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Optimal control for systems with varying sampling rate

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Abstract

The paper addresses the aspects of control of real-time systems with varying sampling rate. To motivate, an example is given in which a stable continuous system is sampled at two different sampling rates. Two controllers are designed to minimize the same continuous quadratic loss function with the same weights. It is shown that although the design leads to stable controlled closed loop systems, for both discretizations, the resulting system can be unstable due to variations in sampling rate. To avoid that problem, we suggest an optimal controller design in which a bound on the cost, for all possible sampling rate variations, is computed. This results in a piecewise constant state feedback control law and guarantees stability regardless of the variations in sampling rate. The controller synthesis is cast into an LMI, which conveniently solves the synthesis problem. To illustrate the procedure, the introduction example is revise using the proposed LMI synthesis method and the stable control law is given, which is robustly stable against variations in sampling rate.

1 Introduction

The paper is concerned with the control of sampled data systems which have variations in the sampling rate. Such systems arise for different reasons. One of them is the optimal usage of central processing unit (CPU) resources [3] [2]. Roughly speaking, several tasks are carried out on the very same CPU, one of them is to compute the control law. When enough computational recourses are available, the control law is computed more frequently than when the recourses are used for other computations. This leads to variations in sampling rate, which can potentially unstabilize the controlled system.

In the following we will give an example of how variations in sampling time can lead to instability. We proceed in proposing a controller design, which results into a piecewise linear state feedback control law and is robustly stable to variations among the prescribed sampling rates. We show how such state feedback controllers can be found using linear matrix inequalities (LMI). We illustrate the design procedure by revisiting the introductory example, where a linear quadratic design approach lead to instability.

1.1 Example 1: Two different sampling times, same continuous loss function in both

As an example of instability by scheduling, the real-time control of the following linear continuous system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx \end{aligned} \tag{1}$$

is considered, where

$$A = \begin{bmatrix} 0 & 1 \\ -10000 & -0.1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0] \tag{2}$$

are the system, input and output matrix. The continuous system is stable with poles in the left hand-side of the complex plane $p_{1,2} = -0.05 \pm 100i$. In the following, two discrete systems are derived from this continuous system. The continuous system is discretized with two different zero order hold circuits, where the sampling rates are $h_1 = 0.002s$, $h_2 = 0.094s$ respectively. The two discretizations are represented by

$$\begin{aligned} x_{k+1} &= \Phi_i x_k + \Gamma_i u_k \\ y_k &= C_i x_k \end{aligned} \tag{3}$$

$$i \in \{1, 2\}$$

where $\Phi_i = e^{Ah_i}$, $\Gamma_i = \int_0^{h_i} e^{As} B ds$ and i denotes the discretized system obtained with sampling time h_i. Both discretizations lead to stable discrete systems with the spectral radius $\rho(\Phi_1) < 1$, $\rho(\Phi_2) < 1$ respectively, where $\rho(\Phi_i)$ gives the largest eigenvalue of Φ_i.

A discrete linear quadratic optimal controller is designed for both discretizations, minimizing the continuous loss function

$$J = \int_0^\infty (x(t)^T Q_c x(t) + u(t)^T R u(t)) dt \tag{4}$$

subject to system (2) sampled at h_1, h_2, where

$$Q_c = \begin{bmatrix} 20000 & 0 \\ 0 & 20000 \end{bmatrix} \quad R = 50$$

The resulting gain matrices are found by discretizing the loss function (4)

$$\begin{aligned} Q_{1,i} &= \int_{kh_i}^{kh_i+h_i} (\Phi_i^T(s, kh_i) Q_c \Phi_i(s, kh_i)) ds \\ Q_{12,i} &= \int_{kh_i}^{kh_i+h_i} (\Phi_i^T(s, kh_i) Q_c \Gamma_i(s, kh_i)) ds \\ Q_{2,i} &= \int_{kh_i}^{kh_i+h_i} (\Gamma_i^T(s, kh_i) Q_c \Gamma_i(s, kh_i) + R) ds \end{aligned}$$

and solving the discrete algebraic Riccati equation.

$$\begin{aligned} P_i = \Phi_i^T P_i \Phi_i + Q_{1,i} - (\Phi_i^T P_i \Gamma_i + Q_{12,i}) \cdot \\ \cdot (\Gamma_i^T S(k+1) \Gamma_i + Q_{2,i})^{-1} (\Gamma_i^T P_i \Phi_i + Q_{12,i}^T) \end{aligned}$$

