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Abstract

The present study contains an analysis of self-tuning control – a case with an object described by the ARMAX model $A(q)y(t) = B(q)u(t) + C(q)v(t)$ and a controller $R(q,t)u(t) = -S(q,t)y(t)$ where R and S are adjusted according to principles of minimum variance direct adaptive control. The parameter estimation is made with recursive least squares identification.

A state space analysis is made where it is shown that there is asymptotically a subspace without noise corruption. Stochastic Lyapunov function theory is used for analysis. It is shown that there are two attraction points θ_o and θ_{MV} for the parameter estimation. A set around the point θ_o is globally attractive while θ_{MV} is only locally attractive. Convergence to θ_{MV} is therefore possible with trajectories proceeding via a neighbourhood of θ_o.

The stability assumption of previous papers is removed in this paper so that stability properties of self-tuning control are shown with explicit bounds on cost functionals. An information theoretical interpretation is also given.

Key words
Adaptation, Convergence, Self-tuning control, Stability.

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Introduction

The purpose of this paper is to show stability properties of discrete time self tuning direct control as it is presented beginning with Åström and Wittenmark[1]. We use a Lyapunov theory approach.

Self-tuning control is based on least-squares identification. Such solutions are systematically biased in the presence of colored noise and convergence towards correct values of the estimated control parameters is not self evident. It was however stated by Åström and Wittenmark[1] that their self tuning controller will converge to a minimum variance regulator if it converges. Ljung[2] formulated positive real conditions for stationary parameter convergence under the assumption that the trajectories are stable and finite.

Nonstationary analysis of the estimation was made by Solo[3] who showed convergence of pseudolinear regression by 'near supermartingale' methods of Neveu[4]. It is then a necessary condition that the regressors are bounded. Landau and Silveira[5], Landau[6,7] used the same outlines together with hyperstability analysis[8,9] to show parameter convergence of least squares based adaptive control in the presence of noise. The stability does however appear also here as an assumption in the proof. Becker et al[9] used a geometric argument to demonstrate convergence points of parameters when simple gradient methods are used.

In all these papers stability appears as an assumption of the proofs in which convergence is shown. Some of problems of stability and convergence have been treated by Goodwin and Sin[11] although their approach lacks more precise bounds on cost functionals. The purpose of this paper is to remove the stochastic stability assumption in the above papers and to

- Establish stability properties for minimum variance adaptive control based on least squares identification.

- Give explicit bounds on cost functionals

- Give an information theoretical interpretation

A stability investigation must be more than a demonstration of convergence. It is necessary to consider different impacts of disturbances. There are many ways to model a disturbance in a stability investigation. A white noise sequence is often used

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in stochastic control theory. An l^2–bounded disturbance in a deterministic system may be modelled by an initial condition of Lyapunov function in Lyapunov stability theory. A stochastic Lyapunov function study of the transient trajectories from an initial state value to an equilibrium point has the advantage to cover both the convergence aspects and the disturbance rejection aspects of stability, see Kushner[12].

A heuristic argument for global stability of adaptation transients was formulated by Ljung and Wittenmark[13] in the following terms. Assume that the signals increase due to instability induced by the adaptation. As the signals increase more information is available and the parameters converge towards their appropriate stabilizing values. The instability behaviour is then stopped.

Such statements need to be formalized and it is then reasonable to consider two competing convergence points namely

- The convergence point θ_o for parameters associated with the purely deterministic noisefree problem.

- The parameter convergence point θ_{MV} for minimum variance control.

The deterministic aspects of stability and the convergence towards θ_o are similar to earlier results on adaptive control which have been presented by Johansson[14,15]. The results are based on the same assumptions as in chapter 11.3 of Goodwin and Sin[11]. It is shown in this paper that a set around θ_o is globally attractive while the point θ_{MV} is only locally attractive. Stable solutions of large initial magnitudes thus start their trajectories by attraction from θ_o. Eventually, these trajectories enter the domain of attraction of θ_{MV} and converge to the minimum variance regulator.

Scenario of convergence

The proofs of stability will demonstrate that there are two convergence points θ_o, θ_{MV} with the following properties.

- There is convergence in a finite time to a set Ω_o in the neighbourhood of θ_0 with a finite $\|x\|_2$. The convergence point θ_o is attractive in the whole state space outside Ω_o. The attractivity is better on some distance from θ_o and the convergence rates in a close neigbourhood of θ_o may be weak.

-- The convergence point θ_{MV} is attractive inside Ω_{MV} when there is noise interference modeled by the C^*—polynomial. The attractivity of θ_{MV} is only local but the fact that θ_o lies close to θ_{MV} inside Ω_o is sufficient for convergence towards θ_{MV} when the trajectories have entered Ω_o. The convergence to Ω_o takes only a finite time. It is therefore assured that Ω_{MV} is reached so that convergence may proceed towards θ_{MV}.
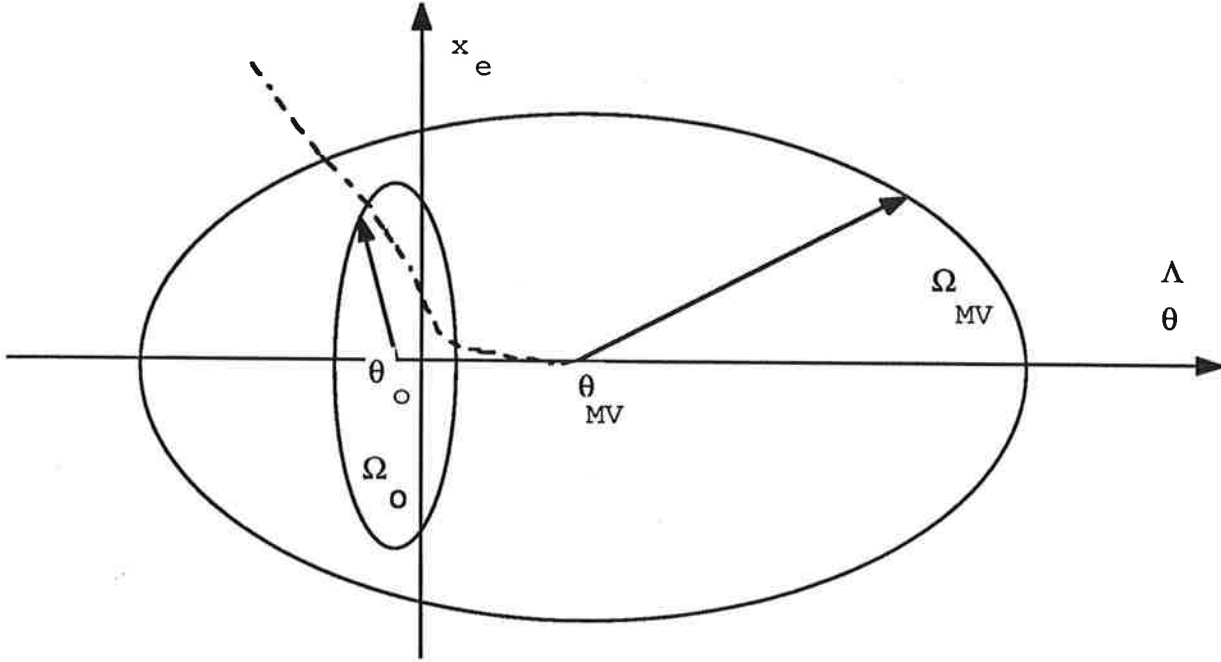
Figure 1. Solution points θ_0 and θ_{MV} with the deterministic attractive set Ω_o and the minimum variance stability domain Ω_{MV}. A typical trajectory from outside Ω_{MV} therefore proceeds via Ω_o towards θ_{MV}.

The convergence towards θ_o is a good model for the behaviour when there are large disturbances or large initial values of signals. Error recovery is then modelled by a convergence from the initial state of the Lyapunov function. The stationary stochastic behaviour is characterized by the convergence towards θ_{MV}.

System Description and Notations

We make a standard system decription[16,17]. Assume that the process model is described by the discrete-time ARMAX-model

$$A^*(q^{-1})y(t) = b_0 q^{-d} B^*(q^{-1})u(t) + C^*(q^{-1})w(t) \tag{1}$$

from the input u and the noise w to the output y with coprime polynomials

$$A^*(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_{n_A} q^{-n_A}$$

$$B^*(q^{-1}) = 1 + b_1 q^{-1} + \cdots + b_{n_A-d} q^{-n_A+d}$$

$$C^*(q^{-1}) = 1 + c_1 q^{-1} + \cdots + c_{n_A} q^{-n_A} \tag{2}$$

The parameter b_0 is a gain factor and d is a time delay. The B^*-polynomial should not have any non-minimum phase zeros. The coprimeness of A^*, B^* and C^* assures that the input-output model (1) also corresponds to a state space realization of order n_A and also the fractional form

$$A^*(q^{-1})\xi(k) = u(k) + v(k) \tag{3}$$

$$y(k) = b_0 q^{-d} B^*(q^{-1})\xi(k) + e(k) \tag{4}$$

with noise components

$$v(k) = \frac{G^*(q^{-1})}{b_0 B^*(q^{-1})} w(k) \qquad e(k) = F^*(q^{-1}) w(k) \tag{5}$$

