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A Bode Sensitivity Integral for Linear Time-Periodic Systems

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Abstract— **For linear time-invariant systems Bode's sensitivity integral is a well-known formula that quantifies some of the limitations in feedback control. In this paper we show that a very similar formula holds for linear time-periodic systems. We use the infinite-dimensional frequency-response operator called the harmonic transfer function to prove the result. It is shown that the harmonic transfer function is an analytic operator and a trace class operator under the assumption that the periodic system has roll-off 2. A periodic system has roll-off 2 if the first time-varying Markov parameter is equal to zero.**

I. Introduction

In recent years there has been an increased interest for the fundamental limitations in feedback control. One reason for this is that in many control design tools these limitations are not clearly visible, and an inexperienced designer can easily specify performance criteria that are not possible to attain. The articles [1] and [2] contain examples of this. There are many of these limitations in control. The connection between amplitude and phase of transfer functions and Bode's sensitivity integral formula are two examples. The limitations come from the fact that the transfer functions are *analytic functions*, and this has strong implications.

In this paper we focus on Bode's sensitivity integral. This is a standard result in control, see for example [3]. The sensitivity function $S = (I + G)^{-1}$ is defined as in Fig. 1. The result says that the sensitivity function cannot be small everywhere. If the transfer function $\hat{G}(s)$ of the open-loop system G has roll-off 2 and is stable, then we have in the Multi-Input-Multi-Output (MIMO) case that

$$\int_0^\infty \log |\det(I + \hat{G}(j\omega))^{-1}| \, d\omega = 0. \tag{1}$$

This is also called the "waterbed effect". If $\hat{S}(j\omega_1) = (I + \hat{G}(j\omega_1))^{-1}$ is made small for some frequency ω_1, then it will be large for some other frequency ω_2. In particular, the modulus of the sensitivity, $|\det \hat{S}(j\omega)|$, cannot be less than 1 for all frequencies ω.

This trade-off holds for time-invariant linear systems. It is known that there are limitations also for linear time-varying and nonlinear systems, see for example [4]. However, frequency-domain methods are then often not applicable. In the paper [5] an analogue to (1) is developed for continuous-time time-varying linear systems. The sensitivity integral is interpreted as an entropy integral in the time domain, i.e., no frequency-domain representation is used. For discrete-time time-varying systems similar results are given in [6].

For time-periodic linear systems there do exist frequency-domain representations. Sampled-data systems are a special type of time-periodic systems. Fundamental limitations for sampled-data systems are studied in [7], [8] using transfer function techniques. In this paper we study general time-periodic systems and we use the *harmonic transfer function* (HTF), see [9], [10], [11], which formally is a MIMO transfer function $\hat{G}(s)$ with an infinite amount of inputs and outputs. Using the convergence and existence results for the harmonic transfer function that are developed in [12] we will be able to write (1) with $\hat{G}(j\omega)$ being the HTF. To do this we need to answer the following questions:

1. What does roll-off 2 mean for a time-periodic system?
2. In what sense is the HTF $\hat{G}(s)$ analytic?
3. What does the determinant mean for the HTF?

We do not consider open-loop unstable systems in this paper. This case is considered in [5] using exponential dichotomies. In the time-invariant case when the open-loop system is unstable, the right hand side of (1) is equal to $\pi \sum_i \operatorname{Re} p_i$, where p_i are the unstable open-loop poles, see [13]. The authors do believe that it will be possible to generalize the method of this paper to cover also the unstable case.

The paper is organized as follows: In section II we give some of the basic results for the harmonic transfer function. The section ends with Proposition 1 which tells what roll-off 2 means. In section III we review what an analytic operator is. In Proposition 2 we show that with roll-off 2 the HTF is in fact an analytic operator. In section IV we review the definition of the trace class operators and the operator determinant. In Proposition 3 we see that the HTF indeed is a trace class operator and that the determinant is well defined. By using the propositions of the previous sections, we can in section V state the main result, which is a direct analogue of (1) for periodic systems. In section VI we give an example of the result.