The sate feedback law $u = -K_i \cdot x$ is then given by

$$K_i = (Q_{2,i} + \Gamma_i^T P_i \Gamma_i)^{-1} (\Gamma_i^T P_i \Phi_i + Q_{12,i}^T)$$

such that we get

$$K_1 = \begin{bmatrix} -195.401 \\ 19.4121 \end{bmatrix} \quad K_2 = \begin{bmatrix} -1296.6 \\ -8.826 \end{bmatrix}$$

For both discretizations the controlled closed loop systems is stable, i.e $\rho(\Phi_1 - \Gamma_1 K_1) < 1$, $\rho(\Phi_2 - \Gamma_2 K_2) < 1$ respectively. However in the case where the system is sampled with h_1 for 1 sampling interval and then the system is sampled with h_2 for 2 sampling intervals repeatedly we find that this sequence is unstable. This can be seen by looking at figure 1 or at the spectral radius of the resulting system $\rho((\Phi_2 - \Gamma_2 K_2)^2 (\Phi_1 - \Gamma_1 K_1)^1) > 1$. We obtain the spectral radius of the resulting system by writing the solution for sampling at h_1 once, $x_{h_1} = (\Phi_1 + \Gamma_1 K_1) x_0$ and sampling at h_2 twice, $x_{2h_2+h_1} = (\Phi_2 + \Gamma_2 K_2)^2 x_{h_1}$. We can now substitute into each other and obtain $x_{h_2+3h_1} = (\Phi_2 + \Gamma_2 K_2)^2 (\Phi_1 + \Gamma_1 K_1) x_0$. Since this is done repeatedly we can think of it as the new system description and take the spectral radius of it, in this case it is larger than one, hence the resulting system is unstable.

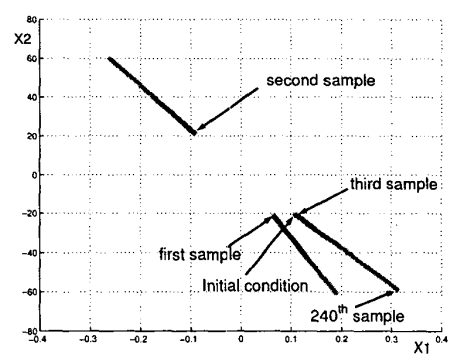

Figure 1: Unstable sequence

The Fig. 1 shows the samples, the discrete points of the continuous trajectory, for the unstable sys-

tem in phase-plane. The system 2 is sampled for one sampling interval with h_1, i.e. small distance between initial and first sample, and twice with h_2, i.e. larger distance between first second and third sample. It can be seen that the trajectory enlarges and it gets further away from the origin, i.e. sample 240 is much further away from the origin then initial sample.
It turns out that this is not the only sequence, which unstabilizes the system, table 1 shows further sequences for which the resulting system is unstable.

	$\rho((\Phi_2 - \Gamma_2 K_2)^{mh_2}(\Phi_1 - \Gamma_1 K_1)^{nh_1}) > 1$				
$n \cdot h_1$	$1 \cdot h_1$	$1 \cdot h_1$	$2 \cdot h_1$	$2 \cdot h_1$	$2 \cdot h_1$
$m \cdot h_2$	$2 \cdot h_2$	$3 \cdot h_2$	$4 \cdot h_2$	$5 \cdot h_2$	$6 \cdot h_2$

Table 1.: unstable sequences

Figure 2 shows another unstable sequence. The system is sampled one time at h_1 and three times at h_2.

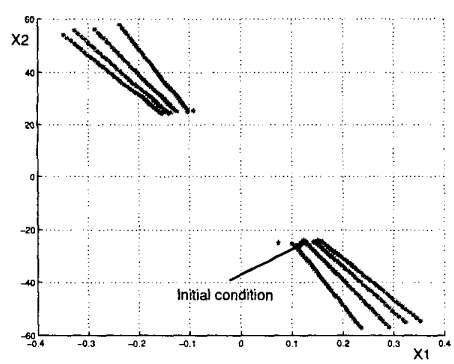

Figure 2: Unstable sequence

We have seen that even when two stable discrete controlled closed loop systems are obtained, from a stable continuous system, with state feedbacks minimizing the same continuous loss function that variation in sampling rate (switching between these two systems) can lead to instability.
In the next section we will propose a controller synthesis which will overcome this problem. An optimal controller design is stated which minimizes the loss function over one sampling time and has a terminal penalty, which is greater or equal than the cost of bringing the states to the origin for the worst case variations in sampling rate. The resulting control law is given by a piecewise linear state feedback controller. We show that this leads to a closed loop system, wich is stable for all variations in sampling rate.

