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Abstract

This is a collection of seven problemsets, used when the author gave a short course on nonlinear system theory at the Department of Automatic Control in Lund, Sweden. The course book was A. Isidori: "Nonlinear Control Systems: An Introduction". However, the problems do not follow the book very closely.

There are student's solutions available in another report. Unfortunately, they are handwritten and in Swedish.

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Problems in Nonlinear Control Theory

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Preface

This is a collection of seven problemsets, used when I gave a short course on nonlinear system theory at the Department of Automatic Control in Lund, Sweden. The course was given during the first half of the spring semester in 1987. The course book was A. Isidori: "Nonlinear Control Systems: An Introduction", but neither the lectures nor the problems followed the book so very closely. During the allotted time—seven weeks—there was not very much possibility to cover the material more than superficially.

The different problemsets was done as homeworks in a weeks time. It should also be noted that problemset 1 was handed out before the start of the course, and due just a few days after the first lecture. It was intended just as a warm-up, without really entering the core of the course.

There are student's solutions available. Unfortunately, they are handwritten and in Swedish.

The references are collected at the last page.

Problemset 1

This is a set of warm-up problem, intended to get used to some simple concepts. These are all in the appendix in Isidori's book. Some new concepts are introduced, but these should hopefully not be too frightening.

1.1 The (scalar) root-locus problem is the following: Static feedback $u = -ky$ is applied to the plant $g(s) = n(s)/d(s) \in \mathbb{R}(x)$. The closed loop characteristic equation will thus be $d(s) + kn(s) = 0$. The root locus is defined as $s : d(s) + kn(s) = 0,\ k \in \mathbb{R}$. Use the implicit function theorem to show that under a certain condition—determine which and when it is satisfied—the root locus locally is a function of k. Determine a differential equation the branches $s_i(k)$ of the root locus satisfies.

1.2 Do the same for the problem of zeros of sampled systems. Here the object of study is the location of the zeros of the sampled system as a function of the sampling interval $h > 0$. (Given an, e.g. strictly proper rational function $G(s)$, then the sampled pulse transfer function can be defined as $H(z) := (1-z^{-1})\mathcal{Z}\mathcal{L}^{-1}G(s)/s$. See e.g. [Åström-Wittenmark 1984], [Åström-Hagander-Sternby 1983], [Mårtensson 1982].)

1.3 For the special case of linear mappings, formulate "the Rank Theorem", p. 255 in Isidori, in matrix language. (In particular, what docs the matrix representation of $H \circ F \circ G$ look like?)

1.4 Consider the function $f : \mathbb{R} \to \mathbb{R}$ discussed on page 254,

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

Draw a figure of f (or even better, plot it). Show Isidori's claim that $f \in C^\infty$. Is there an analytic function with the same property that it vanishes on the negative real axis?

1.5 The first line of the Appendix is "Let A be an open subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ a function." Why is A required to be open? What if $A = \emptyset$?

1.6 Determine in the following cases if the given subsets are open and if they are dense.

a)

$$\mathbb{Q} \subset \mathbb{R}$$

b)

$$(x,0) : x \in \mathbb{Q} \subset \mathbb{R}^2$$

c)

$$\mathbb{R}^2 \setminus (x,0) : x \in \mathbb{Q} \subset \mathbb{R}^2$$

d)

$$(x,y) : x^2 + y^2 < 1 \subset \mathbb{R}^2$$

e)

$$(x,y) : x^2 + y^2 \leq 1 \subset \mathbb{R}^2$$

1.7 Show, e.g. with the rank theorem, that there does not exist a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ if $n \neq m$.

1.8a) Is $\mathbb{R}^3$ together with the scalar product, i.e. with $[x,y] := x \cdot y$, a Lie algebra?

b) Is $\mathbb{R}^3$ together with the vector product, i.e. with $[x,y] := x \times y$, a Lie algebra?

