Fourier Integral Operators

Lectures at the Nordic Summer School of Mathematics

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1969

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Foreword:

These notes were originally distributed among the participants in the Summer School of Mathematics at Tjörn in 1969; later, a copy was given to Johannes Sjöstrand (not himself a participant at the Summer School) who remembers that he received it with instructions that the notes were preliminary and should not be circulated. Not surprisingly then, my father chose to destroy the original before his death and so did not include these notes in the collection of Unpublished Manuscripts he wished to survive him (and which were published by Springer Verlag in 2018). We can never know for sure why he did not want to keep them himself, but a likely reason is that he felt that they were superceded by the published presentation of the theory. Nevertheless, they were quite widely distributed and I understand from discussions with Johannes Sjöstrand and Gerd Grubb (who supplied the copy of the manuscript) that they have significant historical interest, as also discussed in more detail by Johannes Sjöstrand in the commentary appended at the end of the text. I have therefore decided to make these notes available to a wider audience here at LUCRIS despite my father excluding them from his intended mathematical legacy. Enjoy!

Lund, in August 2018

Sofia Broström

Daughter and heir of Lars Hörmander

Introduction

Pseudo-differential operators have been developed as a tool for the study of elliptic differential equations. In suitably extended versions they are also applicable to hypoelliptic equations, but their value is rather limited in genuinely non-elliptic problems. In these lectures we shall therefore discuss some more general classes of operators which should be more useful in such contexts. Their theory is still in a fairly primitive state but seems well worth further development in view of the applications which can already be made. It also seems that one gains some more insight into the theory of pseudo-differential operators by considering them from the point of view of the wider classes of operators to be discussed here.

Pseudo-differential operators as well as our Fourier integral operators are supposed to make it possible to handle differential operators with variable coefficients roughly as one would handle differential operators with constant coefficients using the Fourier transformation. Thus for example the inhomogeneous Laplace equation

$$\Delta u = f \in C_0^{\infty}(\mathbb{R}^n)$$

is for $n > 2$ solved by

$$u(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} |\xi|^{-2}\hat{f}(\xi)\, d\xi.$$

To be able to solve arbitrary elliptic equations with variable coefficients one is led to consider more general operators of the form

$$(0.1) \qquad Af(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} a(x,\xi)\, \hat{f}(\xi)\, d\xi$$

where a behaves as a sum of homogeneous functions when $\xi \to \infty$. These are the (classical) pseudo-differential operators. On the other hand, suppose that we want to solve the Cauchy problem

$$\Delta u - \partial^2 u/\partial t^2 = 0 \; ; \; u = 0, \; \partial u/\partial t = f \in C_0^\infty(\mathbb{R}^n), \; t = 0.$$

Then the solution is given by

$$(0.2) \qquad u(x,t) = (2\pi)^{-n} \int e^{i(\langle x,\xi\rangle + t|\xi|)} (2|\xi|)^{-1} \hat{f}(\xi)\, d\xi - (2\pi)^{-n} \int e^{i(\langle x,\xi\rangle - t|\xi|)} (2|\xi|)^{-1} \hat{f}(\xi)\, d\xi.$$

Each of the terms on the right hand side is similar to (0.1) except for the fact that the function $\langle x,\xi\rangle$ in the exponent has been replaced by $\langle x,\xi\rangle - t|\xi|$. This is a homogeneous function of ξ with critical points as a function of ξ where $x = t\xi/|\xi|$, thus $|x|^2 = t^2$, which is the light cone. The function $\langle x,\xi\rangle$ on the other hand has no critical point except when $x = 0$. These observations reflect the fact that the fundamental solution of the wave equation is singular on the light cone whereas the fundamental solution of the Laplacean is singular only at the origin.

If one introduces the definition of the Fourier transform f, the operators occurring in (0.1) and (0.2) assume the form

$$(0.3) \qquad Af(x) = \iint e^{i\varphi(x,y,\xi)} a(x,y,\xi)\, f(y)\, dy d\xi.$$

Although one has to take some care to give a precise meaning to this integral, we shall see below that it is advantageous to write pseudo-differential operators also in this form. However, our main purpose is of course to discuss the properties of (3) for a general choice of φ.

The first topic in the lectures is to discuss the precise meaning of

(0.3) for suitable amplitude functions a and phase functions φ . We shall then review the theory of pseudo-differential operators from this point of view. We then pass to an analogous though unfortunately very incomplete discussion of general operators of the form (0.3). A section on L^2 estimates was originally intended but has been omitted due to some hopefully minor difficulties which turned up in the proof. We then pass to a further extension of the class of Fourier integral operators where the amplitude function in (0.3) may have singularities. The purpose is to construct fundamental solutions of differential equations of principal type with variable coefficients by imitating the definition of the integral

$$(2\pi)^{-n} \int e^{i\langle x,\xi\rangle} P(\xi)^{-1} \hat{u}(\xi)\, d\xi$$

by complex deformation of the integration contour in the case where P is homogeneous and has no real critical point $\xi \neq 0$. This will yield new regularity and existence theorems for such operators.

Chapter I

Definitions and basic properties of Fourier integral operators

1.1. Symbols. Concerning the amplitudes in the operators (0.3) we shall make the same assumptions as in the study of pseudo-differential operators so we shall only recall the definitions and some fundamental facts.

Definition 1.1.1. Let m, ρ, δ be real numbers with $0 < \rho \leq 1$ and $0 \leq \delta < 1$, let Ω be an open subset of $\mathbb{R}^n$ and N a positive integer. Then we denote by $S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ the set of all $a \in C^\infty(\Omega \times \mathbb{R}^N)$ such that for every compact set $K \subset \Omega$ and all multi-indices α, β we have with a constant $C_{\alpha,\beta,K}$ the estimate

$$(1.1.1) \qquad |D_x^\beta D_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \; x \in K, \; \xi \in \mathbb{R}^N.$$

The elements of $S^m_{\rho,\delta}$ are called symbols of order m and type ρ, δ; if $\rho = 1$, $\delta = 0$ we sometimes drop the subscripts and talk simply about symbols of order m. If (1.1.1) is only valid for large $|\xi|$, we say that $a \in S^m_{\rho,\delta}$ for large $|\xi|$.

Example 1. If $a \in C^\infty$ and if a is a homogeneous function of degree $\leq m$ for large $|\xi|$, then a is a symbol of order m and type 1,0.

Example 2. If a is semi-homogeneous in the sense that

$$a(\xi_1 t^{m_1}, \ldots, \xi_n t^{m_n}) = t^m a(\xi_1, \ldots, \xi_n)$$

for some $m_j > 0$ and m, and if a C^∞ for $\xi \neq 0$, then a is for large $|\xi|$ a symbol of degree $\max m/m_j$ and type $\min m_j/m_k$, 0.

Example 3. If $\chi \in C_0^\infty$, then $a(x,\xi) = \chi(x|\xi|^\varepsilon)$ is of type 1, ε and degree 0 for large $|\xi|$. (Note that $|x|$ can be bounded by $|\xi|^{-\varepsilon}$ in the support.)

Example 4. If $0 < t < 1$, the function $\xi \to \exp i|\xi|^{1-t}$ is of degree 0 and type $\rho, 0$ if and only if $\rho \leqq t$.

Proposition 1.1.2. $S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ is a Fréchet space with the topology defined by taking as seminorms the best constants $C_{\alpha,\beta,K}$ which can be used in (1.1.1). This space increases when δ and m increase and ρ decreases. If $a \in S^m_{\rho,\delta}$ it follows that $a^{(\alpha)}_{(\beta)} = (iD_\xi)^\alpha (iD_x)^\beta a \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}$, and if $b \in S^{m'}_{\rho,\delta}$ it follows that $ab \in S^{m+m'}_{\rho,\delta}$.

The proof is obvious. Note that to prove (1.1.1) for ab one needs to know only that (1.1.1) holds for a and for b when the differentiations involved are of order $\leqq |\alpha|+|\beta|$. This is important for some proofs by induction.

It follows immediately from Definition 1.1.1 that $S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ is invariant for diffeomopphisms in the x variable, so the definition makes sense also if Ω is a manifold. However, it may then be more natural to replace $\Omega \times \mathbb{R}^N$ by a real vector bundle V over Ω. We can do so provided that for every $a \in S^m_{\rho,\delta}$ we have $a(x, A(x)\xi) \in S^m_{\rho,\delta}$ if $x \to A(x)$ is a C^∞ map $\Omega \to GL(N, \mathbb{R})$. Inspection of Example 4 shows that this requires that $1-\rho \leqq \delta$. Conversely, this condition is also sufficient so we can define $S^m_{\rho,\delta}(\Omega, V)$ where V is a real vector bundle over the manifold Ω provided that $1-\rho \leqq \delta$ and only then. Indeed, we have the following proposition which makes it possible to define $S^m_{\rho,\delta}$ for still more general objects:

Proposition 1.1.3. With the notations of Definition 1.1.1 let $A_j(x,\xi) \in S^1_{1,0}(\Omega, \mathbb{R}^N)$, $j = 1, \ldots, N$, and set $A(x,\xi) = (A_1(x,\xi),\ldots,A_N(x,\xi))$. We assume that $(1+|\xi|) \leqq C_K(1+|A(x,\xi)|)$, $x \in K$. If $a \in S^m_{\rho,\delta}$ it follows then that $a(x, A(x,\xi)) \in S^m_{\rho,\delta}$ provided that $1-\rho \leqq \delta$.

Proof. Writing $a^j(x,\xi) = \partial a(x,\xi)/\partial\xi_j$ and $a_j(x,\xi) = \partial a(x,\xi)/\partial x_j$, we have

$$\partial a(x,A(x,\xi))/\partial x_j = a_j(x,A(x,\xi)) + \sum a^k(x,A(x,\xi))\partial A_k(x,\xi)/\partial x_j,$$

$$\partial a(x,A(x,\xi))/\partial\xi_k = \sum a^j(x, A(x,\xi))\partial A_j(x,\xi)/\partial\xi_k.$$

Here $a^j \in S^{m-\rho}_{\rho,\delta}$, $a_j \in S^{m+\delta}_{\rho,\delta}$ and $\partial A_k/\partial\xi_j \in S^0_{1,0} \subset S^0_{\rho,\delta}$, $\partial A_k/\partial x_j \in S^1_{1,0} \subset S^1_{\rho,\delta}$. It is clear that $a(x, A)$ satisfies (1,1.1) when $\alpha+\beta = 0$, so we may prove this estimate in general assuming it to be known for smaller $|\alpha+\beta|$. But then it follows from the formulas above in view of the statement concerning mutliplication of symbols given in Proposition 1.1.2 and the following remarks.

Remark. The proof shows that the statement remains valid if the hypothesis $A_j \in S^1_{1,0}$ is weakened to $A_j \in S^1_{1,\delta}$.

Proposition 1.1.3 makes it possible to define $S^m_{\rho,\delta}(\Omega, V)$ if V and Ω are manifolds with a map $p: V \to \Omega$ having surjective differential such that there is given a free differentiable action of the multiplicative group of positive real numbers on V which commutes with p, for which the quotient of V by $\mathbb{R}_+$ has compact fibers. Roughly speaking this means that the fibers of V are cones and it is natural to call V a cone bundle. However, we shall mainly discuss local results here so we shall not pursue this global setting further.

Next we recall an important completeness property of the space of symbols. For the proof see Theorem 2.7 in Hörmander [2].

Proposition 1.1.4. Let $p_j \in S^{m_j}_{\rho,\delta}(\Omega, \mathbb{R}^N)$, $j = 0, 1, 2, \ldots$ and assume that $m_j \to -\infty$. Set $m'_k = \max_{j \geq k} m_j$. Then one can find $p \in S^{m'_0}_{\rho,\delta}(\Omega, \mathbb{R}^N)$ such that for every k

$$p - \sum_{j<k} p_j \in S^{m'_k}_{\rho,\delta}.$$

The function p is uniquely determined modulo $S^{-\infty}_{\rho,\delta}(\Omega, \mathbb{R}^N) = \bigcap S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ and has the same property relative to any rearrangement of the series $\sum p_j$.

To recognise when p has the properties in the proposition it is useful to have the following result, which is the same as Theorem 2.9 in [2].

Proposition 1.1.4. Let $p_j \in S^{m_j}_{\rho,\delta}(\Omega, \mathbb{R}^N)$, $j = 0, 1, \ldots$ and assume that $m_j \to -\infty$ when $j \to \infty$. Let $p \in C^\infty(\Omega \times \mathbb{R}^N)$ and assume that for all multi-indices α, β and compact sets $K \subset \Omega$ we have for some C and μ depending on α, β and K

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leq C(1+|\xi|)^\mu, \quad x \in K.$$

If there exist numbers $\mu_k \to -\infty$ such that

$$|p(x,\xi) - \sum_{j<k} p_j(x,\xi)| \leq C_{K,k}(1+|\xi|)^{\mu_k}, \quad x \in K,$$

it follows that $p \in S^{m'_0}_{\rho,\delta}(\Omega, \mathbb{R}^N)$ where $m'_0 = \sup m_j$, and that $p \sim \sum p_j$.

Finally we shall make some remarks on the topology of the Fréchet space $S^m_{\rho,\delta}$. Recall that a set $M \subset S^m_{\rho,\delta}$ is bounded if (1.1.1) is valid with $C_{\alpha,\beta,K}$ independent of a when $a \in M$. On a bounded set in $S^m_{\rho,\delta}$ the topology of pointwise convergence, the topology of $C^\infty(\Omega \times \mathbb{R}^N)$ and the topology of $S^{m'}_{\rho,\delta}(\Omega, \mathbb{R}^N)$, $m' > m$, all coincide. This is an immediate consequence of Ascoli's theorem.

Proposition 1.1.5. Let $a \in S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ and let $\chi \in \mathcal{S}(\mathbb{R}^N)$ be equal to 1 at 0. If $a_\varepsilon(x,\xi) = \chi(\varepsilon\xi)a(x,\xi)$ it follows that $a_\varepsilon \in S^{-\infty}_{\rho,\delta}(\Omega, \mathbb{R}^N)$ and that $a_\varepsilon \to a$ in $S^{m'}_{\rho,\delta}(\Omega, \mathbb{R}^N)$ when $\varepsilon \to 0$ if $m' > m$.

Proof. It suffices to note that the functions $(x,\xi) \to \chi(\varepsilon\xi)$ form a bounded set in $S^0_{1,0}$ when $0 \leqq \varepsilon \leqq 1$ (cf. Example 3, page 4).

In particular, we can take χ with compact support. Then we obtain

Corollary 1.1.6. Let L be a linear map from functions in $\mathcal{E}^\infty(\Omega \times \mathbb{R}^N)$ vanishing for large $|\xi|$ to a Fréchet space F such that L is continuous for the topology of $S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ for every m. Then there is a unique extension of L to $S^\infty_{\rho,\delta}(\Omega, \mathbb{R}^N) = \bigcup\limits_m S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ which is continuous on $S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ for every m.