2 Controller design

For controller design we suggest that instead of minimizing a continuous objective function over the infinite horizon, we minimize only over one sampling period. To compensate for the remaining cost we add a terminal penalty. Minimizing only over one sampling rate is more sensible since the sampling rate may change after one sampling period anyway. Since the terminal penalty has to be at least as big as the remaining worst case cost we write the following inequality

$$\begin{aligned} x(kh)^T P x(kh) \;\geq\; & \min_u \int_{kh}^{kh+h} (x^T Q_c x + u^T R u) dt \\ & + x(kh+h)^T P x(kh+h) \end{aligned} \quad (8)$$

$$\forall \quad h \in H = \{h_1, h_2, \ldots, h_n\}$$

The solution gives an optimal, piecewise constant state feedback controller which is stable regardless of the scheduling.
The first step in solving (8) is to discretize the objective function. This is done similarly as in [8]. The discretized objective function over one sampling interval with terminal penalty is

$$\begin{aligned} & x(kh)^T P x(kh) \geq \\ & \min_u (x^T(kh) Q_{1,i} x(kh) + 2x^T(kh) Q_{12,i} u(kh) \\ & + u^T(kh) Q_{2,i} u(kh)) + x(kh+h_i)^T P x(kh+h_i) \end{aligned} \quad (9)$$

$$\forall \quad i \in \{1, 2, \ldots, n\}$$

where

$$Q_{1,i} = \int_{kh_i}^{kh_i+h_i} (\Phi_i^T(s,kh_i)Q_c\Phi_i(s,kh_i))ds \quad (10)$$

$$Q_{12,i} = \int_{kh_i}^{kh_i+h_i} (\Phi_i^T(s,kh_i)Q_c\Gamma_i(s,kh_i))ds \quad (11)$$

$$Q_{2,i} = \int_{kh_i}^{kh_i+h_i} (\Gamma_i^T(s,kh_i)Q_c\Gamma_i(s,kh_i) + R)ds \quad (12)$$

and $\Phi_i = e^{Ah_i}$, $\Gamma_i = \int_0^{h_i} e^{As}Bds$ is the fundamental respectively the input matrix of the discretized system.

Theorem: If there exists $P = P^T > 0$, K_i, $i \in \{1,2,\ldots,n\}$ such that

$$(\Phi_i + \Gamma_i K_i)^T P(\Phi_i + \Gamma_i K_i) - P + \\ + (Q_{1,i} + 2Q_{12,i}K_i + K_i^T Q_{2,i} K_i) \leq 0 \quad (13)$$

$$\forall \quad i \in \{1,2,\ldots,n\}$$

then the sampled system is stable under variations for all admissible sampling rates h_i, $i \in \{1,2,\ldots,n\}$ and its performance is bounded by $x^T P x$.

Proof: Rearranging (9) and taking $x_{k+1} = \Phi_i x_k + \Gamma_i u_k$, $u_k = K_i x_k$ we obtain

$$(\Phi_i + \Gamma_i K_i)^T P(\Phi_i + \Gamma_i K_i) - P + \\ + (Q_{1,i} + 2Q_{12,i}K_i + K_i^T Q_{2,i} K_i) \leq 0. \quad (14)$$

$V(x) = x^T P x$ serves as Lyapunov function since $P = P^T > 0$ and

$$\begin{aligned} \Delta V(x) &= (\Phi_i + \Gamma_i K_i)^T P(\Phi_i + \Gamma_i K_i) - P \\ &\leq [IK_i]Q_i[IK_i]^T \end{aligned} \quad (15)$$

with

$$\forall \quad i \in I = \{1,2,\ldots,n\} \quad Q_i = \begin{bmatrix} Q_{1,i} & Q_{12,i} \\ Q_{12,i}^T & Q_{2,i} \end{bmatrix}$$

where Q_i are positive definite for all i and $[IK_i]$ are full rank, therefore $-[IK_i]Q_i[IK_i]$ is negative definite and hence $\Delta V(x) \leq 0$.