II. The Harmonic Transfer Function and Roll-Off of Time-Periodic Systems

In [12] it is shown how the harmonic transfer function of a time-periodic system G given on impulse-response form

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau \tag{2}$$

can be computed. We repeat some results briefly here. For a *periodic system* there is a period $T \neq 0$ such that

$$g(t,\tau) = g(t+T, \tau+T). \tag{3}$$

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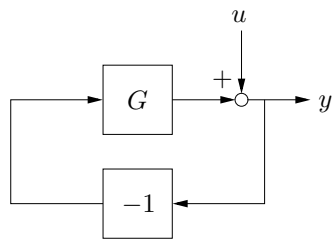

Fig. 1. The sensitivity operator S is defined as $y = Su = (I+G)^{-1}u$.

We assume that $g(t,\tau)$ is real and has uniform exponential decay

$$|g(t,\tau)| \le K \cdot e^{-\alpha(t-\tau)}, \quad t \ge \tau,$$

for some positive constants K and α. The operator G is then bounded on L_2.

To define the HTF of a linear periodic system G we need the following steps: First we expand the periodic impulse response (3) in a Fourier series:

$$\begin{aligned} g(t,\tau) &= \sum_{l=-\infty}^{\infty} g_l(t-\tau)e^{jl\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}, \\ g_l(t-\tau) &= \frac{1}{T}\int_0^T g(r, r-t+\tau)e^{-jl\omega_0 r}dr, \end{aligned} \tag{4}$$

with convergence in L_2, see [12]. Hence we expand the periodic system into a sum of modulated time-invariant impulse responses $g_l(t)$. For exponentially stable systems we can apply the Laplace transform on each time-invariant impulse response $g_l(t)$:

$$\hat{g}_l(s) = \int_0^\infty g_l(t)e^{-st}dt, \quad \operatorname{Re} s > -\alpha. \tag{5}$$

Furthermore, we have that $\hat{g}_l(s)$ is analytic in $\operatorname{Re} s > -\alpha$ and $\hat{g}_l \in H_2 \cap H_\infty$. Now the HTF $\hat{G}(s)$ is defined as the infinite-dimensional matrix

$$\begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \hat{g}_0(s+j\omega_0) & \hat{g}_1(s) & \hat{g}_2(s-j\omega_0) & \\ \ddots & \hat{g}_{-1}(s+j\omega_0) & \hat{g}_0(s) & \hat{g}_1(s-j\omega_0) & \ddots \\ & \hat{g}_{-2}(s+j\omega_0) & \hat{g}_{-1}(s) & \hat{g}_0(s-j\omega_0) & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{6}$$

for complex numbers s in a region J_0:

$$\begin{aligned} J_0 &= \{s : \operatorname{Re} s \ge 0,\ \operatorname{Im} s \in I_0\} \\ I_0 &= (-\omega_0/2, \omega_0/2], \quad \omega_0 = 2\pi/T. \end{aligned}$$

Notice that we only need to define the HTF for frequencies $\omega \in I_0$. The HTF $\hat{G}(s)$ is a linear infinite-dimensional operator, which is a bounded operator on the space of square-summable sequences ℓ_2 (at least for almost all $s \in J_0$).

In [9], [11], [12] it is shown that for stable systems G we can compute the induced L_2-norm as

$$\|G\|_{L_\infty} = \sup_{\|u\|_{L_2} \le 1} \|Gu\|_{L_2} = \operatorname{ess\,sup}_{\omega \in I_0} \|\hat{G}(j\omega)\|_\infty \tag{7}$$

where $\|\cdot\|_\infty$ is the induced ℓ_2-norm.

A. Roll-Off of Periodic Systems

Notice that for all numbers q we can rewrite (2) as

$$y(t)e^{-qt} = \int_{-\infty}^t [g(t,\tau)e^{-q(t-\tau)}]u(\tau)e^{-q\tau}d\tau. \tag{8}$$

We use the notation

$$y_q = G_q u_q$$

where the operator G_q has impulse response $g(t,\tau)e^{-q(t-\tau)}$ and maps input signals of the type $u_q(t) = u(t)e^{-qt}$ into signals $y_q(t) = y(t)e^{-qt}$. For every fixed $q \ge 0$ we may apply the theory developed in [12]. In particular we may apply the time-varying Markov parameter expansions.

