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# Necessary Conditions in Adaptive Control

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NECESSARY CONDITIONS IN ADAPTIVE CONTROL

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NECESSARY CONDITIONS IN ADAPTIVE CONTROL

## 1. INTRODUCTION

Parameter adaptive control is a potentially important extension of the traditional system theoretic design of controllers, necessitated in practice because certain of the system parameters are unknown for a variety of reasons. These include:

(i) the system parameters (e.g. altitude, attitude, or, for example, the value of a resistor) may change;

(ii) certain of the system parameters, especially those related to high frequency modes, cannot be precisely determined experimentally; or

(iii) the inclusion of flexible and/or high frequency modes into the system model is prohibited by the additional complexity (e.g. infinite versus finite dimensional) incurred and thus the system parameters represent only an approximation to true physical parameters.

The resurgence of interest in adaptive control in the past decade has stemmed from the impressive practical applications of self-tuning regulators originating with the research effort at the Lund Institute of Technology ([1] - [9]), from the application of new results in system identification (e.g. [4], [10] - [12]), and also from results in model reference adaptive control (e.g. [13] - [15]).

As one can see from the references cited, the most spectacular successes in practical applications of adaptive control have been in process control. In fact, in 1983 the Swedish firm, ASEA, produced and sold approximately 140 self-tuning regulators for implementation in process control plants. On the other hand, one of the legitimate criticisms [16] of the "universal" applicability of existing adaptive control schemes is that the theoretical assumptions on which many of these schemes are based are very restrictive, typically involving assumptions about the number of system poles, and of system zeroes, and hence about their difference, the relative degree of the system. For many reasons, including (ii) - (iii) above, high frequency information such as the relative degree cannot necessarily be regarded as known in practice. In this sense, existing adaptive control schemes may be expected to function quite well for slow systems like those encountered in process control while these schemes may be expected to have a more limited range of applicability for faster systems such as high performance aircraft. This is well documented; see for example [17], which evaluates important performance aspects such as identification accuracy in steady flight, convergence characteristics, and tracking characteristics for standard flight transitions for an identification-based adaptive controller implemented (in simulation) on NASA's F-8 digital "fly-by-wire" research aircraft. The undesirable characteristics noted in [17] which are due to the "dichotomy" between identification and control, e.g. identification lag or "bursting", have recently been theoretically analyzed in some generality by several authors, see esp. [11].

In the light of this research, it is clear that adaptive control is a promising field which, however, continues to require development of algorithms, a rigorous analysis of algorithm convergence under well-understood hypotheses, and a clear delineation of its applicability. Evidently, what is required is the development of a theory of necessary conditions for adaptive control, comparable in scope to necessary conditions (e.g. Pontryagin's Principle) in optimal control theory. In this paper we initiate such a program giving a derivation (which relies heavily on non-linear dynamics, center manifold theory, etc.) of the necessary conditions for adaptive stabilization. One goal of such a theory is to clearly and unequivocally delineate the scope of applicability of adaptive control. Our program is in the spirit of [18] which focused on necessary conditions in the one-dimensional situation. Since such a theory would be algorithm independent it would also provide:

(i) a rigorous understanding of which desirable or undesirable characteristics any particular adaptive control algorithm must have;

(ii) a basis for evaluating the relative performance of specific adaptive controllers; and

(iii) a potential tool in the design of new adaptive control algorithms.

Examples of (i) would include: How much (if any) high-frequency information is needed to design an adaptive stabilizer, or a model reference adaptive controller? As an illustration, the output feedback-based adaptive stabilizers designed in [19] did not require, as identification-based adaptive stabilizers do, an upper bound on the system-order. On the other hand, these controllers do require that the system has relative degree one, and one of our goals is to determine if such high frequency conditions are necessary for adaptive stabilization.

