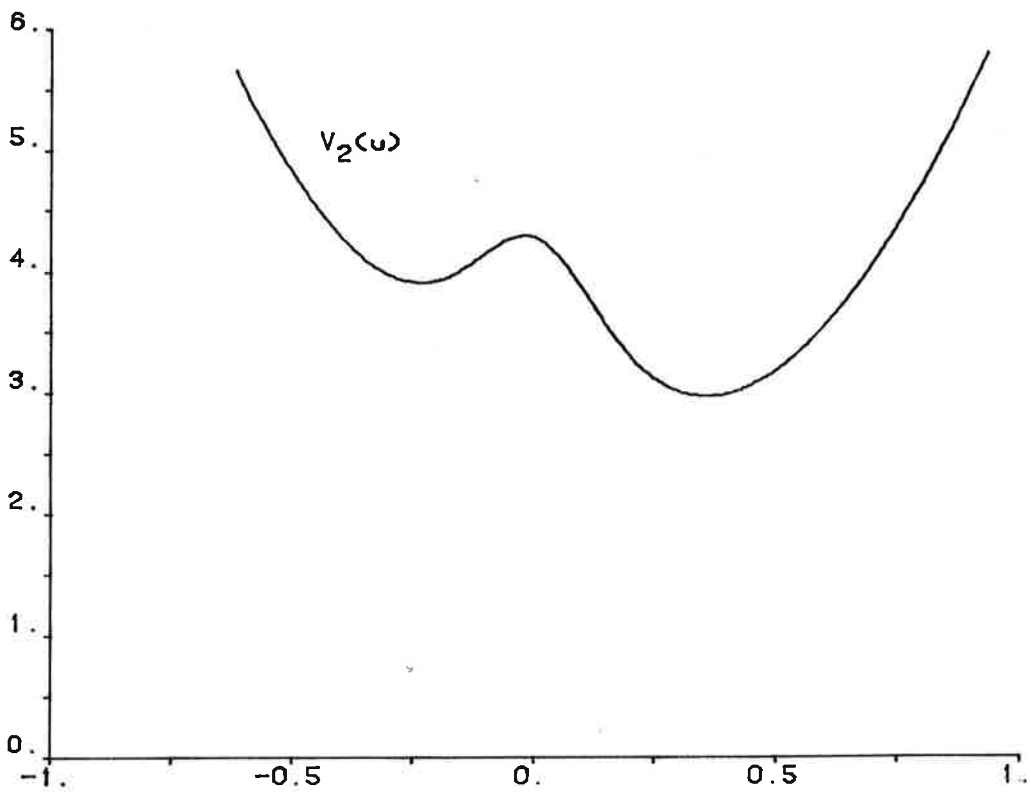

Figure 1. The typical appearance of the loss function V_2.

The loss function can now be rewritten, using f(P), $\tilde{\Theta}(t)$, and ℓ, as:

$$V_2 = [(\tilde{\Theta}(t+1)+\ell u(t))\hat{x}(t+1) - y_r]^2 + (\tilde{\Theta}(t+1)+\ell u(t))P(t+1)(\tilde{\Theta}(t+1)+\ell u(t))^T + \sigma^2 + \lambda L(u(t))$$

with

$$L(u) = \ell\left[\Phi P(t+1)\Phi^T + R_1 - \frac{\Phi P(t+1)(\tilde{\Theta}(t+1)+\ell u(t))^T(\tilde{\Theta}(t+1)+\ell u(t))P(t+1)^T\Phi^T}{((\tilde{\Theta}(t+1)+\ell u(t))P(t+1)(\tilde{\Theta}(t+1)+\ell u(t))^T + \sigma^2}\right]\ell^T$$

$$= \ell R_1 \ell^T + \frac{\Phi_b^2\left[p_{b1}(t+1)[\tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1)^T+\sigma^2] - [\ell P(t+1)\tilde{\Theta}(t+1)^T]^2\right]}{p_{b1}(t+1)u^2(t)+2\ell P(t+1)\tilde{\Theta}(t+1)^T u(t)+\tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1)^T+ \sigma^2} .$$

Series expansion about the point γ, gives the approximation

$$V_2(u(t)) \approx V_2(\gamma) + \frac{\partial V_2(\gamma)}{\partial u}(u(t)-\gamma) + \frac{1}{2}\frac{\partial^2 V_2(\gamma)}{\partial u^2}(u(t)-\gamma)^2 . \quad (6)$$

The partial derivatives of V_2 on the right hand side above are given by

$$\frac{\partial V_2}{\partial u} = 2\left[(\tilde{\Theta}(t+1)\hat{x}(t+1) + \hat{b}_1(t+1)u(t) - y_r)\hat{b}_1(t+1) + \ell P(t+1)\tilde{\Theta}(t+1)^T + p_{b1}(t+1) \right] + \lambda\frac{\partial L}{\partial u}$$

and

$$\frac{\partial^2 V_2}{\partial u^2} = 2(\hat{b}_1^{\,2}(t+1) + p_{b1}(t+1)) + \lambda \frac{\partial^2 L}{\partial u^2} .$$

For convenience introduce the notation

$$d_1 = \ell P(t+1)\tilde{\Theta}(t+1)^T$$

$$d_2 = p_{b1}(t+1)(\tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1)^T+ \sigma^2) - {}^2d_1 .$$

The partial derivatives of the function L(u) are then given by

$$\frac{\partial L}{\partial u} = -\Phi_b^2 d_2 \frac{2(p_{b1}(t+1)u(t) + d_1)}{\left[p_{b1}(t+1)u^2(t)+2d_1u(t)+\tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1)^T+ \sigma^2\right]^2}$$

and

$$\frac{\partial^2 L}{\partial u^2} = \Phi_b^2 d_2 \frac{6(p_{b1}(t+1)u(t) + d_1)^2 - 2d_2}{\left[p_{b1}(t+1)u^2(t)+2d_1u(t)+\tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1)^T + \sigma^2\right]^3} .$$

Taking now the partial derivatives of V_2 and substitute back into the second order Taylor expansion (6). Minimizing this expression leads to the solution for an approximate u(t).

The derivative of (6) is

$$\frac{\partial V_2(u)}{\partial u} \approx 2\left[[(\tilde{\Theta}(t+1)+\ell\gamma)\hat{x}(t+1) - y_r]\hat{b}_1(t+1) + d_1 + p_{b1}(t+1)\gamma\right] + \lambda\frac{\partial L(\gamma)}{\partial u} + \left[2(\hat{b}_1^2(t+1) + p_{b1}) + \lambda\frac{\partial^2 L(\gamma)}{\partial u^2}\right](u-\gamma) .$$

Setting the above equal to zero, gives the control law sought for the approximated loss function

$$u_{as} = \gamma - \frac{2\left[[(\tilde{\Theta}(t+1)+\ell\gamma)\hat{x}(t+1)-y_r]\hat{b}_1(t+1) + d_1 + p_{b1}\gamma\right] + \lambda\frac{\partial L(\gamma)}{\partial u}}{2[\hat{b}_1^2(t+1) + p_{b1}(t+1)] + \lambda\frac{\partial^2 L(\gamma)}{\partial u^2}} . \tag{7}$$

The point γ is so far any point. It is critical that γ be chosen in the appropriate region of V_2. Knowledge about the control law's behavior can lead to an γ which lies near the desired minimum. Jan Sternby's "Topics in Dual Control" [2], offers useful insight into the behavior of V_2.