In the following proposition we use the notation $g_x^{(a)} = \partial^a g/\partial x^a$, and p is the differential operator $pu(t) = du(t)/dt$. Furthermore, the set $\mathcal{S}$ is the set of Schwartz functions, i.e., the set of infinitely differentiable functions $u(t)$ with $t^a p^b u(t)$ bounded for $t \in \mathbb{R}$ and all non-negative a and b. The set $\mathcal{S}$ is dense in L_2.

Proposition 1: Assume that $g(t,t) = 0$ for all t, that $g(t,\tau)$ is twice continuously differentiable in the region $t \ge \tau$, and that all the derivatives have uniform exponential decay. Then G is said to have *roll-off* 2, and for all $q \ge 0$ we may expand (8) in either of the following ways:

$$\begin{aligned} y_q(t) &= -g'_\tau(t,t)\frac{1}{(p+q)^2}u_q(t) \\ &\quad + \int_{-\infty}^t [g''_\tau(t,\tau)e^{-q(t-\tau)}]\frac{1}{(p+q)^2}u_q(\tau)d\tau \end{aligned} \tag{9}$$

$$\begin{aligned} y_q(t) &= \frac{1}{(p+q)^2}g'_t(t,t)u_q(t) \\ &\quad + \frac{1}{(p+q)^2}\int_{-\infty}^t [g''_t(t,\tau)e^{-q(t-\tau)}]u_q(\tau)d\tau. \end{aligned} \tag{10}$$

when $u_q \in \mathcal{S}$.

Proof: We prove (10). (9) may be proven similarly. By the assumptions on $g(t,\tau)$ and since $u_q \in \mathcal{S}$, y_q is absolutely (and hence uniformly) continuous and belongs to L_1. By Barbalat's lemma we conclude that $y_q(t) \to 0$ as $|t| \to \infty$. If we differentiate (8) with respect to t we obtain

$$\begin{aligned} \frac{d}{dt}y_q(t) &= g(t,t)u_q(t) - qy_q(t) \\ &\quad + \int_{-\infty}^t [g'_t(t,\tau)e^{-q(t-\tau)}]u_q(\tau)d\tau. \end{aligned} \tag{11}$$

If we integrate (11) over $(-\infty, t]$ and solve for $y_q(t)$ we obtain

$$y_q(t) = \frac{1}{p+q} g(t,t) u_q(t) + \frac{1}{p+q} \int_{-\infty}^{t} [g_t'(t,\tau) e^{-q(t-\tau)}] u_q(\tau) d\tau.$$

By assumption $g(t,t) = 0$ and the first term disappears. If we repeat the above procedure on the second term we obtain (10). ■

We say that the systems in Proposition 1 have roll-off 2, see also [12]. This can be motivated as follows: We introduce P_Ω as an ideal (non-causal) low-pass filter with the frequency characteristic

$$\hat{P}_\Omega(j\omega) = \begin{cases} 1, & |\omega| \le \Omega \\ 0, & |\omega| > \Omega \end{cases}.$$

Proposition 1 together with the facts that $\mathcal{S}$ is dense in L_2 and that the Fourier transform of a function in $\mathcal{S}$ is again in $\mathcal{S}$, implies that if we filter the input or the output of systems G_q there are positive constants C_1, C_2, δ such that

$$\|G_q(I - P_\Omega)\|_{L_\infty} \le \frac{C_1}{|\delta + q + j\Omega|^2} \tag{12}$$

$$\|(I - P_\Omega)G_q\|_{L_\infty} \le \frac{C_2}{|\delta + q + j\Omega|^2}. \tag{13}$$

To show (12) one uses (9), and to show (13) one uses (10), see [12]. In particular we have that $\|G_q\|_{L_\infty} = O(q^{-2})$ as $q \to \infty$ and $\|G_q(I - P_\Omega)\|_{L_\infty} = O(\Omega^{-2})$ and $\|(I - P_\Omega)G_q\|_{L_\infty} = O(\Omega^{-2})$ for each fixed q as $\Omega \to \infty$.