The polynomials F^* and G^* solve the equation

$$A^* F^* + q^{-d} G^* = C^* \tag{6}$$

A linear control law is decribed by

$$R^*(q^{-1}) u(t) = -S^*(q^{-1}) y(t) + T^*(q^{-1}) u_c(t) \tag{7}$$

where R^*, S^* are polynomials of degrees n_R, n_S, respectively. The closed-loop system should reproduce the reference signal u_c. The appropriate minimum variance regulator is given by

$$R^* = b_0 B^* F^* \qquad S^* = G^* \qquad T^* = C^* \tag{8}$$

in the case of known parameters. The pole polynomial of the closed-loop system is

$$R^* A^* + S^* \left(b_0 q^{-d} B^* \right) = b_0 P^* \tag{9}$$

with

$$P^* = B^* C^* = 1 + p_1 q^{-1} + \cdots + p_{n_P} q^{-n_P} \tag{10}$$

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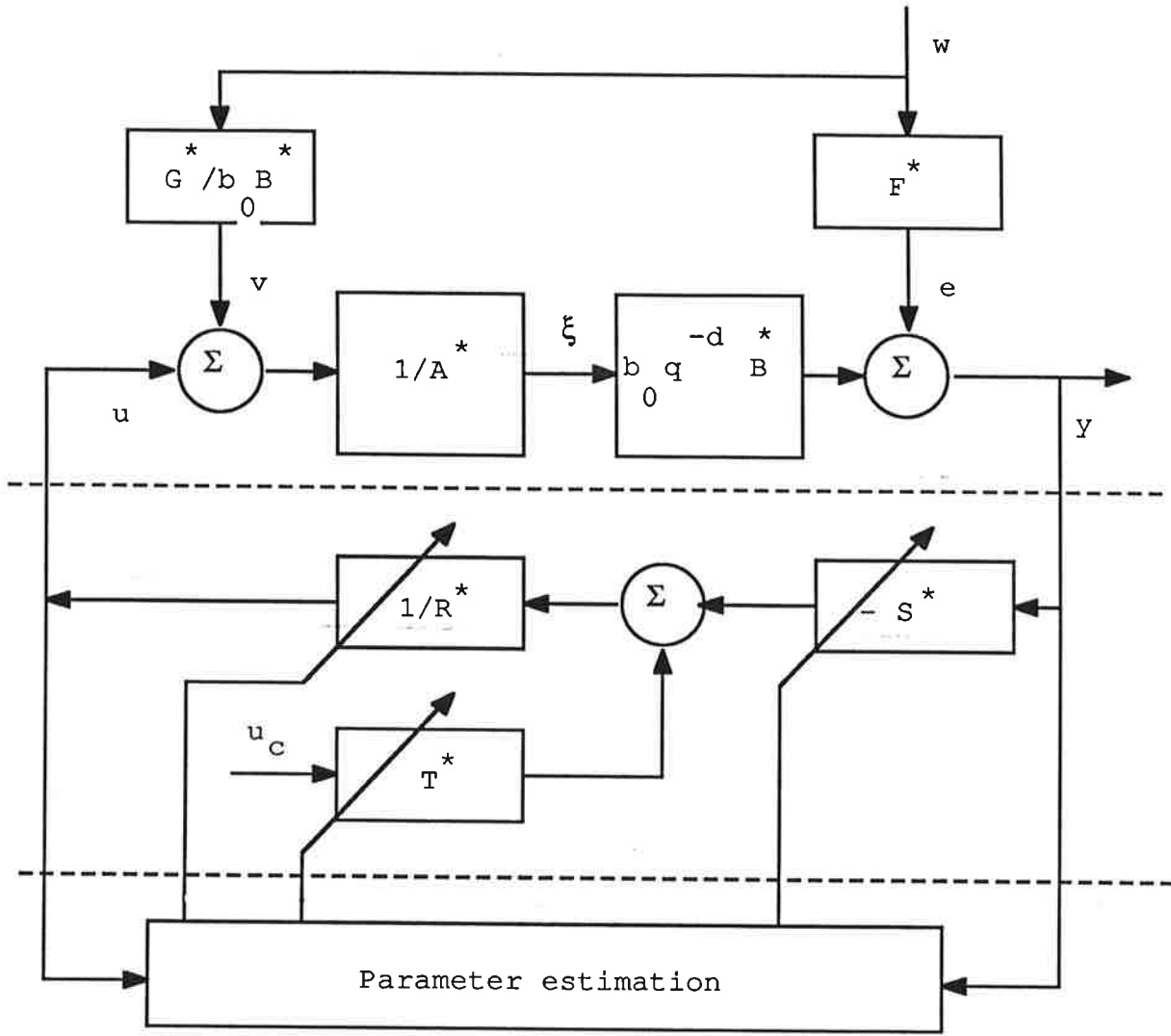
Figure 2. Block diagram of the self-tuning regulator with a noise model according to (3)-(5). Notice that a correctly tuned minimum variance regulator totally decouples ξ from noise interference of w.

Another parallel notation is motivated by the problem of parameter estimation in adaptive control theory. Assume that the parameter vector θ contains the coefficients of at least R^* and S^*. The T^*−polynomial need to be included when tracking a reference value

$$\theta = \left(r_1 r_2 \ldots s_0 s_1 \ldots t_0 t_1 \ldots \right) \tag{11}$$

and φ contains the u and y corresponding to components of θ

$$\varphi(k) = \left(u(k-1)u(k-2)\ldots y(k)y(k-1)\ldots u_c(k)u_c(k-1)\ldots \right) \tag{12}$$

Let also y_c denote the following filtered signals

$$y_c(t) = C^*(q^{-1})u_c(t) \tag{13}$$

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A reformulation of (1) to scalar product form gives a model suitable for estimation. This estimation model is obtained from manipulations[1,11] of (1) with (6), (7).

$$y(k + d) = b_0 u(k) + \theta^T \varphi(k) \tag{14}$$

The correct control law (2) may be reformulated to

$$u(k) = -\frac{1}{b_0}\left(\theta^T \varphi(k)\right) \tag{15}$$

The Adaptive Control Algorithm

The idea of the direct adaptive control algorithm is to find the regulator parameters from input-output data when the process model is unknown. The adaptive control problem is then partitioned into a parameter estimation problem and a control problem. The parameter estimation problem is a standard linear estimation problem and we will study the choice of recursive least squares identification.

The recursive identification algorithm is based on the linear estimation model (14) and is associated with minimization of the least squares criterion

$$J(\widehat{\theta}(k)) = \sum_{i=1}^{k}(y(i) - b_0 u(i-1) - \widehat{\theta}^T(k)\varphi(i-1))^2 \tag{16}$$

The direct adaptive control algorithm then comprises the following steps. The recursive least squares estimation algorithm is given by

$$\widehat{\theta}(k) = \widehat{\theta}(k-1) + P(k)\varphi(k-d)\varepsilon(k) \tag{17}$$

$$P(k) = P(k-1) - \frac{P(k-1)\varphi(k-d)\varphi^T(k-d)P(k-1)}{1 + \varphi^T(k-d)P(k-1)\varphi(k-d)} \tag{18}$$

$$P(0) = P_0 > 0 \tag{19}$$

$$\varepsilon(k) = y(k) - \beta_0 u(k-d) - \widehat{\theta}^T(k-1)\varphi(k-d) \tag{20}$$

where β_0 is a fixed *a priori* estimate of b_0. The adaptive control law should be

$$u(k) = -\frac{1}{\beta_0}\left(\widehat{\theta}^T(k)\varphi(k)\right) \tag{21}$$

where the estimated parameters $\widehat{\theta}$ have replaced the parameters θ of the correct model matching control law.

State Space Model

The control object has been described by the fraction form (3)-(5)

$$A^*(q^{-1})\xi(k) = u(k) + v(k) \tag{22}$$

$$y(k) = b_0 q^{-1} B^*(q^{-1})\xi(k) + e(k) \tag{23}$$

with

$$v(k) = \frac{G^*(q^{-1})}{b_0 B^*(q^{-1})} w(k) \qquad e(k) = F^*(q^{-1})w(k) \tag{24}$$

with G^* and F^* as solutions to (6). The states of the regulator may be represented in a similar way. The regulator makes however use of old input-output data and it is natural to express the control object states as well as the regulator states of φ in terms of ξ. Introduce therefore the state vector

$$x(k) = \Big(\xi(k-1) \quad \xi(k-2) \quad \ldots \quad \xi(k-n) \Big)^T \tag{25}$$

The order of x should be some number $n > n_A$ that is large enough to express $\varphi(k)$ on the form

$$\varphi(k) = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n_A} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \ddots & & & \vdots \\ 0 & \cdots & 1 & a_1 & \cdots & a_{n_A} & \cdots & 0 \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & & 0 \\ 0 & \cdots & 0 & 0 & b_0 & \cdots & & \\ \vdots & & \ddots & \ddots & \ddots & & \vdots & \end{pmatrix} x(k) + \begin{pmatrix} -v(k-1) \\ -v(k-2) \\ \vdots \\ e(k) \\ e(k-1) \\ \vdots \end{pmatrix}$$

or

$$\varphi(k) = M_\varphi x(k) + M_w(k) \tag{26}$$

The matrix M_φ contains the parameters of the A^*- and B^*-polynomials of (22-23). The state x may be decomposed into a desirable reference model state x_m and an error state x_e where

$$x(k) = x_m(k) + x_e(k)$$

The error state x_e should be zero for an adapted closed loop system and one objective of a stability investigation for adaptive control is to show that x_e approaches the origin. A suitable state equation for x is therefore necessary. Define via (9) and (3),(4) the scalar signal ν_{MV}

$$\nu_{MV}(k) = P^*(q^{-1})\xi(k) = u(k) + \frac{1}{b_0}\theta_{MV}^T \varphi(k) \tag{27}$$

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The signal ν_{MV} may be interpreted as the input to the compensated closed loop system. Introduce a Φ_{MV}–matrix and a Γ–vector given by

$$\Phi_{MV} = \begin{pmatrix} -p_1 & -p_2 & \cdots & -p_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} = \begin{pmatrix} & p^T & & 0 \\ & & & \vdots \\ & I_{(n-1)x(n-1)} & & \vdots \\ & & & 0 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (28)$$

The components of the vector p are the coefficients p_i of the polynomial P^* of (10). A state equation of (27) for x on controllable canonical form is given by

$$x(k+1) = \Phi_{MV}x(k) + \Gamma\nu_{MV}(k) \quad (29)$$

$$x(k) = x_m(k) + x_e(k)$$

Notice that the polynomial P^* and the matrix Φ_{MV} represent all poles of the closed-loop system - also those poles which cancel the zeros of the B^*–polynomial.
The noise free system dynamics when θ_o is an attraction point is better described by the following state equation.

$$\nu_o(k) = B^*(q^{-1})\xi(k) = u(k) + \frac{1}{b_0}\theta_o^T\varphi(k)$$

$$x(k+1) = \Phi_o x(k) + \Gamma\nu_o(k) \quad (30)$$

The matrix Φ_o is of the same form as Φ_{MV} but contains the coefficients of B^*.