Going back to equation (5) for V_2 (not approximative), it can be seen that the function has a quadratic part and an additional term. These parts will be called h(u) and g(u), respectively, and have the form

$$\begin{aligned} h(u) = {} & [\hat{b}_1^2(t+1) + p_{b1}(t+1)]u^2(t) \\ & + 2\left[[(\tilde{\Theta}(t+1)\hat{x}(t+1) - y_r]\hat{b}_1^2(t+1) + \ell P(t+1)\tilde{\Theta}(t+1)^T\right]u(t) \\ & + [\tilde{\Theta}(t+1)\hat{x}(t+1)]^2 + y_r^2 - 2\tilde{\Theta}(t+1)\hat{x}(t+1)y_r \\ & + \tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1)^T + \sigma^2 + \lambda\ell R_1\ell^T \end{aligned}$$

$$g(u) = \frac{\lambda\, \Phi_b^2\, d_2}{\left[p_{b1}(t+1)u^2(t) + 2d_1 u(t) + \tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1) + \sigma^2\right]} .$$

Figure 2 depicts h(u) and g(u) together with the loss function. Typically, V_2 has two local minimums and one local maximum. Using the reasoning presented in Sternby [2], Appendix B, two important attributes of V_2 are known immediately. First, the global minimum of V_2 lies on the same side of the origin as h(u)'s minimum. This minimum point is given also by the cautious input, u_c. The point u_c however, often does not give a satisfactory approximation; since u_c can fall at a point where V_2 is concave downward, approximation leads to the local maximum, which always lies close to zero. We may then utilize a second important characteristic of V_2, that V_2 is concave upwards for all $|u| \geq |u_A|$ where u_A satisfies $g''(u)=0$ and $\mathrm{sign}(u_A)=\mathrm{sign}(u_c)$. The point u_A is easily determined analytically. Expansion about the point u_A will always give an approximated loss with a local minimum. However, u_A is also inappropriate for approximation; we may still seek a point

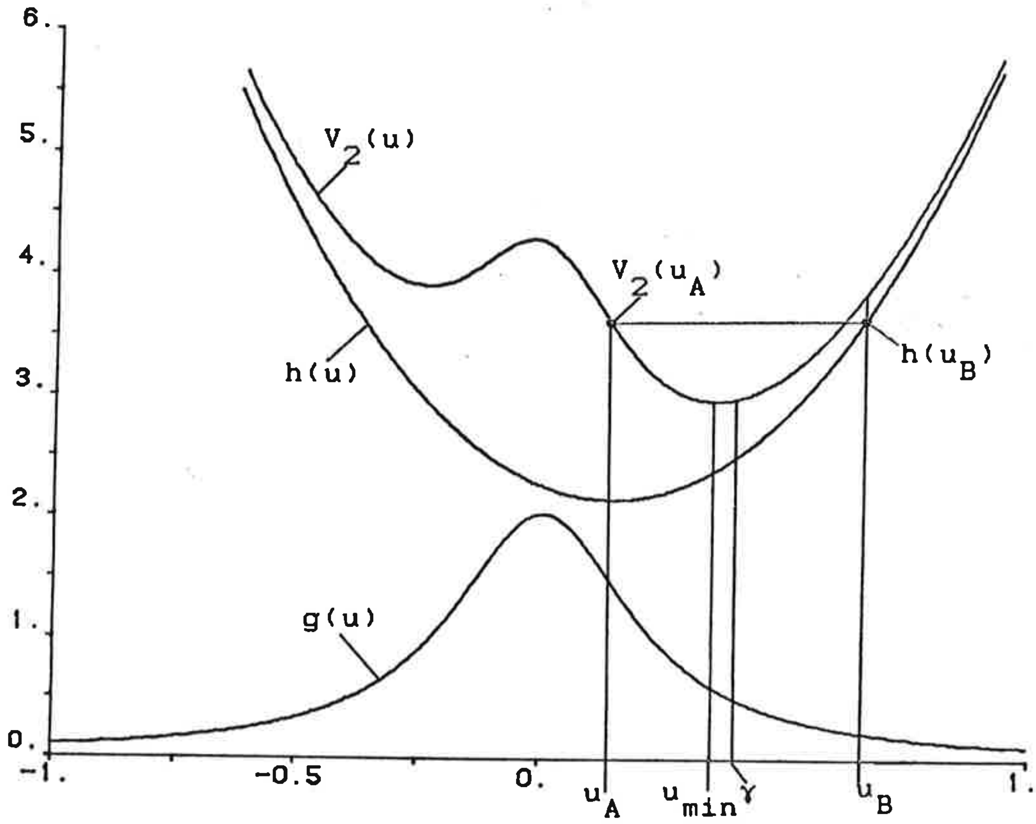

Figure 2. The loss function V_2 together with its quadratic and nonlinear parts, marked respectively h(u) and g(u).

nearer u_{min}. There are two cases to be considered: the case that $|u_{min}| \geq |u_A|$ and that $|u_{min}| < |u_A|$. The case where u_{min} has absolute value greater than $|u_A|$ will first be discussed.

A point u_B is then determined (see Figure 2). It is reasonable to assume that $h(u_B) \approx V_2(u_B)$ since at u_B the function $g(u)$ can be neglected. In typical cases u_{min} lies somewhere in between u_A and u_B. Since this section of the curve approximates a quadratic, take the point to be expanded about to be

$$\gamma = \frac{u_A + u_B}{2} .$$

The point u_A is obtained through solving the equation $h''(u)+g''(u) \geq 0$. Since $h''(u) \geq 0$ for all u, solving the equation $g''(u)=\partial^2 L/\partial u^2=0$ leads to

$$6(p_{b1}(t+1)u(t) + d_1)^2 - 2d_2 \geq 0 .$$

This gives that $h''(u) + g''(u) \geq 0$ for

$$\left.\begin{array}{l} u(t) \geq \dfrac{\sqrt{d_2/3} - d_1}{p_{b1}(t+1)} \\[2ex] u(t) \leq \dfrac{-\sqrt{d_2/3} - d_1}{p_{b1}(t+1)} \end{array}\right\} u_c(t) \geq 0 .$$

Assuming $u_c \geq 0$, then $u(t)$ must be chosen greater than zero, giving

$$u_A = \frac{\sqrt{d_2/3} - d_1}{p_{b1}(t+1)} .$$

The point u_B is easily obtained through solving $h(u_B)-V_2(u_A)=0$. Appendix 1 gives a summary of the steps used in this algorithm.