1.9 Does the associative law hold in a Lie algebra? (Hint: previous problem.)

1.10 Classify all one-dimensional Lie algebras up to isomorphism.

1.11 Show that *any* (associative) algebra (i.e. a vector space that also has the structure of a ring; the reader not familiar with these terms are welcome to think about square matrices) can be made into a Lie algebra by defining the Lie bracket as the commutator, i.e. according to

$$[x,y] := xy - yx$$

1.12 (The skew line on the torus.) The circle S^1 can be defined e.g. as $S^1 = \mathbb{R}/2\pi\mathbb{Z} = \mathbb{R} \bmod 2\pi$. The torus is defined as $\mathbb{T} = S^1 \times S^1$. For $a \in \mathbb{R}$, consider the mapping $f_a : \mathbb{R} \to \mathbb{T}$ defined by

$$f_a(t) = (t \bmod 2\pi, at \bmod 2\pi)$$

Show that

a) $f_a(t)$ is a closed, periodic curve if and only if $a \in \mathbb{Q}$.

b) If $a \notin \mathbb{Q}$ then $f_a(t)$ is injective, and furthermore, $f_a(\mathbb{R})$ is a dense subset of $\mathbb{T}$. (i.e. the curve $f(t)$ comes arbitrarily close to every point $p \in \mathbb{T}$.)

c*) Select a particular $a \notin \mathbb{Q}$, and select an embedding of $\mathbb{T}$ in $\mathbb{R}^3$ (the familiar doughnut). Make a beautiful computer plot of the skew line with perspective projection, hidden line removal etc.

Problemset 2

These problems give some exercise on Lie algebras, especially the classical ones. Some concepts on calculus on manifolds are used. Bilinear systems on the sphere are dealt with. Also, existence and uniqueness questions of differential equation are briefly touched upon.

We will need the following theorem.

Theorem. *Consider the system*

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{m} u_i B_i x \qquad x \in \mathbb{R}^n \\ x(0) &= x_0 \end{aligned} \tag{$\heartsuit$}$$

where $u_i(t)$ are smooth functions of time. Assume that B_i is a (Lie-) subalgebra of $\mathbf{gl}(n)$ *(the $n \times n$ matrices, the bracketing operation being the commutator). Then, for small t, there exist functions $g_1(t), \ldots, g_m(t)$ such that $\Phi(t)$, the fundamental solution of ($\heartsuit$) satisfying $\Phi(0) = I$ can be written*

$$\Phi(t) = e^{g_1(t)B_1} \ldots e^{g_m(t)B_m}$$

Remark. If the B_i's do not form a Lie Algebra, the theorem can still be applied if we first ...

2.1a) Show that $\mathbf{sl}(n)$, the $n \times n$ matrices of trace 0, form a Lie algebra.

b) Show that $\mathbf{so}(n)$, the skew-symmetric $n \times n$ matrices, form a Lie algebra.

2.2a) Show that if A is skew-symmetric (skew-hermitian) then e^A is orthogonal (unitary). Is the converse true?

b) Consider the bilinear system

$$\dot{x} = Ax + \sum_{1}^{m} u_i B_i x \qquad x \in \mathbb{R}^n$$

where A, B_i are skew-symmetric. Give a bound on the reachable set (from $x(0) = x_0$) for the system using smooth inputs.

c) Consider

$$\frac{d}{dt} X = \sum_{1}^{n-1} u_i B_i X \qquad X(0) \in \mathcal{SO}(n) \tag{$\spadesuit$}$$

where

$$B_i = \text{block diag}\left\{\underbrace{0,\ldots,0}_{i-1}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0,\ldots,0\right\}$$

Show that for all smooth inputs u_i, $X(t) \in \mathcal{SO}(n)$. Show, possibly for $n = 4$, that $LA\{B_i\}$, the Lie algebra generated by the B_i's, equals $\mathbf{so}(n)$, the Lie algebra of all $n \times n$ skew-symmetric matrices. Use Euler's theorem (see e.g. [Mårtensson, 1986 p. 86]) to prove that (♠) is a globally reachable control system on $\mathcal{SO}(n)$.