Parameter stability is also needed and state vectors for the parameter estimation error are given by

$$\tilde{\theta}_{MV}(k) = \hat{\theta}(k) - \theta_{MV} \qquad \tilde{\theta}_o(k) = \hat{\theta}(k) - \theta_o \quad (31)$$

A state space representation of the matrix $P(k)$ of (18) is needed. Introduce the vector Π

$$\Pi(k) = \begin{pmatrix} p_{11}(k) & \cdots & p_{1n}(k) & p_{21}(k) & \cdots & p_{nn}(k) \end{pmatrix}^T \quad (32)$$

The full error dynamics state vector X comprising the states of the control object, controller and the parameter estimation is now

$$X_{MV}(k) = \begin{pmatrix} x_e^T(k) & \tilde{\theta}_{MV}^T(k) & \Pi^T(k) \end{pmatrix}^T \quad (33)$$

$$X_o(k) = \begin{pmatrix} x_e^T(k) & \tilde{\theta}_o^T(k) & \Pi^T(k) \end{pmatrix}^T \quad (34)$$

Assumptions

A1: The polynomials A^*, B^* and C^* are mutually prime.

A2: The polynomial B^* has a stable inverse.

A3: Gain b_0 estimated β_0 such that $0 < \frac{b_0}{\beta_0} < 2$

A4: The noise process $\{w(k)\}_{k=0}^\infty$ has the properties:

$$E\{w(k+1)|\mathcal{F}_k\} = 0 \qquad E\{w^2(k+1)|\mathcal{F}_k\} = \sigma^2 \qquad a.s.$$

$$w^2(k) \le C_w^2 < \infty$$

where $\mathcal{F}_k$ is the $\sigma-$algebra of measurements up to time k.

A5: The parameter vector $\widehat{\theta}(k)$ has a correct number of parameters.

A6: There is a constant C_r such that the reference model state is bounded

$$0 \le x_m^T(k)x_m(k) \le C_r^2 \quad \forall k \ge 0$$

Temporary assumptions

Before giving the analysis we first make some simplifying specializations. Generalizations are made later in order to solve the full problem. Assume that

A7: Reference value $u_c = 0$ and

$$\theta = \begin{pmatrix} r_1 & r_2 & \cdots & s_0 & s_1 & \cdots \end{pmatrix}^T$$

A8: $\beta_0 = b_0$

A9: Time delay $d = 1$

A10: The C^*-polynomial has a stable invers.

A Lyapunov function

We will now investigate the growth rate of the state vector X_o and a Lyapunov function candidate is introduced to represent the components of (33-34). Lyapunov function candidates must be continuous at the origin and grow with the magnitude of all state vector components, see LaSalle[18]. We first present the following result

LEMMA 1

Let

$$x(k+1) = \Phi x(k) + \Gamma \nu(k) \qquad \forall k \geq 0 \qquad (35)$$

for a matrix Φ with all eigenvalues within the unit circle and a vector Γ. Then for all positive matrices Q and all positive constants μ there is a positive definite matrix Λ and a constant c such that the growth rate of the function

$$v_x(x(k)) = \log_e(1 + \mu x^T(k)\Lambda x(k)) \qquad (36)$$

is bounded as

$$v_x(x(k+1)) - v_x(x(k)) \leq \mu \frac{-x^T(k)Qx(k) + c^2\Gamma^T\Lambda\Gamma\nu^2(k)}{1 + \mu x^T(k)\Lambda x(k)} \qquad (37)$$

The positive definite matrix Λ and the positive constant c may be chosen as the solutions to the equations

$$\Phi^T\Lambda\Phi - \Lambda = -Q - I$$

$$c^2 = max\left(2, \lambda_{max}(\Phi^T\Lambda\Phi)\right) \qquad (38)$$

where $\lambda_{max}(\cdot)$ denotes the largest eigenvalue of a positive definite matrix. $\square$

Proof: See appendix 1

A Lyapunov function candidate for x_e is

$$v_x(x_e(k)) = \log_e(1 + \mu x_e^T(k)\Lambda_o x_e(k)) \qquad (39)$$

satisfies the conditions of lemma 1 for the case with $\Phi = \Phi_o$ and a solution $\Lambda = \Lambda_o$ to (38). Another Lyapunov function candidate is proposed for the parameter error of (31).

$$v_\theta(\widetilde{\theta}_o(k)) = \widetilde{\theta}_o^T(k)P^{-1}(k)\widetilde{\theta}_o(k) \qquad (40)$$

In order to represent the states it is finally necessary to consider the $P-$matrix or Π of (32). A scalar, positive, radially growing function to represent the states of the $P-$matrix is e.g.

$$v_P(\Pi(k)) = tr(P^T(k)P(k)) \qquad (41)$$

It is shown in appendix 2 that v_P decreases at each recursion. A Lyapunov function for X of (32) may now be composed from (39), (40), and (41) and we claim:

Proposition 1:

There are constants $\mu > 0$, $K > 0$ and a positive definite matrix Λ_o such that the function

$$V_o(X_o(k)) = v_\theta(\tilde{\theta}_o(k)) + K v_x(x_e(k)) + v_P(\Pi(k)) =$$

$$= \tilde{\theta}^T(k)P^{-1}(k)\tilde{\theta}(k) + K \log_e(1 + \mu x_e^T(k)\Lambda_o x_e(k)) + tr(P^T(k)P(k)) \quad (42)$$

decreases in each recursion at least as

$$V_o(X_o(k+1)) - V_o(X_o(k)) \leq -\mu K \frac{x_e^T(k)Q x_e(k)}{1 + \mu x_e^T(k)\Lambda_o x_e(k)} \qquad V_o(X_o(0)) = V_0 \quad (43)$$

The function V_o is a Lyapunov function for the adaptive system (1-21) and the system is stable in the sense of Lyapunov.

Proof: See appendix 3.

Remark:

The choice $\mu = 1/\sigma^2$ gives an interesting information theoretical interpretation to the Lyapunov function candidate v_x. The function v_x is a logarithmic function of the signal to noise ratio and may be interpreted as the information of x_e contained in the output y of a discrete time channel, see Gallager[19] (ch. 8.2)).

Finite Time Convergence to Ω_o

The stability properties imply certain convergence properties. We now show finite time convergence to the neighbourhood of the origin. For all k and some arbitrary radius r such that $\|x_e(k)\|^2 \geq r^2 \geq 0$ it holds in the noise free case that

$$V_o(X_o(k+1)) - V_o(X_o(k)) \leq \mu K \frac{-q_{min}r^2}{1 + \mu\lambda_{max}r^2} \quad (44)$$

The constants q_{min} and λ_{max} denote extremal eigenvalues of the positive definite matrices Q and Λ_o, respectively. It then follows that x_e converges with an exponential rate from the initial value V_0 to a ball with radius r in at most the finite time

$$T(r) = \frac{1}{\mu K}\left(\frac{1}{q_{min}r^2} + \mu\frac{\lambda_{max}}{q_{min}}\right)V_0 \quad (45)$$

It follows from the properties of a Lyapunov function that the magnitude of $x_e \in l^\infty$ and is limited by a function f of the Lyapunov function V

$$x_e^T(k)\Lambda_o x_e(k) \leq f(V_o(X_o(k))) = \frac{1}{\mu}\exp\left(\frac{1}{K}V_o(X_o(k))\right) \qquad (46)$$

It follows from the properties of a Lyapunov function that the magnitude of x_e is limited by an upper bound f which decreases exponentially at least as

$$f(V_o(X_o(k+1))) \leq \exp\left(\frac{-\mu q_{min}r^2}{1+\mu\lambda_{max}r^2}\right)f(V_o(X_o(k))) \qquad (47)$$

From (45) and (46) follow the propositions

Proposition 2:

The error state vector $x_e \in l^\infty$.

Proposition 3:

The error state vector x_e converges in finite time with an exponential rate to any spherical neighbourhood of radius $r > 0$ around the origin.