In the case that $|u_A|$ is greater than $|u_{min}|$, the algorithm presented above also suffices. This case is illustrated in Figure 3. The algorithm still calculates a point $|u_B| > |u_A|$.
By testing the first derivative of V_2 at u_A it may be determined where $|u_A|$ lies in relation to $|u_{min}|$. However, there are several

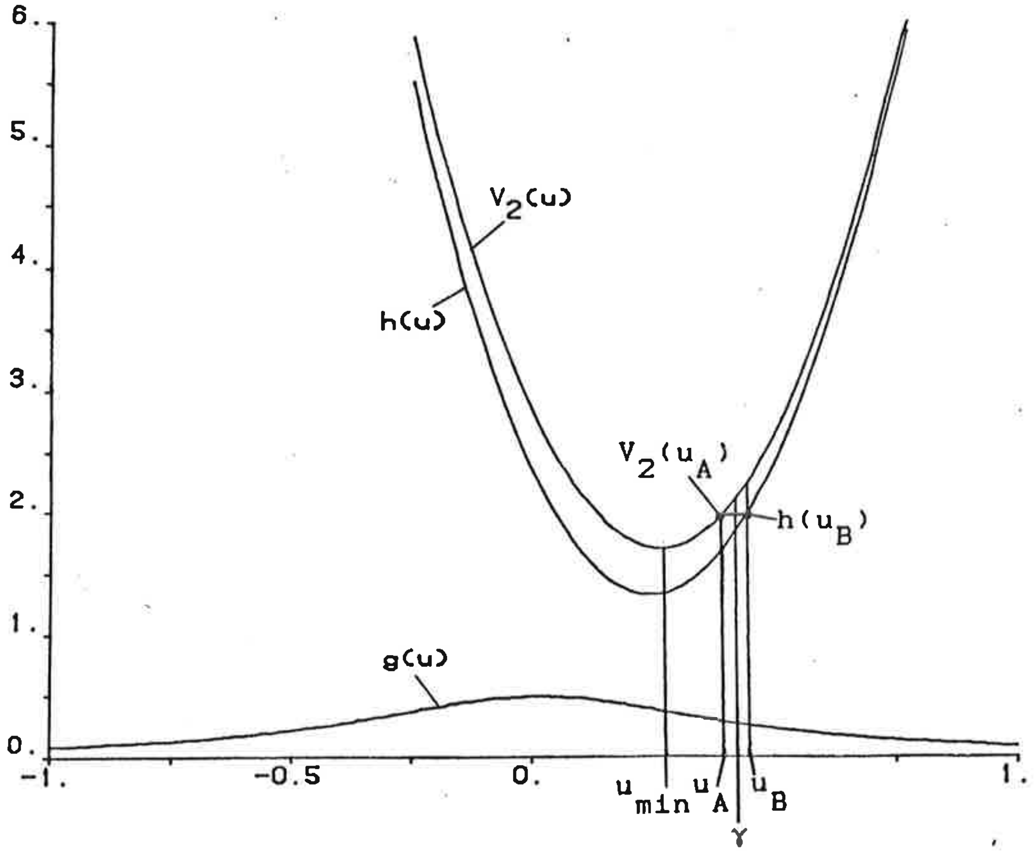

Figure 3. The typical appearance of V_2 when $|u_A| \geq |u_{min}|$.

reasons for maintaining the same algorithm even in this second case. A primary reason is that for small values of u(t) the curve h(u) no longer approximates V_2. It is then possible that γ (as calculated before) lies at a point where V_2 is concave downward, and if γ is not corrected, there may be a period of regulator turn off. If steps are taken to correct γ, additional iterations are necessary. The algorithm begins resembling an iterative numerical algorithm.

Furthermore, even in this second case, expansion about γ as calculated above yields a point u_{as} which falls close to the exact minimum. The case that $|u_A| > |u_{as}|$ often occurs when the slope of g(u) is shallow compared with the slope of h(u), giving a loss curve which is almost quadratic; in this case V_2 has only one local (global) minimum. Therefore the approximation leads even in this case to a $u_{as} \approx u_{min}$.

It has been demonstrated through simulations that the point

$\gamma=(u_A+u_B)/2$ gives results close to those of numerical minimization. There remains some doubt, however, as to whether the second order derivative of L(u) is required in (7). The examples show that a control law for u_{as} which includes only the first order derivative of L(u) also gives satisfactory results.

III. Simulations

The approximate analytic ASOD control law was simulated together with other control strategies on two simple systems. Simulations were carried out using the interactive simulation language Simnon. In each case, cautious control along with ASOD expanded about u_c, γ, and γ without the $\partial^2L/\partial u^2$ term are simulated. For the second system an analytic solution to V_1 which gives near-optimal dual control, is also compared.

1. Simple zero-order system

The approximated ASOD was first tested on a simple system with no dynamics, but only a time-varying gain, and a one-step time delay. The system was chosen for the purpose of comparison, since simulation results are already available in "Problems of Identification and Control" by Karl Johan Åström and Björn Wittenmark [3].

The system has the form:

$$x(t+1) = 0.9x(t) + v(t)$$
$$y(t) = x(t)u(t-1) + e(t).$$

The noise sequences v(t) and e(t) are Gaussian with zero mean and variances 1.0 and 0.25, respectively. The output reference is taken to be 1.0 .

The Kalman filter looks like:

$$K(t) = 0.9P(t)u(t-1)/(0.25 + P(t)u^2(t-1))$$
$$P(t+1)=0.9(0.9-K(t)u(t-1))P(t) + 1$$
$$\hat{x}(t+1)=0.9\hat{x}(t) + K(t)(y(t)-\hat{x}(t)u(t-1)) .$$

The results of 60 simulations are shown below in Table 1. The average accumulated sums, which are measures of average system output, and average losses are given.

These results agree well with those presented in Åström and Wittenmark [3]. The results for cautious control match those

Table 1. The results from a Monte Carlo run consisting of 60 simulations, for the zero order system.

Control law	loss = $\sum_{n=1}^{1000}(y_n - 1)^2$		sum = $\sum_{n=1}^{1000} y_n$	
	Mean	St. Dev	Mean	St. Dev
Cautious Control 1	1113.8	±77.4	133.2	±73.1
ASOD expansion about u_c 2	845.1	±54.2	602.8	±52.6
ASOD expansion about γ with $\partial^2L/\partial u^2$ 3	819.8	±51.2	588.8	±51.4
ASOD expansion about γ without $\partial^2L/\partial u^2$ 4	819.1	±51.2	571.1	±50.8

presented here, i.e. 0.11 average units of loss per step. The results for the ASOD control law minimized with numerical algorithms (see Wittenmark [1]) give an average loss per step of 0.82 and estimated standard deviation of 0.06 units per step, matching results in Table 1 which give a loss/step of 0.82 and estimated standard deviation of 0.05 per step. The optimal loss per step is calculated to be 0.83 units per step, indicating that ASOD in this case performs nearly optimally. Table 1 also indicates that omitting the second order term $\partial^2L/\partial u^2$ about γ suffices, based on the accumulated loss. However, the running sum of outputs is closer to 1.0 when using the full second order approximation, which may point to an advantage in keeping the second order derivative term. Figure 4 depicts the dispersion of the simulation results for the different controllers.

Because each noise sequence employed in the simulations has a different character (i.e. each is independent of the others leading to widely scattered losses), a comparison of average losses may not always be conclusive. The statistical method employed in Wenk - Bar-Shalom [4], Appendix II provides an appropriate basis for comparison.