2.3 Find two non-commuting matrices A and B such that $e^A e^B = e^{A+B}$. (As is well known, this is in general false if $[A, B] \neq 0$.) (*Hint:* previous problem with $n = 3$. Try to make both sides $= I$.)

2.4 Show that from the viewpoint of modeling an input-output behavior, the class of bilinear system contains the linear systems.

2.5 *(Cooperation allowed, but must be reported.)* Simulate a Wiener process on the sphere S^2 using (♠) (with $x \in \mathbb{R}^3$ instead of in $\mathcal{SO}(n)$), Simnon and the standard system `noise1`. Verify that $x(t)$ does live on the sphere by also plotting $\|x\|^2$. If time permits, make a 3D-plot.

2.6 Consider

$$\begin{aligned} \dot{x} &= f(x,t) \qquad x \in \mathbb{R}^n \\ x(0) &= x_0 \end{aligned} \tag{$\star$}$$

Find out what it means to say that f satisfies a (local) Lipschitz-condition. Find a theorem on $(\star)$, provided f satisfies a Lipschitz-condition.

2.7 Consider the differential equation

$$\dot{x} = x^2$$

Discuss existence and uniqueness of the solution.

2.8 Prove that if $(\star)$ satisfies a *global* Lipschitz-condition, then the solution exists for all t. In particular, what can be said about

$$\dot{x} = A(t)x + \sum u_i(t)B_i(t)x$$

where $A(t)$, $B_i(t)$, and $u_i(t)$ are smooth and bounded?

2.9 Exercise 1.5 in [Byrnes].

2.10 Let $\mathcal{V}(O)$ denote the set of vector fields on the open set $O \subset \mathbb{R}^n$. Show that $\mathcal{V}(O)$ is a Lie algebra if we define

$$[f, g] := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

2.11 Which ones of the following "objects" are manifolds (decent motivation required)?

a) $x \in \mathbb{R}^n : x_1 > 0$

b) $x \in \mathbb{R}^n : x_1 \geq 0$

c) $A \in \mathcal{GL}(n) : \det A = \pi$

d) $A \in \mathcal{GL}(n) : \det A \neq \pi$

e) $(x, y) \in \mathbb{R}^2 : y = x^2$

f) (Skip if you have never heard of projective spaces.) $\mathbb{P}(n)$, the projective space over $\mathbb{R}^{n+1}$

Problemset 3

Lie bracket- and Lie derivative computations are exercised with. Controllability with Lie brackets. Solution of the system equation for bilinear systems.

3.1 Consider the system

$$\dot{x} = f(x) + ub \qquad x(0) = x_0$$

where b is a constant n-vector, with the input

$$u_\varepsilon(t) = \eta\,(\delta(t) - \delta(t-\varepsilon))$$

where $\varepsilon, \eta > 0$. Compute $x(\varepsilon+)$ up to first order in ε, and first order in η. Express the results in terms of the Lie bracket between f and b.

3.2 (Compare Isidori p. 282–283.) Let M be a smooth manifold, and $\mathcal{V}(M)$ the set of vector fields on M. Let $\lambda \in \mathcal{C}^\infty(M)$ and $f, g \in \mathcal{V}(M)$. The *Lie derivative of λ along f* is a function $\in \mathcal{C}^\infty(M)$ defined as

$$(L_f\lambda)(p) = (f(p))\,(\lambda)$$

for $p \in M$. The Lie bracket $[f, g]$ is defined by

$$L_{[f,g]}\lambda = L_f L_g \lambda - L_g L_f \lambda$$

a) Let $M = \mathbb{R}^3$, $f = \begin{pmatrix} x_1x_2 & \sin x_3 & 1 \end{pmatrix}^T$, and $\lambda = x_1 \cos(x_2x_3)$. Compute $L_f\lambda$.

b) Show that $[f, g]$ is a vector field, but the "vector field" defined by $\lambda \mapsto L_f L_g \lambda$ is not.

c) Show that in coordinates

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

3.3 Let

$$f = \begin{pmatrix} \cos(x_3) - \sqrt{x_1} \\ \frac{x_2^{14}}{x_3} \\ \frac{e^{x_1}}{x_2} \end{pmatrix} \qquad g = \begin{pmatrix} \sqrt{x_3} \\ x_1x_2x_3^3 \\ \frac{x_1}{x_2x_3} \end{pmatrix}$$

Compute $[g, \mathrm{ad}_f^2\, g]$.