1.2 Oscillatory integrals. We shall now discuss the definition of integrals of the form

$$(1.2.1) \qquad I_\varphi(au) = \iint e^{i\varphi(x,\xi)} a(x,\xi)u(x)\, dx d\xi, \; u \in C_0^\infty(\Omega),$$

where $a \in S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$. For the sake of simplicity it will be assumed that φ is real valued and positively homogeneous of degree 1 with respect to ξ, and that $\varphi \in C^\infty$ for $\xi \neq 0$. However, this hypothesis could easily be relaxed as we shall indicate later on.

The integral (1.2.1) is absolutely convergent for every $a \in S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ provided that $m+N < 0$. In particular, it is well defined if $a(x,\xi) = 0$ for large $|\xi|$. Using Corollary 1.1.6 we wish to extend the definition of (1.2.1) by continuity to arbitrary $a \in S^\infty_{\rho,\delta}(\Omega, \mathbb{R}^N) = \bigcup S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$. This is not always possible - for example φ must not vanish in any open set - but we shall prove that the definition of (1.2.1) is indeed possible if φ has no critical point $\neq 0$. This will follow by means of integrations by parts in (1.2.1).

Lemma 1.2.1. If φ has no critical point, one can find a first order differential operator

$$L = \sum a_j \partial/\partial\xi_j + \sum b_j \partial/\partial x_j + c$$

with $a_j \in S^0_{1,0}(\Omega, \mathbb{R}^N)$ and $b_j, c \in S^{-1}_{1,0}(\Omega, \mathbb{R}^N)$, such that ${}^tL\, e^{i\varphi} = e^{i\varphi}$ if ${}^tL$ is the adjoint of ${}^tL$.

Proof. By hypothesis the sum

$$|\xi|^2 \sum (\partial\varphi/\partial\xi_j)^2 + \sum (\partial\varphi/\partial x_j)^2$$

is homogeneous of degree 2 with respect to ξ and $\neq 0$ for $\xi \neq 0$. Let ψ be the reciprocal of this sum which is then homogeneous of degree -2 and C^∞ for $\xi \neq 0$. With $\chi \in C_0^\infty(\mathbb{R}^N)$ chosen so that $\chi = 1$ near 0, we set

$$M = \sum a_j' \partial/\partial\xi_j + \sum b_j' \partial/\partial x_j +$$

where $a_j' = -i(1-\chi)\psi|\xi|^2 \partial\varphi/\partial\xi_j \in S^0_{1,0}$, $b_j' = -i(1-\chi)\psi\,\partial\varphi/\partial x_j \in S^{-1}_{1,0}$. The coefficients are chosen so that $M\, e^{i\varphi} = e^{i\varphi}$, so $L = {}^tM$ has the required properties, for

$$a_j = -a_j',\ b_j = -b_j',\ c = \chi - \sum \partial a_j'/\partial\xi_j - \sum \partial b_j'/\partial x_j \in S^{-1}_{1,0}.$$

The lemma is proved.

If a vanishes for large $|\xi|$, we can integrate by parts in (1.2.1) after replacing $e^{i\varphi}$ by ${}^tL\, e^{i\varphi}$. This gives

$$I_\varphi(au) = \iint e^{i\varphi(x,\xi)}\, L(a(x,\xi)u(x))\, dx d\xi$$

or after iteration

$$(1.2.2) \qquad I_\varphi(au) = \iint e^{i\varphi(x,\xi)}\, L^k(a(x,\xi)u(x))\, dx d\xi,\ k = 0, 1, 2, \ldots$$

Now L is a continuous map of $S^m_{\rho,\delta}$ into $S^{m-t}_{\rho,\delta}$ if $t = \min(\rho, 1-\delta)$. Hence L^k maps $S^m_{\rho,\delta}$ continuously into $S^{m-kt}_{\rho,\delta}$. If $m - kt < -N$, the integral (1.2.2) is thus defined and continuous on all of $S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$. In view of Corollary 1.1.6 we have therefore proved

Proposition 1.2.2. If φ has no critical points, the definition of the integral (1 .2.1) can be extended in one and only one way to all $a \in S^{\infty}_{\rho,\delta}(\Omega, \mathbb{R}^N)$ so that $I_{\varphi}(au)$ is a continuous function of $a \in S^m_{\rho,\delta}$ for every fixed m. The linear form $u \to I_{\varphi}(au)$ is a distribution of order $\leq k$ if $a \in S^m_{\rho,\delta}$ and $m-k\rho < -N$, $m-k(1-\delta) < -N$. (More precisely, it is a sum of derivatives of order $\leq k$ of continuous functions.)

For the extended form we have the representation (1.2.2). According to Proposition 1.1.5 we also have

$$(1.2.3) \qquad I_{\varphi}(au) = \lim_{\varepsilon \to 0} \iint e^{i\varphi(x,\xi)}\, a(x,\xi)\chi(\varepsilon\xi)\, u(x)\, dx d\xi$$

if $\chi \in \mathcal{S}$ and $\chi(0) = 1$.

It is convenient to use the notation (1.2.1) also for the continuous extension which has now been defined . We shall then refer to the generalised integral as an oscillatory integral.

If φ and a are continuous functions of a parameter t with values in $C^{\infty}(\Omega \times (\mathbb{R}^N \setminus \{0\}))$ and $S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$ respectively, then an obvious modification of the proof of Proposition 1.2.2 shows that $I_{\varphi}(au)$ is a continuous function of t. Note that if a is a continuous function of t with values in $C^{\infty}(\Omega \times \mathbb{R}^N)$ whose range is a bounded subset of $S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$, then a is a continuous function of t with values in $S^{m'}_{\rho,\delta}(\Omega, \mathbb{R}^N)$ when $m' > m$. These remarks make it possible to pass to the limit in the oscillatory integral (1.2.2) and in particular to differentiate with respect to parameters under the integral sign.

1 .3. Definition of Fourier integral operators. Let Ω_j be an open subset of $\mathbb{R}^{n_j}$, $j = 1, 2$, and let φ be a real valued function of $(x,y,\xi) \in \Omega_1 \times \Omega_2 \times \mathbb{R}^N$ which is positively homogeneous of degree 1 with respect to ξ and infinitely differentiable for $\xi \neq 0$. With a symbol $a \in S^m_{\rho,\delta}(\Omega_1 \times \Omega_2, \mathbb{R}^N)$ we wish to consider the operator defined by the integral

$$(1.3.1) \qquad Au(x) = \iint e^{i\varphi(x,y,\xi)}\, a(x,y,\xi)\, u(y)\, dyy\xi, \quad u \in C_0^\infty(\Omega_2), \ x \in \Omega_1,$$

or the corresponding weak form

$$(1.3.2) \qquad \langle Au, v\rangle = \iiint e^{i\varphi(x,y,\xi)}\, a(x,y,\xi)v(x)u(y)\, dxdyd\xi, \quad u \in C_0^\infty(\Omega_2), \ v \in C_0^\infty(\Omega_1).$$

To define these integrals we can apply the results on oscillatory integrals given in section 1.2. The conclusions are as follows:

Theorem 1.3.1. (i) If φ has no critical point (x,y,ξ) with $\xi \neq 0$, then the oscillatory integral (1.3.2) exists and is a continuous bilinear form for the C_0^k topologies on u, v if

$$(1.3.3) \qquad m - k\rho < -N, \quad m-k(1-\delta) < -N.$$

When (1.3.3) is valid we thus obtain a continuous linear map A from $C_0^k(\Omega_2)$ into $\mathcal{D}'^k(\Omega_1)$ which has a distribution kernel $K_A \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ given by the oscillatory integral

$$(1.3.4) \quad K_A(u) = \iiint e^{i\varphi(x,y,\xi)}\, a(x,y,\xi)\, u(x,y)\, dxdyd\xi, \quad u \in C_0^\infty(\Omega_1 \times \Omega_2).$$

(ii) If φ has no critical point (x,y,ξ) with $\xi \neq 0$ considered as a function of (y, ξ) for fixed x, then (1.3.1) is defined as an oscillatory integral. When (1.3.3) is valid we obtain a continuous map $A: C_0^k(\Omega_2) \to C(\Omega_1)$. By differentiation under the integral sign it follows that A is a continuous map of $C_0^k(\Omega_2)$ into $C^j(\Omega_1)$ if

$$(1.3.5) \qquad m+N+j < k\rho, \quad m+N+j < k(1-\delta).$$

(iii) If φ has no critical point (x,y,ξ) with $\xi \neq 0$ considered as a function of (x,ξ) for fixed y, then the adjoint of A has the properties listed in (ii) so A is a continuous map of $\mathcal{E}'^{j}(\Omega_2)$ into $\mathcal{D}'^{k}(\Omega_1)$ when (1.3.5) is fulfilled. In particular, A defines a map from $\mathcal{E}'(\Omega_2)$ into $\mathcal{D}'(\Omega_1)$.

(iv) Let Ω_φ be the open set of all $(x,y)\in \Omega_1\times\Omega_2$ such that $\varphi(x,y,\xi)$ has no critical point $\xi \neq 0$ as a function of ξ. Then the oscillatory integral

$$(1.3.6) \qquad K_A(x,y) = \int e^{i\varphi(x,y,\xi)}\, a(x,y,\xi)\, d\xi, \quad (x,y)\in\Omega_\varphi,$$

defines a function in $C^\infty(\Omega_\varphi)$ which is equal to the distribution (1.3.4) in Ω_φ. If $\Omega_\varphi = \Omega_1\times\Omega_2$, it follows that A is an integral operator with a C^∞ kernel so A is a continuous map of $\mathcal{E}'(\Omega_2)$ into $C^\infty(\Omega_1)$.

The proof is an immediate consequence of the properties of oscillatory integrals listed in section 1.2.

Example 1. Pseudo-differential operators correspond to the function $\varphi(x,y,\xi) = \langle x-y,\ \xi\rangle$ $(n_1=n_2=N)$. Then (i), (ii), (iii) are fulfilled and Ω_φ is the complement of the diagonal, if we take $\Omega_1 = \Omega_2$.

Example 2. In the introduction we saw that the study of the Cauchy problem for the wave equation leads to the function

$$(x,t;\ y,\ \xi) = \langle x-y,\xi\rangle + t|\xi|.$$

Here $n_1-1 = n_2 = N$ and the variable in Ω_1 is denoted by (x,t). Then (i), (ii), (iii) are fulfilled and $\complement\Omega_\varphi$ consists of all $(x,t;\ y)$ with $|x-y|^2 = t^2$, that is, (x,t) shall lie on the light cone with vertex at $(y,0)$.

Definition 1.3.2. A real valued function φ of $(x,y,\xi)\in\Omega_1\times\Omega_2\times\mathbb{R}^N$ which is a C^∞ function for $\xi \neq 0$ and positively homogeneous of degree 1 with respect to ξ will be called a phase function if for fixed x (or y) it has no critical point as a function of (y, ξ) (or (x,ξ)) when $\xi \neq 0$.

When φ is a phase function the requirements in parts (i),(ii),(iii) of Theorem 1.3.1 are thus fulfilled. Let F_φ denote the complement of Ω_φ in $\Omega_1 \times \Omega_2$, that is, the set of all $(x,y) \in \Omega_1 \times \Omega_2$ such that $\operatorname{grad}_\xi \varphi(x,y,\xi) = 0$ for some $\xi \neq 0$. From (iv) in Theorem 1.3.1 it follows then that

$$\text{sing supp } Au \subset F_\varphi \text{ supp } u, \quad u \in \mathcal{E}'(\Omega_2), \tag{1.3.7}$$

where the right hand side is defined by considering F_φ as a relation between points in Ω_2 and in Ω_1. Thus $F_\varphi K = \{x;\ (x,y) \in F_\varphi\}$ for some $y \in K$. If $K_2 = \text{supp } u$ and K_1 is a compact subset of Ω_1 which does not intersect $F_\varphi K_2$, we have $K_1 \times K_2 \in \Omega_\varphi$, so we can find neighborhoods $\Omega_1' \supset K_1$, $\Omega_2' \supset K_2$ such that $\Omega_1' \times \Omega_2' \subset \Omega_\varphi$. This proves (1.3.7). Using (ii) in Theorem 1.3.1 we can improve (1.3.7) further. In fact, for any neighborhood Ω_2' of sing supp u we can make a decomposition $u = v + w$ where supp $v \subset \Omega_2'$ and $w \in C_0^\infty$. Since $Aw \in C^\infty$ we obtain

$$\text{sing supp } Au = \text{sing supp } Av \subset F_\varphi \text{ supp } v,$$

so we have proved

<u>Theorem 1.3.3.</u> If $u \in \mathcal{E}'(\Omega_2)$, then

$$\text{sing supp } Au \subset F_\varphi \text{ sing supp } u. \tag{1.3.8}$$

<u>Example.</u> For pseudo-differential operators this means that

$$\text{sing supp } Au \subset \text{sing supp } u,$$

that is, we have the pseudo-local property.

Finally we note an obvious localization property:

<u>Theorem 1.3.4.</u> If $a(x,y,\xi)$ vanishes when (x,y) belongs to some neighborhood of F_φ, then the distribution K_A defined by (1.3.4) is in $C^\infty(\Omega_1\times\Omega_2)$.

Indeed, the formula (1.3.6) makes sense in $\Omega_1 \times \Omega_2$.

<u>Remark.</u> It is easy to show that the statement remains valid under the weaker hypothesis that $a = 0$ when $|\mathrm{grad}_\xi \varphi||\xi|^\varepsilon < C$ for some C and some $\varepsilon < \min(\rho, 1/2)$. We shall prove a still better result for some φ in Proposition 2.1.7.

Finally we introduce the notation $L^m_{\rho,\delta}(\Omega_1, \Omega_2, \varphi)$ for the class of operators of the form (1.3.1) with $a \in S^m_{\rho,\delta}$ when φ is a given phase function. Note that every operator with a C^∞ kernel belongs to $L^m_{\rho,\delta}$, for if we take a function $\chi \in C_0^\infty(\mathbb{R}^N)$ vanishing in a neighborhood of 0 and with $\int\chi(\xi)\,d\xi = 1$, then $a(x,y,\xi) = e^{-i\varphi(x,y,\xi)}\chi(\xi)\,b(x,y)$ corresponds to the integral operator with the kernel $b(x,y)$.

Chapter II

Pseudo-differential operators

2.1. The calculus of pseudo-differential operators. If Ω is an open set in $\mathbb{R}^n$, we shall write $L^m_{\rho,\delta}(\Omega)$ for the class of operators $L^m_{\rho,\delta}(\Omega, \Omega, \varphi)$ defined with the phase function $\varphi(x,y,\xi) = \langle x-y, \xi\rangle$ $(N = n)$. As we shall see in a moment, this agrees with the definitions given in Hörmander [2] at least when $\delta < \rho$, so we shall call these operators pseudo-differential of type ρ, δ.