Instead of comparing the average loss for one control strategy against the average loss of another, we build first a series of

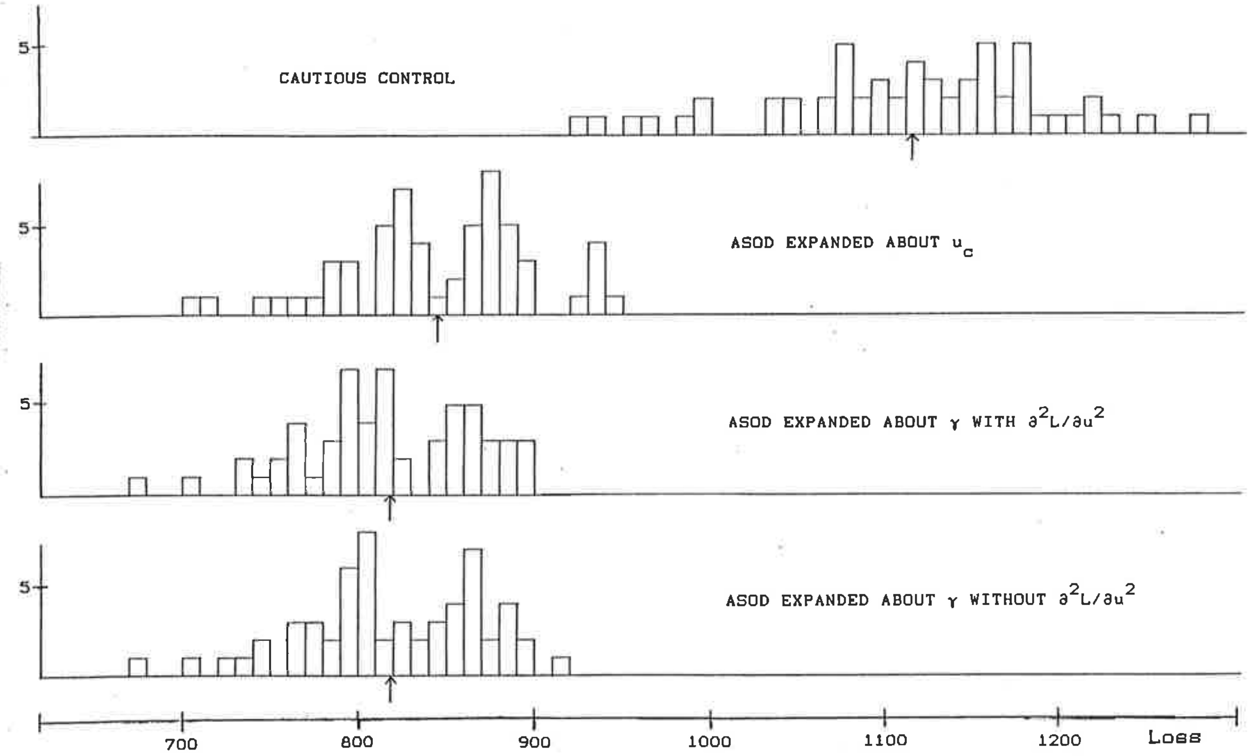

Figure 4. The losses for 60 simulations of the zero-order system. The arrows indicate mean values.

Δ's each of which equals the difference between the loss for controller b and controller a, for a simulation with the same noise sequences, that is,

$$\Delta_i = \text{loss}_i^{(b)} - \text{loss}_i^{(a)} \qquad \text{for simulations } i=1,2,\ldots n\ .$$

Assuming that there are a sufficient number of Δ's available, the central limit theorem can be employed to compare the differences (i.e. the Δ's) and give a probability to whether $\bar{\Delta}$ is positive or negative. Thus we may test the hypotheses

$$H_o:\ \Delta = \text{true } \overline{\text{loss}}^{(b)} - \text{true } \overline{\text{loss}}^{(a)} \leq 0 \quad \text{algorithm a not best}$$

and

$$H_1:\ \Delta = \text{true } \overline{\text{loss}}^{(b)} - \text{true } \overline{\text{loss}}^{(a)} > 0 \quad \text{algorithm a best .}$$

The hypothesis H_1 will be accepted if

$$\bar{\Delta} > \mu\sigma_{\bar{\Delta}}$$

where

$$\bar{\Delta} = \frac{1}{n}\sum_{i=1}^{n} \Delta_i \quad \text{and} \quad \sigma_{\bar{\Delta}}^2 = \frac{1}{n^2}\sum_{i=1}^{n} (\Delta_i - \bar{\Delta})^2 \quad .$$

The value for μ given by the normal distribution tables leads to a corresponding probability α that the hypothesis H_1 holds true. Table 2 shows the results of this statistical comparison on the same simulations used in producing Table 1, with the same numbering for the control laws. Both mean losses and sums are compared. Since the criteria for a "better" mean sum is in this case the reverse of that for losses, that is, a higher mean sum is closer to the reference value of 1 , the hypotheses are adjusted for sums to:

$$H_0: \quad \Delta = \text{true } \overline{\text{sum}}^{(a)} - \text{true } \overline{\text{sum}}^{(b)} \leq 0 \quad \text{algorithm a not best}$$

and

$$H_1: \quad \Delta = \text{true } \overline{\text{sum}}^{(a)} - \text{true } \overline{\text{sum}}^{(b)} > 0 \quad \text{algorithm a best}$$

The statistical comparison does not lead to any unforeseen conclusions, but confirms expectations based on Table 1. For all values of α which are given as >99.9, μ has a value between 10.0 and 20.0, indicating a high degree of certainty. ASOD control strategies yield superior results. Even expanding about the cautious input u_c offers an improvement to cautious control, when the $\partial^2 L/\partial u^2$ term is omitted.

2. System with integrator.

The second example system has been used in "Dual Control of a Low Order System", by Åström and Helmersson [5] and has the form

$$y(t+1) = y(t) + b(t)u(t) + \sigma e(t) \ .$$

The parameter b varies, obeying,

$$b(t+1) = \varphi b(t) + \varrho v(t) \ .$$

As before, the noise sequences e(t) and v(t) are Gaussian with mean zero and variance 1.0. The constants σ and ϱ are known; σ is held at the constant 1.0 for all simulations, and ϱ is varied to constant values between 0.003 and 0.5 for different Monte Carlo runs. In order that the gain for b(t) has a constant variance of

Table 2. The results of the statistical test are shown for Table 1 simulations. The specification >99.9 implies that $|\mu|>3.09$.

