Frequency-Domain Analysis of Linear Time-Periodic Systems

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Frequency-Domain Analysis of Linear Time-Periodic Systems

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***Abstract*—In this paper, we study convergence of truncated representations of the frequency-response operator of a linear time-periodic system. The frequency-response operator is frequently called the harmonic transfer function. We introduce the concepts of input, output, and skew roll-off. These concepts are related to the decay rates of elements in the harmonic transfer function. A system with high input and output roll-off may be well approximated by a low-dimensional matrix function. A system with high skew roll-off may be represented by an operator with only few diagonals. Furthermore, the roll-off rates are shown to be determined by certain properties of Taylor and Fourier expansions of the periodic systems. Finally, we clarify the connections between the different methods for computing the harmonic transfer function that are suggested in the literature.**

***Index Terms*—Convergence analysis, frequency-response operators, linear time-periodic systems, series expansions.**

Notation: Signals $u(t)$ in continuous time on an interval I where $\|u\|_{L_r} = \left(\int_I |u(t)|^r dt\right)^{1/r}$, $r \geq 1$, is finite belongs to $L_r(I)$. When the interval I is clear from the context, it will be left out in the notation. Square-summable sequences $u(k)$ belong to ℓ_2, and the norm $\|u\|_{\ell_2} = \left(\sum_k |u(k)|^2\right)^{1/2}$ is finite. The set of L times continuously differentiable real functions in some open set Ω is denoted by $\mathcal{C}^L(\Omega)$. For $g(t,\tau)$, we denote by $g_t^{(k)}$ and $g_\tau^{(k)}$ the k-th partial derivative of g with respect to its first and second argument, respectively. p denotes the differentiation operator, and $1/p$ integration. $\mathbb{R}$ denotes the real numbers, $\mathbb{Z}$ the integers, and $\mathbb{C}$ the set of complex numbers. j is the imaginary unit, and $j\mathbb{R}$ is the imaginary axis, $\bar{x}$ denotes complex conjugate of x, and G^* is the adjoint of G.

I. Introduction

In this paper, we study linear operators G defined on signals u in L_r, where $r \geq 1$,

$$y = Gu.$$

We restrict ourselves to the set of bounded operators G, i.e., operators with finite induced norm

$$\|G\|_{L_r \to L_r} = \sup_{\|u\|_{L_r} \leq 1} \|Gu\|_{L_r}. \quad (1)$$

We assume in the following that the given operator G is bounded and has a time-domain representation with a *causal* impulse response

$$g(t,\tau) = 0, \qquad \text{for all} \quad t < \tau$$

and

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau \quad (2)$$

where $u(t)$ and $y(t)$ belong to $L_r(\mathbb{R})$. Often we identity the impulse response g with the operator G. Conditions for representability of an operator as an integral equation (2) are given in, for instance, [1]. It is well known that systems with finite-dimensional state-space realizations as well as infinite-dimensional models such as time-delay systems may be written on the form (2). We will often make the assumption that the impulse response has *uniform exponential decay*. This means that there are positive constants κ_1 and κ_2 such that

$$|g(t,\tau)| \leq \kappa_1 \cdot e^{-\kappa_2(t-\tau)}, \qquad \text{for all} \quad t > \tau.$$

In particular, this assumption implies boundedness of G, since $\|G\|_{L_r \to L_r} \leq \kappa_1/\kappa_2$, for all $r \geq 1$. This may be shown by using [2, Th. IV.7.2.22]. If there is a real positive number T such that

$$g(t+T,\tau+T) = g(t,\tau), \qquad \text{for all} \quad t > \tau \quad (3)$$

then the operator (or the impulse response) is said to be *periodic* with period T. The impulse response of a *time-invariant* system satisfies

$$g(t,\tau) = g(t-\tau,0), \qquad \text{for all} \quad t > \tau.$$

A system with a finite-dimensional state-space realization with no direct term can be written as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t). \end{aligned} \quad (4)$$

The impulse response of the system is given by $g(t,\tau) = C(t)\Phi_A(t,\tau)B(\tau)$, $t > \tau$, where $\Phi_A(t,\tau)$ is the transition

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matrix for $\dot{x} = A(t)x$, see [3]. If the matrices $A(t)$, $B(t)$, $C(t)$ are T-periodic, the impulse response satisfies (3). For simplicity, we do not deal with systems with direct terms in this paper. Direct terms may be included in the analysis using techniques similar to those in [4].

In the following, $u(t)$ and $y(t)$ are scalar signals. That is, we treat single-input–single-output systems. However, this is just for notational convenience. Everything can be done for multiple-input–multiple-output systems with only minor modifications.

It is well known, see, for example, [4], [5], that frequency-domain representations of periodic systems are infinite-dimensional operators. The main goal of this paper is to develop intuition for the decay rate of the elements in these operators, and to review some known results in the area.

A. Previous Work

The study of periodic systems has a long history in applied mathematics and control. One reason for the interest in periodic systems is that natural and man-made systems often have the periodicity property (3). Some examples are oscillators used in communication systems, planets and satellites in orbit, rotors of wind mills and helicopters, sampled-data systems, and ac power systems. There is an excellent survey of periodic systems and control in [6].

Frequency-domain analysis of linear time-periodic systems in continuous time has been studied by several authors in the past. A classical reference is the work by Zadeh, see [7], where the steady-state response of a time-varying system to harmonics is used to define a time-varying transfer function, the *parametric transfer function* (PTF). The PTF has been further developed and used in [8], [9]. The PTF is a scalar function that depends on two variables: time and frequency.

A second frequency-domain representation can be obtained by using time-domain lifting on the time-periodic system, and then applying the z-transform, see [5], [10], [11]. Colaneri called this function the *transfer function operator* (TFO). This is an integral operator with a kernel depending on time and frequency. An alternative derivation of the TFO comes from studies of the steady-state response of the system to an *exponentially modulated periodic* (EMP) signal. The EMP signals serve as good test functions, since EMP signals are mapped to EMP signals by periodic systems.

A third approach was taken by Wereley and Hall in [5], [12]. They applied an harmonic balance method to state-space systems with EMP inputs. That is, periodic matrices and signals are expanded into Fourier series and harmonics are equated. This method yields a transfer function that depends only on frequency. The function was called the *harmonic transfer function* (HTF). The HTF is an infinite-dimensional operator. The infinite dimensionality can be seen as the price that is paid for the removal of the time dependence in the transfer function. For example, it was shown in [13] that a Fourier expansion of the PTF in the time direction yields the elements of the HTF.

All of the above transfer functions can be used for studies of periodic systems, for example, to compute norms. It is important to understand that all of these transfer functions are equivalent. The PTF and the TFO are time dependent. If this time dependence is expressed in an harmonic basis of the type $\{e^{jk2\pi t/T}\}_{k\in\mathbb{Z}}$, we essentially obtain the HTF. Relations of this sort are treated in [13]–[15]. In this paper, we investigate what happens when higher harmonics in this basis are truncated. An alternative approximation method is, for example, to use fast sampling in time. Such approximations are discussed in [16]. It should also be mentioned in this context that the frequency-domain operator of a discrete-time periodic system becomes finite dimensional, see [17], [18]. The previous listing of frequency-domain methods is not complete. There are more representations, see, for example, [19], [20].

In the area of sampled-data systems, a lot of related work has been done. The PTF has been applied to sampled-data systems in [8]. A method similar to the TFO, that is, a lifting and z-transform approach, has been used in, for example, [15] and [21]. An approach similar to the HTF has been used in [22] and [23]. A nice property of sampled-data systems is that often closed-form solutions are obtained. This is not the case for generic periodic systems. The literature on sampled-data systems is vast and many more references can be found in the previous work.

We mainly work with the HTF in this paper. From the above discussion, it follows that this is not a severe restriction. The HTF has successfully been used by several authors for different applications in the past. For example, for identification of helicopter dynamics, see [18], for vibration damping in helicopters, see [24], and for stability and robustness analysis in switched power systems, see [25], [26]. A nice feature with the HTF is that we can directly extract Bode-type diagrams that describe the cross-coupling of frequencies from the diagonals of the HTF. This can be used to detect resonances that involve several frequencies, see [26]. Another reason for studies of the HTF is that it has recently obtained a lot of theoretical attention. Formally, we can work with the HTF just as with a standard transfer function. Hence, formulas for H_2/H_∞-norms are completely analogous to the time-invariant formulas, see [4], [5], and [27]. However, the HTF is also useful for studies of attainable H_2 performance, see [28], generalization of the Nyquist criterion [29], and for generalization of Bode's sensitivity integral [30], [31].

B. Computation of the HTF

Despite all of this work, there are still open issues about the HTF. In particular, how the HTF should be computed. Three approaches have been taken, to the authors' knowledge.

In the first approach, see [4] and [5] it is assumed that G has a state-space realization (4), and that a Floquet transformation has been performed. Then the matrix $A(t)$ is time invariant, and explicit formulas for the elements in the HTF can be given as a series of the Fourier coefficients of $B(t)$ and $C(t)$.

The second approach is also a state–space approach, see [28]. The elements of the HTF are given implicitly via an inversion of an unbounded quasi-Toeplitz operator, with the Fourier coefficients $\{A_k\}_{k\in\mathbb{Z}}$ of the state matrix $A(t)$ on the diagonals. It has been claimed that this yields the HTF when the dimension of finite-dimensional truncations of the operator grows toward infinity. However, to the best knowledge of the authors of this paper, how and when this convergence works has not been properly explained. This second approach is interesting since it does not require a Floquet transformation on the state-space model, and allows us to work directly with the Fourier coefficients of the state-space realization. We call this method the *truncated harmonic balance method*.