In this paper we address these issues proving an explicit form of a rather intuitive principle relating the nonlinear complexity of an arbitrary adaptive controller to the linear complexity inherent in any classical controller in the case where the system parameters are known. This in fact impacts the aspects (i) - (iii) mentioned above. Before stating this "principle of adaptive stabilization", we remark that the proof of our technical result (which comprises the latter part of this paper) requires concepts from nonlinear dynamics which are quite new to adaptive control, e.g. center manifolds theory and reduction techniques, but which are equally applicable to other problems in adaptive control. For this reason, we devote section 3 to a brief summary of these methods while the proof itself is contained in section 4.

Our first main result (with definitions provided in section 2 and an illustration in Example 2.6) is:

Theorem 2.2. Suppose a smooth controller of order q is an adaptive stabilizer, universal for a class $\Sigma$ of linear systems. Then the poles of each system in $\Sigma$ can be placed in the closed left half plane by some linear compensator of order q.

Although plausible, this result requires a careful formulation and, of course, a rigorous proof. As immediate corollaries (Corollary 2.3 - 2.4) we give what is, to the best of our knowledge, the first proofs of the nonexistence of smooth, universal, finite-dimensional, adaptive stabilizers for linear systems and for minimum phase linear systems. We also interpret the standard hypotheses in adaptive stabilization as giving estimates

$$q^*(g) \leq q \tag{1.1}$$

on the minimal order of a stabilizing compensator. In this manner, we observe (Remark 2.5) that high-frequency information is not necessarily needed a priori for adaptive stabilization of certain classes $\Sigma$ of linear systems. Indeed, high frequency information is only one way to obtain an estimate (1.1) and does not, e.g., always give $q^*(g)$. This point is made even more forcibly in Mårtensson [24] where it is proved that the necessary condition in the principle of adaptive stabilization is in fact sufficient.

## 2. NECESSARY CONDITIONS FOR ADAPTIVE STABILIZATION

As we stated in the introduction one of our goals is the development of fundamental principles for adaptive control, similar in breadth to the principle of optimality. Specifically, one "candidate" principle from which one can clearly derive necessary conditions might be roughly stated as follows

*Principle of Adaptive Control: If a control objective can be achieved in an adaptive context, it can be achieved using linear compensation if one has knowledge of the system parameters.*

As simple as it may seem, the assertion does require rigorous formulation and verification. In this section, we illustrate this principle and indicate what we expect is involved in this general research program, in the context of adaptive stabilization.

We therefore begin with a definition of adaptive stabilization (cf. [18]). Explicitly, we suppose $\Sigma$ is a class of linear systems

$$\dot{x} = Ax + Bu \tag{2.1a}$$

$$y = Cx \tag{2.1b}$$

where $u \in R^m$, $y \in R^p$. By a smooth controller we mean a $C^\infty$ control system

$$\dot{z} = f(z,y) \tag{2.2a}$$

$$u = g(z,y) \tag{2.2b}$$

where $z \in R^q$ (in what follows, we could also take $z \in M$ with $M$ a smooth manifold of dimension $q$).

We note that, as a consequence of the "ODE method" (see e.g. [4]), self-tuning regulators based on an explicit identification subroutine can be modeled as a closed-loop system arising from (2.1) - (2.2). Model reference adaptive controllers also take the form (2.1) - (2.2) where the dynamics (2.2) consist of (a universal) observer dynamics together with nonlinear parameter adjustment equations for the observer parameters. Finally, the "dynamic-feedback" based adaptive controllers recently introduced in [19], [21], [22], [24] all have the form (2.1) (2.2).

We therefore regard (2.2) as an algorithm independent definition of an (smooth) adaptive controller, provided (2.1) - (2.2) achieves the desired control objective. We will illustrate this in the case where the desired objective is closed-loop (internal) stability.

Definition 2.1. A smooth controller is a universal adaptive stabilizer for $\Sigma$ provided for any fixed system (2.1) (in $\Sigma$) and for all initial data $z_0$, $x_0$, the closed-loop system (2.1) - (2.2) satisfies:

(i) $$\lim_{t\to\infty} x_t = 0 \tag{2.3a}$$

(ii) $$\lim_{t\to\infty} z_t = z_\infty \text{ exists.} \tag{2.3b}$$

With these conventions, we can now give one precise formulation of the principle of adaptive control introduced above. For adaptive stabilization, this takes the form

Theorem 2.2. Suppose a smooth controller of order $q$ is an adaptive stabilizer, universal for a class $\Sigma$ of linear systems. Then the poles of each system in $\Sigma$ can be placed in the closed left half plane by some linear compensator of order $q$.