Proposition 4:

All l^2−bounded disturbances w with

$$\|w\|^2 = \sum_{j=0}^{\infty} w^2(j) < \infty \qquad \|c\|^2 = \sum_{j=0}^{n_A} c_j^2$$

result in an augmented Lyapunov function so that

$$V(k) \leq V_1 = V_0 + \|c\|^2 \cdot \|w\|^2 \leq \infty \qquad \forall k$$

Stochastic Stability Analysis

Both the behaviour and the analysis are more complicated when there is noise acting on inputs and outputs. The Lyapunov functions may be replaced by stochastic Lyapunov functions or by some other supermartingale analysis. The behaviour is changed by the fact that the noise gives a systematic bias of the least squares estimated parameters from the convergence point θ_o to θ_{MV}. Large transients converge towards X_o but the final convergence is towards $X_{MV} = 0$.
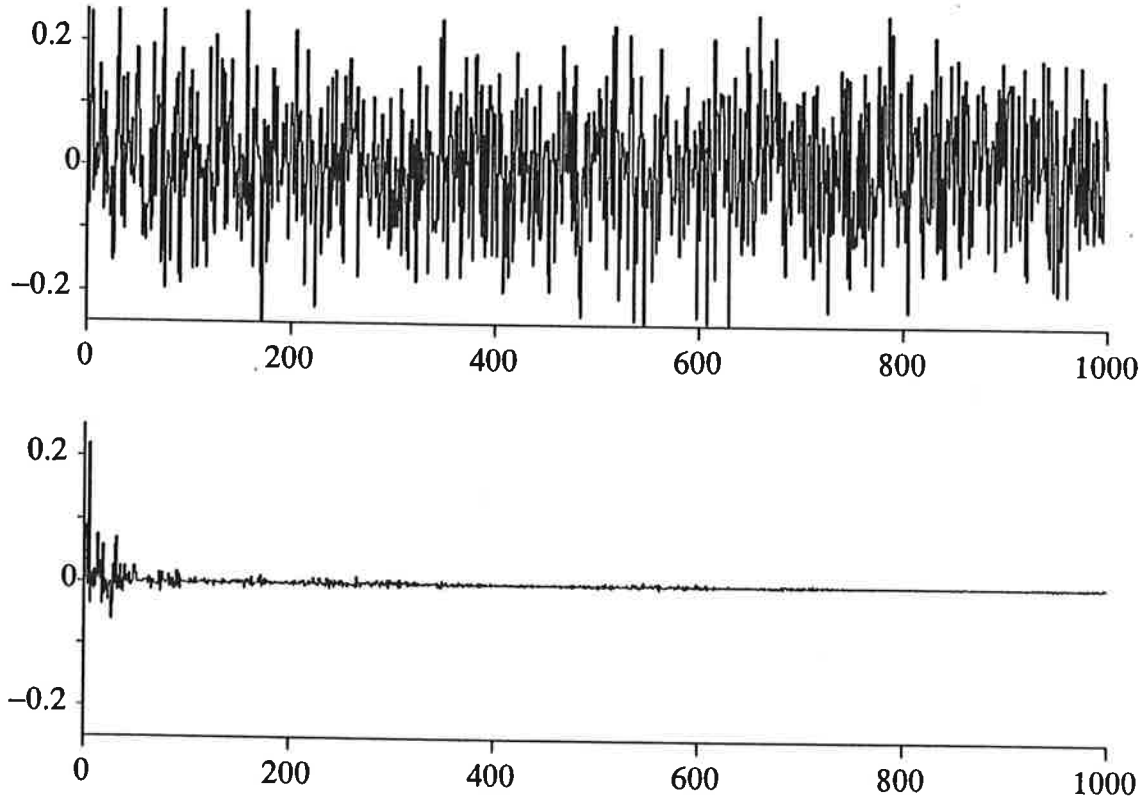
Figure 3. Simulation of a transient of output y *(upper)* and state x *(lower)* vs. time in an adaptive control system with $A^* = 1 - 0.7q^{-1}$, $B^* = q^{-1}$, and $C^* = 1 + 0.3q^{-1}$. Notice that x is asymptotically decoupled from the noise.

Convergence towards θ_o

This convergence point is attractive in a large region of the state space. The attractivity is better on some distance and the convergence rates in a close neigbourhood of θ_o may be weak. Modification of $V_o(X_o(k))$ in presence of noise gives that the expected value of the Lyapunov function develops over time as

$$E\{V_o(X_o(k+1))|\mathcal{F}_k\} - V_o(X_o(k)) \leq$$

$$\leq E\{\left(C^*(q^{-1})w(k+1)\right)^2|\mathcal{F}_k\} - \mu K \frac{x_e^T(k)Qx_e(k)}{1 + \mu x_e^T(k)\Lambda_o x_e(k)} \tag{48}$$

This shows convergence in a finite time to a neighbourhood of θ_o with a finite $\|x_e\|$ for limited noise $w \in l^2$. Large excursions in the state space therefore always finish with convergence to a ball Ω_o around $X_o = 0$.

Convergence within Ω_{MV} towards θ_{MV}

Convergence analysis within Ω_{MV} may be performed with several Lyapunov functions. We continiue with logarithmic functions although a quadratic Lyapunov
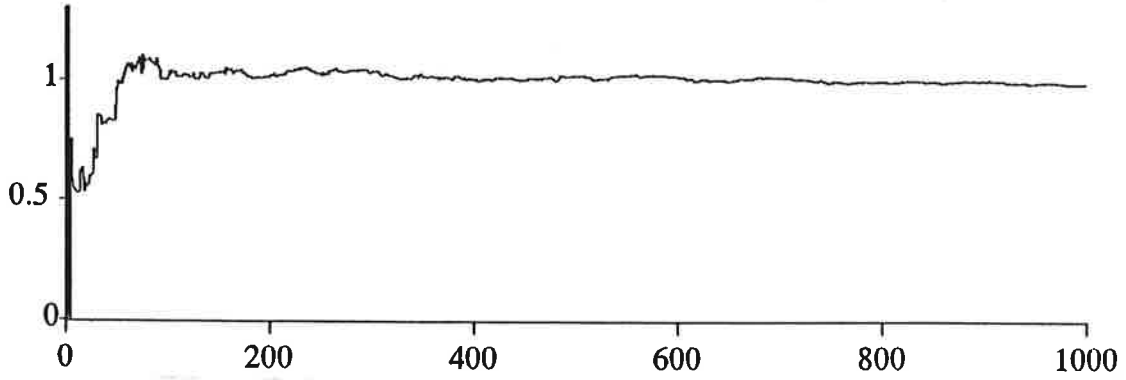
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Figure 4. Parameter convergence of $\widehat{\theta}$ vs. time with $\theta_o = 0.7$ and $\theta_{MV} = 1.0$. Notice that $\widehat{\theta}$ first converges towards θ_o and then proceeds towards θ_{MV}.

function would have been feasible within Ω_{MV}. The systematic bias with $X_o(\infty) \neq 0$ concluded in (48) results in final convergence towards $X_{MV} = 0$. The Lyapunov function with respect to the convergence point θ_{MV} develops as

$$E\{V_{MV}(X_{MV}(k+1))|\mathcal{F}_k\} - V_{MV}(X_{MV}(k)) \leq$$

$$\leq \frac{\varphi^T(k)P(k)\varphi(k)}{1 + \varphi^T(k)P(k)\varphi(k)}\sigma^2 + (c^T x_e(k))^2 - \mu K \frac{x_e^T(k)Q x_e(k)}{1 + \mu x_e^T(k)\Lambda_{MV} x_e(k)} \qquad a.s. \quad (49)$$

The convergence point θ_{MV} is more attractive than θ_o when there is noise interference. There is eventually no disturbance on x_e. The attractivity of θ_{MV} is only local but the fact that θ_o lies close to θ_{MV} is sufficient for global attraction to the neighbourhood of θ_{MV}. We reproduce the result of Ljung[2] and Landau[7] in the setting provided by this paper.

Proposition 5:

There is convergence towards θ_{MV} of least squares estimation within the region Ω_{MV} provided that the transfer function

$$\frac{1}{C^*(q^{-1})} - \frac{1}{2} \tag{50}$$

is strictly positive real. The point θ_o is attractive outside this area.

Proof: See appendix 6.

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Cost Criterion Evaluation

It is of interest to evaluate cost functionals such as different l^2-norms of x_e. The transient noiseless properties for large $\|x\| > r$ according to (46)-(47) is such that

$$J_x(\tau) = \sum_{k=0}^{\tau} x_e^T(k)\Lambda_o x_e(k) \le$$

$$\le \sum_{k=0}^{\tau} exp(\frac{-\mu q_{min}r^2}{1+\mu\lambda_{max}r^2}k)f(V_0) < \frac{1+\mu\lambda_{max}r^2}{1+\mu(\lambda_{max}+q_{min})r^2}\frac{1}{\mu}\exp(V_0/K) \qquad (51)$$

where the properties og geometric series have been used. It is seen that the cost functional depends on V_0. The cost functional after the lapse of a full adaptation is also limited which is claimed in the following proposition.

Proposition 6:

The noise free adaptive system is l^2- stable satisfying the bound

$$\sum_{k=0}^{\infty} x_e^T(k)Q x_e(k) \le \frac{1}{\mu K}\exp(\frac{V_0}{K})V_0 \qquad (52)$$

Proof: This result follows also from the Lyapunov function property with the intermediate steps obtained via (43) as

$$\sum_{k=0}^{\infty} x_e^T(k)Q x_e(k) \le$$

$$\le \sum_{k=0}^{\infty} \frac{1}{\mu K}\Big(1 + \mu x_e^T(k)\Lambda x_e^T\Big)\Big(V(X(k)) - V(X(k+1))\Big) \le$$

$$\le \frac{1}{\mu K}\exp(\frac{V_0}{K})\sum_{k=0}^{\infty}\Big(V(X(k)) - V(X(k+1))\Big) \le \frac{1}{\mu K}\exp(\frac{V_0}{K})V_0$$

$\square$

The cost functional after a disturbance $w \in l^2$ is similar with V_0 replaced by V_1 according to proposition 4. The exponential dependence on V_0 is less dramatic with respect to initial values of x due to the logarithmic Lyapunov function. It implies however sensitivity with respect to parameter errors and disturbances.