A pseudo-differential operator

$$(2.1.1) \qquad Au(x) = \iint e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\, u(y)\, dy d\xi$$

is called properly supported if both projections $\operatorname{supp} K_A \to \Omega$ are proper, that is, if $\{(x,y) \in \operatorname{supp} K_A;\ x \in K \text{ or } y \in K\}$ is compact for every compact set $K \subset \Omega$. (Here we recall that K_A is the distribution kernel of A.) It is then clear that Au can be defined without restrictions on the support of u; thus A maps $C^\infty(\Omega)$ into $C^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ into $\mathcal{D}'(\Omega)$. Furthermore, A maps $C_0^\infty(\Omega)$ into $C_0^\infty(\Omega)$ and $\mathcal{E}'(\Omega)$ into $\mathcal{E}'(\Omega)$. If χ is a function in $C^\infty(\Omega\times\Omega)$ which is equal to 1 in a neighborhood of $\operatorname{supp} K_A$ and the projections $\operatorname{supp}\chi \to \Omega$ are also proper, it is clear that the operator A is also defined by the symbol $\chi(x,y)a(x,y,\xi) = a_1(x,y,\xi)$. Note that

$$\{(x,y);\ (x,y,\xi) \in \operatorname{supp} a_1 \text{ for some } \xi \text{ and } \quad x \text{ or } y \in K\}$$

is then relatively compact in $\Omega\times\Omega$ for every compact set $K \subset \Omega$. We shall also say that such a symbol is proper. Every pseudo-differential operator is the sum of one with a C^∞ kernel and one which is properly supported. This follows immediately if we choose $\chi \in C^\infty(\Omega \times \Omega)$ so that $\chi = 1$ in a neighborhood of the diagonal and χ is properly supported, for the symbol $(1-\chi)a$ defines an operator with C^∞ kernel according to Theorem 1.3.4, and χa is a proper symbol.

When $\delta < \rho$ we shall now derive an expression for a properly supported pseudo-differential operator A which will connect the definition used here with that used in Hörmander [2]. Thus let A be defined by (2.1.1) where a is proper. We apply A to the exponential function $e_\xi(x) = \exp i\langle x,\xi\rangle$ and obtain $Ae_\xi(x) = \sigma_A(x,\xi)\, e_\xi(x)$ where

$$\sigma_A(x,\xi) = \iint a(x,y,\eta)\, e^{i\langle x-y,\eta\rangle + i\langle y-x,\xi\rangle}\, dy d\eta =$$

$$= \iint a(x,\, x+y,\, \xi+\eta)\, e^{-i\langle y,\eta\rangle}\, dy d\eta.$$

The oscillatory integral here may be interpreted as a repeated integral taken first with respect to η, then with respect to y. We set $b(x,y,\xi) = a(x,x+y,\xi)$ and introduce the Fourier transform

$$\hat{b}(x,\,\theta,\,\xi) = \int b(x,y,\xi)\, e^{-i\langle y,\,\theta\rangle}\, dy.$$

For every compact set $K \subset \Omega$ we obtain if $a \in S^m_{\rho,\delta}$ is proper and if $x \in K$

$$(2.1.2) \qquad |D_x^\alpha\, \theta^\beta D_\xi^\gamma\, \hat{b}(x,\,\theta,\,\xi)| \leqq C(1+|\xi|)^{m+\delta(|\alpha|+|\beta|) - \rho|\gamma|},$$

hence for any positive integer ν

$$(2.1.3) \qquad |D_x^\alpha D_\xi^\gamma\, \hat{b}(x,\,\theta,\,\xi)| \leqq C(1+|\xi|)^{m+\delta(|\alpha|+\nu) - \rho|\gamma|}\, (1+|\theta|)^{-\nu}.$$

Now we have

$$\sigma_A(x,\xi) = \int \hat{b}(x,\,\eta,\,\xi+\eta)\, d\eta.$$

Since $\delta < 1$ it follows from (2.1.3) that σ_A and any one of its derivatives can be bounded by some powers of $(1+|\xi|)$. To obtain the asymptotic behavior of σ_A when $\xi \to \infty$ we form the Taylor expansion; by (2.1.3) we have

$$|\hat{b}(x,\eta,\xi+\eta) - \sum_{|\alpha|<N} (iD_\xi)^\alpha \hat{b}(x,\eta,\xi)\, \eta^\alpha/\alpha!| \leqq C|\eta|^N \sup_{0<t<1} \frac{(1+|\xi+t\eta|)^{m+\delta\nu-\rho N}}{(1+|\eta|)^{\nu}}.$$

Here ν is any positive number or 0. With $\nu = N$ we obtain the bound $C(1+|\xi|)^{m+(\delta-\rho)N}$ if $|\eta| < |\xi|/2$, and if we choose ν large we get a bound by any power of $(1+|\eta|)^{-1}$ if $|\xi| < 2|\eta|$. Hence

$$|\sigma_A(x,\xi) - (2\pi)^n \sum_{|\alpha|<N} (iD_\xi)^\alpha D_y^\alpha b(x,y,\xi)/\alpha!_{/y=0}| \leqq C(1+|\xi|)^{m+n+(\delta-\rho)N}.$$

In view of Proposition 1.1.4 it follows that $\sigma_A \in S^m_{\rho,\delta}(\Omega)$ and that

$$(2.1.4) \qquad \sigma_A(x,\xi) \sim (2\pi)^n \sum_\alpha (iD_\xi)^\alpha D_y^\alpha a(x,y,\xi)/\alpha!_{/y=x}.$$

If $u \in C_0^\infty(\Omega)$ we have Fourier's inversion formula

$$u(x) = (2\pi)^{-n} \int e_\xi(x)\hat{u}(\xi)\, d\xi.$$

Since A is continuous from $C^\infty(\Omega)$ to $C^\infty(\Omega)$, we can apply A under the sign of integration and obtain

$$(2.1.5) \qquad Au(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} \sigma_A(x,\xi)\, \hat{u}(\xi)\, d\xi,\ u \in C_0^\infty(\mathbb{R}^n),\ x \in \Omega.$$

In the left hand side we should interpret u as the restriction of u to Ω. Obviously (2.1.5) determines σ_A uniquely. Summing up, we have proved

Theorem 2.1.1. If A is a properly supported operator $\in L^m_{\rho,\delta}(\Omega)$, $\delta < \rho$, then A can be written in one and only one way in the form (2.1.5). Here $\sigma_A \in S^m_{\rho,\delta}(\Omega, \mathbb{R}^n)$ is asymptotically given by (2.1.4) and is called the symbol of A.

Conversely, every operator of the form (2.1.5) with $\sigma_A \in S^m_{\rho,\delta}$ is in $L^m_{\rho,\delta}(\Omega)$ by our present definition. Hence the definitions used here are equivalent with those used in [2] when $\delta < \rho$, which we assume from now on. If we note that $\sigma_A \in S^{-\infty}$ if the kernel belongs to C^∞ and conversely, the preceding theorem shows that the map $A \to \sigma_A$ defined there together with the map $\sigma_A \to A$ given by (2.1.5) leads to an isomorphism

$$L^m_{\rho,\delta}/L^{-\infty}_{\rho,\delta} \to S^m_{\rho,\delta}/S^{-\infty}_{\rho,\delta}.$$

The formula for the symbol of the adjoint operator given in |2| is an immediate consequence of Theorem 2.1.1. Indeed, if A is properly supported and we define the adjoint by $\langle Au, v\rangle = \langle u, {}^tAv\rangle$, then we obtain from (2.1.5)

$$ {}^tAv(x) = (2\pi)^{-n}\int e^{-i\langle x-y,\xi\rangle}\, \sigma_A(y,\xi)\ v(y)\ dyd\xi $$

or

$$ {}^tAv(x) = (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}\, \sigma_A(y,-\xi)\ v(y)\ dyd\xi. $$

It follows that for the properly supported operator ${}^tA \in L^m_{\rho,\delta}$ we have

(2.1.6) $$ \sigma_{{}^tA}(x,\xi) \sim \sum (iD_\xi)^\alpha D_x^{\ \alpha}\, \sigma_A(x,-\xi)/\alpha!. $$

The formula extends immediately to the symbols for arbitrary $A \in L^m_{\ ,\delta}$.

Using the adjoint operator we can also get another useful representation for a properly supported pseudo-differential operator, already used by Kohn and Nirenberg [4]. In fact, if

$$ {}^tAv(y) = (2\pi)^{-n}\int e^{i\langle y,\xi\rangle}\ \sigma'(y,\xi)\ \hat{v}(\xi)\ d\xi, $$

then

$$ \langle Au, v\rangle = (2\pi)^{-n}\iint e^{i\langle y,\xi\rangle}\ \sigma'(y,\xi)\ \hat{v}(\xi)\ u(y)\ d\xi dy, $$

which means that Au is the Fourier transform of

$$ \xi \longrightarrow (2\pi)^{-n}\int e^{i\langle y,\xi\rangle}\ \sigma'(y,\xi)\ u(y)\ dy. $$

Writing $\tilde{\sigma}_A(y, \xi) = \sigma'(y, -\xi)$, we have then with the notation of oscillatory integrals

$$ Au(x) = (2\pi)^{-n}\iint e^{i\langle x-y,\xi\rangle}\, \tilde{\sigma}_A(y,\xi)\ u(y)\ dyd\xi, $$

or equivalently that

$$ \widehat{(Au)}(\xi) = \int e^{-i\langle y,\xi\rangle}\ \tilde{\sigma}_A(y, \xi)\ u(y)\ dy. $$

If A is of the form (2.1.1), then

(2.1.7) $$ \tilde{\sigma}_A(y, \xi) \sim \sum (-iD_\xi)^\alpha\ D_x^{\ \alpha}\ a(x,y,\xi)/\alpha!\,\big/_{x=y}. $$

Now let B be another properly supported pseudo-differential operator. Using (2.1.5) for B and (2.1.7) for A we then obtain

(2.1.8) $$BAu(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} \sigma_B(x,\xi)\, \tilde{\sigma}_A(y,\xi)\, u(y)\, dy d\xi,$$

which proves that BA is a pseudo-differential operator. For the symbol we obtain using (2.1.4)

$$\sigma_{BA}(x,\xi) \sim \sum (iD_\xi)^\alpha\, D_y{}^\alpha\, \sigma_B(x,\xi)\, \tilde{\sigma}_A(y,\xi)/\alpha!_{/y=x}.$$

In view of (2.1.7) we have

$$\tilde{\sigma}_A(y,\xi) \sim \sum (-iD_\xi)^\beta D_y{}^\beta \sigma_A(y,\xi)/\alpha!.$$

Hence

$$\sigma_{BA}(x,\xi) \sim \sum (iD_\xi)^\alpha D_y{}^\alpha \sigma_B(x,\xi)(-iD_\xi)^\beta D_y{}^\beta \sigma_A(y,\xi)/\alpha!\beta!_{/y=x} =$$
$$= \sum (iD_\xi)^\alpha \sigma_B(x,\xi)(-iD_\xi)^\beta D_x^{\alpha+\beta} \sigma_A(x,\xi)/\alpha!\beta!.$$

The right hand side can be simplified by means of the binomial theorem

$$\sum_{\alpha+\beta=\gamma} \eta^\alpha \theta^\beta/\alpha!\beta! = (\eta+\theta)^\gamma/\gamma!$$

if we note that in the preceding formula a factor iD_ξ to the left is equivalent to the sum of a factor iD_ξ acting only on the first factor and one acting only on the second factor. This gives the familiar result

(2.1.9) $$\sigma_{BA}(x,\xi) \sim \sum ((iD_\xi)^\gamma \sigma_B(x,\xi))\, D_x{}^\gamma \sigma_A(x,\xi)/\alpha!.$$

We shall now consider the effect of a change of variables. Let

$$\varkappa : \Omega \to \Omega_1$$

be a diffeomorphism between open sets in $\mathbb{R}^n$, let $A \in L^m_{\rho,\delta}(\Omega)$ and set $A_1 u = (A(u \circ \varkappa)) \circ \varkappa_1$, $u \in C_0^\infty(\Omega_1)$ where $\varkappa_1 : \Omega_1 \to \Omega$ is the inverse of $\varkappa$. This means, if A is of the form (2.1.1), that

$$(A_1 u)(x) = \int e^{i\langle \varkappa_1(x)-y,\, \xi\rangle} a(\varkappa_1(x),\, y,\, \xi)\, u(\varkappa(y))\, dy d\xi$$

or after a change of variables

$$(A_1u)(x) = \iint e^{i\langle \varkappa_1(x) - \varkappa_1(y), \xi\rangle} a(\varkappa_1(x), \varkappa_1(y), \xi)u(y)|d\varkappa_1(y)/dy| \, dy\, d\xi$$

where $d\varkappa_1(y)/dy = \det \varkappa_1'(y)$. This is again a Fourier integral operator but with a different phase function. That $A_1 \in L^m_{\rho,\delta}(\Omega_1)$ follows for suitable ρ and δ from

<u>Theorem 2.1.2.</u> Let φ be a phase function in $\Omega \times \Omega \times \mathbb{R}^n$ such that $\operatorname{grad}_\xi \varphi(x,y,\xi) = 0$ is equivalent to $x = y$ and $\operatorname{grad} \varphi(x,y,\xi)$ is a linear function of ξ when $x = y$. Then $L^m_{\rho,\delta}(\Omega, \Omega, \varphi) = L^m_{\rho,\delta}(\Omega)$ if $1-\rho \leqq \delta < \rho$.

We shall give a simple proof suggested by Kuranishi (see Friedrichs [1]), using the following lemma.

<u>Lemma 2.1.3.</u> Let φ be a phase function satisfying the hypotheses of Theorem 2.1.2. For some neighborhood ω of the diagonal in $\Omega \times \Omega$ one can then find a map $\Phi : \omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi(x,y,\xi)$ is positively homogeneous with respect to $\xi \in \mathbb{R}^n$ and C^∞ when $\xi \neq 0$,

$$\varphi(x,y,\Phi(x,y,\xi)) = \langle x-y, \xi\rangle. \tag{2.1.10}$$

If φ is linear with respect to ξ for all x, y, then Φ can also be chosen linear.

<u>Proof.</u> By hypothesis we have

$$\partial\varphi(x,y,\xi)/\partial x_j = -\partial\varphi(x,y,\xi)/\partial y_j = \sum a_{jk}(x)\xi_k \text{ when } x = y.$$

Since φ is a phase function, the linear forms on the right cannot have a common zero $\xi \neq 0$, so $\det a_{jk} \neq 0$. By Taylor's formula we have in a neighborhood of the diagonal

$$\varphi(x,y,\xi) - \sum a_{jk}(x)(x_j-y_j)\xi_k = \sum (x_j-y_j)R_j(x,y,\xi)$$

where $R_j \in C^\infty$ for $\xi \neq 0$ and is homogeneous of degree 1 with respect to ξ; moreover we have $R_j(x,x,\xi) = 0$ identically. Now the equation (2.1.10) is valid if

(2.1.11) $$\sum a_{jk}(x)\,\Phi_k + R_j(x,\ y,\ \Phi) = \xi_j,\ j = 1,\ \ldots,\ n.$$

This system of equations for Φ satisfies the hypotheses of the implicit function theorem when $x = y$ since the Jacobian of the left hand side with respect to Φ is $\det a_{jk}(x) \neq 0$. Hence (2.1.11) can be solved uniquely for $|\xi| = 1$ and (x,y) in a neighborhood ω of the diagonal. By homogeneity we can extend the solution to all ξ. If φ is linear with respect to ξ we can choose R_j linear, and the solution of the linear system of equations (2.1.11) for Φ is then linear with respect to ξ.