A third approach is used in [26]. This approach is based on an impulse-response model (2) of the periodic system. It is shown how the elements of the HTF can be computed via a Fourier expansion of the impulse response. This approach is interesting since it only uses input-output data of the model. However, the calculations in [26] are formal and many details and possibilities are not treated.

In Section VI of this paper, we clarify the connections between the above approaches, using the tools that are developed here. In the limit, we obtain the same operator no matter what approach that is used. Hence, it is justified to use the term HTF in all of the previous cases.

C. Organization and Contributions

This paper serves the purpose of survey some existing results as well as introducing new results. The new results give intuition for the structure of the HTF, based on two types of series expansions of the systems.

In Section II, we derive Taylor expansions of time-varying systems. The expansions are around "infinite frequency," and the coefficients become time-varying Markov parameters. Two different expansions are studied. We introduce the concepts of input and output roll-off of a time-varying system, and relate the Markov parameters to the roll-off concepts. In Section III, we derive Fourier expansions of time-periodic systems. The generalized Fourier coefficients become time-invariant systems. Similar ideas were suggested in, for example, [7], [26]. Here we apply Hilbert space formalism to the problem. Furthermore, we derive conditions under which truncated Fourier series converge in induced L_r-norm, and introduce the concept of skew roll-off. In Section IV, we define the HTF based on the impulse response, as was done in [26]. Its definition is straightforward after a Fourier expansion. We show that if the periodic system has high input and output roll-off, then it may be well approximated by a finite low-dimensional matrix function. If the system has high skew roll-off, then it may be approximated by an operator with only few diagonals. In Section V, we obtain error bounds for the closed-loop operator $(I+G)^{-1}$, when it is computed from truncated HTFs. These formulas are useful in Section VI, where we show that the HTF defined in Section IV is identical to the HTF defined in [4], [5]. We also study the truncated harmonic balance method. It is seen that by a minor modification of the method, we can show that it converges to the desired operator. The convergence may, however, be quite slow.

An early version of this paper is [32].

II. Taylor Expansions of Time-Varying Systems

We will obtain a frequency-domain description of G. Many times the input signal $u(t)$ and output signal $y(t)$ in $L_2(\mathbb{R})$ are represented by their Fourier transforms $\hat{u}(j\omega)$ and $\hat{y}(j\omega)$, where ω is the angular frequency. This presents no problems since $L_2(\mathbb{R})$ is isomorphic to $L_2(j\mathbb{R})$ under the Fourier transform, see, for example, [33], and $\|u\|_{L_2} = \left(\int_{-\infty}^{\infty} |u(t)|^2 dt\right)^{1/2} = (1/\sqrt{2\pi})\left(\int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega\right)^{1/2} =: \|\hat{u}\|_{L_2}$.

A. Markov Parameters for Time-Varying Systems

As a first step in the analysis, we make an expansion of the convolution integral (2) that resembles a Taylor expansion. This is motivated by the Markov parameters for time-invariant systems. That is, a transfer function $\hat{g}(j\omega)$ of a time-invariant system, where $g(t)$ is the impulse response, can under certain regularity conditions be Taylor expanded as

$$\hat{g}(j\omega) = \frac{g(0)}{j\omega} + \frac{g'(0)}{(j\omega)^2} + \frac{g''(0)}{(j\omega)^3} + \dots, \quad \text{as} \quad |\omega| \to \infty. \quad (5)$$

The Markov parameters are $\{g(0), g'(0), g''(0), \dots\}$ and they determine the response to high-frequency signals. If the first Markov parameters are zero, then high-frequency signals are attenuated quickly. ("The system has high roll-off.") We will establish similar conditions in the time-varying case and start by putting restrictions on the impulse response, for the computations to be justified. The set of continuously differentiable and exponentially bounded impulse responses will appear frequently throughout this article.

Definition 1 (The Set $\mathcal{C}_e^L$): A causal time-varying (not necessarily periodic) real impulse response g belongs to the set $\mathcal{C}_e^L$ if

E1) $g(t,\tau)$ belongs to $\mathcal{C}^L(\Omega)$, where $\Omega = \{(t,\tau) : t > \tau\}$;

E2) $g(t,\tau)$ and all its partial derivatives up to order L have a limit everywhere on the boundary of Ω, that is

$$g_x^{(k)}(t,t) := \lim_{\Omega \ni (\nu,\xi) \to (t,t)} g_x^{(k)}(\nu,\xi)$$

exists for all t, where $x = t$ or τ, and $k = 0 \dots L$ ($g_x^{(0)} = g$);

E3) $g(t,\tau)$ and all its partial derivatives up to order L have uniform exponential decay.

Example 1 (State–Space Models): We can check that the impulse response of state-space models belong to $\mathcal{C}_e^L$ if the model is exponentially stable, $B(t)$ and $C(t)$ belong to $\mathcal{C}^L(\mathbb{R})$, and $A(t)$ belongs to $\mathcal{C}^{L-1}(\mathbb{R})$. Furthermore, all the matrices should be bounded over $\mathbb{R}$.

It will be useful to consider signals in the space of Schwartz functions $\mathcal{S}$,

$$\mathcal{S} = \big\{u(t) : u(t) \in \mathcal{C}^\infty(\mathbb{R}) \text{ and } t^\alpha p^\beta u(t) \text{ is bounded for all } \alpha, \beta \geq 0\big\}$$

where p is the differentiation operator. The set $\mathcal{S}$ is dense in $L_r(\mathbb{R})$, for $1 \leq r < \infty$, and the Fourier transform of an element in $\mathcal{S}$ is again in $\mathcal{S}$, see [34]. To obtain expansions of (2) in the form of (5), we proceed by using integration by parts. If we choose an input signal $u \in \mathcal{S}$, this gives

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(t,\tau) u(\tau) d\tau \\ &= g(t,t)\frac{u(t)}{p} - \int_{-\infty}^{t} g_\tau^{(1)}(t,\tau)\frac{u(\tau)}{p} d\tau. \end{aligned} \quad (6)$$

$1/p$ is the integration operator: $u(t)/p = \int_{-\infty}^{t} u(s)ds$, and $g_\tau^{(1)}$ means the partial derivative of g with respect to its second argument. The above computation is allowed if $g \in \mathcal{C}_e^1$ and $u \in \mathcal{S}$. One should notice that (6) is an expansion of the time-varying system (2) into a sum of a modulated integrator and a new stable time-varying system with $g_\tau^{(1)}(t,\tau)$ as impulse response. After the following definition, we obtain the general expansion formulas.

Definition 2 (Input and Output Markov Parameters): For a system with impulse response g in $\mathcal{C}_e^{L-1}$, the *input Markov parameters*, $\{a_k(t)\}_{k=1}^{L}$, are defined as

$$\begin{aligned}\{a_1(t), a_2(t), \ldots, a_L(t)\} \\ = \left\{g(t,t), -g_\tau^{(1)}(t,t), \ldots, (-1)^{L-1} g_\tau^{(L-1)}(t,t)\right\}\end{aligned} \tag{7}$$

and, the *output Markov parameters*, $\{b_k(t)\}_{k=1}^{L}$, are defined as

$$\begin{aligned}\{b_1(t), b_2(t), \ldots, b_L(t)\} \\ = \left\{g(t,t), g_t^{(1)}(t,t), \ldots, g_t^{(L-1)}(t,t)\right\}\end{aligned} \tag{8}$$

for $t \in \mathbb{R}$.

Remark 1 (Markov Parameters for Time-Invariant Systems): For time-varying systems in $\mathcal{C}_e^L$, the Markov parameters are bounded continuously time-varying functions. For time-invariant systems, with impulse response $g(t,\tau) = g(t-\tau, 0)$, the input and output Markov parameters coincide with the traditional Markov parameters, and are constant and equal.

Theorem 1 (Taylor Expansions of Time-Varying Systems): Assume that g belongs to $\mathcal{C}_e^L$. Then for every input $u \in \mathcal{S}$, the output $y = Gu$ given by (2) can be expressed in either of the following two ways:

— Input Markov parameter expansion

$$\begin{aligned}y(t) = a_1(t)\frac{u(t)}{p} + a_2(t)\frac{u(t)}{p^2} + \ldots + a_L(t)\frac{u(t)}{p^L} \\ +(-1)^L \int_{-\infty}^{t} g_\tau^{(L)}(t,\tau)\frac{u(\tau)}{p^L}d\tau.\end{aligned} \tag{9}$$

— Output Markov parameter expansion

$$\begin{aligned}y(t) = \frac{1}{p}b_1(t)u(t) + \frac{1}{p^2}b_2(t)u(t) + \ldots + \frac{1}{p^L}b_L(t)u(t) \\ +\frac{1}{p^L}\int_{-\infty}^{t} g_t^{(L)}(t,\tau)u(\tau)d\tau.\end{aligned} \tag{10}$$

Proof: To prove (9), use integrations by parts on $\int_{-\infty}^{t} g_\tau^{(1)}(u/p)d\tau$, and substitute into (6). If the procedure is repeated, we obtain (9).