Persistent excitation

An evaluation of cost criteria for non-stationary sporadically occurring disturbances $w \in l^2$ can be made by Lyapunov analysis. More severe noise conditions may

be analysed by using stochastic Lyapunov functions. A common method in the literature, see Anderson[20], is to consider the case of persistent excitation. A cost functional for persistant noise excitation may now be evaluated. It is known from above that there is finite time convergence to a neighbourhood of X_o. For $\|x(k)\|^2 \leq r^2$ it follows that

$$\varphi^T(k)\varphi(k) \leq 2\|x(k)\|^2 + 2M_w^T(k)M_w(k)C_w^2 \leq 2r^2 + 2C_w^2 = c_\varphi^2$$
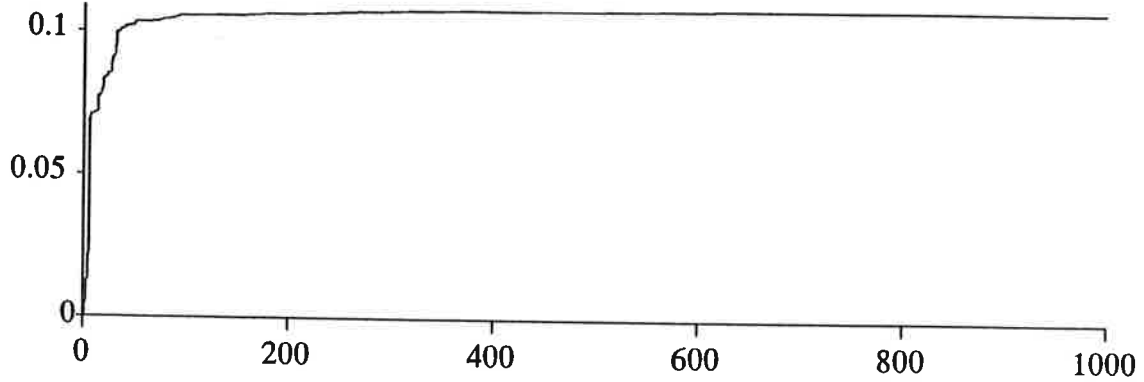

Figure 5. Cost functional $J_x(k) = \sum_{i=0}^{k} x^T(i)Qx(i)$ vs. time k in persistent excitation.

Strong conclusions may be drawn for the cases when it can be shown that the following condition of persistency of excitation holds:

$$0 < \alpha c_\varphi^2 k I_{nxn} \leq P^{-1}(k) \quad \Rightarrow \quad P(k) \leq \frac{1}{\alpha c_\varphi^2 k} I_{nxn} \quad \forall k \geq 1 \quad (53)$$

The evaluation of the cost functional up to time k may be derived from (49). In the case of a strict positive real transfer function (50) it follows that the second and third terms are together nonpositive and

$$E\{V(\tau)|\mathcal{F}_k\} - V(k) \leq \sum_{j=k}^{\tau} \frac{\varphi^T(j)P(j)\varphi(j)}{1 + \varphi^T(j)P(j)\varphi(j)}\sigma^2 \leq$$

$$\leq \sum_{j=k}^{\tau} \frac{1}{1 + j\alpha} \leq \sigma^2 \int_{k-1}^{\tau+1} \frac{dt}{1 + \alpha t} \leq \frac{\sigma^2}{\alpha}\log\left(\frac{1 + \alpha(\tau+1)}{1 + \alpha(k-1)}\right) \quad a.s. \quad (54)$$

so that

$$E\{V(k)|\mathcal{F}_0\} \leq V_0 + \frac{1}{\alpha}\log\left(1 + \alpha(k+1)\right)\sigma^2 \quad a.s. \quad (55)$$

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Proposition 7:

We conclude from (55) that noise asymptotically does not affect the state x. □

For low signal to noise ratios with $\|x_e\|^2 \leq \sigma^2$ with $Q = Q_1 + (2/\mu K) \cdot cc^T$ and $\mu = 1/\sigma^2$ it follows from (49) that

Proposition 8:

The cost functional

$$E\{J_x(k)|\mathcal{F}_0\} = E\{\sum_{i=0}^{k} x^T(i)Q_1 x(i)|\mathcal{F}_0\} \leq c_0 + c_1 \log(1 + \alpha(k+1)) \qquad a.s. \quad (56)$$

with

$$c_0 = \frac{2\sigma^2}{K}V_0; \qquad c_1 = \frac{2\sigma^4}{\alpha K}$$

□

The cost functional for high signal to noise ratios is similar to that of (51) where V_0 is augmented by the finite noise energy entering the system during the transient.

Remark:

The choice of a quadratic Lyapunov function within Ω_{MV} is feasible and gives similar bounds.

No stable inverse of C^*

It was seen that a set around θ_o is attractive globally. The point θ_{MV} is however no longer attractive when (50) does not hold. Consider now the case where assumption A10 is not valid.
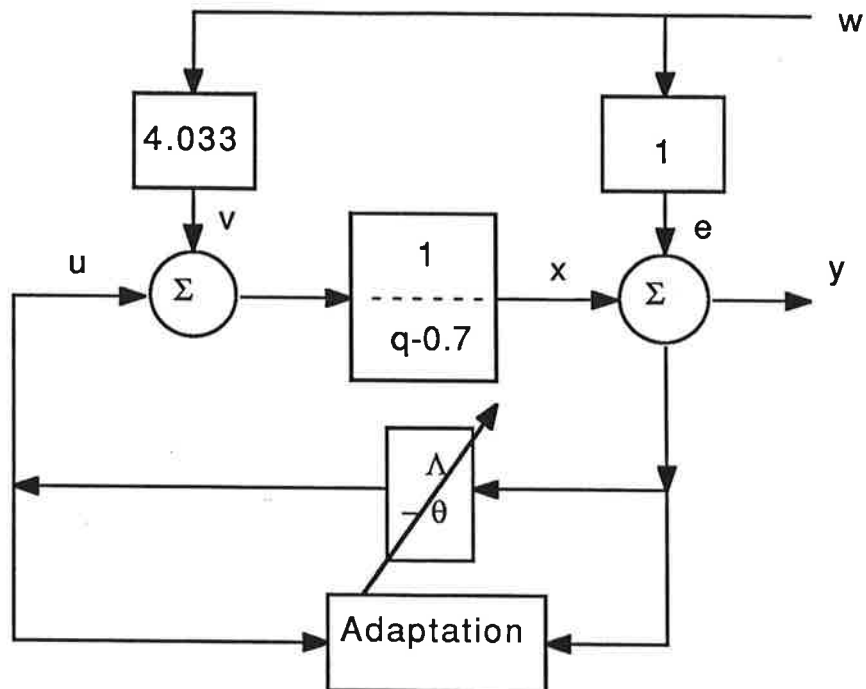
Figure 6. An adaptive control system with $A^* = 1 - 0.7q^{-1}$, $B^* = q^{-1}$, and $C^* = 1 + 3.33q^{-1}$.
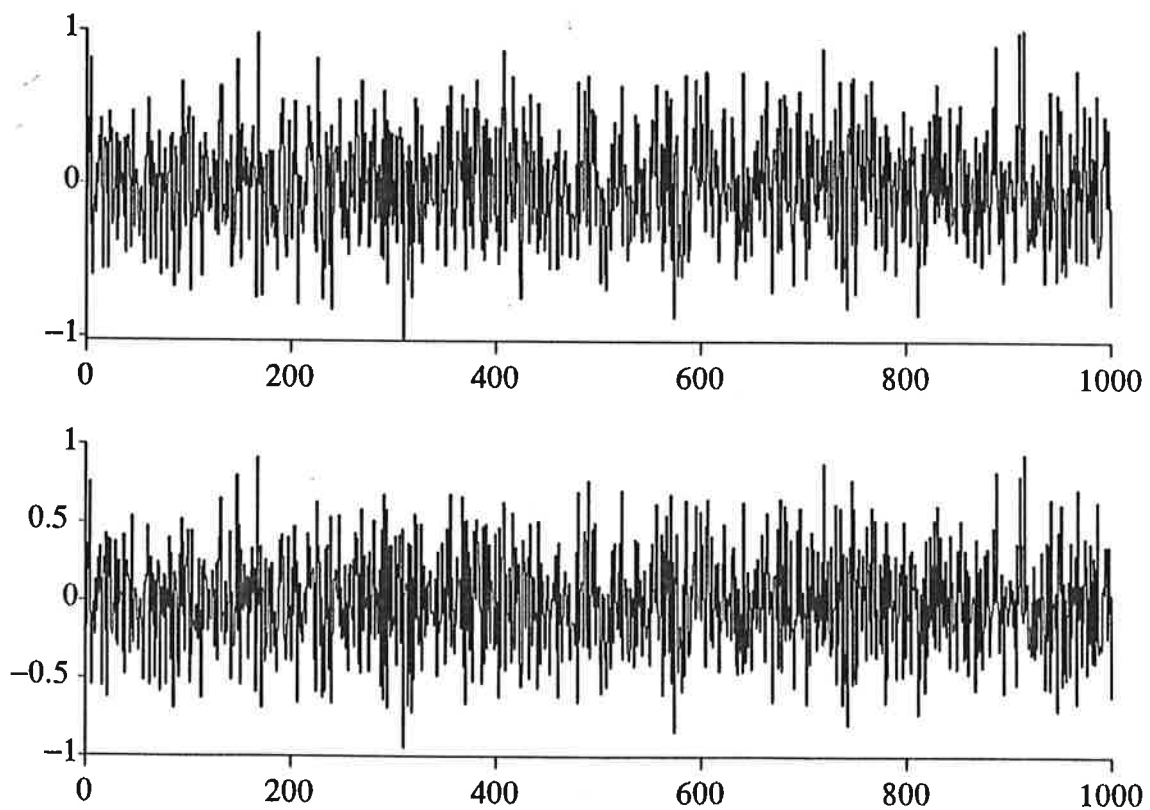

Figure 7. A transient of output y *(upper)* and state x *(lower)* vs. time in the adaptive control system with $A^* = 1 - 0.7q^{-1}$, $B^* = q^{-1}$, and $C^* = 1 + 3.33q^{-1}$. Notice that x remains noisy.
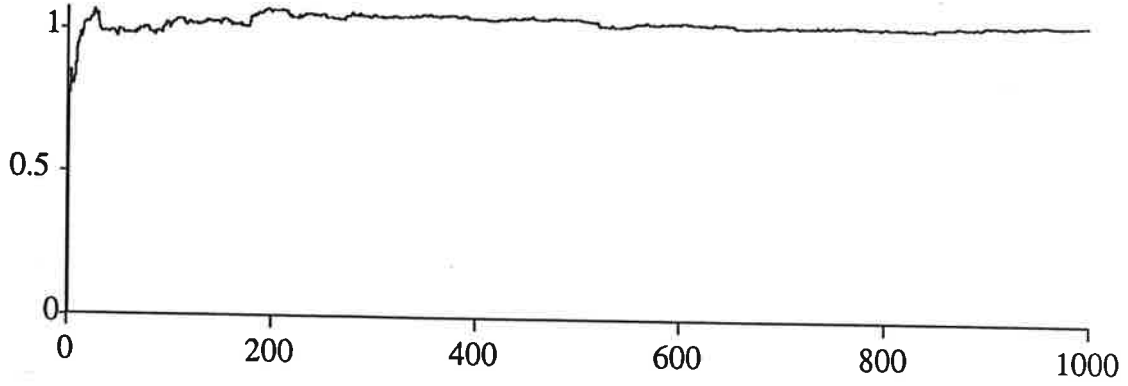
Figure 8. Parameter convergence of $\hat{\theta}$ vs. time with $\theta_o = 0.7$ and $\theta_{MV} = 1.0$. Notice that $\hat{\theta}$ first converges towards θ_o and then proceeds towards θ_{MV}.