The relation between the HTF of G and G_q is simple:

$$\hat{G}(q + j\omega) = \hat{G}_q(j\omega), \quad q + j\omega \in J_0, \tag{14}$$

so it is enough to speak of $\hat{G}(s)$. The high-pass filtering of G_q with $(I - P_\Omega)$ means that rows or columns are truncated (replaced by zeros) in $\hat{G}(s)$. If we choose $\Omega = (N+1/2)\omega_0$ for some non-negative integer N, then $G_q(I - P_\Omega)$ has an HTF where the $2N+1$ middle columns of $\hat{G}(s)$ are replaced by zeros. $(I - P_\Omega)G_q$ has an HTF where the $2N+1$ middle rows of $\hat{G}(s)$ are replaced by zeros, see [12] for details. This has consequences for the roll-off of the individual transfer functions $\hat{g}_l(s)$ as is shown in the next section.

Remark 1: For a stable time-invariant system with smooth impulse response $g(t,\tau) = g(t-\tau)$, $t \ge \tau$, the Markov parameters are equal to $\{g(0), g'(0), g''(0), \ldots\}$. If $g(t,t) = g(0) = 0$ then we have that

$$|\hat{g}(s)| = O(|s|^{-2}),$$

as $|s| \to \infty$ and $\operatorname{Re} s \ge 0$. This is called roll-off 2 for a time-invariant system.

III. Analytic Operators

To prove Bode's integral theorem for time-invariant systems one uses that the transfer function is analytic and Cauchy's integral theorem. We will do something similar. The HTF is an infinite-dimensional operator and therefore we will need some of the theory for *analytic operators*.

There are several equivalent definitions of an analytic operator, see for example [14]. We say that a bounded linear operator $\hat{G}(s)$ is analytic in an open set $\Omega \subseteq \mathbb{C}$ if it can be expanded in a power series around each $s_0 \in \Omega$:

$$\hat{G}(s) = \sum_{k=0}^{\infty} (s - s_0)^k \hat{G}_k, \quad s \in \Omega(s_0) \subseteq \Omega$$

with uniform convergence in the open disc $\Omega(s_0)$ in the induced ℓ_2-norm, $\|\cdot\|_\infty$. The constant operators $\hat{G}_k$ are linear bounded operators on ℓ_2. To prove that the HTF $\hat{G}(s)$ is an analytic operator we can check the following sufficient conditions [14]:

K1 All the elements of $\hat{G}(s)$ are analytic functions in Ω

K2 There is a positive constant K such that $\|\hat{G}(s)\|_\infty \le K$ for all $s \in \Omega$.

The property K1 follows by (5) and (6). The property K2 needs some extra attention. We will use the Hilbert-Schmidt norm to prove it. It is well known that the Hilbert-Schmidt norm $\|\cdot\|_2$ gives an upper bound to the induced ℓ_2-norm, i.e., $\|\hat{G}(s)\|_\infty \le \|\hat{G}(s)\|_2$. Now by definition,

$$\|\hat{G}(s)\|_2^2 = \sum_{k,l=-\infty}^{\infty} |\hat{g}_l(s + jk\omega_0)|^2. \tag{15}$$

We will show that we can bound the sum (15) for all $s \in J_0$.

By using the roll-off formulas and the discussion about the truncation of rows and columns in section II-A, we can conclude that for all non-negative integers N, and $\operatorname{Re} s \ge 0$:

$$|\hat{g}_l(s)| \le \frac{C_1 + C_2}{N^2\omega_0^2 + \delta^2}, \quad l \in \mathbb{Z}, \quad |l| \ge 2N+1, \tag{16}$$

$$|\hat{g}_l(s)| \le \frac{C_1}{|\delta + s|^2}, \quad l \in \mathbb{Z}, \tag{17}$$

The first bound follows as $\|G_q - P_\Omega G_q P_\Omega\|_{L_\infty} \le \|(I - P_\Omega)G_q)\|_{L_\infty} + \|G_q(I - P_\Omega)\|_{L_\infty} \le (C_1 + C_2)/(N^2\omega_0^2 + \delta^2)$ when $\Omega = (N + 1/2)\omega_0$. The modulus of the analytic elements of the HTF of $G_q - P_\Omega G_q P_\Omega$ must be less or equal to the L_∞-norm according to (7). As the transfer functions $\hat{g}_l(s)$, $|l| \ge 2N+1$, are not truncated with this choice of Ω, (16) follows. The second bound follows as the modulus of the analytic functions $\hat{g}_l(s)$ must be less than the L_∞-norm bound in (12), and then we can choose $s = q + j\Omega$.