3.4 Consider the system

$$\dot{x}_1 = u$$
$$\dot{x}_2 = x_1^2$$

Compute the controllability Lie algebra and its span at 0.

3.5 *(Cooperation allowed if reported.)* Consider

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{z} = xy$$

a) Compute the controllability Lie algebra and its span at 0.

b) Given $t > 0$ and an initial condition $\begin{pmatrix} x(0) & y(0) & z(0) \end{pmatrix}^T$, find a control (possibly using delta-functions) that takes the initial state to 0 in time t.

c) By simulation, try some reasonable feedback control law for stabilization of the system around 0.

3.6 Show that, in order to check the Lie algebra rank condition at 0, no finite number of bracket computations suffices uniformly. I.e. there is no algorithm of the type "check these N brackets".

3.7a) Write a CTRL-C-function `LIE(A,B)` that computes the commutator between A and B.

b) Write a CTRL-C-function `AD(A,K,B)` that computes $\mathrm{ad}_A^k B$. (*Hint:* Simpler by recursion...)

3.8 Consider the bilinear system

$$\dot{x} = Ax + u_1(t)B_1x + u_2(t)B_2x \qquad x(0) = x_0$$

where

$$A = \begin{pmatrix} 0 & -4 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 6 & 6 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & -10 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & 2 & 8 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & -12 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 18 & 0 & 0 & 6 \\ 0 & -3 & 0 & 0 & -1 \\ 0 & 3 & 0 & 0 & 1 \end{pmatrix}$$

Solve the equation!

Problemset 4

Some stuff on distributions. Different controllability concepts are played around with.

4.1 Prove Lemma 2.7 in Isidori.

4.2 Prove Lemma 2.9 in Isidori.

4.3 Consider the case of Δ being a one-dimensional distribution on the manifold M. Investigate what smoothness, regularity, and involutivity mean in this case. What does Frobenius' theorem say?

4.4 A system is said to be *small time locally reachable* from x_0 if

$$\bigcup_{0 \le T \le T_0} \mathcal{R}(x_0, T)$$

contains an open neighborhood of x_0 for all $T_0 > 0$.

Now consider the system on the cylinder $S^1 \times \mathbb{R}$ described by

$$\dot{z} = u$$
$$\dot{\theta} = 1$$

where $\theta \in S^1$ is considered only modulo 2π. (This is really a sloppy way of expressing things, but...) Determine if this system is locally accessible, locally accessible at time T, locally reachable, locally reachable at time T, small time locally reachable.

4.5 Give a heuristic "proof" of the following theorem.

Theorem. *Consider the system*

$$\dot{x} = f(x) + \sum_i u_i g_i(x)$$

and assume that R, the smallest distribution closed under bracketing with f, g_i containing f and the g_i's, is nonsingular. Further, assume that for all initial conditions, the flow corresponding to just the drift term is periodic with a least period. More precisely, $x(t, x_0) = \Phi_t^f x_0$ is periodic with period $p(x_0)$, where $p(x_0) \ge c > 0$ for some c.

Under these conditions, the system is locally reachable from x_0 if and only if the Lie algebra rank condition is satisfied at x_0.

4.6 Consider the system

$$\frac{d}{dt}X = B_1 X + \sum_{2}^{n-1} u_i B_i X \qquad X(0) \in \mathcal{SO}(n) \qquad (\spadesuit)$$

where

$$B_i = \text{block diag}\left\{ \underbrace{0,\ldots,0,}_{i-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0,\ldots,0 \right\}$$

Show that ($\spadesuit$) is locally and globally reachable. What about time-T-reachability?