The optimal solutions found by the adaptive control algorithm are modified in the case where C^* has no stable inverse. The solution is given by spectral decomposition and reflections of zeros in the unit circle. Find a decomposition of C^* into

$$C^*(q^{-1}) = C^{*+}(q^{-1})C^{*-}(q^{-1}) \qquad (57)$$

so that C^{*-} contains nothing but the non-invertible zeros. The minimum variance adaptive control parameters converges towards the regulator

$$R^* = b_0 B^* F^*_{mod} \qquad S^* = G^*_{mod} \qquad T^* = C^- C^{+*} \qquad (58)$$

with F^*_{mod} and G^*_{mod} as solutions from

$$A^*(q^{-1})F^*_{mod}(q^{-1}) + q^{-d}G^*_{mod}(q^{-1}) = C^-(q^{-1})C^{+*}(q^{-1}) \qquad (59)$$

The closed loop pole polynomial will be

$$B^* C^{*+} C^- \qquad (60)$$

Minimum variance control of the output is indeed no well-posed optimal control problem in this case. The solution gives minimal variance of the output by variation of the state x. The stability will guarantee that $\|x\|$ does not grow beyond a certain limit but x is excited by noise also in the well-tuned loop. It is thus not expected that $\|x\| \to 0$. Stability will hold but the cost functional J_x will be proportional to elapsed time k rather than to $\log(1 + k)$.

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Generalization to arbitrary delay d

The assumptions A9 was made to simplify the presentation. A generalization to an arbitrary but known time delay $d > 1$ is made by modifying (17-20) to a d–step prediction error method, see Ljung and Söderström[21]. The recursive least squares estimation is then modified to

$$\widehat{\theta}(k) = \widehat{\theta}(k-1) + P(k)\varphi(k)\varepsilon(k) \tag{61}$$

$$P(k) = P(k-1) - \frac{P(k-1)\varphi(k-d)\varphi^T(k-d)P(k-1)}{1 + \varphi^T(k-d)P(k-1)\varphi(k-d)} \quad P(0) = P_0 > 0 \tag{62}$$

$$\varepsilon(k) = y_f(k) - b_0 u(k-d) - \widehat{\theta}^T(k-1)\varphi(k-d) \tag{63}$$

The parameter error state (31) and v_θ of (40) must be modified as follows:

$$\Xi(k) = \left(\begin{array}{ccc} \widetilde{\theta}^T(k+d-1) & \ldots & \widetilde{\theta}^T(k) \end{array} \right)^T \tag{64}$$

$$v_\theta(\Xi(k)) = \sum_{i=k}^{k+d-1} \widetilde{\theta}^T(i) P^{-1}(i) \widetilde{\theta}(i) \tag{65}$$

With these modifications it can be shown that

$$v_\theta(\Xi(k+1)) - v_\theta(\Xi(k)) \leq -\frac{(\widetilde{\theta}^T(k)\varphi(k))^2}{1 + \varphi^T(k)P(k)\varphi(k)} \tag{66}$$

A proof of (66) is found in Johansson[15] and is based on the properties of a least squares solution. All other arguments follow as in the previously presented proofs.

Generalization to cases with unknown b_0

The assumption A8 on a known gain b_0 is often relaxed in the literature to a requirement on a fixed gain estimate β_0 such that $0 < \frac{b_0}{\beta_0} < 2$, see Goodwin and Sin[11]. It is for example common to base adaptive control algorithms on simple estimation algorithms where (17)-(20) is replaced by gradient algorithms of the type

$$\widehat{\theta}(k) = \widehat{\theta}(k-1) + \gamma(k-1)\varphi(k-1)\varepsilon(k) \tag{67}$$

$$\gamma(k-1) = \frac{1}{1 + \varphi^T(k-1)\varphi(k-1)} \tag{68}$$

$$\varepsilon(k) = y_f(k) - \beta_0 u(k-1) - \widehat{\theta}^T(k-1)\varphi(k-1)$$

$$u(k) = -\frac{1}{\beta_0}\widehat{\theta}^T(k)\varphi(k) \tag{69}$$

(The reference value has been excluded for simplicity.) The parameter error Lyapunov function candidate is

$$v_\theta(\widetilde{\theta}(k)) = \widetilde{\theta}^T(k)\widetilde{\theta}(k) \qquad \widetilde{\theta}(k) = \widehat{\theta}(k) - \frac{\beta_0}{b_0}\theta \tag{70}$$

and

$$v_\theta(\widetilde{\theta}(k+1)) - v_\theta(\widetilde{\theta}(k)) \leq -\left(2\frac{\beta_0}{b_0} - 1\right)\frac{\varepsilon^2(k+1)}{1 + \varphi^T(k)\varphi(k)} \tag{71}$$

The function v_θ is non-increasing for $0 < \frac{b_0}{\beta_0} < 2$ and all other proof details for global Lyapunov stability follow the same outline as for least squares estimation. Another similar case is treated in Johansson[22].

An analysis of the recursive least squares estimation counterpart of (67)-(69) does not promise global results of Lyapunov stability and l^2-stability. Instead it is possible to show local results if the initial value V_0 of the Lyapunov function V_o in (43) is such that

$$V_0 < c_1 - c_2 \log\left|1 - \frac{b_0}{\beta_0}\right| \tag{72}$$

for some positive constants c_1 and c_2. This is a local result but the stability region is very large when β_0 approximates b_0 well.

Another common modification is to estimate b_0 or its inverse along with the other parameters θ of (11). The Lyapunov analysis gives only local stability and convergence results for this case, see Johansson[15] for details of analysis. Stability and convergence properties are promised to be good only as long as β_0 is a good estimate of b_0 or as long as the input u is bounded. The Lyapunov analysis gives a warning that this modification is not globally stable.

Generalization to non-zero reference value

Let now assumption A7 be removed. The state vector x may be decomposed into one component x_m which corresponds to the desirable reference model state and a state x_e of the model matching error dynamics.

$$x(k) = x_m(k) + x_e(k) \tag{73}$$

Similar decompositions of ξ, ν, φ are defined via the expressions

$$\xi = \xi_m + \xi_e \quad \nu = \nu_m + \nu_e \quad \varphi = \varphi_m + \varphi_e \tag{74}$$

with (27) reformulated to the

$$v_m(k) = P^*(q^{-1})\xi_m(k) = \frac{1}{b_0}y_c(k) \tag{75}$$

$$v_e(k) = P^*(q^{-1})\xi_e(k) = u(k) + \frac{1}{b_0}\theta^T\varphi(k) - \frac{1}{b_0}y_c(k) \tag{76}$$

The state vector components are given by

$$x_m(k) = \left(\begin{matrix} \xi_m(k-1) & \xi_m(k-2) & \dots & \xi_m(k-n) \end{matrix} \right)^T \tag{77}$$

$$x_e(k) = \left(\begin{matrix} \xi_e(k-1) & \xi_e(k-2) & \dots & \xi_e(k-n) \end{matrix} \right)^T \tag{78}$$

and (29) holds for both x_m and x_e with obvious changes in subscripts. The model state x_m is associated with the reference trajectory and is bounded in magnitude for stable reference models and bounded reference inputs u_c. The boundedness may be formalized by assumption $A6$.

Connections to information theory

There are properties of the Lyapunov function that provide some connections to information theory.

$$V(X(k)) = \widetilde{\theta}^T(k)P^{-1}(k)\widetilde{\theta}(k) + K\log\left(1 + \mu x_e^T(k)\Lambda x_e(k)\right) + tr(P^T(k)P(k)) \tag{79}$$

with

$$\mu = \frac{1}{\sigma^2} \qquad K = \frac{1}{\det(P(0))} \tag{80}$$

The first term of V is a least-squares based criterion with a matrix $P^{-1}(k)$ which grows with time. It is quite standard to argue that P^{-1} contains the accumulated (Fisher-) information collected since initial time $k = 0$, see Goodwin and Payne[23]. The covariance of $\widetilde{\theta}(k)$ is often estimated via the Cramer-Rao bound

$$E\{\widetilde{\theta}(k)\widetilde{\theta}^T(k)\} \geq \sigma^2 P(k) \tag{81}$$

The development of V is however also determined by its second term

$$\log\left(1 + \frac{x_e^T(k)\Lambda x_e(k)}{\sigma^2}\right) \tag{82}$$

which involves the state x_e and the noise variance σ^2. The expression is a logarithm of a signal to noise ratio (S/N) of $\|x_e\|^2$ and σ^2 on the form

$$\log(S + N) - \log N = \log(1 + \frac{S}{N}) \tag{83}$$

This is a measure of the information of the signal S contained in the perturbed signal (S+N). The information theoretical interpretation is that of delivered (Shannon-) information or the *channel capacity* when S is limited in magnitude, see Gallager[19] or Shannon[24].