The first step in proving (10) is to differentiate (2)

$$y'(t) = g(t,t)u(t) + \int_{-\infty}^{t} g_t^{(1)}(t,\tau)u(\tau)d\tau$$

and then to integrate over $(-\infty, t]$ in the t-direction

$$y(t) = \frac{1}{p}g(t,t)u(t) + \frac{1}{p}\int_{-\infty}^{t} g_t^{(1)}(t,\tau)u(\tau)d\tau. \tag{11}$$

Repeat this procedure on the virtual output $y_m = \int g_t^{(m)} u d\tau$ for $m = 1 \ldots L-1$ and substitute into (11). The above computations are allowed under the given assumptions, since the outputs are continuously differentiable and $y_m \to 0$, as $|t| \to \infty$. Equation (10) may also be proven from (9) by a duality argument, see Remark 2. ■

Remark 2 (Duality of Input and Output Markov Parameters): The adjoint G^* of $G: L_2 \to L_2$, where $g \in \mathcal{C}_e^L$, is given by (the anti-causal relation) $z(\tau) = \int_\tau^\infty g(t,\tau)w(t)dt$, and one can make Taylor expansions of this relation as well

$$\begin{aligned}z(\tau) &= g(\tau,\tau)\frac{w(\tau)}{p^*} + g_t^{(1)}(\tau,\tau)\frac{w(\tau)}{(p^*)^2} + \ldots \\ z(\tau) &= \frac{1}{p^*}g(\tau,\tau)w(\tau) - \frac{1}{(p^*)^2}g_\tau^{(1)}(\tau,\tau)w(\tau) + \ldots\end{aligned}$$

where $w(\tau)/p^* = \int_\tau^\infty w(s)ds$. So with the interchange $t \leftrightarrow \tau$ the input Markov parameters of G are the output Markov parameters of G^* (with the obvious changes from causality to anticausality).

B. Input Roll-Off and Output Roll-Off

Equation (5) shows that there is a relation between the Markov parameters and high-frequency behavior for time-invariant systems. This relation will be further explored for time-varying systems. We need the projection operator P_Ω on L_2 that is defined by

$$(\widehat{P_\Omega y})(j\omega) = \begin{cases} \hat{y}(j\omega), & |\omega| \le \Omega \\ 0, & |\omega| > \Omega. \end{cases}$$

Notice that P_Ω is not causal in the time domain, and $\|P_\Omega\|_{L_2 \to L_2} = 1$. It is also convenient to define $Q_\Omega = I - P_\Omega$. This term is motivated in Section IV-A.

Definition 3 (Rectangular Truncation): Assume that $G : L_2 \to L_2$. Then $P_{\Omega_1} G P_{\Omega_2}$ is called a rectangular truncation of G.

The systems in the following are strictly proper (they have no direct term) and can be arbitrarily well approximated by its low-frequency part:

$$\|G - P_{\Omega_1} G P_{\Omega_2}\|_{L_2 \to L_2} \to 0 \qquad \text{as} \quad \Omega_1, \Omega_2 \to \infty.$$

To quantify the rate of convergence, we use the bound

$$\begin{aligned}\|G - P_{\Omega_1} G P_{\Omega_2}\|_{L_2 \to L_2} \\ \le \|(I - P_{\Omega_1})G\|_{L_2 \to L_2} + \|G(I - P_{\Omega_2})\|_{L_2 \to L_2}.\end{aligned}$$

Definition 4 (Input and Output Roll-Off): Assume that G is a bounded operator on L_2. If there are positive constants C_1 and k_1 such that

$$\|(I - P_\Omega)G\|_{L_2 \to L_2} \le C_1 \cdot \Omega^{-k_1}$$

then G is said to have *output roll-off* k_1. The largest such k_1 is called the *maximum output roll-off*. If there are positive constants C_2 and k_2 such that

$$\|G(I-P_\Omega)\|_{L_2\to L_2} \leq C_2 \cdot \Omega^{-k_2}$$

then G is said to have *input roll-off* k_2. The largest such k_2 is called the *maximum input roll-off*.

Remark 3: The roll-off rates k_1 and k_2 are not necessarily integers, but sometimes the maximum roll-off rates are, see Theorem 2. If a system has output roll-off k_1, then it also has output roll-off $\tilde{k}_1$, where $0 < \tilde{k}_1 < k_1$, and similarly for input roll-off.

Some simple properties for calculations with systems with input/output roll-off are stated in the following proposition.

Proposition 1 (Input and Output Roll-Off): The following rules apply to systems with roll-off.

i) If H has output roll-off k_1 and G is bounded, then HG has output roll-off of k_1. If G has input roll-off k_2 and H is bounded, then HG has input roll-off of k_2.

ii) Input and output roll-off reduce to the standard notion of roll-off for time-invariant G. That is, $g(t,\tau) = g(t-\tau,0) =: g_0(t-\tau)$ and $|\hat{g}_0(j\omega)| \leq C \cdot |\omega|^{-k}$, where k is either the input or the output roll-off of G.

iii) If H is a time-invariant system with output roll-off k_1 and G has output roll-off k_2, then HG has output roll-off k_1+k_2. If H is a time-invariant system with input roll-off k_1 and G has input roll-off k_2, then GH has input roll-off k_1+k_2.

Proof: In this proof, $\|\cdot\|$ means the induced L_2-norm. i) follows from Definition 4 and the induced norm property $\|GH\| \leq \|G\|\|H\|$. ii) Q_Ω commutes with a time-invariant G. We have $\|Q_\Omega G\| = \|GQ_\Omega\| = \operatorname{ess\,sup}_{\omega\in\mathbb{R}\setminus[-\Omega,\Omega]} |\hat{g}_0(j\omega)| \leq C \cdot \Omega^{-k}$, where k could be either the input or output roll-off. iii) We have $Q_\Omega H = Q_\Omega H Q_\Omega$ and $\|Q_\Omega(HG)\| = \|(Q_\Omega H)(Q_\Omega G)\| \leq \|Q_\Omega H\|\|Q_\Omega G\|$, which proves the first statement. The second statement follows from a dual argument. ■

Since Definition 4 may be hard to check for a given operator G, it simplifies if we decompose the system into terms that are easier to analyze. The Taylor expansions in Theorem 1 are such decompositions. We use them in the following theorem to show that the Markov parameters in Definition 2 determine the roll-off.

Theorem 2 (Markov Parameters and Roll-Off): Assume that the impulse response g of G belongs to $\mathcal{C}_e^L$. Let $a_{l_1}(t)$ be the first nonzero input Markov parameter and $b_{l_2}(t)$ be the first nonzero output Markov parameter. Then

i) G has input roll-off l_1. If $L > l_1$ and $|a_{l_1}(t)| \geq \epsilon > 0$ for all t, then G has maximum input roll-off l_1;

ii) G has output roll-off l_2. If $L > l_2$ and $|b_{l_2}(t)| \geq \epsilon > 0$ for all t, then G has maximum output roll-off l_2.

Proof: We start with the second statement of i). We need a bound on $\|GQ_\Omega\|_{L_2\to L_2}$. By definition, we have

$$\|GQ_\Omega\|_{L_2\to L_2} = \sup_{\|u_\Omega\|_{L_2}\leq 1} \|Gu_\Omega\|_{L_2} \qquad u_\Omega = Q_\Omega u. \quad (12)$$

TABLE I
INPUT AND OUTPUT MARKOV PARAMETERS OF A TIME-VARYING STATE–SPACE MODEL (4)

#	Input Markov parameter	Output Markov parameter
1.	$C(t)B(t)$	$C(t)B(t)$
2.	$C(t)[A(t)B(t) - \frac{d}{dt}B(t)]$	$[\frac{d}{dt}C(t) + C(t)A(t)]B(t)$
3.	$C(t)[\frac{d^2}{dt^2}B(t) - 2A(t)\frac{d}{dt}B(t)$ $-(\frac{d}{dt}A(t))B(t) + A^2(t)B(t)]$	$[\frac{d^2}{dt^2}C(t) + 2(\frac{d}{dt}C(t))A(t)$ $+C(t)\frac{d}{dt}A(t) + C(t)A^2(t)]B(t)$

To bound (12), we make an input Markov parameter expansion of Gu_Ω, for $u_\Omega \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $Q_\Omega L_2$, such a bound holds for inputs u in L_2 as well. Using the assumption that the first Markov parameters are zero and $L > l_1$, we have

$$\begin{aligned} y(t) &= GQ_\Omega u(t) \\ &= a_{l_1}(t)\frac{u_\Omega(t)}{p^{l_1}} + a_{l_1+1}(t)\frac{u_\Omega(t)}{p^{l_1+1}} \\ &\quad + (-1)^{l_1+1}\int_{-\infty}^{t} g_\tau^{(l_1+1)}(t,\tau)\frac{u_\Omega(\tau)}{p^{l_1+1}}d\tau \\ &= a_{l_1}(t)\frac{Q_\Omega}{p^{l_1}}u(t) + \tilde{G}Q_\Omega u(t). \end{aligned}$$

The first term in the expansion can be bounded as

$$\frac{\epsilon}{\Omega^{l_1}} \leq \left\| a_{l_1}(t)\frac{Q_\Omega}{p^{l_1}} \right\|_{L_2\to L_2} \leq \frac{K_1}{\Omega^{l_1}}$$

where $K_1 = \sup_t |a_{l_1}(t)|$. For the remaining terms, we can make the upper bound

$$\|\tilde{G}Q_\Omega\|_{L_2\to L_2} \leq \frac{K_2+K_3}{\Omega^{l_1+1}}$$

where $K_2 = \sup_t |a_{l_1+1}(t)|$ and $K_3 = \kappa_1/\kappa_2$, where $|g_\tau^{(l_1+1)}(t,\tau)| \leq \kappa_1 e^{-\kappa_2(t-\tau)}$. Hence, using the triangular inequality we have

$$\frac{\epsilon}{\Omega^{l_1}} - \frac{K_2+K_3}{\Omega^{l_1+1}} \leq \|GQ_\Omega\|_{L_2\to L_2} \leq \frac{K_1}{\Omega^{l_1}} + \frac{K_2+K_3}{\Omega^{l_1+1}} \quad (13)$$

and the system has input roll-off l_1. To see that l_1 is the maximum input roll-off, assume G has a larger roll-off $k_1 > l_1$. Then, we obtain a contradiction for sufficiently large Ω in (13) due to the lower bound.