Hence, roll-off 2 for a time-periodic system as defined in Proposition 1 implies that the transfer functions $\hat{g}_l(s)$ on the diagonals of $\hat{G}(s)$ have roll-off 2 in the classical sense (Remark 1). Now we can prove that $\hat{G}(s)$ is analytic:

Proposition 2: If the periodic system G fulfills the assumptions of Proposition 1, then its harmonic transfer function $\hat{G}(s)$ is an analytic operator for $s \in J_0$.

Proof: If we use the bounds (16)–(17) for $s = j\omega$ in (15) we show, see below, that the sum converges uniformly and there is a constant K such that

$$\|\hat{G}(j\omega)\|_\infty \le \|\hat{G}(j\omega)\|_2 \le K, \quad \omega \in I_0. \tag{18}$$

By the maximum modulus theorem $|\hat{g}_l(q+j\omega)| \le |\hat{g}_l(j\omega)|$, $q > 0$. We then also show that $\|\hat{G}(s)\|_2 \le K$, $s \in J_0$.

We now prove (18): To compute the Hilbert-Schmidt norm we shall sum over the indices k and l. By using (17) the sum in the k-direction converges for each l:

$$S(l) = \sum_{k=-\infty}^{\infty} |\hat{g}_l(j\omega + jk\omega_0)|^2 < \infty. \tag{19}$$

We need to show that $\sum_{l=-\infty}^{\infty} S(l) \le K^2$. From (16)–(17) we have

$$|\hat{g}_{\pm(2N+1)}(j\omega)| \le (C_1 + C_2) \min\left\{\frac{1}{N^2\omega_0^2}, \frac{1}{|\omega|^2}\right\}.$$

For $\omega \in I_0$ and a fixed $N > 0$ we have

$$\begin{aligned} S(\pm(2N+1)) &\le (C_1 + C_2)^2 \\ &\times \left\{\frac{2N-1}{N^4\omega_0^4} + 2\sum_{k=N}^{\infty} \frac{1}{|\omega + k\omega_0|^4}\right\} \le \frac{C}{N^3} \end{aligned} \tag{20}$$

where C is a constant independent of ω. We can derive a similar bound for $S(\pm 2(N+1))$. Hence we have that $S(l) = O(|l|^{-3})$ as $|l| \to \infty$. Hence the sum $\sum_l S(l)$ converges and there exists a constant K as in (18).

Since $\hat{G}(s)$ fulfills the conditions K1 and K2, it is analytic and the proposition follows. ■

IV. Trace Class Operators and Determinants

In the linear time-invariant MIMO Bode integral (1) the determinant of the transfer function matrix is used. We need to define a determinant for infinite-dimensional operators also. This can be done for so-called trace class operators, see [15], [16]. For a trace class operator $\hat{G}$ the determinant is defined as

$$\det(I + \hat{G}) = \prod_k \left(1 + \lambda_k(\hat{G})\right), \tag{21}$$

where $\lambda_k(\hat{G})$ are the eigenvalues of $\hat{G}$. Trace class operators are compact operators and have a countable number of eigenvalues. The possibly infinite product (21) converges for trace class operators, see (24). Note that for finite matrices, (21) coincides with the regular determinant.

For the definition of a trace class operator we need the s-numbers (or singular numbers) of an operator $\hat{G}$:

$$s_k(\hat{G}) = \inf\{\|\hat{G} - \hat{G}_k\|_\infty : \operatorname{rank} \hat{G}_k \le k\}.$$

The numbers s_k tell how well $\hat{G}$ may be approximated by a finite-rank operator. If $\hat{G}$ is compact we have that $s_k \to 0$ as $k \to \infty$ and $s_0 = \|\hat{G}\|_\infty$. The trace class operators are those operators for which

$$\|\hat{G}\|_1 = \sum_{k=0}^{\infty} s_k < \infty. \tag{22}$$

With the norm $\|\cdot\|_1$ the trace class operators form a complete normed space, see [15]. In particular we have that:

$$\operatorname{trace} \hat{G} = \sum_k \lambda_k(\hat{G}) \le \|\hat{G}\|_1, \tag{23}$$

$$|\det(I + \hat{G})| \le \exp(\|\hat{G}\|_1). \tag{24}$$

Next we will see that under the assumptions of Proposition 1, the HTF $\hat{G}(s)$ is in fact a trace class operator for all $s \in J_0$.