Since a linear system with an eigenvalue on the imaginary axis of geometric multiplicity > 1 is unstable, Theorem 2.2 does not quite imply that any system in $\Sigma$ can be stabilized by a linear compensator of order $q$. A proof that this non-generic situation can be excluded requires more subtle arguments and can be given under auxiliary technical hypotheses. We will treat this case in a future, expanded version of the present paper.

It appears that a necessary condition for adaptive stabilization is an a priori knowledge of the order of a stabilizing linear compensator for each system in $\Sigma$. For example, Theorem 2.2 gives an interesting interpretation of the standard assumptions for adaptive stabilization:

(i) each system in $\Sigma$ is minimum phase;

(ii) each system in $\Sigma$ has relative degree $r \leq r^*$.

Using the classical root-locus theory hypotheses (i) - (ii) yield the estimate

$$q \leq r^* - 1$$

for the minimal order of a stabilizing compensator, in harmony with Theorem 2.2. We remark that B. Mårtensson has just shown [24] that the necessary condition in the principle of adaptive stabilization is in fact also sufficient for adaptive stabilization, justifying our interpretation of the hypotheses (i) - (ii).

We now wish to point out an immediate corollary of Theorem 2.2. To our knowledge, this is the first rigorous proof of a very plausible folklore result.

Corollary 2.3. There is no finite dimensional, smooth controller which is a universal adaptive stabilizer for all m input, p output linear systems.

Similarly, we conclude:

Corollary 2.4. There is no finite dimensional smooth controller which is a universal adaptive stabilizer for all scalar, minimum phase systems.

On the other hand, if one fixes (say) an upper bound on the McMillan degree n of the system then there is a universal adaptive stabilizer. For scalar systems with $n = 1$, this is a celebrated construction due to Nussbaum [21], while for multivariable systems of order $n \geq 1$ this follows from Mårtensson's Theorem [24].

Remark 2.5 (High Gain/High Frequency Assumptions). Another important consequence of Theorem 2.2 is that the question of how much high frequency information must be tacitly assumed in the design of an adaptive stabilization scheme is now reduced to a purely linear system-theoretic question: Given a system g(s) what information is needed to compute an upper bound

$$q^*(g) \leq q \tag{2.4}$$

on the minimum order $q^*(g)$ of a stabilizing compensator. Important steps have recently been taken ([25] - [28]) towards a computation of $q^*(g)$ and we expect this classical problem will ultimately be resolved. In any case, we remark that classical lead-lag compensation, which is based on high-gain techniques, does not always give the computation of $q^*(g)$. From this point of view, it is therefore not clear that high-gain methods are necessary. To the contrary, as we have just seen above, there are many classes $\Sigma$ of systems where the estimate (2.4) does not require high frequency information (see esp. [24]).

Example 2.6. We now illustrate Definition 2.1 and Theorem 2.2 in a simple case, also motivating our later use of nonlinear dynamics. Consider the unstable first-order linear system

$$\dot{y} = y + bu \tag{2.1)'$$

where b is a positive but unknown constant. If b were known, one simply would choose the constant gain controller

$$u = -ky, \; k > 1/b$$

giving rise to the stable closed-loop system

$$\dot{y} = (1-bk)y.$$

Returning to the adaptive case, of course no fixed choice of k will stabilize (2.1)' if $k \leq \frac{1}{b}$.

On the other hand, the smooth controller

$$\dot{k} = y^2, \; u = -ky \qquad (2.2)'$$

increases k, on line, as long as $y_t$ is not going to zero. Indeed, the closed-loop system (2.1)' - (2.2)'

$$\begin{aligned} \dot{y} &= (1-bk)y \\ \dot{k} &= y^2 \end{aligned} \qquad (2.5)$$

satisfies the conditions (2.3) for all initial data, $y_0$, $k_0$. In other words, (2.2)' is an adaptive stabilizer which is universal for the class $\Sigma$ of systems (2.1)'. The phase portrait of (2.1)' - (2.2)' is depicted in Figure 2.1, using the fact that the function

$$E(k,y) = y^2 + bk^2 - 2k \qquad (2.6)$$

is an invariant integral for (i.e. is constant on the trajectories of) (2.1)' - (2.2)'. Thus the integral curves are "semi-ellipsoids".