The lemma means that apart from the bundle maps discussed in section 1.1 there is only one function φ satisfying the hypotheses of Theorem 2.1.2. Note also that if we consider the global situation where Ω is a manifold and φ is defined in a real vector bundle V over a neighborhood of the diagonal in $\Omega \times \Omega$, then the stated conditions give rise to an identification of V with the cotangent bundle $T^*(\Omega)$ lifted to $\Omega \times \Omega$ by one of the projections (in a neighborhood of the diagonal). Thus we see that the pair V, φ is unique up to isomorphism. It is clear that this observation can be used to give a direct global definition of pseudo-differential operators on a manifold though we shall not pursue this aspect here.

Proof of Theorem 2.1.2. In view of Theorem 1.3.4 it is no restriction to consider only operators $A \in L^m_{\rho,\delta}(\Omega, \Omega, \varphi)$ of the form (1.3.1) where $a(x,y,\xi)$ vanishes for (x,y) outside a closed subset of $\Omega \times \Omega$ contained in the set ω of Lemma 2.1.3. Now a change of variables gives

$$Au(x) = \iint e^{i\langle x-y,\xi\rangle}\, a(x,y,\Phi(x,y,\xi))\,|d\Phi(x,y,\xi)/d\xi|\, u(y)\, dy d\xi.$$

In view of Proposition 1.1.3 this proves the theorem.

We shall now return to the phase function

$$\varphi(x,y,\xi) = \langle \varkappa_1(x) - \varkappa_1(y), \xi\rangle,$$

which occurred in the change of variables, in order to determine the transformation law for the symbol. The proof of Lemma 2.1.3 then reduces to writing $\varkappa_1(x) - \varkappa_1(y) = H(x,y)(x-y)$ where H is an $n\times n$ matrix, $H(x,x) = \varkappa_1'(x)$, so that $\varphi(x,y,\xi) = \langle x-y, {}^tH(x,y)\xi\rangle$. Then we can take $\Phi(x,y,\xi) = {}^tH(x,y)^{-1}\xi$. Note that ${}^tH(x,x)^{-1} = {}^t\varkappa'(x)$. Now we have

$$(A_1u)(x) = \iint e^{i\langle x-y,\xi\rangle}\, a(\varkappa_1(x), \varkappa_1(y), {}^tH(x,y)^{-1}\xi)\, D(x,y)u(y)\,dy\,d\xi$$

where $D(x,y) = |\det \varkappa_1'(x)|/|\det H(x,y)|$, thus $D(x,x) = 1$. (That $D(x,x)=1$ means precisely that $dx\,d\xi$ is an invariant measure on the cotangent space which is of course very well known.) If we take $a = \sigma_A(x,\xi)$ it follows that

(2.1.12) $$\sigma_{A_1}(x,\xi) \sim \sum_\alpha (iD_\xi)^\alpha D_y^\alpha\, \sigma_A(\varkappa_1(x), {}^tH(x,y)^{-1}\xi)D(x,y)/\alpha!\,\Big/_{x=y}.$$

If we set

$$\sigma_A^{(\beta)}(x,\xi) = (iD_\xi)^\beta \sigma_A(x,\xi),$$

the general term in (2.1.12) will be a linear combination of terms of the form $c(x,y)\,\xi^\gamma \sigma_A^{(\beta)}(\varkappa_1(x), {}^tH(x,y)^{-1}\xi)$ with

(2.1.13) $$|\gamma| + |\alpha| \leqq |\beta| \leqq 2|\alpha|.$$

The second inequality is obvious. To prove the first we note that application of D_y does not change $|\beta| - |\gamma|$ while application of D_ξ increases this difference by 1. From (2.1.12) we obtain in view of (2.1.13)

(2.1.14) $$\sigma_{A_1}(\varkappa(x), \xi) \sim \sum \sigma_A^{(\beta)}(x, {}^t\varkappa'(x)\xi)\, \varphi_\beta(x,\xi)/\alpha!$$

where φ_β is a polynomial in ξ of degree $\leqq |\alpha|/2$, (2.1.15) $$\varphi_0(x,\xi) = 1.$$

Since φ_β does not depend on A, we can determine φ_β by choosing A to be a differential operator. Then we have

$$\sigma_{A_1}(y,\xi)_{y=\varkappa(x)} = e^{-i\langle y,\xi\rangle} A_1 e^{i\langle y,\xi\rangle}/_{y=\varkappa(x)} =$$
$$= e^{-i\langle \varkappa(z),\xi\rangle} \sigma_A(x,D) e^{i\langle \varkappa(z),\xi\rangle}/_{z=x}.$$

Here we introduce the Taylor expansion

$$\varkappa(z) = \varkappa(x) + \varkappa'(x)(z-x) + \varkappa''_x(z)$$

where $\varkappa''_x(z)$ vanishes to the second order when $z = x$. We have

$$e^{i\langle \varkappa(z),\xi\rangle} = e^{i\langle \varkappa(x),\xi\rangle} e^{-i\langle x,{}^t\varkappa'(x)\xi\rangle} e^{i\langle z,{}^t\varkappa'(x)\xi\rangle} e^{i\langle \varkappa''_x(z),\xi\rangle}.$$

In view of Leibniz' formula we obtain (2.1.14) with

(2.1.16) $$\varphi_\beta(x,\xi) = D_z^\beta e^{i\langle \varkappa''_x(z),\xi\rangle}/_{z=x},$$

and with no other polynomials φ_β. Note in particular the first few polynomials

(2.1.17) $$\varphi_\beta(x,\xi) = 0,\ |\beta| = 1;\ \varphi_\beta(x,\xi) = D_x^\beta i\langle \varkappa(x),\xi\rangle,\ |\beta| = 2.$$

We have now developed all the calculus of pseudo-differential operators as given in Hörmander [2]. The derivation given here is somewhat simplified by the use of amplitude functions depending on (x,y,ξ) and not only on (x,ξ), and also by the straightforward substitution used to prove Theorem 2.1.2. However, in Chapter III we shall also need certain precise estimates related to Theorem 2.6 in [2], on which the calculus was based there, so we shall end this section by proving these also.

<u>Proposition 2.1.4.</u> Let A be a properly supported operator $\in L^m_{\rho,\delta}(\Omega)$ $(\delta < \rho)$ and let N be an integer $\geq (m^+ + n)/\rho$ where $m^+ = \max(m, 0)$. For every compact set $K \subset \Omega$ one can then find a constant C such that for $x \in K$

(2.1.18) $$\sup_\xi (1+|\xi|)^{\rho N - m^+ - n} |e^{-i\langle x,\xi\rangle} A(ue^{i\langle x,\xi\rangle}) - \sum_{|\alpha|<N} \sigma_A^{(\alpha)}(x,\xi) D^\alpha u(x)/\alpha!|$$
$$\leq C \sum_{|\alpha|\leq N} \sup |D^\alpha u|,\ u \in C^\infty(\Omega).$$

Proof. Since A is properly supported, the left hand side of (2.1.18) does not change if u is modified outside a suitable compact subset of Ω, so we may assume that $\operatorname{supp} u \subset K'$ where K' is a compact set $\subset \Omega$. We write A in the form (2.1.1) with a proper symbol a. Then

$$(2.1.19) \qquad e^{-i\langle x,\xi\rangle} A(ue^{i\langle x,\xi\rangle}) = \iint e^{i\langle x-y,\eta-\xi\rangle} a(x,y,\eta)\, u(y)\, dy d\eta.$$

We shall estimate separately the contributions when $|\eta|$ is small or large compared with $|\xi|$; we may assume that $|\xi|$ is large. First we prove that for $|\theta| > 1$

$$(2.1.20) \qquad \left|\int e^{-i\langle y,\theta\rangle} a(x,y,\eta) u(y) dy\right| \leqq C|\theta|^{m^+-N} |u|_N, \; x \in K, \; |\eta| < |\theta|,$$

where $|u|_N$ denotes the C^N norm of u on the right hand side of (2.1.18). This is obvious when $N = 0$ so we may assume in the proof that (2.1.20) is already known for smaller values of N and all symbols a with given support. Choose j so that $|\theta| = |\theta_j|$. (We may use the maximum norm in $\mathbb{R}^n$.) Since

$$\theta_j \int e^{-i\langle y,\theta\rangle} a(x,y,\eta) u(y)\, dy = \int e^{-i\langle y,\theta\rangle} (D_{y_j} a(x,y,\eta)) u(y) dy + \\ + \int e^{-i\langle y,\theta\rangle}\, a(x,y,\eta)\, D_{y_j} u(y)\, dy,$$

we obtain by the inductive hypothesis

$$|\theta|^N \left|\int e^{-i\langle y,\theta\rangle} a(x,y,\eta)\, u(y)\, dy\right| \leqq C|\theta|^{m^+} |u|_N + \\ + |\theta|^{N-1} \left|\int e^{-i\langle y,\theta\rangle} (D_{y_j} a(x,y,\eta)) u(y) dy\right|.$$

Here a is of degree $\leqq m+\delta \leqq m^+ + \delta$. Iterating this argument k times we obtain

$$|\theta|^N \left|\int e^{-i\langle y,\theta\rangle} a(x,y,\eta) u(y) dy\right| \leqq C|\theta|^{m^+} |u|_N + \\ + |\theta|^{N-k} \left|\int e^{-i\langle y,\theta\rangle} (D_{y_j}^{\,k} a(x,y,\eta)) u(y)\, dy\right|.$$

The last term can be estimated by $C|\theta|^{N-k+m^+ + k\delta} |u|_N$, and if $k(1-\delta) > N$ we obtain (2.1.20).

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\chi(\xi) = 1$ for $|\xi| < 1/4$, $\chi(\xi) = 0$ for $|\xi| > 1/2$. We split (2.1.19) into two parts by introducing a factor $\chi(\eta/|\xi|)$ or

$\chi_1(\eta/|\xi|)$ where $\chi_1 = 1 - \chi$. In view of (2.1.20) we have

(2.1.21) $$\left|\int e^{i\langle x-y,\eta-\xi\rangle}\chi(\eta/|\xi|)a(x,y,\eta)\, u(y)\, dy d\eta\right| \leq C|\xi|^{m^+ +n-N}|u|_N,$$

for if $\Theta = \eta-\xi$ we have $|\eta| \leq |\xi|/2 \leq |\Theta|$ in the support of the integrand. To study the other part of (2.1.19) we introduce Taylor's formula as im the proof of Theorem 2.1.1:

$$u(y) = u(x+y-x) = \sum_{|\alpha|<N} (i(y-x))^\alpha D^\alpha u(x)/\alpha! + \sum_{|\alpha|=N}(i(y-x))^\alpha R_\alpha(x,y)$$

where $R_\alpha(x,y)$ are C^∞ functions which can be bounded in absolute value by $|u|_N$. By an integration by parts we now obtain

$$\iint e^{i\langle x-y,\eta-\xi\rangle}\chi_1(\eta/|\xi|)a(x,y,\eta)u(y)dyd\eta =$$

$$= \sum_{|\alpha|<N} D^\alpha u(x)/\alpha! \iint e^{i\langle x-y,\eta-\xi\rangle}(iD_\eta)^\alpha(\chi_1(\eta/|\xi|)a(x,y,\eta))dyd\eta +$$

$$+ \sum_{|\alpha|=N} e^{i\langle x-y,\eta-\xi\rangle}(iD_\eta{}^\alpha)(\chi_1(\eta/|\xi|)a(x,y,\eta))R_\alpha(x,y)\, dyd\eta.$$

The integrals in the last sum can be estimated by $C|\xi|^{n+m-\rho N}$ if we recall that the functions $\chi_1(\eta/t)$, $t > 1$, form a bounded set in $S^0_{1,0}$. From the proof of Theorem 2.1.1 we find that

$$\iint e^{i\langle x-y,\eta-\xi\rangle}(iD_\eta)^\alpha \chi_1(\eta/|\xi|)a(x,y,\eta)dyd\eta = \sigma_A^{(\alpha)}(x,\xi) -$$

$$- \iint e^{-i\langle x-y,\eta-\xi\rangle}\, (iD_\eta)^\alpha\chi(\eta/|\xi|)a(x,y,\eta)\, dyd\eta.$$

In view of (2.1.20) with $u = 1$ the last integral can be estimated by any power of $|\xi|^{-1}$. This completes the proof.

Proposition 2.1.4 is analogous to Leibniz' rule. In fact, we can write (2.1.18) in the form

$$\left|A(ue^{i\langle x,\xi\rangle}) - \sum_{|\alpha|<N} e^{i\langle x,\xi\rangle}\sigma_A^{(\alpha)}(x,\xi)\, D^\alpha u(x)/\alpha!\right| \leq C(1+|\xi|)^{-\nu}\, |u|_N .$$

Here $u \in C^\infty(\Omega)$ and $x \in K$, and we have used the notation $\nu = \rho N - m^+ - n > 0$. If $v \in C_0^\infty$ we obtain after multiplication by $\hat{v}(\xi)$ and integration, also in K,

(2.1.22) $$\left|A(vu) - \sum_{|\alpha|<N} (A^{(\alpha)}v)\, D^\alpha u/\alpha!\right| \leq C|u|_N|v|_{-\nu}$$

where

$$|v|_{-\nu} = \int |\hat{v}(\xi)|(1+|\xi|)^{-\nu} d\xi .$$

Thus

$$|v|_{-\nu} = \sup |\int \hat{v}(\xi)V(\xi)d\xi| = \sup |\int v(x)\hat{V}(x)\, dx|$$

where the supremum is taken over all V with $\int |V(\xi)|(1+|\xi|)^{\nu} d\xi \leq 1$. If k is a non-negative integer with $k+n < \nu$, it follows that

$$|v|_{-\nu} \leq C \sup_{|W|_k<1} |\int v(x)W(x)\, dx| .$$

In particular we can take for v a rapidly oscillating function of a more general type than the one which occurs in (2.1.18). Thus let $\chi \in C_0^\infty(\Omega)$ and $\psi \in C^\infty(\Omega, \mathbb{R})$, grad $\psi \neq 0$ in supp χ. If $v = \chi e^{i\lambda\psi}$, we obtain

$$(2.1.23) \qquad |\int v(\xi)W(\xi)d\xi| \leq C\lambda^{-k}|W|_k .$$

In fact, this is obvious if ψ is one of the coordinates. To prove (2.1.23) in general it suffices therefore to use a partition of unity. By combining (2.1.22) and (2.1.23) we now obtain since the hypothesis that has compact support is irrelevant

<u>Proposition 2.1.5.</u> Let A be a properly supported operator $\in L^m_{\rho,\delta}(\Omega)$ $(\delta < \rho)$ and let N be an integer $\geq (m^+ +1+2n)/\rho$ where $m^+ = \max(m, 0)$. Let $\psi \in C^\infty(\Omega, \mathbb{R})$ and $\chi \in C^\infty(\Omega, \mathbb{C})$ be fixed functions with grad $\psi \neq 0$ in supp χ. Then we have

$$(2.1.24) \qquad |A(e^{i\lambda\psi}\chi u) - \sum_{|\alpha|<N} A^{(\alpha)}(e^{i\lambda\psi}\chi)\, D^\alpha u/\alpha!| \leq$$
$$\leq C|u|_N \lambda^{m^+ +1+2n-\rho N}, \quad u \in C^\infty(\Omega),\ x \in K,\ \lambda \geq 1,$$

if K is a compact subset of Ω.