Control law a	Control law b		Better Control law	α (percent)
1	3	loss:	3	>99.9
		sum:	3	>99.9
1	4	loss:	4	>99.9
		sum:	4	>99.9
1	2	loss:	2	>99.9
		sum:	2	>99.9
2	3	loss:	3	>99.9
		sum:	2	>99.9
2	4	loss:	4	>99.9
		sum:	2	>99.9
3	4	loss:	inconclusive	
		sum:	3	>99.9

1.0 for all values of ϱ, the parameter φ is calculated to satisfy

$$\varphi = \sqrt{1 - \varrho^2} .$$

The terms needed for approximation have the form

$$\hat{x}(t)=\begin{bmatrix} -1 \\ \hat{b}(t) \end{bmatrix}, \qquad \Phi=\begin{bmatrix} 1 & 0 \\ 0 & \varphi \end{bmatrix}, \qquad R_1=\begin{bmatrix} 0 & 0 \\ 0 & \varrho^2 \end{bmatrix},$$

$$\Theta(t)=[\ -y(t-1),\ u(t-1)], \qquad \tilde{\Theta}(t)=[\ -y(t-1),\ 0],$$

$$\text{and} \qquad \ell = [\ 0 \ 1\ ] .$$

The Kalman filter is then given by

$$K_b(t) = \varphi p_{b1}(t)u(t-1)/[\sigma^2 + p_{b1}(t)u^2(t-1)]$$

$$p_{b1}(t+1) = \varphi(\varphi - K_b(t)u(t-1))p_{b1}(t) + R_{b1}$$

$$\hat{b}(t+1) = \varphi\hat{b}(t) + K_b(t)[y(t) - y(t-1) - \hat{b}(t)] .$$

In this case the reference value for the output is set at zero.

The four control strategies used in the previous example were also used here, that is, the cautious and approximate ASOD controllers for a first order system. Additionally, an analytic solution to the optimal dual control loss function is simulated. The control algorithm is taken from Åström and Helmersson [5]. This control law is derived for a constant b(t), but is expected to be a good approximation for a b(t) which varies slowly, i.e. when ϱ is small. The losses for different values of ϱ are shown in Table 3. The average running output sums were all within 40 units of zero, and could not be distinguished significantly. Therefore output sums will be omitted from discussion.

Figure 5 shows the dispersion of results for cautious, approximate ASOD (with $\partial^2L/\partial u^2$), and near-optimal dual (for ϱ=0.0) controllers. The constant λ=0.5 gave satisfactory results for all values of ϱ except for ϱ=0.5, where a value of λ=2.5 was found to give better results.

ASOD expanded about γ again gives results superior to those of the cautious controller. The near-optimal controller 5 and ASOD controllers 3 and 4 give comparable results for small values of ϱ. As is to be expected, the performance of the analytic expression for optimal dual control becomes worse for higher ϱ's. The near-optimal analytic control law loses validity, and must be reformulated.

The statistical comparisons carried out for the Monte Carlo runs for this system are shown in Table 4. As expected, the control laws 3 and 4 are statistically indistinguishable. To save calculation time, omission of the $\partial^2L/\partial u^2$ term may be considered here.

Figures 6 and 7 illustrate typical simulations for cautious and approximate ASOD (with the $\partial^2L/\partial u^2$ term) controllers, respectively. Both simulations were obtained using the same noise sequences, and the losses for both are close to average; ϱ is equal to 0.3. Figure 6 exemplifies the turn off phenomenon occuring during cautious control where parameter estimation ceases and the control signal nears zero.

In Figure 7, ASOD control shows no tendency toward turn off. Parameter estimation and good control are maintained. A large portion of the loss takes place when b(t) changes sign. The estimated b(t) is delayed slightly, and the control signal may

Table 3. The results from six Monte Carlo runs for the first order system each for a different ϱ. The runs with 60 simulations are used in the statistical comparison shown in Table 4.

$$\text{Loss} = \sum_{n=1}^{1000} (y_n)^2$$

Control law	$\varrho=0.003$ $\lambda=0.5$ (60 sim)	$\varrho=0.1$ $\lambda=0.5$ (60 sim)	$\varrho=0.2$ $\lambda=0.5$ (10 sim)	$\varrho=0.3$ $\lambda=0.5$ (60 sim)	$\varrho=0.4$ $\lambda=0.5$ (10 sim)	$\varrho=0.5$ $\lambda=2.5$ (60 sim)
Cautious Control 1	2316.6 ±775.8	2846.1 ±798.5	4145.1 ±978.6	5223.1 ±919.5	6473.7 ±905.0	8197.0 ±1301.2
ASOD expansion about u_c 2	1869.0 ±565.7	2250.1 ±597.8	3158.1 ±728.6	3631.0 ±675.9	4331.2 ±726.6	4572.1 ±744.32
ASOD expansion about γ with $\partial^2L/\partial u^2$ 3	1658.7 ±390.8	2038.6 ±489.0	2585.7 ±491.0	3128.3 ±618.2	3702.5 ±385.7	4082.1 ±708.5
ASOD expansion about γ without $\partial^2L/\partial u^2$ 4	1658.7 ±390.8	2040.0 ±493.6	2582.5 ±482.8	3122.7 ±619.9	3721.1 ±399.0	4064.4 ±690.0
near optimal dual dual analytic for $\varrho=0.0$ 5	1677.6 ±394.7	2077.4 ±504.1	3041.7 ±821.0	3328.8 ±749.0	4135.2 ±723.3	5107.0 ±1288.3

have the wrong sign. The example simulation in Figure 7 illustrates this phenomenon.

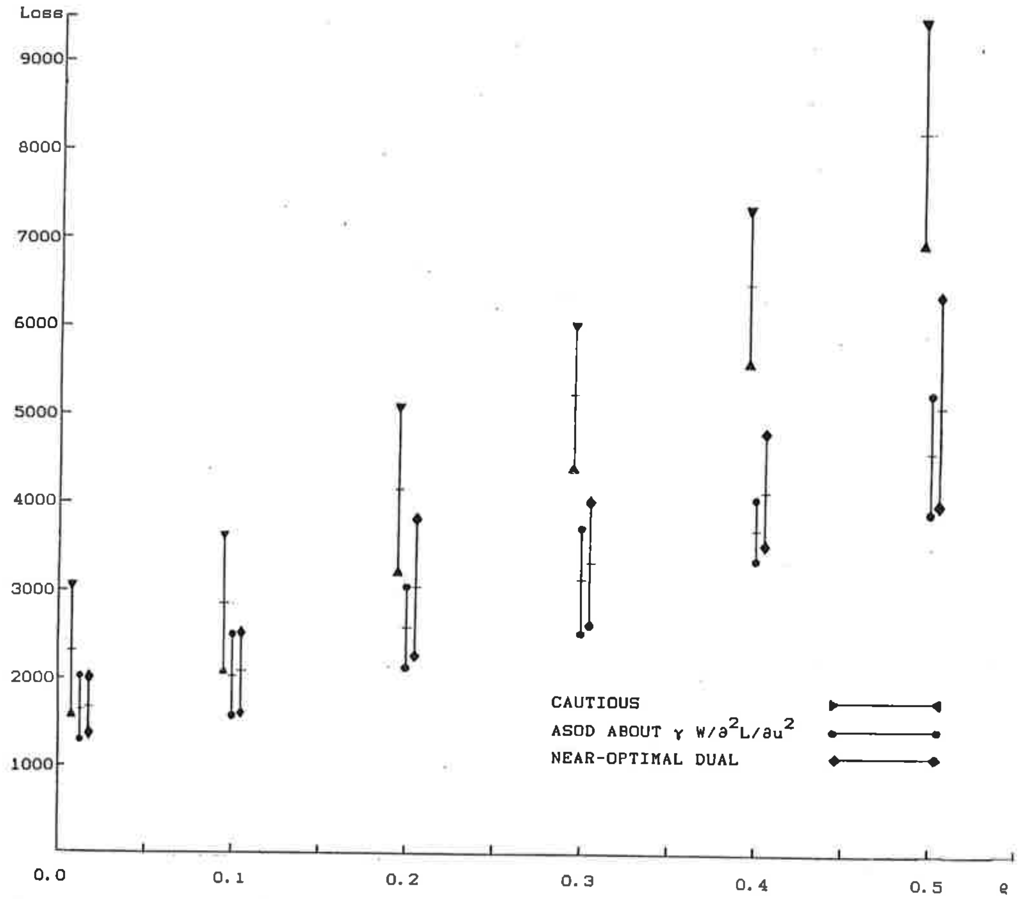

Figure 5. The dispersion of results obtained for six Monte Carlo runs with different values of ϱ. The vertical axis is tracking loss, and the horizontal axis is ϱ.