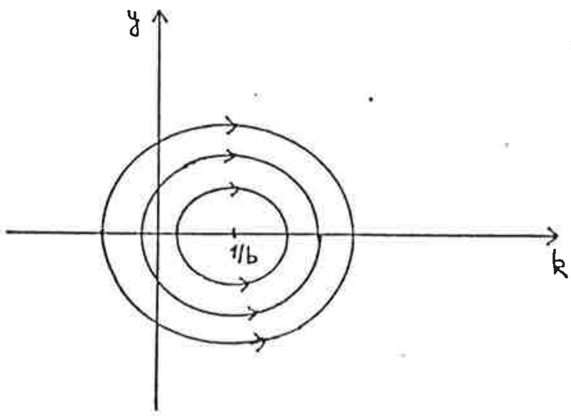



Figure 2.1 The Phase Portrait of the (universal) adaptive stabilizer.

As for Theorem 2.2, the assertion that (2.1)' can be stabilized using constant gain is of course correct, but it is useful to see this from Figure 2.1. For any initial condition $(k_0, y_0)$ the forward limit $(k_\infty, y_\infty)$ satisfies

$$(k_\infty, y_\infty) = (k_\infty, 0) \qquad k_\infty \geq 1/b. \qquad (2.7)$$

The Jacobian of (2.1)' - (2.2)' at the equilibrium (2.7) is therefore

$$J = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 - bk \end{array}\right]. \tag{2.8}$$

From (2.7) we know spec(J) $\subset \overline{\mathbb{C}^-}$, but we can also deduce this from the local form (cf. section 3) of the flow in a neighborhood of (2.7). In general, Theorem 2.2 would follow by an elementary linearization argument if it were true that J is nonsingular, i.e. that the equilibrium of (2.5) were isolated. As examples show, however, this is not always the case. Thus, the presence of nonisolated equilibria, i.e. a nontrivial center manifold, requires a more subtle argument which we will sketch in section 3 - 4.

## 3. RESUMÉ OF RESULTS FROM NONLINEAR DYNAMICS

Let $X : M \to TM$ be a $C^r$ vectorfield defined on a smooth n-dimensional manifold M. We denote by $\phi_t$ the flow generated by X, i.e. $x(t) \equiv \phi_t(x)$ is the unique solution of the ordinary differential equation

$$\dot{x}(t) = X(x(t)),\ x(0) = x,$$

which passes through x at time t = 0.

The points $a \in M$ where the vectorfield vanishes, $X(a) = 0$, are the equilibrium points of X. If $\phi_t(x)$ converges to some $a \in M$ for $t \to \infty$, a is necessarily an equilibrium point of X.

The <u>inset of X</u> is the set

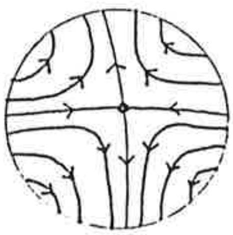
$$W^s(a) := \{x \in M \mid \phi_t(x) \to a \text{ for } t \to \infty\},$$

while the <u>outset of X</u> is defined as

$$W^u(a) := \{x \in M \mid \phi_t(x) \to a \text{ for } t \to \dot{\infty}\}.$$

Even locally, these sets can look rather complicated.

If a is an equilibrium point of X, the linearization $DX_a : T_aM \to T_aM$ of X at a is well defined. Let c, resp. s, resp. u denote the number of eigenvalues of $DX_a$ (counted with multiplicities) whose real part is equal to 0, resp. < 0, resp. > 0. Let $E^c$, resp. $E^s$, resp. $E^u$ denote the corresponding generalized eigenspaces of $DX_a$. The equilibrium point a is called <u>hyperbolic</u>, if $DX_a$ has no eigenvalues on the imaginary axis, i.e. of u + s = n.