The uncertainty or *entropy* represented by V may be increased by noise and decreased by a non-zero $\|x_e\|$. The stability may be interpreted as follows: Any information in the signal x_e results in a decrease in the parameter uncertainty v_θ so that the adaptive system entropy represented by V decreases in the noise free case.

This information theoretical interpretation of the Lyapunov function also explains the poor final convergence towards the minimum variance solution θ_{MV}. The signal to noise ratio is approaching zero when getting close to θ_{MV} so that no information is obtained at the solution point.

Conclusions

The stability properties of minimum variance adaptive control has been investigated for cases with large disturbances and stationary noise.

The time variant non-ergodic behaviour has motivated the use of stochastic Lyapunov functions or supermartingale methods. It was shown that there are two convergence points θ_o and θ_{MV}. A set Ω_o around θ_o is globally attractive for large transient trajectories but θ_{MV} is locally attractive. Recovery from a large disturbance starts with initial convergence towards the deterministic solution point. The final convergence is towards θ_{MV} when the trajectory has reached the minimum variance solution domain of attraction.

This work confirms earlier results of Ljung[2] and Landau[7] on convergence and contributes new results by removing the stability condition. The cost functionals and the stability in the case of l^2−disturbances has been investigated. Another contribution is the explicit expressions on cost functions with at most a logarithmic growth in the case of persistent excitation and an invertible C^*.

The slow final convergence to the minimum variance solution is explained with an information theoretical argument.

Appendix 1 — Proof of Lemma 1

For any choice of $\Lambda > 0$ it found that

$$x_e^T(k+1)\Lambda x_e(k+1) - x_e^T(k)\Lambda x_e(k) =$$

$$= (\Phi x_e(k) + \Gamma\nu_e(k))^T\Lambda(\Phi x_e(k) + \Gamma\nu_e(k)) - x_e^T(k)\Lambda x_e(k) =$$

$$= x_e^T(k)(\Phi^T\Lambda\Phi - \Lambda)x_e(k) + 2\Gamma^T\Lambda\Phi x_e(k)\nu_e(k) + \Gamma^T\Lambda\Gamma\nu_e^2(k) \qquad (A1.1)$$

Let now Λ be the positive definite solution to the Lyapunov equation (38) where Q is an arbitrary positive definite matrix and I is the identity matrix. Make the factorization

$$\Lambda = L^T L \qquad (A1.2)$$

and define the constant

$$c^2 = max\left(2, \ \lambda_{max}(\Phi^T\Lambda\Phi)\right) \qquad (A1.3)$$

where $\lambda_{max}(\cdot)$ means the largest eigenvalue of a positive definite matrix. Introduce

$$F = \frac{1}{\kappa}L\Phi \quad G = \kappa L\Gamma \quad \kappa^2 = c^2 - 1 \qquad (A1.4)$$

The cross term of (A1.1) may be reformulated as

$$2\Gamma^T\Lambda\Phi x_e\nu_e = 2G^T F x_e\nu_e = -(Fx_e - G\nu_e)^2 + x_e^T F^T F x_e + G^T G\nu_e^2 \qquad (A1.5)$$

Substitution of this expression into (A1.1) gives

$$x_e^T(k+1)\Lambda x_e(k+1) - x_e^T(k)\Lambda x_e(k) = \qquad (A1.6)$$

$$x_e^T(k)(\Phi^T\Lambda\Phi - \Lambda + F^T F)x_e(k) - (Fx_e(k) - G\nu_e(k))^2 + (G^T G + \Gamma^T\Lambda\Gamma)\nu_e^2(k) \qquad (A1.7)$$

From (A1.3-4) it is found that

$$F^T F \leq I \qquad (A1.8)$$

and it follows that a term of (A1.6) may be simplified to

$$x_e^T(\Phi^T\Lambda\Phi - \Lambda + F^T F)x_e \leq -x_e^T Q x_e \qquad (A1.9)$$

The function v_x of the lemma formulation develops as

$$v_x(x_e(k+1)) - v_x(x_e(k)) =$$

$$= \log\left(1 + \mu \frac{x_e^T(k+1)\Lambda x_e(k+1) - x_e^T(k)\Lambda x_e(k)}{1 + \mu x_e^T(k)\Lambda x_e(k)}\right) \le$$

$$\le \mu \frac{x_e^T(k+1)\Lambda x_e(k+1) - x_e^T(k)\Lambda x_e(k)}{1 + \mu x_e^T(k)\Lambda x_e(k)} \qquad (A1.10)$$

Substitution of (A1.6) and (A1.8) into (A1.9) finally proves the lemma.

$$v_e(x_e(k+1)) - v_x(x_e(k)) \le \mu \frac{-x_e^T(k)Q x_e(k) + c^2 \Gamma^T \Lambda \Gamma \nu_e^2(k)}{1 + \mu x_e^T(k)\Lambda x_e(k)} \qquad (A1.11)$$

□

Appendix 2

Introduce the short notation

$$\Delta = 1 + \varphi^T(k)P(k)\varphi(k); \qquad \psi = P(k)\varphi(k) \qquad (A2.1)$$

The updating algorithm for P of (18) turns out as

$$P(k+1) = P(k) - \frac{1}{\Delta}\psi\psi^T \qquad (A2.2)$$

The function v_P of (41) develops in one recursion as

$$v_P(\Pi(k+1)) - v_P(\Pi(k)) = tr\left(P^T(k+1)P(k+1) - P^T(k)P(k)\right) =$$

$$= \frac{1}{\Delta}tr\left(-P^T(k)\psi\psi^T - \psi\psi^T P(k) + \frac{1}{\Delta}\psi\psi^T\psi\psi^T\right) =$$

$$= \frac{1}{\Delta}\psi^T(-2P(k) + \frac{1}{\Delta}\psi\psi^T)\psi \qquad (A2.3)$$

Substitution with (A2.2) and the observation that $P(k) \ge 0$ for all k finally gives that

$$v_P(\Pi(k+1)) - v_P(\Pi(k)) = -\frac{1}{\Delta}\psi^T\left(P(k) + P(k+1)\right)\psi \le 0 \qquad (A2.4)$$

□

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Appendix 3

The positive function V of (42) is radially growing with $\|X\|$ and continuous at $X = 0$. It remains to show that this Lyapunov function candidate never increases in a recursion. It is found from (42) that

$$\Delta V(k) = V(X(k+1)) - V(X(k)) = v_\theta(\widetilde{\theta}(k+1)) - v_\theta(\widetilde{\theta}(k)) +$$

$$+ K_\theta(\widetilde{\theta}(k+1)) + K_\theta(\widetilde{\theta}(k)) + tr(P^T(k+1)P(k+1) - P^T(k)P(k)) \qquad (A3.1)$$

Manipulations of (40) with (17-20) shows that the function v_θ decreases as

$$v_\theta(\widetilde{\theta}_o(k+1)) - v_\theta(\widetilde{\theta}_o(k)) = \frac{-\varepsilon^2(k+1)}{1 + \varphi^T(k)P(k)\varphi(k)} \qquad (A3.2)$$

From (37), (A3.2), and appendix 2 it is found that

$$\Delta V(k) < \frac{-\varepsilon^2(k+1)}{1 + \varphi^T(k)P(k)\varphi(k)} + \mu K \frac{-x_e^T(k)Qx_e(k) + c^2\Gamma^T\Lambda\Gamma\nu_e^2(k)}{1 + \mu x_e^T(k)\Lambda x_e(k)} \qquad (A3.3)$$

Start by investigating the terms of the denominators of (A3.3). It is found from (17-19) that

$$\varphi^T(k)P(k)\varphi(k) \leq \varphi^T(k)P(0)\varphi(k) \leq \varphi^T(k)P_0\varphi(k) \qquad (A3.4)$$

It follows from (18) and (21) that

$$\varphi^T P_0 \varphi = x^T M_\varphi^T P_0 M_\varphi x = (x_e + x_m)^T M_\varphi^T P_0 M_\varphi(x_e + x_m) \leq$$

$$\leq 2x_e^T M_\varphi^T P_0 M_\varphi^T x_e + 2x_m^T M_\varphi^T P_0 M_\varphi x_m \qquad (A3.5)$$

Define a constant μ_1 such that

$$\mu_1 \Lambda > M_\varphi^T P_0 M_\varphi \qquad (A3.6)$$

The bound C_r on x_m of (26) gives

$$\varphi^T(k)P(k)\varphi(k) \leq 2\mu_1 x_e^T(k)\Lambda x_e(k) + 2\mu_1\lambda_{max}C_r^2 \qquad (A3.7)$$

where λ_{max} is the largest eigenvalue of Λ, see (38).

$$\frac{-1}{1 + \varphi^T(k)P(k)\varphi(k)} \leq \frac{-1}{1 + 2_1\lambda_{max}C_r^2 + 2\mu_1 x_e^T(k)\Lambda x_e(k)} \qquad (A3.8)$$

Let the constants of V be chosen as

$$\mu = \frac{2\mu_1}{1 + 2\mu_1\lambda_{max}C_r^2} \qquad K = \frac{b_0^2}{2\mu_1 c^2\Gamma^T\Lambda\Gamma} \qquad (A3.9)$$

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It is found from (20) and (27) that

$$\nu_e^2(k) = \frac{1}{b_0^2}(\tilde{\theta}^T(k)\varphi(k))^2 \qquad (A3.10)$$

From (20) and (21) follows that

$$\varepsilon^2(k+1) = (\tilde{\theta}^T\varphi(k))^2 \qquad (A3.11)$$

Substitution of (A3.6-9) into (A3.3) gives

$$\Delta V(k) \leq \mu K \frac{-x_e^T(k)Qx_e(k)}{1 + \mu x_e^T(k)\Lambda x_e(k)} \qquad (A3.12)$$

which means that V decreases in each recursion for $\|x_e\| \neq 0$. Stability in the sense of Lyapunov may be concluded. $\qquad \square$