Table 4. A statistical comparison of loss results obtained for the first order system.

Control law a	b	ϱ=0.003 Better control law	α (%)	ϱ=0.1 Better control law	α (%)	ϱ=0.3 Better control law	α (%)	ϱ=0.5 Better control law	α (%)
1	2	2	>99.9	2	>99.9	2	>99.9	2	>99.9
1	4	4	>99.9	4	>99.9	4	>99.9	4	>99.9
1	5	5	>99.9	5	>99.9	5	>99.9	5	>99.9
2	3	3	>99.9	3	>99.9	3	>99.9	3	>99.9
3	4	inconcl.		inconcl.		inconcl.		inconcl.	
3	5	inconcl.		inconcl.		3	99.9	3	>99.9
4	5	inconcl.		inconcl.		4	99.9	4	>99.9

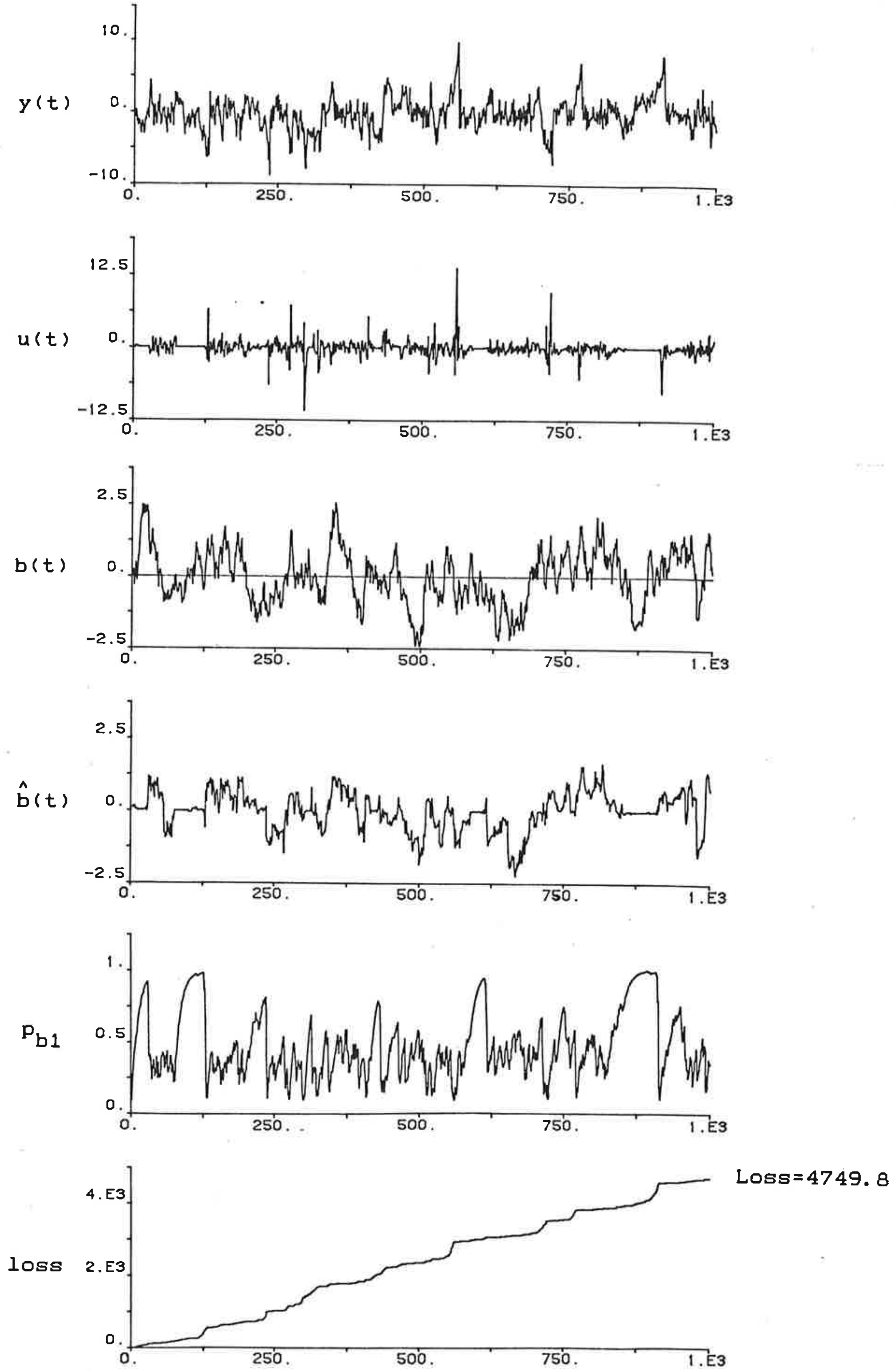

Figure 6. Representative simulation for the first order system (ϱ=0.3) with cautious control. Turn off occurs when the control signal u(t) nears zero, and identification of the parameter b(t) is impeded. Here the variance of b(t) increases to its maximum. Variables are plotted versus time.