<u>Figure 3.1.</u> Hyperbolic equilibrium point.

Definition 3.1. A center manifold of X at the equilibrium point $a \in M$ is a c-dimensional submanifold $W^c$ of M with

(1) $a \in W^c$, $T_a W^c = E^c$

(2) $X(x)$ is tangent to $W^c$ for each $x \in W^c$.

Similarly, a center-stable manifold of X is defined as a c+s-dimensional submanifold $W^{cs}$ of M, tangent to $E^c \oplus E^s$ at a, such that the vectorfield X is tangent to $W^{cs}$ at any point $x \in W^{cs}$. Center manifolds (resp. center-stable manifolds) do exist:

Center Manifold Theorem 3.2. If X is a $C^r$ vectorfield on M, then there exists a $C^r$ center manifold $W^c(a)$ resp. a $C^r$ center-stable manifold $W^{cs}(a)$ of X at a. $W^c(a)$ (resp. $W^{cs}(a)$) is locally invariant: there is a neighborhood V of a in M with:

$$x \in W^c(a),\ t > 0 \text{ and } \phi_t(x) \in V \Rightarrow \phi_t(x) \in W^c(a).$$

For proofs see e.g. [30] or the Appendix written by Al Kelley in [29].

A very useful tool from bifurcation theory is the following reduction theorem, due to [31] and [32]. It implies that the only interesting local recurrence phenomena of a vectorfield occur on the center manifold.

Reduction Theorem 3.3. Let X be a $C^r$ vectorfield on M and $a \in M$ an equlibirium point. X is locally topological equivalent to a $C^r$ vectorfield of the form

$$\dot{x}_1 = X_1(x_1)$$
$$\dot{y} = y$$
$$\dot{z} = -z$$

where $x_1 \in \mathbf{R}^c$, $y \in \mathbf{R}^u$, $z \in \mathbf{R}^s$.

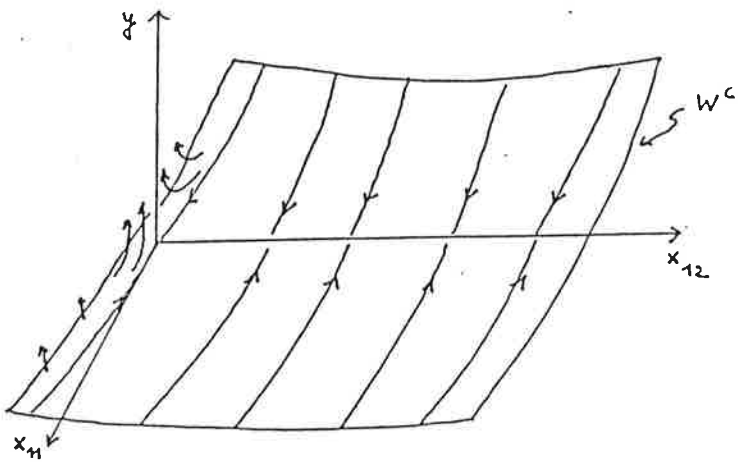



Figure 3.2. A Center Manifold

Example 3.4. (Example 2.6 continued.) Consider the adaptive stabilizer (2.2)' which is universal for the class $\Sigma$ of systems (2.1)' with $b > 0$. This defines a vector field X on $R^2$ via

$$X(k,y) = (y^2,(1-bk)y)$$

and therefore has the entire k-axis as its set E of equilibrium points

$$X(k,y) = (0,0).$$

Since the linearization of $X(k,y)$ at $(k,0) \in E$ is given by (2.7), X has a 2-dimensional center manifold at $(1/b,0)$, compare Figure 2.1, and a 1-dimensional center manifold at any $(k_\infty,0)$, $k_\infty > 1/b$.