The HTF of $G_q P_\Omega$, with $\Omega = (N+1/2)\omega_0$, has elements equal to zero everywhere except for its $2N+1$ middle columns which are identical to the $2N+1$ middle columns of $\hat{G}(s)$ defined by (6). Hence, the truncated HTF has at most rank $2N+1$. We know that $G_q P_\Omega$ converges to G_q as $O(\Omega^{-2}) = O(N^{-2})$ from (12). Using the norm formula (7) and the continuity of $\hat{G}_q(j\omega)$ ($\hat{G}(s)$ is analytic), we conclude that for each $q + j\omega \in J_0$ we have that

$$\begin{aligned} s_{2N+1}(\hat{G}_q(j\omega)) &\le \|\hat{G}_q(j\omega)(I - \hat{P}_\Omega(j\omega))\|_\infty \\ &\le \|G_q(I - P_\Omega)\|_{L_\infty} \\ &\le \frac{C_1}{|\delta + q + j\Omega|^2} \\ &\le \frac{C_1}{(\delta+q)^2 + N^2\omega_0^2}. \end{aligned} \tag{25}$$

For each fixed s the singular numbers $s_k(\hat{G}(s))$ decay as $O(k^{-2})$ for systems with roll-off 2. We are now ready to state the proposition of this section:

Proposition 3: If the periodic system G fulfills the assumptions of Proposition 1, then its harmonic transfer function $\hat{G}(s)$ is a bounded trace class operator in J_0:

$$\|\hat{G}(q + j\omega)\|_1 \le \frac{K_1}{K_2 + q}, \quad q + j\omega \in J_0,$$

for some positive constants K_1, K_2.

Proof: We have that

$$\begin{aligned} s_0(\hat{G}(q + j\omega)) &= \|\hat{G}(q + j\omega)\|_\infty \\ &\le \|G_q\|_{L_\infty} \le \frac{C_1}{(\delta + q)^2}, \end{aligned} \tag{26}$$

and for $N = 0, 1, 2, \ldots$ we have that $s_{2N+1}(\hat{G}(q + j\omega))$ is bounded as in (25). The singular numbers form a decreasing sequence and hence we can make the upper estimate

$$s_{2N+2}(\hat{G}(q + j\omega)) \le s_{2N+1}(\hat{G}(q + j\omega)).$$

Now we can use these estimates to bound the trace norm (22):

$$\begin{aligned} \|\hat{G}(q + j\omega)\|_1 &\le \sum_{k=0}^{\infty} \frac{2C_1}{(\delta+q)^2 + \omega_0^2 k^2} \\ &\le \frac{K_1}{K_2 + q}, \end{aligned}$$

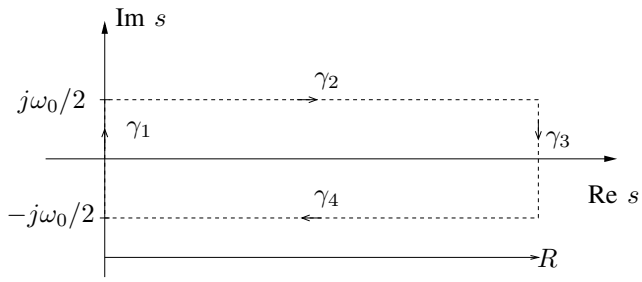

Fig. 2. The integration path Γ_R.

for some constants K_1, K_2. ■

Before stating the main result, we need the following lemma:

Lemma 1 ([15]): If Ω is an open set in $\mathbb{C}$ and if $\hat{G}(s)$ is an analytic trace-class-operator-valued function for $s \in \Omega$, then $\det(I + \hat{G}(\cdot)) : \Omega \to \mathbb{C}$ is an analytic function.