For later reference we point out that C can be taken independent of ψ and χ if these functions vary in compact subsets of C^∞ and $|\text{grad}\, \psi(x)| \geq c(x)$ for $x \in$ supp χ where c is a positive continuous function. The operator $A^{(\alpha)}$ corresponds to $\sigma_A^{(\alpha)}$ so its kernel is equal to that of A multiplied by $(i(y-x))^\alpha$.

When applying Proposition 2.1.5 we shall need an asymptotic expansion of $e^{-i\lambda\psi}A^{(\alpha)}(e^{i\lambda\psi}\ \ )$ valid in the compact set K. This we shall now derive from the formulas for changing variables. To simplify notations we take $\alpha=0$.

It is clear that $e^{-i\lambda\psi}A(e^{i\lambda\psi}(\chi_1-\chi_2)) = O(\lambda^{-N})$ on K for every N if $\chi_1 = \chi_2 = 1$ in a neighborhood of K and ψ has no critical point in supp $(\chi_1-\chi_2)$; this follows from the fact that the kernel of A is a C^∞ function outside the diagonal. To determine the asymptotic behavior of $e^{-i\lambda\psi}A(e^{i\lambda\psi}\chi)$ near a point x_0 in K we can therefore take χ with support in a small neighborhood of x_0 and equal to 1 in another neighborhood. If the support of χ is small enough and $\xi_0 = \operatorname{grad}\psi(x_0)$, we can find a diffeomorphism $\varkappa : \Omega\to\Omega$ such that $\varkappa(x_0) = x_0$ and $\psi(x) - \psi(x_0) = \langle\varkappa(x),\xi_0\rangle - \langle\varkappa(x_0),\xi_0\rangle$ in supp χ. For example, if $\varphi\in C_0^\infty(\mathbb{R}^n)$ is 1 near 0, we can set for small $\varepsilon > 0$

$$\varkappa(x) = x + \varphi((x-x_0)/\varepsilon)(\psi(x)-\psi(x_0)-\langle x-x_0,\xi_0\rangle)\xi_0|\xi_0|^{-2}.$$

In fact, we have $\varkappa(x) = x$ when $\varphi((x-x_0)/\varepsilon) = 0$, and $\langle\varkappa(x),\xi_0\rangle = \psi(x)-\psi(x_0) + \langle x_0,\xi_0\rangle$ when $\varphi((x-x_0)/\varepsilon) = 1$. Since $\psi(x)-\psi(x_0)-\langle x-x_0,\xi_0\rangle$ vanishes to the second order at x_0, we have $\partial\varkappa_j/\partial x_k - \delta_{jk} = O(\varepsilon)$ so the implicit function theorem gives that $\varkappa$ is a diffeomorphism also in a neighborhood of x_0 for small ε. Now we have in supp χ

$$e^{-i\lambda\psi}A(e^{i\lambda\psi}\chi) = e^{-i\langle\varkappa(x),\lambda\xi_0\rangle}A(e^{i\langle\varkappa(x),\lambda\xi_0\rangle}\chi) = e^{-i\langle y,\lambda\xi_0\rangle}A_\varkappa(e^{i\langle y,\lambda\xi_0\rangle}\chi_1(y))_{/y=\varkappa(x)},$$

where $\chi_1 = \chi\circ\varkappa^{-1} = 1$ in a neighborhood of $\varkappa(x_0) = x_0$. In view of (2.1.18) with $u = \chi_1$ and (2.1.14), (2.1.16), we obtain near x_0

$$e^{-i\lambda\psi}A(e^{i\lambda\psi}\chi) - \sum_{|\beta|<N}\sigma_A^{(\beta)}(x,{}^t\varkappa'(x)\lambda\xi_0)D_z^\beta e^{i\langle\varkappa''_x(z),\lambda\xi_0\rangle}/\beta!_{/z=x} + O(\lambda^{m+(1/2-\rho)N}).$$

Now the fact that $\psi(x)-\psi(x_0) = \langle \varkappa(x)-\varkappa(x_0), \xi_0\rangle$ implies that $\langle y, \psi'(x)\rangle = \langle \varkappa'(x)y, \xi_0\rangle = \langle y, {}^t\varkappa'(x)\xi_0\rangle$ so that ${}^t\varkappa'(x)\xi_0 = \psi'(x)$. Similarly,

$$\langle \varkappa_x''(z), \xi_0\rangle = \psi(z) - \psi(x) - \langle z-x, \psi'(x)\rangle = \psi_x''(z)$$

is the non-linear part of ψ at x. Hence we have in a neighborhood of x_0

$$e^{-i\lambda\psi}A(e^{i\lambda\psi}\chi) - \sum_{|\beta|<N} \sigma_A^{(\beta)}(x,\lambda\psi'(x))\, D_z^\beta e^{i\lambda\psi_x''(z)}/\beta!\,\big|_{z=x} =$$
$$= O(\lambda^{m+(1/2-\rho)N}),\ \lambda\to\infty.$$

Summing up, we have proved

<u>Proposition 2.1.6.</u> Let A be a properly supported operator $\in L^m_{\rho,\delta}(\Omega)$, $1-\rho \leqq \delta < \rho$. If $\psi\in C^\infty(\Omega,\mathbb{R})$, $\chi\in C^\infty(\Omega,\mathbb{C})$, $\operatorname{grad}\psi\neq 0$ in $\operatorname{supp}\chi$ and $\chi = 1$ in a neighborhood of a compact set $K\subset\Omega$, then we have on K

$$(2.1.25)\qquad e^{-i\lambda\psi}A(e^{i\lambda\psi}\chi) - \sum_{|\beta|<N}\sigma_A^{(\beta)}(x,\lambda\psi'(x))D_z^\beta e^{i\lambda\psi_x''(z)}/\beta!\,\big|_{z=x}$$
$$= O(\lambda^{m+(1/2-\rho)N}).$$

<u>Remark.</u> It follows easily from this that

$$e^{-i\lambda\psi}A(e^{i\lambda\psi}\chi) \sim \sum \sigma_A^{(\beta)}(x,\lambda\psi'(x))D_z^\beta\, e^{i\lambda\psi_x''(z)}/\beta!\,\big|_{z=x}$$

in an open set where $\chi = 1$. However, we do not carry out the proof since only (2.1.25) is required in the applications. On the other hand, it is then important to know that the proof shows that (2.1.25) is valid uniformly with respect to ψ and χ if these functions vary in compact subsets of C^∞ while satisfying the hypotheses.

As a preparation for section 2.2 we shall finally give for pseudo-differential operators a more precise version of Theorem 1.3.4.

<u>Proposition 2.1.7.</u> Let $A\in L^m_{\rho,\delta}(\Omega)$, $0\leqq\delta<\rho\leqq 1$, and assume that A is defined by (2.1.1) where $a\in S^m_{\rho,\delta}(\Omega\times\Omega, \mathbb{R}^n)$ and $a(x,y,\xi) = 0$ when $|x-y||\xi|^\varepsilon \leqq c(x,y)$ where c is a positive continuous function on $\Omega\times\Omega$. If

$\varepsilon < \rho$ it follows that A has a C^∞ kernel.

Proof. With $a_j(x,y,\xi) = (x_j-y_j)a(x,y,\xi)/|x-y|^2$, where $|w|^2 = \sum |w_j|^2$, we have $a(x,y,\xi) = \sum (x_j-y_j)a_j(x,y,\xi)$. Assuming as we may that $\varepsilon \geqq \delta$, we have $a_j \in S^{m+\varepsilon}_{\rho,\delta}(\Omega\times\Omega, \mathbb{R}^n)$. In fact, differentiation of a_j with respect to y either leads to differentiation of the factor a or the factor $(x_j-y_j)|x-y|^{-2}$, giving rise to a factor which is homogeneous in x-y of one lower degree of homogeneity. Since $|x-y| \geqq c(x,y)|\xi|^{-\varepsilon}$, the assertion follows. (We may assume a = 0 when $|\xi| < 1$.) Now we know that a defines the same operator as $i \sum \partial a_j/\partial\xi_j \in S^{m+\varepsilon-\rho}_{\rho,\varepsilon}(\Omega\times\Omega, \mathbb{R}^n)$. Iteration of the argument shows that A can be defined with an amplitude function in $S^{m+k(\varepsilon-\rho)}_{\rho,\varepsilon}$ for every k, and since $\varepsilon < \rho$ it follows that the kernel of A is in C^∞.

2.2. The continuity of pseudo-differential operators. In this section we shall give a somewhat different proof of Theorem 3.1 in [2].

Theorem 2.2.1. Let $A \in L^0_{\rho,\delta}(\Omega)$, $0 \leqq \delta < \rho \leqq 1$. If K is a compact set in Ω there is a constant C such that

(2.2.1) $$\int_K |Au|^2 dx \leqq C \int |u|^2 dx, \quad u \in C_0^\infty(K).$$

Proof. If $\chi \in C_0^\infty(\Omega)$ and $\chi = 1$ on K, we can replace A by $\chi A \chi$ so we may assume from the start that A is given by (2.1.1) with a vanishing for (x,y) outside some compact set $\subset \Omega\times\Omega$. We have for $u,v \in C_0^\infty$

$$\langle Au, v\rangle = \iiint e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\, u(y)\, v(x)\, dxdyd\xi.$$

To estimate A we shall localize. The partition of unity to be used is somewha t cruder than that used in [2], and following Friedrichs [1] we choose it depending on a continuous parameter which is somewhat more convenient notationally. Thus let $\psi_1 \in C_0^\infty(\mathbb{R}^n)$ be a function of $|\xi|$ with support where $1/2 < |\xi| < 1$ and such that $\int_0^\infty \psi_1(\xi/t)^2 dt/t = 1$ for $\xi \neq 0$, thus $\int_1^\infty \psi_1(\xi/t)^2 dt/t = 1$ for $|\xi| \geq 1$. Choose $\psi_2 \in C_0^\infty$ with support in the unit ball and $\int \psi_2 dx = 1$. Then we have for $|\xi| > 1$

$$1 = \iiint_{t>1} \psi_1(\xi/t)^2 \psi_2((x-x')t^\varepsilon)\, \psi_2((y-y')t^\varepsilon)\, t^{2n\varepsilon-1} dt dx' dy'.$$

Here we choose ε so that $\delta < \varepsilon < \rho$. The integral is still equal to 1 when $|x-y||\xi|^\varepsilon \leq 1$ if we restrict the integration to the set where $|x'-y'|t^\varepsilon \leq 4$, for in the support of the integrand we have

$$|x-x'| < t^{-\varepsilon}, \quad |y-y'| < t^{-\varepsilon}, \quad t/2 < |\xi| < t,$$

so if $|x-y||\xi|^\varepsilon < 1$ it follows that $|x'-y'| \leq 2t^{-\varepsilon} + |\xi|^{-\varepsilon} \leq 4t^{-\varepsilon}$. For arbitrary x, y, ξ we obtain

$$\iiint_{|x'-y'|t^\varepsilon<4} \psi_1(\xi/t)^2\, \psi_2((x-x')t^\varepsilon)\, \psi_2((y-y')t^\varepsilon)\, t^{2n\varepsilon-1} dt dx' dy' = \\ = \chi((x-y)|\xi|^\varepsilon)$$

where

$$\chi(x) = \iiint_{|x'-y'|<4} \psi_1(1/t)^2\, \psi_2(x-x')\, \psi_2(-y')\, t^{-1} dt dx' dy'$$

is a C^∞ function with compact support which is 1 for $|x| < 1$. Assuming as we may that $a(x,y,\xi) = 0$ for $|\xi| < 1$, we can now write $A = A_1 + A_2$ where

$$\langle A_1 u, v\rangle = \iiiint\!\!\iint_{|x'-y'|t^\varepsilon<4,\, t>1} e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\, \psi_1(\xi/t)^2\, \psi_2((x-x')t^\varepsilon) v(x) \times \\ \times \psi_2((y-y')t^\varepsilon) u(y) dx' dy' t^{2n\varepsilon-1} dt dx dy d\xi$$

and A_2 is defined by the amplitude function $a_2(x,y,\xi) = (1-\chi((x-y)|\xi|^\varepsilon))$ $a(x,y,\xi)$. By example 3 on page 1.1 we have $a_2 \in S^0_{\rho,\varepsilon}(\Omega\times\Omega, \mathbb{R}^n)$, and since $a_2 = 0$ when $|x-y||\xi|^\varepsilon \leq 1$ it follows from Proposition 2.1.7 that A_2 has a C^∞ kernel. Thus it only remains to estimate A_1. To do so we pass to amplitude functions independent of x, y by means of a Taylor expansion

$$|a(x,y,\xi) - \sum_{|\alpha+\beta|<N} (x-x')^\alpha (y-y')^\beta a_{\alpha\beta}(x',y',\xi)/\alpha!\beta!| \leqq$$
$$\leqq C|\xi|^{N\delta}(|x-x'|+|y-y'|)^N$$

where $a_{\alpha\beta} = (iD_{x'})^\alpha (iD_{y'})^\beta a(x',y',\xi)$. Since $|x-x'|+|y-y'| = O(t^{-\varepsilon})$ in the integral defining A_1, the contribution to $\langle A_1 u, v\rangle$ from the error term can be estimated by

$$C\int |u|dx \int |v|\, dy \iint_{|\xi|>1,t>1} |\xi|^{N\delta}\, t^{-N\varepsilon}\psi_1(\xi/t)^2 dt/t\, d\xi \leqq$$
$$\leqq C'\int |u|\, dx \int |v|\, dy \int_{|\xi|>1} |\xi|^{N(\delta-\varepsilon)} d\xi \leqq C''||u||_{L^2}||v||_{L^2}$$

when the supports of u and of v lie in a compact set and $N(\varepsilon-\delta) > n$. It remains to estimate for fixed α and

$$(2.2.2)\quad \langle A_{\alpha\beta}u, v\rangle = \iiiint\!\!\int_{|x'-y'|t^\varepsilon<4,t>1} e^{i\langle x-y,\xi\rangle} a_{\alpha\beta}(x',y',\xi)\,\psi_1(\xi/t)^2$$
$$\psi_2^\alpha((x-x')t^\varepsilon)v(x)\,\psi_2^\beta((y-y')t^\varepsilon)u(y)dx'dy't^{(2n-|\alpha+\beta|)\varepsilon-1}dtdxdyd\xi$$

where $\psi_2^\alpha(x) = x^\alpha\psi_2(x)$. Note that there is no contribution for x' or y' outside a certain compact set K'. Since $|a_{\alpha\beta}(x',y',\xi)| \leqq Ct^{\delta|\alpha+\beta|}$ in the support of $\psi_1(\xi/t)$, we obtain by Parseval's formula with L^2 norms

$$|\langle A_{\alpha\beta}u, v\rangle| \leqq C \iiint_{|x'-y'|t^\varepsilon<4,t>1} (||\psi_1(D/t)\psi_2^\alpha((x-x')t^\varepsilon)v(x)||^2 +$$
$$+ ||\psi_1(D/t)\,\psi_2^\beta((y-y')t^\varepsilon)u(y)||^2)\, t^{2n\varepsilon-1}dx'dy'dt.$$

In fact,

$$|\iiint e^{i\langle x-y,\xi\rangle}a(\xi)b(\xi)^2U(x)V(y)dxdyd\xi| = |\int \hat{U}(-\xi)\hat{V}(\xi)a(\xi)b(\xi)^2d\xi| \leqq$$
$$\leqq \sup_{\text{supp } b}|a|\; ||b(-\xi)\hat{U}(\xi)||\,||b(\xi)\hat{V}(\xi)|| = (2\pi)^n \sup_{\text{supp } b}|a|\; ||b(-D)U||\,||b(D)V||.$$

Hence

$$(2.2.3)\quad |\langle A_{\alpha\beta}u, v\rangle| \leqq C\Big\{\iiint_{t>1} |\psi_1(D/t)\psi_2^\alpha((x-x')t^\varepsilon)v(x)|^2 t^{n\varepsilon-1}dxdx'dt +$$
$$+ \iiint_{t>1} |\psi_1(D/t)\,\psi_2((y-y')t^\varepsilon)u(y)|^2 t^{n\varepsilon-1}dydy'dt\Big\}.$$

To show that $|\langle A_{\alpha\beta}u, v\rangle|$ is bounded when $||u|| \leqq 1$ and $||v|| \leqq 1$, it remains to establish a bound for the integrals on the right hand side. This we shall do using a variant of Friedrichs' lemma which allows us to let the two factors $\psi_1(D/t)$ and $\psi_2^\alpha((x-x')t^\varepsilon)$ change places.