To prove the first statement of i), we make an input Markov parameter expansion as before, but we stop the expansion when we have terms with $1/p^{l_1}$. Then we can prove an upper bound similar to (13) (but not a lower bound) and the result follows. To prove ii), the output Markov expansion is used instead of the input Markov expansion. ■

Example 2 (Finite-Dimensional State–Space Models): Let us assume that the system has a state–space realization (4) and that the impulse response belongs to $\mathcal{C}_e^L$, see Example 1. According to Theorem 2 the roll-off of the state–space system can be determined by checking which of the Markov parameters are zero. The first few Markov parameters of (4) are given in Table I. In particular, they coincide with the normal Markov parameters for time-invariant systems: $CB, CAB, CA^2B, \ldots$

The Markov parameter conditions for output roll-off k_1 correspond to the conditions for relative degree k_1 of a periodic system, defined in [35]. Also notice that the k-th Markov parameters may be written as $C[(L_A^*)^{k-1}B]$ and $[L_A^{k-1}C]B$, where $L_A^*B = AB - (d/dt)B$ and $L_AC = CA + (d/dt)C$, using notation from [35].

III. Fourier Expansions of Time-Periodic Systems

Until now we have not used the periodicity condition $g(t,\tau) = g(t+T,\tau+T)$. The periodicity condition can be used for Fourier expansions. The possibility of Fourier expansions of periodic systems was discussed already in [7]. Here, we apply Hilbert space formalism to the problem. We define the space H_2 for periodic impulse responses

$$H_2 = \{g(t,\tau) : \|g\|_{H_2} < \infty, g(t,\tau) \text{ is } T\text{- periodic and causal}\}$$

where

$$\|g\|_{H_2}^2 = \frac{1}{T}\int_{\tau=0}^{T}\int_{r=0}^{\infty} |g(\tau+r,\tau)|^2\, dr d\tau = \frac{1}{T}\int_{t=0}^{T}\int_{r=0}^{\infty} |g(t,t-r)|^2\, dr dt.$$

The previous equality follows from the periodicity condition (3). The H_2-norm is also used in, for example, [4], [10], [28]. H_2 is a Hilbert space with the scalar product

$$\langle g,h\rangle_{H_2} = \frac{1}{T}\int_{\tau=0}^{T}\int_{r=0}^{\infty} \overline{g(\tau+r,\tau)}h(\tau+r,\tau)dr d\tau = \frac{1}{T}\int_{t=0}^{T}\int_{r=0}^{\infty} \overline{g(t,t-r)}h(t,t-r)dr dt. \tag{14}$$

H_2 is composed of the well-known separable Hilbert spaces $L_2[0,T]$ and $L_2[0,\infty)$. Orthonormal basis functions in H_2 are, for example

$$\tilde{e}_{k,n}(t,\tau) = e^{jk\omega_0\tau}e_n(t-\tau) \quad \text{or} \quad e_{k,n}(t,\tau) = e^{jk\omega_0 t}e_n(t-\tau)$$

where $\omega_0 = 2\pi/T$ and $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis in $L_2[0,\infty)$. We can choose $e_n(r) = e^{-r/2}L_n(r)$, where L_n are the Laguerre polynomials, for example. By standard results from functional analysis, see, for example [36], all functions in H_2 can be represented by a generalized Fourier series

$$g = \sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty}\langle \tilde{e}_{k,n}, g\rangle_{H_2}\tilde{e}_{k,n} = \sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty}\langle e_{k,n}, g\rangle_{H_2} e_{k,n} \tag{15}$$

where the series converge in H_2-norm. Instead of using the expansion (15), it is more useful for us to only use the expansion in the k-direction. This is expressed in the following theorem.

Theorem 3 (Fourier Expansion in H_2): Assume that the impulse response of G belongs to H_2. Then

$$g(t,\tau) = \sum_{k=-\infty}^{\infty} \tilde{g}_k(t-\tau)e^{jk\omega_0\tau} \tag{16}$$

$$g(t,\tau) = \sum_{k=-\infty}^{\infty} g_k(t-\tau)e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} \tag{17}$$

with convergence in H_2. The Fourier coefficients are given by

$$\tilde{g}_k(r) = \frac{1}{T}\int_0^T e^{-jk\omega_0\tau}g(\tau+r,\tau)d\tau, \tag{18}$$

$$g_k(r) = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}g(t,t-r)dt. \tag{19}$$

Proof: We show (16). A similar calculation gives (17). Using the Cauchy–Schwartz inequality on (18), we have for all k

$$|\tilde{g}_k(r)|^2 \le \frac{1}{T}\int_0^T |g(\tau+r,\tau)|^2\, d\tau \Rightarrow \|\tilde{g}_k\|_{L_2} \le \|g\|_{H_2}.$$

Since $\tilde{g}_k \in L_2[0,\infty)$, we can expand it in the generalized Fourier series

$$\tilde{g}_k(t-\tau) = \sum_{n=0}^{\infty}\langle e_n, \tilde{g}_k\rangle_{L_2} e_n(t-\tau).$$

If we insert this in the sum in (16), we see that it is equal to (15). ■

Some immediate properties of the Fourier coefficients in (16), (17) are stated in the following corollary.

Corollary 1 (Properties of Fourier Coefficients): Assume that g and h belong to H_2. Then

i) the Fourier coefficients $\tilde{g}_k$ and g_k satisfy $\tilde{g}_k(r) = e^{jk\omega_0 r}g_k(r)$;
ii) if $g(t,\tau)$ is real, then $g_{-k}(r) = \overline{g_k(r)}$ for all k;
iii) the Fourier coefficients $\tilde{g}_k$ and g_k belong to $L_2[0,\infty)$ for all k;
iv) the scalar product (14) can be expressed as

$$\langle g,h\rangle_{H_2} = \sum_{k=-\infty}^{\infty}\langle \tilde{g}_k, \tilde{h}_k\rangle_{L_2} = \sum_{k=-\infty}^{\infty}\langle g_k, h_k\rangle_{L_2}.$$

By Corollary 1 it follows that there is no essential difference between the expansions (16) and (17). They are essentially the same. In the following sections, we will mostly work with the expansions in g_k. It is also useful to introduce the orthogonal projection $P_{[N]} : H_2 \to H_2$

$$g_{[N]}(t,\tau) = P_{[N]}g(t,\tau) = \sum_{k=-N}^{N}\tilde{g}_k(t-\tau)e^{jk\omega_0\tau} = \sum_{k=-N}^{N} g_k(t-\tau)e^{jk\omega_0 t}. \tag{20}$$

We make the following definition.

Definition 5 (Skew Truncation): Assume that the impulse response of G is in H_2. Then the system $G_{[N]}$, with impulse response $g_{[N]}$ given by (20), is called an Nth-order *skew truncation* of G.

The reason for the term "skew truncation" will be more clear once we have constructed the harmonic transfer function in Section IV-B. A simple application of Corollary 1 iv) gives that the skew truncations converge in H_2, $\|g - g_{[N]}\|_{H_2} \to 0$, as $N \to \infty$.

Remark 4 (Optimal Approximations in H_2): By Corollary 1 iv), impulse responses that do not contain the same Fourier coefficients are orthogonal in H_2. Hence, $g_{[N]} \perp (g - g_{[N]})$ in H_2. From standard results for approximation in Hilbert spaces, $g_{[N]}$ is the optimal H_2-approximation of g in the subspace $P_{[N]}H_2$. In particular, $g_{[0]}$ is the optimal *time-invariant* impulse-response approximation of g in H_2.

A. Input–Output Properties

The interpretation of the convergence in the above Fourier expansions is that for impulse inputs, the outputs converge in "mean energy sense." This is a quite weak form of convergence. By strengthening the assumptions on g, we can show stronger forms of convergence, in induced norms. This is the topic of the rest of this section.