V. Main Result

Using Propositions 1–3 and Lemma 1 we are finally ready to state the analogue of Bode's sensitivity integral, applicable to time-periodic systems:

Theorem 1 (Sensitivity integral): Assume that a stable linear time-periodic system G has roll-off 2 in the sense of Proposition 1. Assume furthermore that the sensitivity operator $S = (I + G)^{-1}$ is stable, i.e., there is an ϵ such that

$$|\det(I + \hat{G}(s))| \geq \epsilon > 0, \quad s \in J_0. \tag{27}$$

Then

$$\int_0^{\omega_0/2} \log |\det(I + \hat{G}(j\omega))^{-1}| \, d\omega = 0. \tag{28}$$

Proof: We have that $\det(I + \hat{G}(s))^{-1} = 1/\det(I + \hat{G}(s))$, see [16]. From Proposition 3 we know that $\|\hat{G}(s)\|_1 \leq K_1/K_2$. Using (24) and (27) we then have that

$$\frac{1}{\exp(K_1/K_2)} \leq |\det(I + \hat{G}(s))^{-1}| \leq \frac{1}{\epsilon}$$

and hence $\det(I + \hat{G}(s))^{-1}$ is a bounded function which does not become zero for $s \in J_0$.

As $\det(I + \hat{G}(s))^{-1}$ is nonzero in J_0 we can define a complex logarithm there. Now,

$$\log \det(I + \hat{G}(s))^{-1} = -\log \det(I + \hat{G}(s)). \tag{29}$$

From Propositions 1–3 and Lemma 1 we know that $\det(I + \hat{G}(s))$ is an analytic function in J_0. Then for any simply closed curve $\Gamma \subset J_0$:

$$\int_\Gamma \log \det(I + \hat{G}(s))^{-1} ds = 0, \tag{30}$$

by Cauchy's integral formula. To prove the theorem we choose the curve Γ_R shown in Fig. 2 and let $R \to \infty$.

First we evaluate the integral (30) along γ_2 and γ_4 notice that:

$$\int_0^R \log \det(I + \hat{G}(q + j\omega_0/2))^{-1} dq + \int_R^0 \log \det(I + \hat{G}(q - j\omega_0/2))^{-1} dq = 0$$

for all R. The cancellation is because

$$\det(I + \hat{G}(q - j\omega_0/2)) = \det(I + \hat{G}(q + j\omega_0/2))$$

for all q. This follows by the structure (6) of the HTF and the definition of the determinant.

Next we evaluate the integral along γ_3. The complex logarithm is defined as

$$\log \det(I + \hat{G}(s)) = \log|\det(I + \hat{G}(s))| + j \arg \det(I + \hat{G}(s)).$$

When the impulse response $g(t, \tau)$ is real we have that $\hat{g}_l(s) = \overline{\hat{g}_{-l}(\bar{s})}$ and by the structure (6) and the definition of the determinant that

$$\begin{aligned} \arg \det(I + \hat{G}(s)) &= -\arg \det(I + \hat{G}(\bar{s})) \\ |\det(I + \hat{G}(s))| &= |\det(I + \hat{G}(\bar{s}))|. \end{aligned} \tag{31}$$

The argument is an anti-symmetric function, so when we integrate over the symmetric interval γ_3 it disappears:

$$\begin{aligned} &\left| \int_{\omega_0/2}^{-\omega_0/2} \log \det(I + \hat{G}(R + j\omega)) d(j\omega) \right| \\ &= \left| \int_{\omega_0/2}^{-\omega_0/2} \log|\det(I + \hat{G}(R + j\omega))| d(j\omega) \right| \\ &\leq \int_{\omega_0/2}^{-\omega_0/2} \|\hat{G}(R + j\omega)\|_1 \, d\omega, \end{aligned}$$

for each fixed R. The last bound follows by (24). Now $\|\hat{G}(R + j\omega)\|_1$ converges uniformly to zero as $R \to \infty$ according to Proposition 3. The integral along γ_3 then goes to zero as $R \to \infty$.

The only term remaining of (30) is the integral along γ_1:

$$\int_{-\omega_0/2}^{\omega_0/2} \log \det(I + \hat{G}(j\omega))^{-1} d(j\omega) = 0. \tag{32}$$

Using (31) on the interval $[-\omega_0/2, \omega_0/2]$ we obtain

$$\int_0^{\omega_0/2} \log |\det(I + \hat{G}(j\omega))^{-1}| d\omega = 0,$$

and the result is shown. ■

Remark 2 (Time-invariant systems): The integral in (1) is over the interval $[0, \infty)$ whereas the integral in (28) is over $[0, \omega_0/2]$. This might seem strange, but notice that for a time-invariant system with transfer function $\hat{g}(s)$, the HTF is given by

$$\hat{G}(s) = \text{diag}\{\ldots, \hat{g}(s + j\omega_0), \hat{g}(s), \hat{g}(s - j\omega_0), \ldots\}$$

for any $\omega_0 > 0$, and we see that (1) and (28) are identical if we use that $\hat{g}(\bar{s}) = \overline{\hat{g}(s)}$.