Lemma 2.2.2. If $\varphi, \psi \in C_0^\infty$, we have if N is a positive integer

$$(2.2.4)\quad \int |\psi(D)\varphi(x)v(x) - \sum_{|\alpha|<N} (D^\alpha\varphi(x))\psi^{(\alpha)}(D)v(x)/\alpha!|^2 dx \leqq$$
$$\leqq C\Big(\int |\xi|^N |\hat{\varphi}(\xi)|d\xi \sum_{|\alpha|=N} \sup |D^\alpha\psi|\Big)^2 \int |v|^2 dx,\ v \in C_0^\infty .$$

Proof. The Fourier transform of $\psi(D)\varphi(x)v(x)$ is

$$(2\pi)^{-n}\int \psi(\xi)\hat{\varphi}(\xi-\eta)\hat{v}(\eta)d\eta = \sum_{|\alpha|<N}(2\pi)^{-n}\int \psi^{(\alpha)}(\eta)(\xi-\eta)^\alpha/\alpha!\,\hat{\varphi}(\xi-\eta)\hat{v}(\eta)d\eta +$$
$$+ (2\pi)^{-n}\int \rho(\xi,\eta)\hat{\varphi}(\xi-\eta)\hat{v}(\eta)\,d\eta$$

where $|\rho(\xi,\eta)| \leqq C|\xi-\eta|^N \sum_{|\alpha|=N} \sup |D^\alpha\psi|$. Hence the Fourier transform of $\psi(D)\varphi(x)v(x) - \sum_{|\alpha|<N}(D^\alpha\varphi(x))\psi^{(\alpha)}(D)v(x)/\alpha!$ is bounded by $C\sum_{|\alpha|=N} \sup |D^\alpha\psi| \int |\xi-\eta|^N |\hat{\varphi}(\xi-\eta)||\hat{v}(\eta)|\,d\eta$, so the L^2 norm can be estimated by

$$C \sup_{|\alpha|=N} |D^\alpha\psi| \int |\xi|^N |\hat{\varphi}(\xi)|d\xi\ ||v||.$$

If we replace $\psi(D)$ by $\psi(D/t)$ and $\varphi(x)$ by $\varphi((x-x')t^\varepsilon)$ in (2.2.4), we obtain

$$(2.2.5)\ \int dx\,|\psi(D/t)\varphi((x-x')t^\varepsilon)v(x) - \sum_{|\alpha|<N} t^{|\alpha|(\varepsilon-1)}(D^\alpha\varphi)((x-x')t^\varepsilon)\psi^{(\alpha)}(D/t)v/\alpha!|^2$$
$$\leqq C\,t^{2N(\varepsilon-1)}\int |v|^2 dx.$$

If $n\varepsilon-1 + 2N(\varepsilon-1) < -1$, that is, if $N > n\varepsilon/2(1-\varepsilon)$, we conclude that

$$(2.2.6)\quad \iiint_{t>1\ x'\in K'} |\psi(D/t)\varphi((x-x')t^\varepsilon)v(x)|^2 dx dx' t^{n\varepsilon-1}dt \leqq C\int |v(x)|^2 dx$$

for such an estimate is valid for each term in (2.2.5) as well as the error term. Since we need only integrate in (2.2.3) for $x', y' \in K'$, it follows that $|\langle A_{\alpha\beta}u, v\rangle| \leqq C$ when $||u|| = ||v|| = 1$, which completes the proof of Theorem 2.1.1.

Theorem 2.2.2. Let $A \in L^0_{\rho,\delta}(\Omega)$, $0 \leqq \delta < \rho \leqq 1$. Then A is for every compact set $K \subset \Omega$ a compact operator from $L^2(K) = \{u \in L^2,\ \text{supp}\ u \subset K\}$ to $L^2(K)$ if and only if

(2.2.7) $$a(x,x,\xi) \to 0,\ \xi \to \infty,$$

uniformly for x in compact subsets K of Ω.

Proof. To verify that (2.2.7) is sufficient we need only inspect the proof of Theorem 2.2.1 noting that

$$|a_{00}(x',y',\xi)| \leqq |a_{00}(x',x',\xi)| + C|\xi|^{\delta}|x'-y'| \leqq$$
$$\leqq |a_{00}(x',x',\xi)| + C|\xi|^{\delta-\varepsilon} \to 0,\ \xi \to \infty,$$

in the support of the integrand in (2.2.2). For then the contributions to $\langle Au, v\rangle$ for large values of $|\xi|$ can be estimated by a small constant times $||u||\,||v||$ while the contributions from bounded values of $|\xi|$ always give a compact operator. The details are left for the reader to supply. To prove the necessity of (2.2.7) we assume that $x_\nu \to x_0 \in \Omega$, $\xi_\nu \to \infty$ and that $a(x_\nu, x_\nu, \xi_\nu) \to c$, and we have to prove that $c = 0$ if A is compact. Choosing ε with $\delta < \varepsilon < \rho$ and a non-trivial function $u \in C_0^\infty$ we set

$$u_\nu(x) = |\xi_\nu|^{n\varepsilon/2} u((x-x_\nu)|\xi_\nu|^{\varepsilon}) \exp i\langle x,\xi_\nu\rangle.$$

The norm of u_ν is then independent of ν and $u_\nu \to 0$ weakly when $\nu \to \infty$. If A is compact we must have $\langle Au_\nu, u_\nu\rangle \to 0,\ \nu \to \infty$. Now

$$\langle Au_\nu, \bar{u}_\nu\rangle = |\xi_\nu|^{n\varepsilon} \iiint e^{i\langle x-y,\xi-\xi_\nu\rangle} a(x,y,\xi) u((y-x_\nu)|\xi_\nu|^{\varepsilon}) \bar{u}((x-x_\nu)|\xi_\nu|^{\varepsilon})\, dx dy d\xi$$
$$= \iiint e^{i\langle x-y,\xi\rangle} a(x_\nu + x|\xi_\nu|^{-\varepsilon}, x_\nu + y|\xi_\nu|^{-\varepsilon}, \xi_\nu + |\xi_\nu|^{\varepsilon}\xi) u(y)\overline{u(x)}\, dx dy d\xi.$$

Here $a(x_\nu + x|\xi_\nu|^{-\varepsilon}, x_\nu + y|\xi_\nu|^{-\varepsilon}, \xi_\nu + |\xi_\nu|^{\varepsilon}\xi) \to c$ uniformly on every compact set when $\nu \to \infty$ since $\delta < \varepsilon < \rho$. Since $(2\pi)^n a(x,x,\xi) - \sigma_A(x,\xi) \to 0,\ \xi \to \infty$, we may assume in proving (2.2.7) that a is equal to σ_A near the diagonal. Then we can integrate with respect to y and obtain

$$\langle Au_\nu, \bar{u}_\nu\rangle = \iint e^{i\langle x,\xi\rangle} a(x + x|\xi_\nu|^{-\varepsilon}, \xi_\nu + |\xi_\nu|^{\varepsilon}\xi) \hat{u}(\xi)\overline{u(x)}\, dx d\xi.$$

The existence of an integrable majorant now allows us to let $\nu \to \infty$ under the sign of integration, which gives

$$\langle Au_\nu, \overline{u}_\nu \rangle \to c \int e^{i\langle x,\xi\rangle} u(\xi)\overline{u(x)}\, dx d\xi = (2\pi)^n c \int |u(x)|^2 dx.$$

Since the limit must be 0 we conclude that $c = 0$, which proves Theorem 2.2.2.

Finally we recall briefly how Theorem 2.2.1 can be improved to give precise information concerning the norm of A modulo compact operators. (Cf. Theorem 3.3.in |2|.) Thus let A be defined by (2.1.1) with $a \in S^0_{\rho,\delta}$ vanishing for (x,y) outside a compact set in $\Omega \times \Omega$, and let

$$M > \overline{\lim}\, (2\pi)^n |a(x,x,\xi)|.$$

Let $b(x,\xi)$ be the symbol of A^*A. Then

$$b(x,\xi) = |a(x,x,\xi)|^2 + O(|\xi|^{\delta-\rho}),$$

so $\overline{\lim}_{\xi\to\infty} b(x,\xi) < M^2$. With the square root which is defined outside the negative real axis and is positive on the positive real axis, we can now introduce $c(x,\xi) = \sqrt{M^2 - b(x,\xi)}$ for large $|\xi|$. It is easy to see that $c \in S^0_{\rho,\delta}$. Let Q be a properly supported pseudo-differential operator with symbol c. Then $Q^*Q + A^*A - M^2 = C$ is a compact operator. We have

$$||Au||^2 \leqq M^2||u||^2 + (Cu, u).$$

Now take $\chi \in C^\infty(\mathbb{R}^n)$ with $\chi = 0$ near 0 and $\chi = 1$ outside a compact set, and write $A_j = A\chi(D/j)$. Then $A - A_j$ has a C^∞ kernel. The norm of $\chi(D/j)C$ tends to 0 as $j \to \infty$ since C maps the unit ball into a compact set and $\chi(D/j)$ tends to 0 strongly. Hence $\overline{\lim}\, ||A_j|| \leqq M$. Modulo operators with C^∞ kernel the norm of A as an operator from $L^2(K)$ to $L^2(K)$ is therefore at most

$$\overline{\lim_{\substack{x\in K\\ \xi\to\infty}}}\, (2\pi)^n |a(x,x,\xi)|. \tag{2.2.8}$$

From the second part of the proof of Theorem 2.2.2 it follows that this result cannot be improved.

Chapter III

Properties of Fourier integral operators

3.1. Multiplication by pseudo-differential operators. In section 2.1 we have seen that properly supported pseudo-differential operators form an algebra. We shall now prove that the general class $L^m_{\rho,\delta}(\Omega_1, \Omega_2, \varphi)$ is a module over these algebras.

Theorem 3.1.1. Let $P \in L^m_{\rho,\delta}(\Omega_1, \Omega_2, \varphi)$ and let $A_j \in L^{m_j}_{\rho,\delta}(\Omega_j)$ be properly supported pseudo-differential operators. If $1-\rho \leqq \delta < \rho \leqq 1$ it follows that $A_1 P A_2 \in L^{m+m_1+m_2}_{\rho,\delta}(\Omega_1, \Omega_2, \varphi)$.

The proof will also give an expression for an amplitude function of $A_1 P A_2$.

Proof of Theorem 3.1.1. It suffices to prove that $A_1 P \in L^{m+m_1}_{\rho,\delta}(\Omega_1, \Omega_2, \varphi)$. For then we have ${}^tA_2\, {}^tP \in L^{m+m_2}_{\rho,\delta}(\Omega_2, \Omega_1, \varphi)$, thus $PA_2 \in L^{m+m_2}_{\rho,\delta}(\Omega_1, \Omega_2, \varphi)$ and $A_1 P A_2 \in L^{m+m_1+m_2}_{\rho,\delta}(\Omega_1, \Omega_2, \varphi)$.

We have by definition

$$Pu(x) = \iint e^{i\varphi(x,y,\xi)}\, p(x,y,\xi)\, u(y)\, dy d\xi, \quad u \in C_0^\infty(\Omega),$$

where $p \in S^m_{\rho,\delta}(\Omega_1 \times \Omega_2, \mathbb{R}^N)$ may be chosen so that $p = 0$ when $|\xi| < 1$. Here we want to apply the pseudo-differential operator A_1 under the sign of integration. This will be possible by the results of section 2.1 if we make sure that φ has no critical points as a function of x in the support of p.

We recall tha t by Definition 1.3.2 we have $|\mathrm{grad}_x \varphi| \neq 0$ when $\mathrm{grad}_\xi \varphi = 0$. Hence we can choose a positive continuous function c such

$|\mathrm{grad}_x \varphi| > c(x,y)\ |\xi|$ when $\mathrm{grad}_\xi \varphi = 0$. Then

$$V = \{(x,y,\xi);\ |\mathrm{grad}_x \varphi| > c(x,y)|\xi|\}$$

is a conical neighborhood of $\{(x,y,\xi);\ \mathrm{grad}_\xi \varphi = 0\}$. Choose a function $\chi \in C^\infty(\Omega \times \Omega \times \mathbb{R}^n)$ which is homogeneous of degree 1 with respect to the last variable so that $\chi = 0$ outside a closed cone contained in V and $\chi = 1$ in a neighborhood of $\{\varphi(x,y,\xi);\ \mathrm{grad}_\xi \varphi = 0\}$. Then $P = P_1 + P_2$ where with $p_1(x,y,\xi) = \chi(x,y,\xi)p(x,y,\xi)$, $p_2(x,y,\xi) = (1-\chi(x,y,\xi))p(x,y,\xi)$ we have

$$(3.1.1)\quad P_j u(x) = \iint e^{i\varphi(x,y,\xi)}\ p_j(x,y,\xi)\ u(y)\ dy d\xi.$$

The operator P_2 has a C^∞ kernel since (1.3.6) with a replaced by p_2 makes sense to define the kernel of P_2. The product $A_1 P_2$ is therefore a continuous map from $\mathcal{E}'$ to C^∞ so it has a C^∞ kernel.