SKETCH OF THE PROOF

In order to prove Theorem 2.2, it suffices to consider a fixed n-th order linear system (2.1) and a fixed q-th order smooth controller (2.2) such that the closed-loop system (2.1) - (2.2) satisfies the conditions of Definition 2.1. Recalling Examples 2.6 - 3.4, we want to first prove that there exist some $(z_\infty,0) \in E$, the equilibrium set, such that

$$\dim W^{cs}(z_\infty,0) = n + q.$$

From the linearization of (2.1) - (2.2) at this center-stable equilibrium we will then construct a linear compensator, of order q, for (2.1) which places the closed loop poles in $\overline{C^-}$. Suppose a is an equilibrium point of a $C^r$ vectorfield $X : M \to TM$ and let u denote the number of eigenvalues of the linearization $DX_a$ at a with positive real part.

Ball Lemma 4.1. Let $u \geq 1$. Then there exists an open neighborhood $V_a$ of a and a residual subset $\Omega_a$ of points $x \in M$ such that $\phi_t(x) \notin V_a$ for infinitely many $t > 0$.

Proof. Let $W^{cs}$ be the center stable manifold of X at a. Since $W^{cs}$ is locally closed there exists an open neighborhood V of a such that $W := W^{cs} \cap V$ is a closed submanifold of V. If $u \geq 1$ then $\dim W = \dim W^{cs} = c + s \leq n - 1$. Thus W is nowhere dense in M and the complement $M \setminus \overline{\phi_t(W)}$ is open and dense in M.

Define

$$A := \bigcup_{t>0} \phi_{-t}(W) \quad , \quad B := \bigcup_{\substack{t>0 \\ t \text{ rational}}} \phi_{-t}(W).$$

We show that $A = B$. Obviously $B \subset A$. For $x \in A$, $\phi_t(x) \in W$ for some $t > 0$. Since $W^{cs}$ is locally invariant also $\phi_s(x) \in W$ for $|t-s|$ small. In particular there is a rational approximation s of t with $\phi_s(x) \in W$. Hence $x \in B$.

Q.E.D.

Since $M\setminus\overline{\phi_{-t}(W)}$ is open and dense, the set

$$\Omega_a := M \setminus A \subset \bigcap_{\substack{t>0 \\ t \text{ rational}}} M\setminus\overline{\phi_{-t}(W)}$$

is residual (and therefore dense, by the Baire Category Theorem).

For $x \in \Omega_a$ the Reduction Theorem 3.5 implies that the trajectory $t \to \phi_t(x)$ has to leave a fixed neighborhood of a an infinite number of times.

Q.E.D.

Lemma 4.2. Under the hypothesis of Theorem 2.2, there exists an equilibrium $a = (z_\infty,0)$ for which $u = 0$.

Proof. Suppose not. Since $E \subset R^n \times M^q$ is closed, it is countably compact. Choose then a countable covering of E by balls $V_{a_i}$ about $a_i \in E$, as in the Ball Lemma. By the Baire Category Theorem

$$\Omega = \bigcap_i \Omega a_i$$

is residual and hence non-empty. However, if $(z_0,x_0) \in \Omega$ then

$$(z_t,x_t) \not\to E$$

contrary to hypothesis.

Q.E.D.

Now consider the closed-loop system (2.1) - (2.2) in (z,x) coordinates near an equilibrium $(z_\infty,0)$ as in Lemma 4.2.:

$$\dot{z} = f(z,Cx)$$
$$\dot{x} = Ax + Bg(z,Cx) \qquad (4.1)$$

The linearization (4.1) at $(z_\infty,0)$ takes the form

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} f_z & f_y C \\ Bg_z & A+Bg_y C \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \qquad (4.2)$$

where $f_z, f_y$ (etc.) denote the partial Jacobians of f with respect to z,y evaluated at $(z_\infty,0)$. Thus (4.2) is the transition matrix for a linear compensation in closed-loop around (2.1). Moreover, by Lemma 4.2 we know

$$\text{spec} \begin{bmatrix} f_z & f_y C \\ Bg_z & A+Bg_y C \end{bmatrix} \subset \overline{C^-}$$

Therefore, the linear closed-loop system has spectrum in $\overline{C^-}$.

Q.E.D.

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