The input–output map of $y_N = G_{[N]}u$ is given by

$$\begin{aligned} y_N(t) &= \sum_{k=-N}^{N} \int_{-\infty}^{t} \tilde{g}_k(t-\tau)e^{jk\omega_0\tau}u(\tau)d\tau \\ &= \sum_{k=-N}^{N} \int_{-\infty}^{t} g_k(t-\tau)e^{jk\omega_0 t}u(\tau)d\tau \end{aligned} \tag{21}$$

where we have interchanged the order of integration and summation. The output $y_N(t)$ is given by a parallel connection of input- or output-modulated time-invariant systems. We associate with the kth Fourier coefficients of g causal and time-invariant system $\tilde{G}_k$ and G_k

$$\begin{aligned} \tilde{G}_k: &\quad \tilde{g}_k(t,\tau) = \tilde{g}_k(t-\tau) \\ G_k: &\quad g_k(t,\tau) = g_k(t-\tau). \end{aligned}$$

Hence, we can represent $G_{[N]}$ by the formal Fourier series

$$G_{[N]} = \sum_{k=-N}^{N} \tilde{G}_k e^{jk\omega_0\tau} = \sum_{k=-N}^{N} e^{jk\omega_0 t}G_k \tag{22}$$

where the Fourier coefficients are *time-invariant* systems. We will show that these series converge in induced norms. Let us again use the class of causal continuously differentiable and exponentially bounded impulse responses, $\mathcal{C}_e^L$, defined in Section II.

Lemma 1 (Bounded Fourier Coefficients): Assume that a periodic system G has an impulse response g that belongs to $\mathcal{C}_e^L$. Then

i) g belongs to H_2;
ii) there are positive constants α and C such that the Fourier coefficients are bounded

$$|\tilde{g}_k(r)| = |g_k(r)| \leq \frac{C \cdot e^{-\alpha r}}{|k\omega_0|^L}, \qquad |k| > 0.$$

Proof:

i) We will prove that g belongs to H_2. Since g belongs to at least $\mathcal{C}_e^0$, by E3) we have $|g(t,\tau)| \leq \kappa_1 e^{-\kappa_2(t-\tau)}$. Hence, we can bound the H_2-norm

$$\|G\|_{H_2}^2 \leq \frac{1}{T}\int_{\tau=0}^{T}\int_{r=0}^{\infty} \kappa_1^2 e^{-2\kappa_2 r}dr d\tau = \frac{\kappa_1^2}{2\kappa_2}.$$

ii) By assumption E1) we have that $g(\cdot + r, \cdot) \in \mathcal{C}^L(\mathbb{R})$ for all $r \geq 0$. Make a Fourier expansion in the τ-direction of $(d^L/d\tau^L)g(\tau + r, \tau)$, and notice that

$$(jk\omega_0)^L \tilde{g}_k(r) = \frac{1}{T}\int_{\tau=0}^{T} \left[\frac{d^L}{d\tau^L}g(\tau + r, \tau)\right] e^{-jk\omega_0\tau}d\tau.$$

Hence, we have the bound

$$|k\omega_0|^L |\tilde{g}_k(r)| \leq \max_{\tau} \left|\frac{d^L}{d\tau^L}g(\tau + r, \tau)\right| \leq C \cdot e^{-\alpha r} \tag{23}$$

for some positive constants C and α. Such constants exist by assumption E3). The result follows. ■

Using Lemma 1 we can show the following theorem.

Theorem 4 (Convergence of Skew Truncations): Assume that a periodic system G has an impulse response g that belongs to $\mathcal{C}_e^L$, where $L > 1$. Then, for all inputs $u \in L_\infty(\mathbb{R})$, the output $y_N(t)$ in (21) converges, uniformly in t, to $y(t)$ in (2), as $N \to \infty$.

Under the same assumptions on g, we have the following convergence bound on the Fourier series (22)

$$\left\|G - G_{[N]}\right\|_{L_r \to L_r} \leq K \cdot N^{-(L-1)}, \qquad N > 0,\ 1 \leq r \leq \infty \tag{24}$$

for a system-dependent constant K.

Proof: We start with the first statement. Using Lemma 1 ii) we have that

$$\begin{aligned} |y(t) - y_N(t)| &\leq 2\left|\int_{-\infty}^{t} \sum_{k=N+1}^{\infty} g_k(t-\tau)e^{jk\omega_0 t}u(\tau)d\tau\right| \\ &\leq 2\int_{-\infty}^{t} \sum_{k=N+1}^{\infty} \frac{C \cdot e^{-\alpha(t-\tau)}}{(k\omega_0)^L}|u(\tau)|d\tau \\ &\leq 2\frac{C\gamma}{\alpha\omega_0^L N^{L-1}}\|u\|_{L_\infty}, \end{aligned} \tag{25}$$

where γ is a constant such that

$$\sum_{k=N+1}^{\infty} \frac{1}{k^L} \leq \frac{\gamma}{N^{L-1}}, \qquad L > 1. \tag{26}$$

Since the upper bound in (25) is independent of t and tends to zero as $N \to \infty$, the convergence is uniform. Furthermore, we have shown (24) when $r = \infty$, with $K = 2C\gamma/\alpha\omega_0^L$.

It remains to prove (24) for $1 \leq r < \infty$. We use that the Schwartz functions $\mathcal{S}$ are in L_∞, and are dense in L_r, $1 \leq r < \infty$. If the domain of G is restricted to $\mathcal{S}$, we can by (25) interchange the order of integration and summation and represent G

by the Fourier series $G = \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} G_k$, where G_k are time-invariant systems. Furthermore, we have the bound

$$\|G_k\|_{L_r \to L_r} \leq \|g_k\|_{L_1} \leq \frac{C}{\alpha |k\omega_0|^L}$$

using first Young's inequality and then Lemma 1 ii). Now, since $\mathcal{S}$ is dense in L_r, $1 \leq r < \infty$, we can conclude that

$$\left\|G - G_{[N]}\right\|_{L_r \to L_r} \leq 2 \sum_{k=N+1}^{\infty} \|G_k\|_{L_r \to L_r} \leq 2 \frac{C\gamma}{\alpha \omega_0^L N^{L-1}} \tag{27}$$

where γ is a given by (26). Hence, $K = 2C\gamma/\alpha\omega_0^L$ also in this case. ■

We can apply Theorem 4 to input signals that are harmonics and thereby see the connection between the above Fourier expansions and the classical analysis by Zadeh [7].

Example 3 (Harmonic Response and the PTF): The response of a periodic system to an harmonic $u(t) = e^{j\omega t}$ is now easy to obtain. Under the assumptions of Theorem 4 and by the definition of the Fourier transform we obtain

$$\begin{aligned} y(t) &= G e^{j\omega t} \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{t} g_k(t-\tau) e^{jk\omega_0 t} e^{j\omega\tau} d\tau \\ &= \left(\sum_{k=-\infty}^{\infty} \hat{g}_k(j\omega) e^{jk\omega_0 t} \right) e^{j\omega t}, \end{aligned} \tag{28}$$

where $\hat{g}_k$ is the Fourier transform of g_k. Since $g_k \in L_1[0,\infty) \cap L_2[0,\infty)$, we know that $\hat{g}_k$ is uniformly continuous and belongs to $L_2(j\mathbb{R})$, see [33]. Equation (28) shows that the response includes a countable number of frequencies separated by multiples of ω_0. The *parametric transfer function* (PTF) $\hat{G}(j\omega, t)$ that is used in [7] and [9] may be defined as the steady-state response of the periodic system G to an harmonic, $\hat{G}(j\omega, t) := (Ge^{j\omega t})/e^{j\omega t}$. By the previous analysis, we realize that

$$\hat{G}(j\omega, t) = \sum_{k=-\infty}^{\infty} \hat{g}_k(j\omega) e^{jk\omega_0 t}.$$

The PTF can also be computed directly from the impulse response $g(t,\tau)$, see [7] and [9]. The HTF that we obtain in Section IV, is also a frequency-domain representation of G. The difference is that the PTF is scalar but depends on time and frequency, whereas the HTF only depends on frequency. The price is that the HTF becomes infinite dimensional. The relation between the PTF and the HTF has also been discussed in [13].

B. Skew Roll-Off

In analogy with input roll-off and output roll-off we define skew roll-off.

Definition 6 (Skew Roll-Off): If there are positive constants C and k such that

$$\left\|G - G_{[N]}\right\|_{L_2 \to L_2} \leq C \cdot N^{-k}, \qquad N > 0$$

then G has *skew roll-off* k.

From Theorem 4 we see that G has skew roll-off k if g belongs to $\mathcal{C}_e^{k+1}$.

Remark 5 (Skew Roll-Off of Time-Invariant Systems: For a time-invariant system G, it holds that $G_k = 0$ for all $k \neq 0$ and $G = G_0$. Hence, a time-invariant system has *infinite skew roll-off.*

IV. Harmonic Transfer Function

Time-periodic systems can be lifted to formally time-invariant systems using various techniques, see, for example, [21], [37]. In this section, we review one such representation, the *harmonic transfer function* (HTF). All lifted representations have one thing in common: they are infinite-dimensional operators. Here we apply the Taylor and the Fourier expansions of the previous sections to show how the HTF can be approximated. It also turns out that the roll-off concepts have a clear interpretation for the HTF.