We have seen if we manage to find a controller which satisfies (8) and therefore also (13) we can guarantee that the controlled closed loop system is stable for all variations among h_i, $i \in \{1,2,\ldots,n\}$. We will now show how we can formulate the controller synthesis into a LMI, such that we obtain P and K_i.

3 Controller synthesis using LMI

We have seen that a system in form (1) with its discretizations (2) are robustly stable for variations among the prescribed sampling rates h_i, $\forall\ i \in \{1,2,\ldots,n\}$ and its cost is bounded by $P = P^T > 0$, when we find the state feedback gains K_i, $i \in \{1,2,\ldots,n\}$ which satisfy (13). The remaining problem is to obtain the P and the K_is. In order to obtain P and the K_is, we formulate the controller synthesis into a LMI. We take (13)

$$(\Phi_i + \Gamma_i K_i)^T P(\Phi_i + \Gamma_i K_i) - P + \\ + (Q_{1,i} + 2Q_{12,i}K_i + K_i^T Q_{2,i} K_i) \leq 0$$

which we can write as

$$\begin{bmatrix} \Phi_i + \Gamma_i K_i \\ I \\ K_i \end{bmatrix}^T \times \begin{bmatrix} P & 0 & 0 \\ 0 & Q_{1,i} & Q_{12,i} \\ 0 & Q_{12,i}^T & Q_{2,i} \end{bmatrix} \\ \times \begin{bmatrix} \Phi_i + \Gamma_i K_i \\ I \\ K_i \end{bmatrix} - P \leq 0 \quad (16)$$

$$\forall \quad i \in \{1,2,\ldots,n\}$$

Applying Schur's complement to the above expression we obtain

$$\begin{bmatrix} P & (\Phi_i + \Gamma_i K_i)^T & \begin{bmatrix} I & K_i^T \end{bmatrix} \\ (\Phi_i + \Gamma_i K_i) & P^{-1} & 0 \\ \begin{bmatrix} I \\ K_i \end{bmatrix} & 0 & Q_i^{-1} \end{bmatrix} \geq 0$$

$$\forall \quad i \in \{1,2,\ldots,n\}$$

where

$$Q_i = \begin{bmatrix} Q_{1,i} & Q_{12,i} \\ Q_{12,i}^T & Q_{2,i} \end{bmatrix}$$

Multiplying the above inequality from left and right with

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and setting $W_0 = P^{-1}$, $W_i = K_i P^{-1}$ we obtain the controller synthesis LMIs

$$\begin{bmatrix} W_0 & (\Phi_i W_0 + \Gamma_i W_i)^T & [W_0 \quad W_i^T] \\ \Phi_i W_0 + \Gamma_i W_i & W_0 & 0 \\ \begin{bmatrix} W_0 \\ W_i \end{bmatrix} & 0 & Q_i^{-1} \end{bmatrix} \geq 0 \tag{20}$$

$$\forall \quad i \in \{1, 2, \ldots, n\}$$

in $W_0 = W_0^T > 0$ and W_i. The solution of the LMI (20) gives the state feedback gains $K_i = W_i W_0^{-1}$ $\forall \quad i \in \{1, 2, \ldots, n\}$. Applying the state feedbacks gives a stable closed loop system which is robust against variations among the sampling times h_i $\forall \quad i \in \{1, 2, \ldots, n\}$. However we would not only like to stabilize the system we would further like to minimize the cost for driving the states to the origin for a given objective function (8). Therefore we would like to minimize the trace of W_0^{-1}. Unfortunately this is a non-convex optimization problem. Instead of minimizing $\min Trace(W_0^{-1})$ we minimize

$$\min \log \det W_0^{-1}$$

subject to (20) which is a convex optimization problem.

We have shown how the state feedback synthesis problem is cast into a LMI. In the following we give an illustrative example.