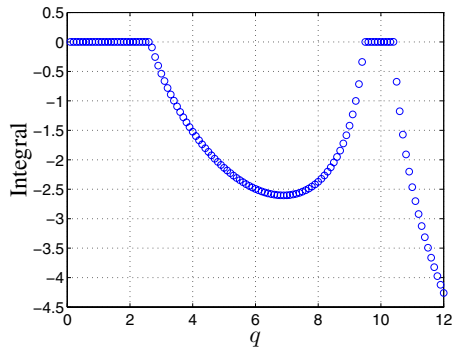

Fig. 3. The values of the integral (28) for different values of q in the Mathieu equation (33). By Theorem 1 the integral must equal zero for stable closed-loop systems. It can be verified by, for instance, Floquet analysis that the system indeed is stable for $q \in [0.2.6] \cup [9.4, 10.4]$.

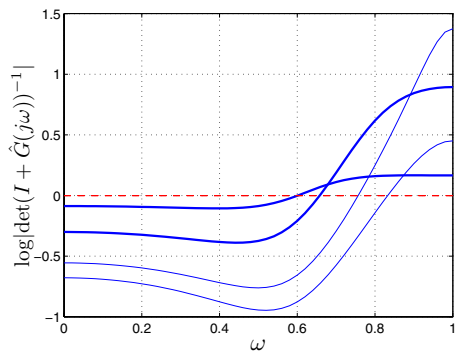

Fig. 4. The logarithm of the sensitivity function for the Mathieu equation (33) is plotted for $q = 1.0, 2.0, 3.0$, and 3.5. For the bold curves (the stable systems) the conservation law in Theorem 1 applies. When q increases the sensitivity decreases for low frequencies. The sensitivity must then increase for high frequencies to keep the areas below and above the zero level equal. This is the waterbed effect.

VI. An Example: The Mathieu Equation

Now we verify the main result on an example. We choose an open-loop system G with dynamics given by

$$\ddot{y}(t) + 0.4\dot{y}(t) + 2y(t) = q\cos(2t)w(t), \qquad (33)$$

where q is a parameter and $w(t)$ the input. The impulse response is given by

$$g(t,\tau) = \frac{q}{1.4}e^{-0.2(t-\tau)}\sin(1.4(t-\tau))\cos(2\tau). \qquad (34)$$

Clearly the system has roll-off 2 in the sense of Proposition 1 and is exponentially stable. To obtain the closed-loop system in Fig. 1 the feedback $w(t) = -(y(t) + u(t))$ is applied. Notice that when $u(t) = 0$ the dynamics of the closed-loop system is given by a damped Mathieu equation, see for example [9].

Next we compute the HTF of G using (4)–(6). Here $\omega_0 = 2$. After this we may compute the integral (28) for different values of q. For $q \in [0.2.6] \cup [9.4, 10.4]$, the closed loop is stable. This can be shown by, for instance, Floquet analysis. According to Theorem 1 the integral should then equal zero. In Fig. 3 this is verified. It is also seen that when the closed loop is unstable, the integral is strictly less than zero.

Furthermore, we can visualize the waterbed effect for periodic systems. This is done in Fig. 4. When the sensitivity decreases for some frequencies, it must increase for other frequencies.

VII. Conclusion

We have seen that there are fundamental limitations for feedback control of linear time-periodic systems. The modulus of the determinant of the harmonic transfer function $\hat{S}(j\omega) = (I + \hat{G}(j\omega))^{-1}$ cannot be made small for every frequency ω. The result is a direct generalization of Bode's sensitivity integral. To prove the result we have defined roll-off for a time-periodic system, and used some of the theory for analytic operators and trace class operators. The case with unstable open-loop systems was not considered here, but this is an interesting problem for future research.

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