By including a sufficient amount of frequencies in the Fourier expansion of G, we can come arbitrarily close to G itself, as discussed in Section III. Since we have decomposed the periodic system into time-invariant terms, the frequency-domain analysis is now straightforward. Assume in the following that the assumptions of Theorem 4 hold. Notice that $y_N(t)$ from (2) may be written as

$$y_N(t) = \sum_{k=-N}^{N} g_k(t) e^{jk\omega_0 t} * u(t) e^{jk\omega_0 t} \tag{29}$$

where $*$ is the standard convolution product. Now pick an input u in L_2. We can apply the Fourier transform on both sides of (29), and get

$$\hat{y}_N(j\omega) = \sum_{k=-N}^{N} \hat{g}_k(j\omega - jk\omega_0) \hat{u}(j\omega - jk\omega_0). \tag{30}$$

All the Fourier transforms are well defined, since by the assumptions $g_k \in L_1[0,\infty) \cap L_2[0,\infty)$ and $u \in L_2(\mathbb{R})$. By Theorem 4, y_N converges to y in L_2. Furthermore, $L_2(\mathbb{R})$ and $L_2(j\mathbb{R})$ are isomorphic under the Fourier transform. Hence, for all inputs $u \in L_2$, $\hat{y}_N$ converges to $\hat{y}$ in $L_2(j\mathbb{R})$ as $N \to \infty$. Therefore we can put $N = \infty$ in (30) if we mean convergence in L_2-sense, and not point-wise convergence.

Next we rewrite the summation (30) by using lifting on $L_2(j\mathbb{R})$. In [22] this lifting was called the Sample-Data Fourier transform (SD-transform). The SD-transform is an isometric isomorphism between $L_2(j\mathbb{R})$ and a Hilbert space we denote by $L_2^Z(jI_0)$. It maps the Fourier transform into an infinite-dimensional column-vector-valued function. The SD-transform of $\hat{u}$ is denoted by $\hat{U}$ and is defined as

$$\hat{U}(j\omega) := [\dots \ \hat{u}(j\omega + j\omega_0) \ \hat{u}(j\omega) \ \hat{u}(j\omega - j\omega_0) \ \dots]^T.$$

Since the vector contains repeated versions of $\hat{u}(j\omega)$, it is enough to define $\hat{U}(j\omega)$ for $\omega \in I_0 = (-\omega_0/2, \omega_0/2]$ to be able to take the inverse SD-transform. We define the norm in $L_2^Z(jI_0)$ as

$$\|\hat{U}\|_{L_2^Z} := \frac{1}{\sqrt{2\pi}}\left(\int_{I_0}\left\|\hat{U}(j\omega)\right\|_{\ell_2}^2 d\omega\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}\left(\int_{I_0}\sum_{k=-\infty}^{\infty}|\hat{u}(j\omega+jk\omega_0)|^2 d\omega\right)^{\frac{1}{2}}.$$

For signals $u \in L_2$, there are now three representations: $u(t)$, $\hat{u}(j\omega)$, and $\hat{U}(j\omega)$, and the following extended Plancherel's theorem holds,

$$\|u\|_{L_2} = \|\hat{u}\|_{L_2} = \|\hat{U}\|_{L_2^Z}.$$

If u has finite L_2-norm, then $\hat{U}(j\omega)$ is in ℓ_2 (its elements are square summable) for almost all $\omega \in I_0$, that is $\|\hat{U}(j\omega)\|_{\ell_2} < \infty$ almost everywhere.

Using the SD-transform, (30) can be written in matrix-vector form as

$$\hat{Y}(j\omega) = \hat{G}(j\omega)\hat{U}(j\omega) \quad \omega \in I_0 = \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right] \tag{31}$$

when $N = \infty$ if we put

$$\hat{G}(j\omega) = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \hat{g}_0(j\omega+j\omega_0) & \hat{g}_1(j\omega) & \hat{g}_2(j\omega-j\omega_0) & \\ \ddots & \hat{g}_{-1}(j\omega+j\omega_0) & \hat{g}_0(j\omega) & \hat{g}_1(j\omega-j\omega_0) & \ddots \\ & \hat{g}_{-2}(j\omega+j\omega_0) & \hat{g}_{-1}(j\omega) & \hat{g}_0(j\omega-j\omega_0) & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

We call $\hat{G}(j\omega)$ the HTF of G and it can be regarded as a linear operator on ℓ_2 for each ω. HTF was the term used by Wereley in [5]. A similar object was called the FR operator in [22] in the case of sampled-data systems. The difference between these efforts is the way the elements of $\hat{G}(j\omega)$ are computed. In the sampled-data case, explicit formulas are given in [22]. In the time-periodic state-space case formulas are given in [4], [5], and in the impulse response case formulas are given here and in [26]. The relation between the impulse-response and state-space approaches is further discussed in Section VI.

It was assumed in the above discussion that the impulse response belongs to $\mathcal{C}_e^L$. This was done to motivate the construction of the HTF from an input-output view. However, the HTF $\hat{G}(j\omega)$ is a meaningful construction as soon as its elements, the Fourier transforms of the Fourier coefficients of G, are well defined. This is the case, for example, when G is in H_2. We have the following well-known results, which are derived in [4], [5], [22] under slightly different assumptions.

Proposition 2 (Norm Formulas): Assume that the impulse response of the periodic system G belongs to H_2. Then $\hat{G}(j\omega)$ can be defined as in (31). We have then

$$\|g\|_{H_2}^2 = \frac{1}{2\pi}\int_{I_0} \text{trace}\left[\hat{G}^*(j\omega)\hat{G}(j\omega)\right] d\omega \tag{32}$$

and

$$\|G\|_{L_2\to L_2} = \operatorname*{ess\ sup}_{\omega\in I_0}\left\|\hat{G}(j\omega)\right\|_{\ell_2\to\ell_2}. \tag{33}$$

A. Rectangular Truncations Revisited

By looking at the structure of the HTF's of rectangular and skew truncated systems we make useful connections to the work in [27]. The reasons for the terms "rectangular" and "skew" will also be obvious. To compute the induced L_2-norm (1) of a system G with input and output roll-off k_1 and k_2, we have the bound

$$0 \le \|G\|_{L_2\to L_2} - \|P_{\Omega_1}GP_{\Omega_2}\|_{L_2\to L_2} \le C_1\cdot\Omega_1^{-k_1} + C_2\cdot\Omega_2^{-k_2}. \tag{34}$$

Proposition 2 gives us a way to compute the induced L_2-norm, given a HTF $\hat{G}(j\omega)$. It is not essential that $\hat{G}(j\omega)$ corresponds to a causal operator for (33) to hold, it is true for every frequency-domain relation (31). Hence, we can apply it to $P_{\Omega_1}GP_{\Omega_2}$. This is favorable, since the HTF of $P_{\Omega_1}GP_{\Omega_2}$ is simple.

Proposition 3 (Rectangular Truncated HTFs): Assume that G has an HTF $\hat{G}(j\omega)$. If we choose $\Omega_1 = (N_1 + 1/2)\omega_0$ and $\Omega_2 = (N_2+1/2)\omega_0$, and introduce the intervals $A = [-\Omega_1, \Omega_1]$ and $B = [-\Omega_2, \Omega_2]$, the HTF of $P_{\Omega_1}GP_{\Omega_2}$ is given by

$$\hat{P}_{\Omega_1}(j\omega)\hat{G}(j\omega)\hat{P}_{\Omega_2}(j\omega) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{G}_{jB\to jA}(j\omega) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\hat{G}_{jB\to jA}(j\omega) \in \mathbb{C}^{(2N_1+1)\times(2N_2+1)}$ is the part of $\hat{G}(j\omega)$ that maps frequencies in B to frequencies in A, that is

$$\hat{G}_{jB\to jA}(j\omega) = \begin{bmatrix} \hat{g}_{N_1-N_2}(j\omega+jN_2\omega_0) & \dots & \hat{g}_{N_1+N_2}(j\omega-jN_2\omega_0) \\ \vdots & & \vdots \\ \hat{g}_{-N_2}(j\omega+jN_2\omega_0) & \dots & \hat{g}_{N_2}(j\omega-jN_2\omega_0) \\ \vdots & & \vdots \\ \hat{g}_{-N_1-N_2}(j\omega+jN_2\omega_0) & \dots & \hat{g}_{-N_1+N_2}(j\omega-jN_2\omega_0) \end{bmatrix}$$

where $\omega \in I_0$.

Proposition 3 shows that we can represent the linear periodic system G arbitrarily well with finite-dimensional functions and estimate its norm as

$$\|P_{\Omega_1}GP_{\Omega_2}\|_{L_2\to L_2} = \max_{\omega\in I_0}\overline{\sigma}\left(\hat{G}_{jB\to jA}(j\omega)\right) \tag{35}$$

using (33) and assuming continuous elements $\hat{g}_k(j\omega)$. The guaranteed accuracy depends upon the matrix size and the roll-off of G according to (34). To use the rectangular truncation (35) to estimate the norm of a periodic system has been suggested by many authors. A similar idea is to use a compression operator, see [38]. In [22], it is shown that the rectangular truncation converges at least at a rate $K\cdot N^{-1/2}$, for $(2N+1)\times(2N+1)$ truncations and some constant K. In [22], sampled-data systems are studied, but similar techniques are used in [4], [27]. As seen in (34), the bound on convergence rate may be improved by checking the Markov parameters. Moreover, nonsquare ($N_1 \neq N_2$) truncations can be used to improve convergence and computation time, adapting N_1 and N_2 to the input and output roll-off rates.