4 Example

We will now demonstrate the synthesis procedure by controlling system (1) again, however since we use the synthesis procedure above we will be certain that the controlled closed loop system is stable and robust against variations among all h_i. We sample the system again with the two very same sampling rates $h_1 = 0.002s$, $h_2 = 0.094s$. Using $\Phi_i = e^{Ah_i}$, $\Gamma_i = \int_0^{h_i} e^{As} B ds$ $\forall \quad i \in \{1, 2\}$ we obtain

$$\begin{aligned} x_{k+1} &= \Phi_i x_k + \Gamma_i u_k \\ y_k &= C_i x_k \end{aligned}$$

$$i \in \{1, 2\}$$

where

$$\Phi_1 = \begin{bmatrix} 0.9801 & 0.0020 \\ -19.8649 & 0.979 \end{bmatrix} \quad \Gamma_1 = \begin{bmatrix} 0.000 \\ 0.020 \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} -0.995 & 0.0002 \\ -2.4660 & -0.9950 \end{bmatrix} \quad \Gamma_2 = \begin{bmatrix} 0.0001995 \\ 0.0002466 \end{bmatrix}$$

For the controller design were we want to satisfy

$$\begin{aligned} x(kh)^T P x(kh) \quad \geq \quad & \min_u \int_{kh}^{kh+h} (x^T Q_c x + u^T R u) dt \\ & + x(kh+h)^T P x(kh+h) \end{aligned} \tag{25}$$

$$\forall \quad h \in H = \{h_1, h_2\}$$

we take the very same weights as in the introduction example

$$Q_c = \begin{bmatrix} 20000 & 0 \\ 0 & 20000 \end{bmatrix} \quad R = 50.$$

We then obtain and obtain $Q_{1,i}$, $Q_{2,i}$ and $Q_{3,i}$ by solving (10)-(12), such that we can write

$$Q_i = \begin{bmatrix} Q_{1,i} & Q_{12,i} \\ Q_{12,i}^T & Q_{2,i} \end{bmatrix}$$

$$\forall \quad i \in \{1, 2\}$$

$$Q_1 = \begin{bmatrix} 5329.5 & -394.6 & -0.529 \\ -394.6 & 39.5 & 0.0395 \\ -0.529 & 0.0395 & 0.1001 \end{bmatrix} \tag{28}$$

$$Q_1 = \begin{bmatrix} 9381400 & -6.0714 & -938.1423 \\ -6.0714 & 933.2359 & 0.0010 \\ -938.1423 & 0.0010 & 4.7938 \end{bmatrix} \tag{29}$$

we can now solve the state feedback synthesis LMI

$$\begin{bmatrix} W_0 & (\Phi_i W_0 + \Gamma_i W_i)^T & [W_0 \quad W_i^T] \\ \Phi_i W_0 + \Gamma_i W_i & W_0 & 0 \\ \begin{bmatrix} W_0 \\ W_i \end{bmatrix} & 0 & Q_i^{-1} \end{bmatrix} \geq 0$$

$$\forall \quad i \in \{1,2\}$$

and obtain $W_0 = W_0^T > 0$ and W_1, W_2, which gives the state feedback gains $K_i = W_i W_0^{-1} \forall \quad i \in \{1,2\}$

$$K_1 = \begin{bmatrix} 0.5784 \\ -0.0570 \end{bmatrix} \quad K_2 = \begin{bmatrix} 1765.5 \\ 0.0109 \end{bmatrix}$$

applying these state feedback gains guarantees stability and robustness against variations in sampling between h_1 and h_2, further the cost is bounded by $P = W_0^{-1}$

$$P = \begin{bmatrix} 2870500 & 12.812 \\ 12.812 & 286.9774 \end{bmatrix} \tag{32}$$

5 Conclusion

The paper has given an example for sample data control of a stable continuous system which was sampled at two different sampling rates. A controller was designed minimizing the same continuous loss function for both sampling rates. This leads to two stable closed loop systems. However it was shown that for various sequences where the sampling rate was changed repeatedly, the resulting system was unstable.

In order to overcome this shortcoming a different controller design was proposed. It was suggested that the objective function had to be minimized only over one sampling rate instead of minimizing over the infinite horizon. It was shown that when a terminal penalty was added which is greater or equal than the remaining cost for the worst case variations in sampling rate, the system was robustly stable against these variations.

The synthesis procedure was then formulated in terms of a LMI. In a second example the synthesis procedure using the proposed LMI was carried out on the introduction example. The state feedback gains, which are the solutions to the LMI, were given as well as the performance bound.

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