We wish to apply A_1 under the oscillatory integral sign in (3.1.1) for $j = 1$. Before doing so we consider

$$(3.1.2)\quad q(x,y,\xi) = e^{-i\varphi(x,y,\xi)} A_1(e^{i\varphi(x,y,\xi)} p_1(x,y,\xi))$$

where A_1 operates with respect to x. The continuity properties of A_1 show that q is a C^∞ function with every derivative bounded by some power of $(1+|\xi|)$ when (x,y) belongs to a compact set. We can therefore apply Proposition 1.1.4 when studying q. Set $\xi = \lambda\eta$, where $|\eta|=1$, and apply (2.1.24) with $\psi = \varphi(x,y,\eta)$, $u = p_1(x,y,\xi) = p_1(x,y,\lambda\eta)$. For (x,y,η) in a compact subset of $\complement V$ we then obtain for every N

$$|q(x,y,\xi)| \leqq C\lambda^{m+m_1^+ +1+2n-\rho N+\delta N},\quad \lambda = |\xi|,$$

which proves that q is rapidly decreasing outside V. Now let $(x_0,y_0,\eta_0) \in \overline{V}$ and choose $\psi \in C_0^\infty(\Omega_1)$ with $\psi(x) = 1$ in a neighborhood U of x_0 so that $|\mathrm{grad}_x \varphi| > c|\xi|/2$ for $x \in \mathrm{supp}\,\psi$ and (y,η) in some neighborhood U' of

(y_0,η_0). For $(x,y,\eta)\in U\times U'$ it follows from Propositions 2.1.5 and 2.1.6 that

$$|q(x,y,\lambda\eta) - \sum_{|\alpha|<\nu} e^{-i\lambda\varphi(x,y,\eta)}A_1^{(\alpha)}(\psi e^{i\lambda\varphi(x,y,\eta)})D_x^{\alpha}p_1(x,y,\eta)/\alpha!| \leq C\lambda^{1+2n+m_1^{+}+m-(\rho-\delta)}$$

where

$$|e^{-i\lambda\varphi(x,y,\eta)}A_1^{(\alpha)}(\psi(x)e^{i\lambda\varphi(x,y,\eta)}) - \sum_{|\beta|<\nu}\sigma_{A_1}^{(\beta+\alpha)}(x,\varphi'_x)\times$$
$$\times D_z^{\beta}e^{i\varphi''_x(z,y,\eta)}/\beta!_{/z=x}| \leq C\lambda^{m_1+(1/2-\rho)|\beta|-\rho|\alpha|}.$$

It follows that for $(x,y,\xi)\in V$ and (x,y) in a compact subset of $\Omega\times\Omega$ the difference

$$q(x,y,\xi) - \sum\sigma_{A_1}^{(\alpha+\beta)}(x,\operatorname{grad}_x\varphi)D_x^{\alpha}p_1(x,y,\xi)\,D_z^{\beta}e^{i\varphi''_x(z,y,\xi)}/\alpha!\beta!_{/z=x}$$

can be estimated by any power of $(1+|\xi|)^{-1}$ if we sum for $|\alpha|<\nu$ and $|\beta|<\nu$, choosing ν sufficiently large. Now the terms in the sum belong to $S^{\mu}_{\rho,\delta}$ where $\mu = m_1-\rho|\alpha+\beta| + m + \delta|\alpha|+|\beta|/2$, thus $\to -\infty$ when $|\alpha+\beta|\to\infty$ for $\delta<\rho$ and $\rho>1/2$. By Proposition 1.1.4 we conclude that

(3.1.3) $$q(x,y,\xi) \sim \sum\sigma_{A_1}^{(\beta+\alpha)}(x,\operatorname{grad}_x\varphi)\,D_x^{\alpha}\,p_1(x,y,\xi)D_z^{\beta}e^{i\varphi''_x(z,y,\xi)}/\alpha!\beta!_{/z=x}.$$

It is clear that we can let A_1 operate under the integral sign in (3.1.1) after introducing a convergence factor $\chi(\varepsilon\xi)$ with compact support as in Proposition 1.1.5. Letting $\varepsilon\to 0$ we obtain

$$A_1P_1u = \iint e^{i\varphi(x,y,\xi)}q(x,y,\xi)\,u(y)\,dyd\xi.$$

Hence $A_1P_1\in L^{m+m_1}_{\rho,\delta}(\Omega_1,\Omega_2,\varphi)$, and an amplitude function representing A_1P_1 modulo an operator with C^{∞} kernel is given by (3.1.3). This formula can also be written as follows

(3.1.4) $$q(x,y,\xi)\sim\sum\sigma_{A_1}^{(\alpha)}(x,\operatorname{grad}_x\varphi)\,D_z^{\alpha}(p_1(z,y,\xi)e^{i\varphi''_x(z,y,\xi)})/\alpha!_{/z=x}.$$

We recall that

$$\varphi''_x(z,y,\xi) = \varphi(z,y,\xi) - \varphi(x,y,\xi) - \langle z-x,\ \mathrm{grad}_x\,\varphi(x,y,\xi)\rangle$$

is the error term of second order in Taylor's formula. (3.1.4) is of course obtained formally by an application of Leibniz' rule.

In the applications we need to know the first few terms explicitly when $\rho = 1$ and $\delta = 0$. Note that the order of the general term in (3.1.3) is then $\leqq m+m_1-|\alpha|-|\beta|+[|\beta|/2]$ where [ ] denotes integral part. This is $< m+m_1-1$ unless $\alpha = 0$ and $|\beta| \leqq 2$ or $|\alpha| = 1$ and $\beta = 0$. Moreover, the terms with $|\beta| = 1$ are 0. Thus we obtain in this case

$$q(x,y,\xi) - \sum_{|\alpha|\leqq 1} \sigma_{A_1}^{(\alpha)}(x,\ \mathrm{grad}_x\,\varphi)D_x^{\alpha}p_1(x,y,\xi) + \tag{3.1.5}$$
$$+ \sum_{|\beta|=2} \sigma_{A_1}^{(\beta)}(x,\ \mathrm{grad}_x\,\varphi)(D_x^{\beta}\, i\,\varphi(x,y,\xi)/\beta!)p_1(x,y,\xi) \in S^{m+m_1-2}_{1,0}.$$

3.2. L^2 estimates.

Chapter IV

Singular Fourier integral operators

4.1. Definitions and basic properties. The definition of the oscillatory integral

$$\iint e^{i\varphi(x,\xi)}a(x,\xi)u(x)dxd\xi$$

given in section 1 .2 did not fully use the hypothesis that a is a symbol in the sense of section 1.1. Indeed, we only used the fact that for some first order differential operator L with the properties stated in Lemma 1.2.1 $L^k(au)$ is an integrable function for sufficiently large values of k. This we shall exploit in what follows.

Our purpose is to define operators of the form

$$\iint e^{i\varphi(x,y,\xi)}a(x,y,\xi)/q(x,y,\xi)\ u(y)\ dyd\xi \tag{4.1.1}$$

where q is homogeneous with respect to ξ of degree m, say, and may have real simple zeros. The real zeros of q form an obvious obstacle to defining the integral. This we shall bypass by integrating over a suitable cycle in the complex domain instead. Assume for simplicity that q and a are analytic with respect to ξ in a neighborhood of the real domain and that there is a vector η such that $\langle \operatorname{grad}_\xi \varphi(x,y,\xi), \eta\rangle \neq 0$ when $q(x,y,\xi) = 0$ and $\xi \in \mathbb{R}^N$, $x \in \Omega_1$, $y \in \Omega_2$. (It will in fact be necessary to let the direction of η vary and in case the data a, q, φ are not analytic make suitable "almost analytic" continuations of them. These questions will be dealt with in the next section.) Our hypotheses imply that

$$q(x,y,\xi+i\eta) = |\xi|^m q(x,y,\xi/|\xi|+i\eta/|\xi|) \geqq c|\xi|^{m-1}$$

for large $|\xi|$, so if we replace ξ by $\xi+i\eta$ in (4.1.1) we shall, for large ξ, no longer have any infinities in the integrand. The next question in defining the integral is if $a(\xi+i\eta)/q(\xi+i\eta)$ has the properties of a symbol. Now we have for example

$$\partial q^{-1}(x,y,\xi+i\eta)/\partial x_j = -\partial q/\partial x_j \; q^{-2}$$

and we can only be certain that this can be bounded by $|\xi|^{m-2(m-1)} = |\xi|^{2-m}$. Similarly $|\partial q^{-1}(x,y,\xi+i\eta)/\partial \xi_j| \leqq C|\xi|^{1-m}$, so if one pursues this argument one will find that $q^{-1}(\xi+i\eta) \in S^{1-m}_{0,1}$ (Note that $\rho = 0$, $\delta = 1$) which does not suffice for the definition of (4.1.1). However, we can fortunately say more about the action on q^{-1} of those differential operators which act along the surface where q vanishes. To prove this we first note that if p is a homogeneous C^∞ function of degree μ and $q = 0$ implies $p = 0$, then p/q is a C^∞ function of degree $m-\mu$. In fact, to prove that p/q is a C^∞ function it suffices to note that locally one can take q as a local coordinate and apply Taylor's formula. Now let

$$L = \sum a_j(x,y,\xi)\partial/\partial\xi_j + \sum b_j(x,y,\xi)\partial/\partial x_j + c(x,y,\xi)$$

where for large $|\xi|$ the functions a_j are homogeneous of degree 0 and the functions b_j, c are homogeneous of degree -1. If now $Lq = 0$ when $q = 0$, we obtain $|L\,q(x,y,\xi)^{-1}| = |Lq||q^{-2}| \leqq C|\xi|^{-1}|q|^{-1}$, and it follows easily that

$$L(q(x,y,\xi+i\eta)^{-1}) \in S^{-m}_{0,1}.$$

More generally, if $L_1, \ldots, L_k$ are operators of this type, then

$$L_1 \ldots L_k q(x,y,\xi+i\eta)^{-1} \in S^{1-m-k}_{0,1} .$$

We have discussed the properties of q^{-1} rather briefly since they have to be considered again after the discussion in section 4.2 of complex continuation of symbols, but we hope that the preceding discussion motivates the following developments.

Definition 4.1.1. Let I be an $S^0_{1,0}(\Omega, \mathbb{R}^N)$ ideal in the space of all first order differential operators

$$(4.1.2) \qquad L = \sum a_j \,\partial/\partial\xi_j + \sum b_j \,\partial/\partial x_j + c$$

with $a_j \in S^0_{1,0}$, b_j, $c \in S^{-1}_{1,0}$. We shall then say that $a \in {}^I S^m_\rho(\Omega, \mathbb{R}^N)$ if $a \in S^m_{0,1}(\Omega, \mathbb{R}^N)$ and for arbitrary $L_1, \ldots, L_k \in I$ we have

$$(4.1.3) \qquad L_1 \ldots L_k a \in S^{m-\rho k}_{0,1}(\Omega, \mathbb{R}^N).$$

Remark 1. It suffices to assume (4.1.3) when $L_1, \ldots, L_k$ run over a set of generators for the ideal I.

Remark 2. If I consists of all operators (4.1.2) with the stated conditions on the coefficients, then we can take $L = \partial/\partial\xi_j$ and $L = |\xi|^{-1}\,\partial/\partial x_j$ and conclude that ${}^I S^m_\rho$ is equal to $S^m_{\rho,1-\rho}$. In general we have of course

$$(4.1.4) \qquad {}^I S^m_\rho \supset S^m_{\rho,1-\rho}$$

for the left hand side decreases with increasing I.

Remark 3. The constant term c in L obviously does not play any essential role but it is convenient to have it occasionally.

The following is an analogue of Proposition 1.1.2.

Proposition 4.1.2. If $a \in {}^I S^m_\rho$, it follows that $a^{(\alpha)}_{(\beta)} \in {}^I S^{m+|\beta|}_\rho$, and if $b \in {}^I S^{m'}_\rho$ it follows that $ab \in {}^I S^{m+m'}_\rho$.

Proof. We begin with the last statement. By Proposition 1.1.2 we have $ab \in S^{m+m'}_{0,1}$. Assuming as we may that L_j has no constant term, we have an identity of the form

$$L_1 \ldots L_k(ab) = \sum_{k=\nu+\mu} (L_{i_1} \ldots L_{i_\nu} a)(L_{j_1} \ldots L_{j_\mu} b),$$

which implies (4.1.3) for ab. In proving the first statement we may assume that $|\alpha+\beta| = 1$, for example that $|\alpha| = 1$ and $\beta = 0$. Then we have $a^{(\alpha)} \in S^m_{0,1}$. Since $La \in S^{m-\varrho}_{0,1}$, differentiation with respect to ξ_α gives

$$La^{(\alpha)} + i[D_\xi^{\alpha}, L]\, a \in S^{m-\varrho}_{0,1}.$$

Here $i[D_\xi^{\alpha}, L]$ is again an operator of the form (4.1.2) but it has coefficients with order lowered by one unit. Hence $La^{(\alpha)} \in S^{m-\varrho}_{0,1}$. We can continue inductively to verify (4.1.3); the details are omitted.

Remark. The proof shows that the action on a in any order of k_1 operators in I, k_2 operators $\partial/\partial\xi_j$ and k_3 operators $\partial/\partial x_j$ can be estimated by $(1+|\xi|)^{m-k_1\varrho+k_3}$. This could also have been taken as a definition of ${}^I S^m_\varrho$.

If now φ is a real valued function $\in S^1_{1,0}(\Omega, \mathbb{R}^N)$ and for some $L \in I$ we have ${}^tLe^{i\varphi} = e^{i\varphi}$, we can define the oscillatory integral

$$(4.1.5) \qquad \iint e^{i\varphi(x,\xi)} a(x,\xi) u(x)\, dx d\xi, \quad u \in C_0^\infty(\Omega),$$

for all $a \in {}^I S^m_\varrho(\Omega, \mathbb{R}^N)$ and it is a distribution of order $\leq k$ if $m-k\varrho < -N$.

Example. Let $q \in C^\infty(\Omega \times (\mathbb{R}^N \setminus \{0\}))$ be homogeneous with respect to the last variable, grad $q \neq 0$ when $q = 0$, and let I_q be the ideal generated by the operators of the form (4.1.2) where for large $|\xi|$ the coefficients a_j are homogeneous of degree 0 while b_j, c are homogeneous of degree -1 and

$$(4.1.6) \qquad \sum a_j\, \partial q/\partial\xi_j + \sum b_j\, \partial q/\partial x_j = 0 \text{ when } q = 0.$$

Let φ be homogeneous. Then the proof of Lemma 1.2.1 shows that to construct L with the properties required to define (4.1.5) we have to satisfy in addition to (4.1.6) the equation

(4.1.7) $\sum a_j \partial/\partial\xi_j + \sum b_j \partial/\partial x_j = -1.$

This is possible if and only if φ has no critical point with $\xi \neq 0$ and in addition grad φ, grad q are linearly independent when $q = \varphi = 0$. (Note that $\varphi = 0$ at a critical point by Euler's homogeneity relation.)

After this discussion of enlarged classes of symbols we pass to a discussion of the corresponding generalized Fourier integral operators. The following analogue of Theorem 1.3.1 follows immediately.

Theorem 4.1.3. Let I be an ideal of first order differential operators in $\Omega_1 \times \Omega_2 \times \mathbb{R}^N$ as considered in Definition 4.1.1 (with $\Omega = \Omega_1 \times \Omega_2$), and let φ be a real valued function $\in S^1_{1,0}(\Omega_1 \times \Omega_2 \times \mathbb{R}^N)$.

(i) If for some $L \in I$ we have ${}^tL\, e^{i\varphi} = e^{i\varphi}$, then the oscillatory integral

$$\langle Au, v\rangle = \iiint e^{i\varphi(x,y,\xi)} a(x,y,\xi) u(y) v(x)\, dx\, dy\, d\xi$$

is defined for all $a \in {}^I S^m_\rho(\Omega_1 \times \Omega_2, \mathbb{R}^N)$, $u \in C^k_0(\Omega_2)$, $v \in C^k_0(\Omega_1)$ provided that $m - k\rho < -N$, and A is then a continuous map from $C^k_0(\Omega_2)$ to $\mathcal{D}'^k(\Omega_1)$.