Input and output roll-off can be visualized in the HTF as follows. Introduce the interval $D = [\Omega, \infty)$ and look at the HTFs corresponding to the truncated operators in the definition of input and output roll-off, Definition 2, 4, and use notation from Proposition 3

$$\hat{Q}_\Omega(j\omega)\hat{G}(j\omega) = \begin{bmatrix} \hat{G}_{j\mathbb{R}\to jD}(j\omega) \\ 0 \\ \hat{G}_{j\mathbb{R}\to -jD}(j\omega) \end{bmatrix},$$
$$\hat{G}(j\omega)\hat{Q}_\Omega(j\omega) = \begin{bmatrix} \hat{G}_{jD\to j\mathbb{R}}(j\omega) & 0 & \hat{G}_{-jD\to j\mathbb{R}}(j\omega) \end{bmatrix}.$$

Then

$$\left\| \hat{G}_{j\mathbb{R}\to \pm jD}(j\omega) \right\|_{\ell_2\to\ell_2} \le C_1 \cdot \Omega^{-k_1} \tag{36}$$
$$\left\| \hat{G}_{\pm jD\to j\mathbb{R}}(j\omega) \right\|_{\ell_2\to\ell_2} \le C_2 \cdot \Omega^{-k_2} \tag{37}$$

for almost all ω from (33). Since the modulus of the elements in an operator on ℓ_2 cannot be larger than the induced ℓ_2-norm, this also bounds the sizes of individual elements. To conclude, if the system has high output roll-off, then $\hat{G}(j\omega)$ decays quickly asymptotically in the up-down direction (36), and if it has high input roll-off it decays quickly asymptotically in the left-right direction (37). In Theorem 2 we have given the conditions for the decay rates: the more input and output Markov parameters that are zero, the higher roll-off.

B. Skew Truncations Revisited

Let us now look at the skew truncated HTF's.

Proposition 4 (Skew Truncated HTF's): The HTF of the skew truncation $G_{[N]}$, $\hat{G}_{[N]}(j\omega)$, consists of the $2N+1$ diagonals $\hat{g}_{-N}(j\omega), \ldots, \hat{g}_N(j\omega)$ of $\hat{G}(j\omega)$ and is zero elsewhere.

Hence, as N increases, more diagonals are added to the skew truncated HTF. Furthermore, we know that each diagonal represents a Fourier coefficient of G, see Theorem 3. For instance, the middle diagonal $\hat{g}_0(j\omega)$ corresponds to the time-invariant component, see Remark 4. From Theorem 4 we know how quickly this HTF converges in induced norms. That is, we can quantify how much accuracy there is, at least, to gain by including an extra diagonal in the HTF. If the system has skew roll-off k, we can conclude that

$$\operatorname{ess\,sup}_{\omega} |\hat{g}_M(j\omega)| \le \left\| G - G_{[N]} \right\|_{L_2\to L_2} \le C \cdot N^{-k}$$

where $|M| \ge N+1$, using Definition 6 and (33). Hence, with high skew roll-off, the diagonals decay quickly for large N. The constants k and C can be determined from the smoothness of g, see Theorem 4.

Remark 6 (Relation to [27]): Skew truncations are used to compute the H_2-norm of periodic systems in [27]. However, it is not the HTF of $G_{[N]}$ that is used there. Instead the state-space matrices $B(t)$ and $C(t)$ are skew truncated. That means that the H_2-norm is computed for the system $\bar{G}_{[N]}$, with impulse response

$$\bar{g}_{[N]}(t,\tau) = C_{[N]}(t)e^{Q(t-\tau)}B_{[N]}(\tau)$$
$$B_{[N]}(\tau) = \sum_{k=-N}^{N} B_k e^{jk\omega_0\tau}$$
$$C_{[N]}(t) = \sum_{k=-N}^{N} C_k e^{jk\omega_0 t}$$

where Q is the (Floquet-transformed) state matrix $A(t)$, see the discussion in Section I-B. The HTF of $\bar{G}_{[N]}$ has $4N+1$ diagonals, but notice that in general $\bar{G}_{[N]} \ne G_{[2N]}$. Hence, $\bar{g}_{[N]}$ are in general not H_2-optimal approximations of g, see Remark 4.

V. Approximate Inverses

Inverses appear for closed-loop systems in mappings such as $(I+G)^{-1}G$ and $(I+G)^{-1}$. Here, we study what can be said about the approximate inverses $(I + P_{\Omega_1}GP_{\Omega_2})^{-1}$ and $(I+G_{[N]})^{-1}$, using the machinery developed thus far. This is of interest by itself, in studies of closed-loop systems, for instance, but in this paper we only use it for state–space models in Section VI. We should first remember that even if G is causal, the approximation $P_{\Omega_1}GP_{\Omega_2}$ is noncausal, even if it gets "less noncausal" as it converges to G. For this reason it is difficult to prove causality of $(I+G)^{-1}$ by studying $(I + P_{\Omega_1}GP_{\Omega_2})^{-1}$. However, to compute $\|(I+G)^{-1}\|_{L_2\to L_2}$, we have the following result:

Proposition 5 (Approximation of $(I+G)^{-1}$ Using $P_{\Omega_1}GP_{\Omega_2}$): Assume that $(I+G)^{-1}$ and $(I + P_{\Omega_1}GP_{\Omega_2})^{-1}$ are bounded operators on L_2, and that G has output roll-off k_1 and input roll-off k_2. Then the relative L_2-induced norm error is bounded

$$\frac{\left\|(I+G)^{-1} - (I + P_{\Omega_1}GP_{\Omega_2})^{-1}\right\|_{L_2\to L_2}}{\left\|(I + P_{\Omega_1}GP_{\Omega_2})^{-1}\right\|_{L_2\to L_2}} \le \left\|(I+G)^{-1}\right\|_{L_2\to L_2} \cdot \left(C_1 \cdot \Omega_1^{-k_1} + C_2 \cdot \Omega_2^{-k_2}\right).$$

Proof: All norms in this proof denote L_2-induced norms. First, we make an orthogonal decomposition of the Hilbert space L_2, so that $L_2 = P_\Omega L_2 \oplus Q_\Omega L_2$. In this basis, G takes the operator-matrix form

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

where $G_{11} : P_{\Omega_2}L_2 \to P_{\Omega_1}L_2$, $G_{12} : Q_{\Omega_2}L_2 \to P_{\Omega_1}L_2$, $G_{21} : P_{\Omega_2}L_2 \to Q_{\Omega_1}L_2$, and $G_{22} : Q_{\Omega_2}L_2 \to Q_{\Omega_1}L_2$. Next, we notice that

$$\left\|(I+G)^{-1} - (I + P_{\Omega_1}GP_{\Omega_2})^{-1}\right\| \le \left\|(I+G)^{-1}\right\| \left\| I - (I+G)(I + P_{\Omega_1}GP_{\Omega_2})^{-1}\right\|$$

and that second factor on the right-hand side in matrix form becomes

$$I - \begin{bmatrix} I+G_{11} & G_{12} \\ G_{21} & I+G_{22} \end{bmatrix} \begin{bmatrix} (I+G_{11})^{-1} & 0 \\ 0 & I \end{bmatrix} = -\begin{bmatrix} 0 & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} (I+G_{11})^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

The norm of the first factor is bounded by $C_1\Omega_1^{-k_1} + C_2\Omega_2^{-k_2}$, and the second factor is bounded by $\|(I+P_{\Omega_1}GP_{\Omega_2})^{-1}\|$, and the result follows. ■

The approximation $G_{[N]}$ is a finite sum of modulated time-invariant causal systems. If we decompose G as $G_{[N]} = G+\Delta$, we have bounds on Δ from Theorem 4. Assume that $(I+G)^{-1}$ is causal and bounded. Then, notice that

$$\begin{aligned} \left(I+G_{[N]}\right)^{-1} &= (I+G+\Delta)^{-1} \\ &= \left(I+(I+G)^{-1}\Delta\right)^{-1}(I+G)^{-1}. \end{aligned} \tag{38}$$

If N is large enough we have $\|(I+G)^{-1}\Delta\|_{L_r\to L_r} < 1$, $1 \le r \le \infty$, and we can make the following Neumann series expansion with absolute convergence in induced L_r-norm:

$$\left(I+(I+G)^{-1}\Delta\right)^{-1} = \sum_{k=0}^{\infty}(I+G)^{-k}(-\Delta)^k. \tag{39}$$

If the approximation error Δ is small, then boundedness of $(I+G_{[N]})^{-1}$ follows. We formalize this in the following proposition.

Proposition 6 (Approximation of $(I+G)^{-1}$ Using $G_{[N]}$): Assume that $(I+G)^{-1}$ is a bounded and causal operator on L_r, where $1 \le r \le \infty$, and that

$$\kappa = \left\|(I+G)^{-1}\right\|_{L_r\to L_r} \|\Delta\|_{L_r\to L_r} < 1$$

where $\Delta = G_{[N]} - G$. Then $(I+G_{[N]})^{-1}$ given by (38), (39) is a bounded and causal operator on L_r, and the relative error is bounded

$$\frac{\left\|\left(I+G_{[N]}\right)^{-1} - (I+G)^{-1}\right\|_{L_r\to L_r}}{\|(I+G)^{-1}\|_{L_r\to L_r}} \le \frac{\kappa}{1-\kappa}. \tag{40}$$

Proof: That $(I+G_{[N]})^{-1}$ is bounded and causal follows since it is a product of the bounded and causal operators in (38). The bound (40) follows from a simple bound on the geometric series (39). ■

Proposition 6 is stronger than Proposition 5 in the sense that we do not need to assume existence of the approximate inverse, the existence follows by the Neumann series expansion. On the other hand, $(I+P_{\Omega_1}GP_{\Omega_2})^{-1}$ is easier to compute, since it can be represented by matrices using Proposition 3.