Appendix 4 - The state space model

Proposition:

The partial state ξ is related to θ_{MV} and φ via the equation

$$\nu_{MV}(k) = P^*(q^{-1})\xi(k) = u(k) + \frac{1}{b_0}\theta_{MV}^T\varphi(k) \qquad (27)$$

Proof:

Via (6), (7) is found that the partial state is

$$\xi(k) = \frac{A^*(q^{-1})R^*(q^{-1}) + b_0q^{-1}B^*(q^{-1})S^*(q^{-1})}{b_0B^*(q^{-1})C^*(q^{-1})}\xi(k) \qquad (A4.1)$$

or

$$\xi(k) = \frac{1}{b_0B^*(q^{-1})C^*(q^{-1})}\left(R^*(q^{-1})(A^*(q^{-1})\xi(k)) + S^*(q^{-1})(b_0q^{-1}B^*(q^{-1})\xi(k))\right) \qquad (A4.2)$$

Via (3)-(5) is found how ξ depends on input-output data

$$\xi(k) = \frac{1}{b_0B^*(q^{-1})C^*(q^{-1})}\left(R^*(q^{-1})u(k) + S^*(q^{-1})y(k)\right) +$$

$$+ \frac{1}{b_0B^*(q^{-1})C^*(q^{-1})}\left(R^*(q^{-1})\frac{G^*(q^{-1})}{b_0B^*(q^{-1})} - S^*(q^{-1})F^*(q^{-1})\right)w(k) \qquad (A4.3)$$

The second term disappears when substituting above for minimum variance control polynomials R^*, S^*. The noise dependent term is thus zero for minimum variance

polynomials and the state is decoupled from noise influence. The partial state depends only on outputs y and inputs u. Noise affects only the output via measurement noise e in an adequately controlled system so that

$$\xi(k) = \frac{1}{b_0 B^*(q^{-1})C^*(q^{-1})}\left(R^*(q^{-1})u(k) + S^*(q^{-1})y(k)\right) \qquad (A4.4)$$

Reformulation gives the proposition

$$\nu_{MV}(k) = P^*(q^{-1})\xi(k) = u(k) + \frac{1}{b_0}\theta^T_{MV}\varphi(k) \qquad (A4.5)$$

□

Appendix 5 - Convergence of parameter estimates

The modified algorithm is

$$\hat{\theta}(k+1) = \hat{\theta}(k) + P(k+1)\varphi(k)\varepsilon(k+1)$$

$$P(k+1) = P(k) - \frac{P(k)\varphi(k)\varphi^T(k)P(k)}{1+\varphi^T(k)P(k)\varphi(k)}$$

$$\varepsilon(k+1) = y(k+1) - \beta u(k) - \hat{\theta}^T(k)\varphi(k) \qquad (A5.1)$$

A positive, radially growing function v_θ

$$v_\theta(\tilde{\theta}(k)) = \tilde{\theta}^T(k)P^{-1}(k)\tilde{\theta}(k) \qquad (A5.2)$$

The development of v_θ one step ahead is determined by

$$\Delta v_\theta = v_\theta(\tilde{\theta}(k+1)) - v_\theta(\tilde{\theta}(k)) = \tilde{\theta}^T(k+1)P^{-1}(k)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)P^{-1}(k)\tilde{\theta}(k) =$$

$$= \left(\tilde{\theta}^T(k)\varphi(k)\right)^2 + 2\tilde{\theta}^T(k)\varphi(k)\varepsilon(k) + \frac{\varphi^T(k)P(k)\varphi(k)}{1+\varphi^T(k)P(k)\varphi(k)}\varepsilon^2(k+1) =$$

$$= \left(\tilde{\theta}^T(k)\varphi(k) + \varepsilon(k+1)\right)^2 - \frac{1}{1+\varphi^T(k)P(k)\varphi(k)}\varepsilon^2(k+1) \qquad (A5.3)$$

The convergence around θ_{MV} is determined by

$$\varepsilon(k+1) = \frac{1}{C^*(q^{-1})}[-\tilde{\theta}^T_{MV}(k)\varphi(k)] + w(k+1) \qquad (A5.4)$$

so that

$$\left[\tilde{\theta}^T(k)\varphi(k) + \varepsilon(k+1)\right]^2 = \left[\left(1 - \frac{1}{C^*(q^{-1})}\right)(\tilde{\theta}^T(k)\varphi(k)) + w(k+1)\right]^2 =$$

$$= \left[\left(\frac{C^*(q^{-1}) - 1}{C^*(q^{-1})} \right) (\tilde{\theta}^T(k)\varphi(k)) + w(k+1) \right]^2 =$$

$$= \left(\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + w(k+1) \right)^2 \qquad (A5.5)$$

The expectation with respect to $\mathcal{F}_k$ is

$$E\{\Delta v_\theta(k+1)|\mathcal{F}_k\} = (c^T x(k))^2 + \sigma^2 \qquad a.s. \qquad (A5.6)$$

$\square$

Appendix 6 - The positive real condition

Let the state equation be

$$x(k+1) = \Phi x(k) + \Gamma \nu(k) \qquad (A6.1)$$

The prediction error is a sum of a systematic error ξ and the noise w

$$\varepsilon(k+1) = \xi(k+1) + w(k+1) = \frac{1}{C^*(q^{-1})}(\nu(k)) + w(k+1) \qquad (A6.2)$$

The expected prediction error one step ahead is

$$E\{\varepsilon^2(k+1)|\mathcal{F}_k\} = (\frac{1}{C^*(q^{-1})}\nu(k))^2 + E\{w^2(k+1)|\mathcal{F}_k\} \qquad a.s. \qquad (A6.3)$$

The growth of x is determined by

$$x^T(k+1)\Lambda x(k+1) - x^T(k)\Lambda x(k) =$$

$$\begin{pmatrix} \nu(k) & x^T(k) \end{pmatrix} \begin{pmatrix} \Gamma^T \Lambda \Gamma & \Gamma^T \Lambda \Phi \\ \Phi^T \Lambda \Gamma & \Phi^T \Lambda \Phi - \Lambda \end{pmatrix} \begin{pmatrix} \nu(k) \\ x(k) \end{pmatrix} \qquad (A6.4)$$

$$C^*(q^{-1})\xi(k) = \nu(k) \qquad \xi(k) = \nu(k) - c^T x(k) \qquad (A6.5)$$

$$\xi^2(k) = (\nu(k) - c^T x(k))^2 = \begin{pmatrix} \nu(k) & x^T(k) \end{pmatrix} \begin{pmatrix} 1 & -c^T \\ -c & cc^T \end{pmatrix} \begin{pmatrix} \nu(k) \\ x(k) \end{pmatrix} \qquad (A6.6)$$

The decrement of v_θ is determined by ε and balances the possible increment of v_x. The condition for stability with a weighting coefficient ϱ is

$$-\varrho\xi^2 + x^T(k+1)\Lambda x(k+1) - x^T(k)\Lambda x(k) \le 0 \qquad \forall x, \nu$$

$$\begin{pmatrix} \nu & x \end{pmatrix}^T \left(-\varrho \begin{pmatrix} 1 & -c^T \\ -c & cc^T \end{pmatrix} + \begin{pmatrix} \Gamma^T \Lambda \Gamma & \Gamma^T \Lambda \Phi \\ \Phi^T \Lambda \Gamma & \Phi^T \Lambda \Phi - \Lambda \end{pmatrix} \right) \begin{pmatrix} \nu \\ x \end{pmatrix} \le 0 \qquad (A6.7)$$

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This condition is equivalent to positive definiteness of the matrix

$$\begin{pmatrix} \varrho - \Gamma^T \Lambda \Gamma & -\varrho c^T - \Gamma^T \Lambda \Phi \\ -\varrho c - \Phi^T \Lambda \Gamma & \varrho c c^T - (\Phi^T \Lambda \Phi - \Lambda) \end{pmatrix} \geq 0 \qquad (A6.8)$$

According to positive real lemma, see Hitz and Anderson[25], there are matrices K and L and a positive definite matrix Q such that

$$\begin{pmatrix} \varrho - \Gamma^T \Lambda \Gamma & -\varrho c^T - \Gamma^T \Lambda \Phi \\ -\varrho c - \Phi^T \Lambda \Gamma & \varrho c c^T - (\Phi^T \Lambda \Phi - \Lambda) \end{pmatrix} = \begin{pmatrix} K^T \\ L \end{pmatrix} \begin{pmatrix} K & L^T \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$$
$$(A6.9)$$

Termwise identification gives

$$K^T K = \varrho - \Gamma^T \Lambda \Gamma$$

$$LK = -\varrho c - \Phi^T \Lambda \Gamma$$

$$LL^T = \varrho c c^T \qquad (A6.10)$$

The matrices K and L are given by solutions to the equations

$$K^T K = \frac{\varrho}{2} + \frac{\varrho}{2} - \Gamma^T \Lambda \Gamma$$

$$\Gamma^T \Lambda \Phi + K^T L^T = -\varrho c^T$$

$$\Phi^T \Lambda \Phi - \Lambda = -Q - \varrho c c^T \qquad (A6.11)$$

if and only if

$$-\varrho c^T [zI - \Phi]^{-1} \Gamma + \frac{\varrho}{2} \quad s.p.r. \qquad (A6.12)$$

$$\varrho \left(-c^T [zI - \Phi]^{-1} \Gamma + \frac{1}{2} \right) \quad s.p.r. \qquad (A6.13)$$

$$\frac{1}{C^*(q^{-1})} - \frac{1}{2} \quad s.p.r. \qquad (A6.14)$$

irrespective of the value of

$$\varrho = K^T K + \Gamma^T \Lambda \Gamma > 0 \qquad (A6.15)$$

□

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