4.7 Consider the following theorem.

Theorem (Brunovsky). *Consider the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

and assume that R, the smallest distribution closed under bracketing with f, g_i containing the g_i's is nonsingular. Further, assume that it has the following "symmetry property": For all $u = (u_1, \ldots, u_m)$ there is $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)$ such that

$$f(-x) + \sum_{i=1}^{m} \tilde{u}_i g_i(-x) = -\left(f(x) + \sum_{i=1}^{m} u_i g_i(x) \right)$$

Then the system is locally reachable from 0 if and only if the Lie algebra rank condition is satisfied at 0.

Use the theorem to show that the linear system

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i b_i$$

is (locally) reachable (from 0) if and only if $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$, where $B = \begin{pmatrix} b_1, \ldots, b_m \end{pmatrix}$.

4.8 *(Cooperation allowed if reported.)* Consider

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= xv - yu \end{aligned}$$

a) Compute the controllability Lie algebra and its span at 0.

b) Show that it is locally reachable and locally time T reachable from 0.

c) By simulation, try some reasonable feedback control law for stabilization of the system around 0.

Problemset 5

Some aspects on observability.

5.1 Formulate a theorem like Theorem I.7.8, but for the case of observability without inputs (i.e. with $u_i(t) \equiv 0$).

5.2 With the results and terminology of Theorem I.7.8, derive (*in great detail!*) the well known result on observability of the system

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i$$

$$y_j = c_j x \qquad j = 1, \ldots, p$$

In particular, show—still using same methods and terminology—that observability with zero inputs are equivalent to observability with arbitrary inputs.

5.3 Find a system on the usual form, which is globally observable, but for which it is not possible (always) to reconstruct $x(0)$ from $y^{(i)}(0), i = 0, 1, \ldots$

5.4 Consider the system

$$\dot{x} = f(x) \qquad x \in M$$

$$y = h(x)$$

where f is an analytic vector field. Write

$$F = \sum_i f_i \frac{\partial}{\partial x_i}$$

This is an operator on the ring $C^\infty(M)$. By its *formal adjoint* F^* we shall mean

$$F^* = -\sum_i \frac{\partial}{\partial x_i} f_i$$

Also the (scalar) function h constitutes an operator on $C^\infty(M)$ by multiplication. Let $\mathcal{O}$ denote the Lie algebra of operators on $C^\infty(M)$ generated by F^* and h. The bracketing operation is defined as the commutator.

Formulate and prove a sufficient condition for local observability around the point x_0, expressed in terms of $\mathcal{O}$. Do not assume that $f(x_0) = 0$. Specialize to linear systems.

5.5 Consider

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + u \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x \qquad x \in \mathbb{R}^3$$
$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

Compute Q for the two cases of $u(t) \equiv 0$ and $u(t)$ arbitrary. Interpretation?

5.6 It is well known that a linear, scalar system has relative degree $r \geq 0$ if and only if it holds that the step response $y(t)$ satisfies $y(0) = \cdots = y^{(r-1)}(0) = 0$ and $y^{(r)}(0+) \neq 0$. Using this property, try to define the relative degree at x_0 for the single-input, single-output system

$$\dot{x} = f(x) + ug(x) \qquad x \in M$$
$$y = h(x)$$

in terms of the vector fields f and g and the function h.

Problemset 6

Input-output relationships and realization theory.

6.1 Consider a body with rest mass m_0, the position described by the spatial coordinate x, and let f an applied force. In special relativity, Newton's second law is replaced by $\frac{d}{dt}p = f$, where

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

As usual, c is the velocity of light, and $v = \dot{x}$. Compute the first three terms of the Volterra series for this dynamical system with input f and output x. Compare with the classical results, i.e. Newton's second law $m\ddot{x} = f$.