VI. State-Space Models

Now, we return to the state–space systems described in (4), and show how the results in the previous sections can be applied to this situation. Here it is useful to allow signals to be vectors or matrices. To use the results of the previous sections, $|\cdot|$ should be interpreted as the (induced) Euclidean norm.

A. Floquet-Transformed State–Space Models

Assume that a Floquet transformation, see, for example [3], has been performed on the state–space realization (4) of G, and that the Fourier series

$$B(t) = \sum_{l=-\infty}^{\infty} B_l e^{jl\omega_0 t} \quad C(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

are absolutely convergent. The state matrix is constant, $A(t) = Q$. Then, the Fourier series of the impulse response g is given by

$$\begin{aligned} g(t,t-r) &= C(t)e^{Qr}B(t-r) \\ &= \sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty} C_{k-l}e^{Qr}B_l e^{-jl\omega_0 r}\right)e^{jk\omega_0 t} \end{aligned}$$

after interchange of summation order. Hence, the Fourier coefficients of G, see Section III, are given by

$$g_k(r) = \sum_{l=-\infty}^{\infty} C_{k-l}e^{Qr}B_l e^{-jl\omega_0 r}.$$

Using the definition (31) of the HTF, we see that $\hat{G}(j\omega)$ is identical to the HTF defined in [4] and [5].

B. Convergence of the Truncated Harmonic Balance Method

As discussed in Section I-B, it is of interest to compute the HTF of a state–space model without first applying the Floquet transform. Truncated harmonic balance was suggested as a method for this in, for example, [28]. However, to the authors' knowledge, no analysis of how and when this method converges has been presented. We will do an attempt to analyze the method here. Define the multiplication operator A as

$$y = Ax: \quad y(t) = A(t)x(t)$$

and B and C similarly. The input–output relation of the state–space model (4) is then given by

$$\begin{aligned} y &= Gu \\ &= C(pI-A)^{-1}Bu \\ &= C\left(I - \frac{1}{p+\epsilon}(A+\epsilon)\right)^{-1}\frac{1}{p+\epsilon}Bu \end{aligned} \tag{41}$$

for all $\epsilon > 0$, where p is the differentiation operator. The reason for introducing ϵ is to make all operators bounded. Equation (41) is decomposed of three simple operators

$$(\mathcal{C},\mathcal{A},\mathcal{B}) = \left(C, \frac{1}{p+\epsilon}(A+\epsilon), \frac{1}{p+\epsilon}B\right). \tag{42}$$

The operators $\mathcal{A}$ and $\mathcal{B}$ have impulse responses of the type

$$g_{\mathcal{A}}(t,\tau) = e^{-\epsilon(t-\tau)}\left(A(\tau)+\epsilon\right)$$

and the HTF $\hat{\mathcal{A}}(j\omega)$ becomes

$$\hat{\mathcal{A}}(j\omega) = \begin{bmatrix} \ddots & & & & 0 \\ & \frac{1}{\epsilon+j(\omega+\omega_0)} & & & \\ & & \frac{1}{\epsilon+j\omega} & & \\ & & & \frac{1}{\epsilon+j(\omega-\omega_0)} & \\ 0 & & & & \ddots \end{bmatrix}$$
$$\times \begin{bmatrix} \ddots & \ddots & \ddots & & \ddots \\ \ddots & A_0+\epsilon & A_1 & A_2 & \\ \ddots & A_{-1} & A_0+\epsilon & A_1 & \ddots \\ & A_{-2} & A_{-1} & A_0+\epsilon & \ddots \\ \ddots & & \ddots & \ddots & \ddots \end{bmatrix}$$

where the second factor is a Toeplitz operator and $\{A_k\}_{k\in\mathbb{Z}}$ are the Fourier coefficients of $A(t)$. It is now straightforward to check the input and output Markov parameters. If $A(t)$ and $B(t)$ belong to $\mathcal{C}^1(\mathbb{R})$, the first Markov parameters of $\mathcal{A}$ and $\mathcal{B}$ are $A(t)+\epsilon$ and $B(t)$, respectively, and hence both $\mathcal{A}$ and $\mathcal{B}$ have input and output roll-off 1 by Theorem 2. If both $A(t)$ and $B(t)$ belong to $\mathcal{C}^L(\mathbb{R})$, then $\mathcal{A}$ and $\mathcal{B}$ have skew roll-off $L-1$, by Theorem 4.

Let us focus on the rectangular truncations, since these are suggested in [28]. Under the assumptions of Proposition 5 and using Proposition 3, we have bounds

$$\begin{aligned} \left\|(I-\mathcal{A})^{-1}-(I-P_\Omega\mathcal{A}P_\Omega)^{-1}\right\|_{L_2\to L_2} &\le K_{\mathcal{A}}(N)=O(N^{-1}), \\ \|\mathcal{B}-P_\Omega\mathcal{B}P_\Omega\|_{L_2\to L_2} &\le K_{\mathcal{B}}(N)=O(N^{-1}) \end{aligned}$$

as $N\to\infty$, and $\Omega=(N+1/2)\omega_0$. To approximate $\mathcal{C}$, we use skew truncations $\mathcal{C}_{[N]}$, since $\mathcal{C}$ is a multiplication operator and has no input and output roll-off. If $C(t)$ is periodic and in $\mathcal{C}^L(\mathbb{R})$, we have that an Nth-order skew truncation converges as

$$\begin{aligned} \left\|\mathcal{C}-\mathcal{C}_{[N]}\right\|_{L_2\to L_2} &= \sup_t \left|\sum_{|k|>N} C_k e^{jk\omega_0 t}\right| \\ &\le K\cdot N^{-(L-1)} = K_{\mathcal{C}}(N) \end{aligned}$$

for some constant K. The HTF of $\mathcal{C}_{[N]}$ is a Toeplitz operator with the Fourier coefficients $C_{-N}\ldots C_N$ on the diagonals, see [4]. Hence, by rectangular truncations of each of the operators in (42), we can approximate $P_\Omega GP_\Omega$ with

$$\widetilde{P_\Omega GP_\Omega} := \left[P_\Omega\mathcal{C}_{[N]}\right](I-P_\Omega\mathcal{A}P_\Omega)^{-1}[P_\Omega\mathcal{B}P_\Omega]$$

and

$$\begin{aligned} &\|P_\Omega GP_\Omega-\widetilde{P_\Omega GP_\Omega}\|_{L_2\to L_2} \\ &\quad\le c_1K_{\mathcal{A}}(N)+c_2K_{\mathcal{B}}(N)+c_3K_{\mathcal{C}}(N) \\ &\quad= O(N^{-1}) \end{aligned} \tag{43}$$

as $N\to\infty$, for some constants c_1, c_2, c_3, using the triangular inequality. Notice that (43) is a worst-case bound. If the system is time invariant, all the operators are diagonal and $\widetilde{P_\Omega GP_\Omega}=P_\Omega GP_\Omega$. If we approximate G, we obtain

$$\begin{aligned} \|G-\widetilde{P_\Omega GP_\Omega}\|_{L_2\to L_2} \le{}& C_1\cdot\Omega^{-k_1}+C_2\cdot\Omega^{-k_2} \\ &+c_1K_{\mathcal{A}}(N)+c_2K_{\mathcal{B}}(N)+c_3K_{\mathcal{C}}(N). \end{aligned}$$

The first part of the error bound depends on the input and output roll-off of G, which are determined by the Markov parameters in Table I. The second part depends on the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, as discussed previously.

To summarize, using techniques developed in this paper, we have shown convergence of the truncated harmonic balance method suggested in [28]. Notice that the method suggested here and in [28] are the same, since P_Ω and $1/(p+\epsilon)$ commute. We have also seen that the worst-case convergence rate is slow, only $O(N^{-1})$ for matrix dimensions $(2N+1)\times(2N+1)$. The advantage with the method is that we only work with the Fourier coefficients and simple matrix algebra. No knowledge of the transition matrix $\Phi_A(t,\tau)$ is needed. If the transition matrix is known, we can compute the elements in $P_\Omega GP_\Omega$ exactly by our results in Section IV. Then the convergence of rectangular truncations may be much faster, and depends only on the input and output roll-off. Since the convergence rate of the rectangular-truncation method may be slow, it is an interesting problem for future research to study how the method may be improved.

VII. Conclusion

We have studied linear time-periodic systems from a frequency-domain point of view in this paper. We started to study Taylor expansions of time-varying systems and defined input and output Markov parameters. We also introduced the concepts of input and output roll-off. These roll-off rates are determined by the Markov parameters. Next we studied Fourier expansions of periodic systems in H_2. We also gave sufficient conditions for convergence rates of truncated Fourier expansions in induced L_r-norm, and introduced the concept of skew roll-off.

After the Fourier expansion, it was straightforward to define the frequency-response operator that is called the HTF. The roll-off concepts were shown to determine the decay rates of elements in different directions of the HTF, and we were able to strengthen available convergence bounds. After studies of inverses, we applied the results to systems given in state-space form. This allowed us to give conditions under which the truncated harmonic balance method converges. This method is interesting since only the Fourier coefficients of the realization are needed. Most other methods that apply to periodic systems require knowledge of the transition matrix. However, the convergence rate of the method can be quite slow.

This paper has provided a systematic convergence analysis for the HTF. This is important since in all applications listed in Section I-A, some sort of truncation is used. We have analyzed the most common approaches of truncation here. However, it is still unclear how the HTF is best approximated.

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