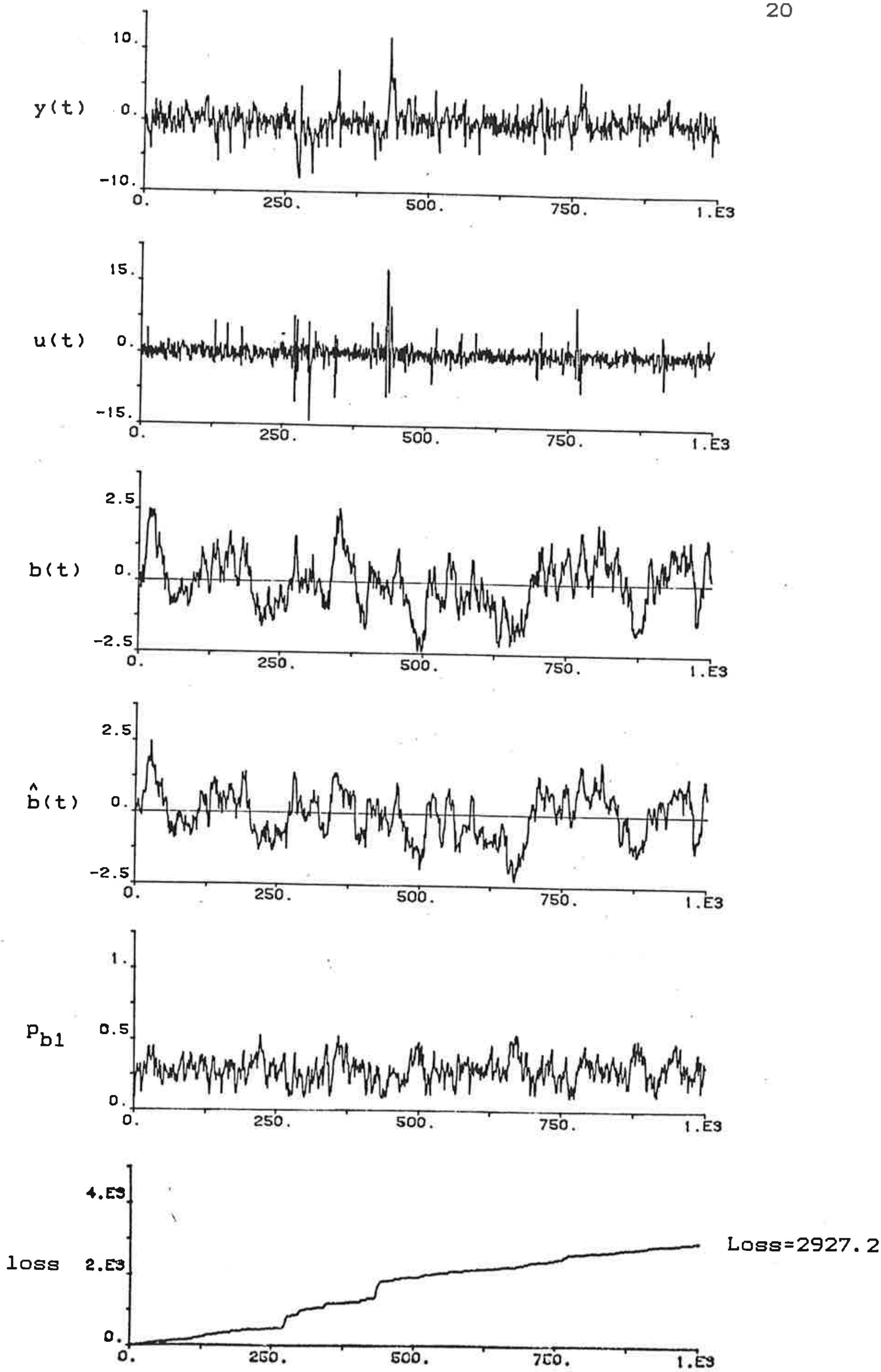

Figure 7. Approximative ASOD control for the integrator (ϱ=0.3) is illustrated, with the same noise sequence used in Figure 6. No turn off occurs. A large portion of the loss takes place when b(t) changes sign. Variables values are plotted versus time.

V. Conclusion

The purpose of this paper has been to describe how approximation of the ASOD loss function, while decreasing calculations in comparison to iterative methods, maintains good results. The loss function V_2 is readily approximated, and an analytic ASOD control law can be calculated.

The two example systems illustrate how the active sub-optimal controller's dual effect can lead to good control while maintaining good parameter estimation. Although cautious control involves few calculations, the ASOD controller has given superior results. The first example shows that this analytic solution leads to results which are equivalent to those obtained using numerical minimization algorithms at least in the case of the simple, zero order system. An approximation omitting the $\partial^2 L/\partial u^2$ term should also be considered.

For the system with an integrator, the approximate ASOD also performed well. In comparing ASOD and the near-optimal analytic dual control law, results show that for low ϱ, ASOD and the near-optimal dual solution gave losses which could not be distinguished with statistical significance. For higher variances ($\varrho \geq 0.3$), the performance of the ASOD controller is better with high statistical significance. This result indicates that the optimal dual solution for constant b(t) does not suffice for a rapidly varying b(t). Additionally, approximations for ASOD including or omitting the second order derivative of L(u) gave statistically indistinguishable results, both in loss and running sum.

For all three controllers, a large portion of the loss takes place when the parameter $b_1(t)$ nears zero. In cautious control, a low $b_1(t)$ leads to the turn-off phenomenon. In dual control, parameter identification continues for small $b_1(t)$. Even so, the wrong sign of the gain can lead to large losses in short periods.

There is some indication that better results can be obtained for the ASOD controller by using a variable (or higher constant) λ, especially for systems with large parameter variation. It is clear that parameter variation and the need for parameter identification increase simultaneously. For a high ϱ (=0.5) in the first order system, a higher λ gave improved results. A suggestion for a variable λ used in a special case of the ASOD controller is found in Sternby [2]. Through improved parameter identification, losses which occur when $b_1(t)$ changes sign may be decreased.

APPENDIX 1

The calculation of $u_{as}(t)$ involves the following 3 steps which are broken down into component calculations. These steps involve no iterations. The estimated parameters are taken from the Kalman filter. The cautious regulator involves only step one.

Step 1: Calculate $u_c(t)$ since $\mathrm{sign}(u_{min}) = \mathrm{sign}(u_c)$.

$$u_c(t) = -\frac{\tilde{\Theta}(t+1)\hat{x}(t+1)\hat{b}_1(t+1) - y_r\hat{b}_1(t+1) + \ell P(t+1)\tilde{\Theta}(t+1)^T}{\hat{b}_1(t+1)^2 + p_{b1}(t+1)}$$

Step 2: Calculate γ for approximation.

$$\gamma = \frac{u_A + u_B}{2}.$$

This involves the calculations:

2a. $d_1 = \ell P(t+1)\tilde{\Theta}(t+1)^T$

2b. $d_2 = p_{b1}(t+1)(\tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1)^T + \sigma^2) - d_1^2$,

2c. $u_A = \dfrac{\sqrt{d_2/3} - d_1}{p_{b1}(t+1)}$, $u_c(t) \geq 0$ $(-u_A$ for $u_c(t) < 0)$

2d.
$$V_2(u_A) = [(\tilde{\Theta}(t+1) + \ell u_A)\hat{x}(t+1) - y_r]^2 + (\tilde{\Theta}(t+1) + \ell u_A)P(t+1)(\tilde{\Theta}(t+1) + \ell u_A)^T + \sigma^2 + \lambda L(u_A)$$

with

$$L(u_A) = \ell R_1 \ell^T + \frac{\Phi_b^2 d_2}{p_{b1}(t+1)u_A^2 + 2d_1 u_A + \tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1) + \sigma^2}$$

2e. $a = \hat{b}_1^2(t+1) + p_{b1}(t+1)$

2f. $b = 2\{[(\tilde{\Theta}(t+1)\hat{x}(t+1) - y_r]\hat{b}_1^2(t+1) + \ell P(t+1)\tilde{\Theta}(t+1)^T\}$

2h.

$$u_B = \begin{cases} \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} & u_c(t) \geq 0 \\ \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} & u_c(t) < 0 \end{cases}$$

Step 3: Calculate ASOD control signal.

$$u_{as} = \gamma - \frac{\dfrac{\partial V(\gamma)}{\partial u}}{\dfrac{\partial^2 V(\gamma)}{\partial u^2}}$$

This involves the calculations:

3a. $$\frac{\partial L}{\partial u} = -\Phi_b^2 d_2 \frac{2(p_{b1}(t+1)\gamma + d_1)}{\left[p_{b1}(t+1)\gamma^2 + 2d_1\gamma + \tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1)^T + \sigma^2\right]^2}$$

3b. $$\frac{\partial^2 L}{\partial u^2} = \Phi_b^2 d_2 \frac{6(p_{b1}(t+1)\gamma + d_1)^2 - 2d_2}{\left[p_{b1}(t+1)\gamma^2 + 2d_1\gamma + \tilde{\Theta}(t+1)P(t+1)\tilde{\Theta}(t+1)^T + \sigma^2\right]^3}$$

3c. $$\frac{\partial V_2}{\partial u} = 2\Big[(\tilde{\Theta}(t+1)\hat{x}(t+1) + \hat{b}_1(t+1)\gamma - y_r)\hat{b}_1(t+1) + \ell P(t+1)\tilde{\Theta}(t+1)^T + p_{b1}(t+1) \Big] + \lambda \frac{\partial L(\gamma)}{\partial u}$$

3d. $$\frac{\partial^2 V_2}{\partial u^2} = 2(\hat{b}_1^{\,2}(t+1) + p_{b1}(t+1)) + \lambda \frac{\partial^2 L(\gamma)}{\partial u^2}$$

As mentioned above, simulations have shown that setting $\partial^2 L/\partial u^2 = 0$ in step 3b, can give satisfactory results. Even though the output losses are comparable, omission of the $\partial^2 L/\partial u^2$ term gives higher losses according to the loss function V_2.

APPENDIX 2

Simnon programs used in simulating the system with an integrator. The system "Newdual2" is the regulator for the system, containing the control algorithms for cautious, approximated ASOD, and near-optimal dual (for ϱ=0.0) controllers. "Syst" is the first order system, and "Condual" connects the system and regulator.

```
DISCRETE SYSTEM Newdual2
"Dual regulator for a system of the form:
"y(t) - y(t-1)=b1(t+1)u(t-1) + sigma*e(t)
"where b1(t+1)=fi*b1(t) + ro*v(t)
"This system component gives a choice between three controllers:
"cautious, ASOD, and an approximate optimal dual controller for syst
"with ro=0.0 (R1b=sqr(ro)). The ASOD control signal approximately
"minimizes approximated ASOD loss function V2.
"(Connected by cdual to syst.)
"Craig Elevitch   August 11, 1983

state xcap2  Pb  uo  ytm1  Psum  V2sum  track
new  nxcap2 nPb nuo nytm1 nPsum nV2sum ntrack
input y R1b
output u
time t
tsamp ts
                                            "Kalman filter:
Pb:0.25                                     "initializations
xcap2:0.1
K2= fi*Pb*uo/(sig2 + Pb*uo*uo)              "Kalman gain
Pbt1= fi*(fi-K2*uo)*Pb + R1b                "variance for b1(t+1)
xbt1= fi*xcap2 + K2*(y-1*ytm1-xcap2*uo)"estimate for b1(t+1)
ts= t + h

uc= xbt1*(yr-y)/(xbt1*xbt1 + Pbt1)          "cautious control

"Find a good point to expand about:
"V2 minimum is on the same side of zero as uc
fi=sqrt(1-R1b)
d1=0.0
d2=Pbt1*sig2
point=(sqrt(d2/3)-d1)/Pbt1
uA=if uc>0.0 then point else -point
V2uA=(y+xbt1*uA-yr)*(y+xbt1*uA-yr)+Pbt1*uA*uA+sig2+lam*LuA
LuA=R1b+(fi*fi*d2)/(Pbt1*uA*uA+sig2)
a=xbt1*xbt1 + Pbt1
b=2*(y*xbt1-xbt1*yr)
c=y*y+yr - 2*y*yr + sig2 + lam*R1b - V2uA
point2=(-b+sqrt(b*b-4*a*c))/(2*a)
point3=(-b-sqrt(b*b-4*a*c))/(2*a)
uB=if uc>0.0 then point2 else point3
best= (uA + uB)/2
```

```
"Possible expansion pts. (must set variables of previous exppts to 1):
"uc (with dL), best ((uA+uB)/2, with or without d2L/du2 term)

alfa=if aver<0.5 then uc else best

dV2= 2*(y + alfa*xbt1 - yr)*xbt1 + Pbt1*alfa
dL=-2*fi*fi*d2*Pbt1*alfa/(sig2+Pbt1*alfa*alfa)^2
d2V2= 2*(xbt1*xbt1 + Pbt1)
glup1=(6*(Pbt1*alfa+d1)*(Pbt1*alfa+d1)-2*d2)/(sig2+Pbt1*alfa*alfa)^3
d2L= if id2L<0.5 then 0.0 else fi*fi*d2*glup1
L=R1b+(fi*fi*d2)/(Pbt1*uo*uo+sig2)
target=alfa-(dV2+lam*dL)/(d2V2+lam*d2L)   "target point


"Compare even dual control algoritm from Åström Helmersson paper
""Dual Control of a Low Order System," Coden:lutfd2/(tfrt-7249)/1-18/(1983
"(note symmetry considerations)
ae=y/sqrt(sig2)
be=abs(xbt1/sqrt(Pbt1))   "normalized gain
factor=sig2/sqrt(Pbt1)
nu=(0.56+be)*abs(ae)/(2.2+0.08*be+be*be)+(1.9/(1.7+be*be*be*be))
v1=if xbt1>0.0 then nu*factor else -nu*factor
v=if y>0.0 then  -v1 else v1

u=if dual<0.5 then uc else if astah<0.5 then target else v

nxcap2=xbt1
nPb=Pbt1
nuo=u
nytm1=y
nPsum= Psum + Pb
nV2sum=V2sum+(y+xbt1*u-yr)*(y+xbt1*u-yr)+u*u*Pbt1+sig2+lam*L
ntrack=track + (y-yr)*(y-yr)

"parameter values
h:1.0
sig2:1.0
lam:0.5
yr:0.0
id2L:0
dual:0
astah:0
aver:0

END
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

DISCRETE SYSTEM syst
"Discrete time system with integrator
"Craig Elevitch   August 11, 1983

state b  x  loss  sum
new   nb nx nloss nsum
```

```
input e v u R1b
output y
time t
tsamp ts

ts=t+h
y=x

nx=x+b*u+e*sqrt(sig2)
nb=fi*b+v*sqrt(R1b)
nloss=loss+(y-yr)*(y-yr)
nsum= sum + y

fi=sqrt(1-R1b)     "syst has unit variance
h:1
yr:0.0
sig2:1.0

END
```

- -

```
CONNECTING SYSTEM cdual
"Connects newdual and syst
e[syst]=e1[noise1]
v[syst]=e2[noise1]
u[syst]=u[newdual2]
y[newdual2]=y[syst]
R1b[syst]=R1b
R1b[newdual2]=R1b
R1b:0.01   "R1b=sqr(ro) where ro is s.d. of v(t)
END
```

- -

```
macro gonew
let n.noise1=2
let nodd.noise1=19
syst newdual syst noise1 condual
par same:1
store u[syst] y[syst] b[syst] xcap2 Pb loss
END
```

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