6.2 Does Picard iteration have anything to do with the input-output descriptions in Chapter 3?

6.3 Find a necessary and sufficient criterion for a Volterra series to describe a time-invariant input-output relationship. In particular, what about the term w_0? How does the formulas III.2.12, describing the Taylor series expansion of the Volterra kernels, simplify?

To what extent can this be done to the Fliess expansion?

6.4 In Lemma III.1.3, show that "$\max |u_i(\tau)| \leq 1$" can be replaced by the formally weaker requirement "$\max |u_i(\tau)| \leq C$ for an arbitrary constant C".

6.5 Find, if possible, a Volterra series and a Fliess expansion of the dynamical system described by the transfer function

$$g(s) = \frac{1}{s} e^{-s}$$

6.6 Do the details in Example III.1.16.

6.7 Compute the Volterra series and the Fliess expansion for

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1^2 \\ y &= x_2 \end{aligned}$$

6.8 Discuss the possibilities for symbolic or numeric computation of the Volterra kernels. Outline a program. Do the same for the Fliess expansion.

6.9 (Optional.) Write a TEXmacro that typesets a truncated Volterra series (or Fliess expansion). It should take two arguments, N, the desired order for the expansion, and m.

6.10 Consider the bilinear system

$$\dot{x} = Ax + uBx$$

Obtain the Volterra series directly by applying the Peano-Baker expansion on $z := e^{-At}x$.

6.11 Give an example of an input-output relation (on *any* form) that has a input-affine, but not a bilinear realization.

6.12 Give an example of an input-output relation (on *any* form) that has both a input-affine and a bilinear realization but where any bilinear realization must be non-minimal.

6.13 Give a strict meaning and a proof to the statement "*From input-output point of view, the set of bilinear systems is dense in the set of input-affine systems*".

Problemset 7

Disturbance decoupling and exact state-space linearization.

7.1 Consider systems of the form

$$\dot{x} = f(x) + ug(x) \qquad x \in U \subset \mathbb{R}^2$$

for which the linearizability condition $\text{span}\{g(x), [f,g](x)\} = T_x(\mathbb{R}^2)$ is satisfied for all $x \in U$. Under what conditions can an explicit formula

$$\begin{aligned} z &= \varphi(x) \\ v &= \psi(x,u) \end{aligned}$$

be found which takes the system to the form

$$\dot{z} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v$$

In particular, show that the mapping $(x,u) \mapsto (z,v)$ is a diffeomorphism from U if and only if the linearizability condition is satisfied. (We shall say that the system is globally linearizable if this condition is satisfied for all $x \in \mathbb{R}^2$).

7.2 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + u \\ \dot{x}_2 &= x_1 x_2 \end{aligned}$$

Show that this system is not globally linearizable. Discuss approximation of the system by a linearizable system of the form $\dot{x} = f(x) + ug(x)$.

7.3 Discuss similarities and differences between gain scheduling and exact linearization.

7.4 In linear system theory, a subspace $\mathcal{V}$ is called (A,B)-*invariant* if $A\mathcal{V} \subset \mathcal{V} + \text{Im } B$. In linear system theory books the following algorithm for computing the maximal (A,B)-invariant subspace $\mathcal{V}^*_{\mathcal{K}}$ contained in a subspace $\mathcal{K}$ is given:

$$\begin{aligned} \mathcal{V}^0 &= \mathcal{K} \\ \mathcal{V}^k &= \mathcal{V}^{k-1} \bigcap A^{-1}(\text{Im } B + \mathcal{V}^{k-1}) \end{aligned}$$

(Note that A^{-1} denotes the inverse image, and is therefore well defined also for singular A's.) Discuss in detail the relationship between this algorithm and the algorithm in Lemma IV.2.4.

7.5 Consider

$$\dot{x} = Ax + bu$$
$$y = cx$$

where $cb \neq 0$. Compute $\mathcal{V}^*_{\mathbf{ker}\, c}$.

7.6 Discuss linearization of $\dot{x} = f(x,u)$ versus $\dot{x} = f(x) + ug(x)$ where u is scalar.

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