(ii) If ${}^tL\, e^{i\varphi} = e^{i\varphi}$ for some $L \in I$ which only involves differentiations with respect to y and ξ, then the oscillatory integral

$$Au(x) = \iint e^{i\varphi(x,y,\xi)} a(x,y,\xi)\, u(y)\, dy\, d\xi, \quad a \in {}^I S^m_\rho(\Omega_1 \times \Omega_2, \mathbb{R}^N),$$

defines a continuous map from $C^k_0(\Omega_2)$ into $C^j(\Omega_1)$ provided that

(4.1.8) $$m + N + j < k\rho .$$

(iii) If ${}^tL\, e^{i\varphi} = e^{i\varphi}$ for some $L \in I$ which only involves differentiations with respect to x and ξ, then A is a continuous map from $\mathcal{E}'^j(\Omega_2)$ into $\mathcal{E}'^k(\Omega_1)$ when (4.1.8) is fulfilled.

(iv) Let Ω_φ be the largest open set $\subset \Omega_1 \times \Omega_2$ such that over Ω_φ there is an operator L which is locally in I such that L only involves differentiations with respect to ξ and ${}^tL\, e^{i\varphi} = e^{i\varphi}$. Then the oscillatory integral

$$K_A(x,y) = \int e^{i\varphi(x,y,\xi)}\, a(x,y,\xi)\, d\xi, \quad (x,y) \in \Omega_\varphi,$$

defines a function in $C^\infty(\Omega_\varphi)$ which is equal to the kernel of the operator A.

In condition (iv) the phrase "L is locally in I" means that $\chi L \in I$ for every $\chi \in C_0^\infty(\Omega_\varphi)$. That there exists a largest set Ω_φ where (iv) is valid follows immediately by application of a partition of unity.

Example. Let $q \in C^\infty(\Omega_1 \times \Omega_2 \times (\mathbb{R}^N \setminus \{0\}))$ be homogeneous with respect to the last variable, grad $q \neq 0$ when $q = 0$, and let I_q be the corresponding ideal defined in an example after Proposition 4.1.2. Then the conditions in Theorem 4.1.3 reduce to

grad $\varphi \neq 0$, and the vectors grad φ, grad q are linearly independent where $q = 0$.

Here gradients are to be taken with respect to x,y,ξ in condition (i), y,ξ in condition (ii), x,ξ in condition (iii) and ξ in condition (iv). In particular, let φ be a phase function in $\Omega \times \Omega \times \mathbb{R}^N$ such that $\mathrm{grad}_\xi \varphi = 0 \Longleftrightarrow x = y$, and assume that $\mathrm{grad}_\xi q \neq 0$ when $q = 0$. The conditions (i),(ii),(iii) are then fulfilled over a neighborhood of the diagonal.

4.2. Almost analytic continuation of symbols. Let $a \in S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$. We wish to extend a to a function in $C^\infty(\Omega \times \mathbb{C}^N)$ in such a way that the Cauchy-Riemann equations are fulfilled of infinite order in $\mathbb{R}^N$. At first we shall do so ignoring the parameter $x \in \Omega$, so we assume that we are given a function $a \in C^\infty(\mathbb{R}^N)$ with

$$(4.2.1) \qquad |D_\xi^\alpha a(\xi)| \leq C_\alpha (1+|\xi|)^{m-\rho|\alpha|}, \quad \xi \in \mathbb{R}^N.$$

If $a(\xi+i\eta)$ were an analytic continuation of a, we would have for small $|\eta|$

$$a(\xi+i\eta) = \sum_\alpha D_\xi^\alpha a(\xi) \; (-\eta)^\alpha/\alpha!.$$

In general the right hand side will of course exist only as a formal power series, so we shall modify it to ensure convergence without changing the formal expansion when $\eta = 0$ so that the Cauchy-Riemann equations will be valid there. The procedure is parallel to the usual proof of the Borel theorem that there exist C^∞ functions with a given formal power series as Taylor expansion at a given point.

Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be equal to 1 in a neighborhood of 0. Choosing a sufficiently rapidly increasing sequence t_j, $j = 0, 1, 2, \ldots$, we shall set

$$(4.2.2) \qquad a(\xi+i\eta) = \sum_\alpha D_\xi^\alpha a(\xi)(-\eta)^\alpha \chi(t_{|\alpha|}\eta(1+|\xi|^2)^{-\rho/2})/\alpha!,$$

and prove

Theorem 4.2.1. Given a function $a \in C^\infty(\mathbb{R}^N)$ satisfying (4.2.1) one can extend a to a function in $C^\infty(\mathbb{C})$ such that

$$(4.2.3) \qquad |D_\xi^\alpha D_\eta^\beta a(\xi+i\eta)| \leq C_{\alpha\beta} (1+|\xi|)^{m-\rho|\alpha+\beta|},$$

$$(4.2.4) \qquad a(\xi+i\eta) = 0 \text{ when } \quad |\eta| \geq C(1+|\xi|)^\rho,$$

$$(4.2.5) \qquad |\bar\partial a(\xi+i\eta)| = \Big(\sum |\partial a/\partial\xi_j + i\,\partial a/\partial\eta_j|^2/4\Big)^{1/2} \leq$$
$$\leq C_k(1+|\xi|)^{m-\rho(1+k)}|\eta|^k$$

for any integer $k \geq 0$. Finally, $a(\xi+i\eta) = 0$ when $\xi \notin \operatorname{supp} a$.

Note that (4.2.5) implies that when $|\eta| < C(1+|\xi|)^{\sigma}$ for some $\sigma < \rho$, then $\bar\partial a$ is a rapidly decreasing function. For our purposes this will be as useful as knowing that a is an analytic function for we do not pay attention to rapidly decreasing symbols or the corresponding operators with C^{∞} kernel.

Proof of Theorem 4.2.1. First note that each term in (4.2.2) satisfies (4.2.4) with a fixed C if $t_j \geqq 1$ for every j, for example. Write

$$\Phi(\xi, \eta) = \eta(1+|\xi|^2)^{-\rho/2} \quad \text{and} \quad \chi_\alpha(\eta) = (-\eta)^\alpha \chi(\eta).$$

The general term in (4.2.2) can then be written $c_\alpha(\xi,\eta)/\alpha!$ where

$$c_\alpha(\xi, \eta) = (D_\xi^{\alpha} a(\xi))(1+|\xi|^2)^{\rho|\alpha|/2}\, t_{|\alpha|}^{-|\alpha|}\, \chi_\alpha(t_\alpha \Phi(\xi, \eta)).$$

Here $(D_\xi^{\alpha} a(\xi))(1+|\xi|^2)^{\rho|\alpha|/2} \in S^m_\rho$, and for the derivatives of Φ we have when (4.2.4) is valid

$$|D_\xi^{\alpha} D_\eta^{\beta} \Phi(\xi,\eta)| \leqq C_{\alpha\beta}(1+|\xi|)^{-|\alpha|-\rho|\beta|} \leqq C_{\alpha\beta}(1+|\xi|)^{-\rho|\alpha+\beta|}.$$

It follows that c_α satisfies (4.2.3). Moreover, if we choose $t_{|\alpha|}$ large enough, we have

(4.2.6) $$|D_\xi^{\beta} D_\eta^{\gamma} c_\alpha(\xi,\eta)| \leqq (1+|\xi|)^{m-|\beta+\gamma|-t}\, |\eta|^t;\ t \geqq 0,\ t+|\beta|+|\gamma| \leqq |\alpha|-1.$$

(We have to make sure only that this is true for the extreme cases $t = 0$ and $t = |\alpha|-1-|\beta+\gamma|$.) Since the series $\sum 1/\alpha!$ converges, it follows from (4.2.6) that (4.2.3) is valid. To prove (4.2.5) we first note that it follows from (4.2.3) when $|\eta|/(1+|\xi|^\rho)$ is bounded from below. On the other hand, when $(t_{|\alpha|}\eta(1+|\xi|^2)^{-\rho/2}) = 1$ for $|\alpha| \leqq k$, we have

$$(\partial/\partial\xi_j + i\partial/\partial\eta_j)a(\xi+i\eta) = \sum_{|\alpha|=k} iD_\xi^{\alpha+1_j} a(\xi)(-\eta)^\alpha/\alpha! +$$
$$+ (\partial/\partial\xi_j + i\partial/\partial\eta_j)\sum_{|\alpha|>k} c_\alpha(\xi,\eta)/\alpha!.$$

The first sum has the required bound, and so has the second by (4.2.6) with $t = k$ (for $|\alpha| > k+1$). This completes the proof.

Remark 1. We introduced the argument $t_\alpha\eta(1+|\xi|^2)^{-\rho/2}$ instead of $t_\alpha\eta|\xi|^{-\rho}$ in the cutoff function just to obtain differentiability at the origin. If a is a homogeneous function, C^∞ for $\xi \neq 0$, the latter choice is more natural for the extension $a(\xi+i\eta)$ is then homogeneous of the same degree and 0 outside a conical neighborhood of $\mathbb{R}^N$.

Remark 2. If $\sum_1^J a_j = 1$ we can choose the same numbers t_ν for the extension of each a_j and concludes that

$$\sum_1^J a_j(\xi+i\eta) = \chi(t_0\eta/(1+|\xi|^2)^{\rho/2}) = 1 \text{ if } |\eta| < C(1+|\xi|)^\rho$$

for a certain constant C.

The proof of Theorem 4.2.1 can be repeated in the presence of parameters with no essential change except that the notations become heavier, so we have also

Theorem 4.2.2. Let $a \in S^m_{\rho,\delta}(\Omega, \mathbb{R}^N)$. Then one can find an extension $a \in C^\infty(\Omega \times \mathbb{C}^N)$ such that for every compact set K we have when $x \in K$

(4.2.7) $\quad |D_\xi^\alpha D_\eta^\beta D_x^\gamma a(x,\xi+i\eta)| \leq C_{\alpha,\beta,\gamma,K}(1+|\xi|)^{m+\delta|\gamma|-\rho|\alpha+\beta|}$,

(4.2.8) $\quad a(x,\xi+i\eta) = 0$ when $|\eta| > C_K(1+|\xi|)^\rho$,

(4.2.9) $\quad |\bar\partial a(x,\xi+i\eta)| \leq C_{k,K}(1+|\xi|)^{m-\rho(1+k)}|\eta|^k$, $k = 1, 2, \ldots$

If a_1 and a_2 are two such extensions and $b = a_1 - a_2$, then

(4.2.10) $\quad |b(x,\xi+i\eta)| \leq C_{K,k}(1+|\xi|)^{m-\rho k}|\eta|^k$, $x \in K$, $k = 1, 2, \ldots$

To prove the last statement we note that (4.2.9) implies that $\partial b/\partial\xi_j + i\partial b/\partial\eta_j$ vanishes of infinite order when $\eta = 0$. Since $b = 0$ when $\eta=0$ we conclude that $\partial b/\partial\eta_j = 0$ when $\eta = 0$; since $\partial/\partial\eta_k(\partial b/\partial\xi_j + i\partial b/\partial\eta_j) = 0$ when $\eta = 0$ we then obtain $\partial^2 b/\partial\eta_j\partial\eta_k = 0$ when $\eta = 0$. Inductively it follows that all derivatives of b vanish when $\eta = 0$. Hence (4.2.10) follows by Taylor expansion of $b(x,\xi+i\eta)$ at $\eta = 0$ using (4.2.7).

Note that (4.2.10) implies that b is rapidly decreasing when for some $\sigma < \rho$ we have $|\eta| < C(1+|\xi|)^{\sigma}$. The same is true for every derivative of b. Different choices of "almost analytic" continuation of a will therefore always lead to operators differing by one with a C^{∞} kernel.

One advantage of the almost analytic continuations introduced here is that they can be applied to partitions of unity so that these can be used as if they were analytic functions. We shall need the following existence theorem for a partition of unity.

Proposition 4.2.3. Let F_j, $j = 1, \ldots, J$, be closed sets in $\Sigma = \{(x,\xi) \in \Omega \times \mathbb{R}^N,\ |\xi| = 1\}$, and assume that $\cup F_j = \Sigma$. Then one can find $\psi_j \in S^0_{\rho,1-\rho}(\Omega, \mathbb{R}^N)$ with $\sum \psi_j = 1$ so that on any compact set K in Ω we have $\psi_j(x,\xi) = 0$ when the distance from $(x, \xi/|\xi|)$ to F_j is $> c_K|\xi|^{\rho-1}$ and $|\xi|$ is sufficiently large; c_K denoting a positive constant.

Proof. Let $\varphi_j(x,\xi,\varepsilon)$ be the convolution of the characteristic function of $\{(x,\xi);\ (x,\xi/|\xi|) \in F_j\}$ by $\varepsilon^{-n}\chi(\xi/\varepsilon)$ where $\chi \in C_0^{\infty}$, $\int\chi\, d\xi = 1$ and χ vanishes outside the unit ball. This is well defined if $\varepsilon <$ the distance from x to $\complement\Omega$. The derivatives of φ_j of order k with respect to x, ξ, ε can be bounded by $C_k\varepsilon^{-k}$. Let $0 < c \in C^{\infty}(\Omega)$ be smaller than the boundary distance function and set $\varepsilon = (1+|\xi|^2)^{(\rho-1)/2}c(x)$, $\psi_j(x,\xi) = \varphi_j(x,\xi/|\xi|, \varepsilon) / \sum\varphi_k(x,\xi/|\xi|,\varepsilon)$. The easy verification of the properties stated in the proposition is left for the reader.

4.3. <u>Definition of simply singular Fourier integral operators.</u> We shall now complete the program of giving a precise meaning to the operator

$$(4.3.1) \qquad Au(x) = \iint e^{i\varphi(x,y,\xi)}\, a(x,y,\xi)/q(x,y,\xi)\, u(y)\, dy d\xi, \quad u \in C_0^\infty(\Omega_2)$$

which was begun in section 4.1. Our hypotheses are

(4.3.2) q is real valued and homogeneous of degree μ with respect to ξ, and $q \in C^\infty$ when $\xi \neq 0$;

(4.3.3) φ is a phase function (Definition 1.3.2);

(4.3.4) grad φ and grad q are linearly independent when $q = 0$ if grad is taken either with respect to $x\xi$ or $y\xi$;

(4.3.5) $a \in S^m_{\rho,1-\rho}(\Omega_1, \Omega_2, \mathbb{R}^N)$, $0 < \rho \leqq 1$.

We choose almost analytic continuations of a, φ and q using Theorem 4.2.2. In the case of φ and q though we use Remark 1 following Theorem 4.2.1 to choose the continuations homogeneous.

Let $\eta = \eta(x,y,\xi)$ be a homogeneous C^∞ function of degree 0 when $\xi \neq 0$ with values in $\mathbb{R}^n$, which satisfies the condition

$$(4.3.6) \qquad \langle \mathrm{grad}_\xi\, q, \eta\rangle \neq 0 \text{ when } q(x,y,\xi) = 0,\ \xi \neq 0.$$

For instance, we can take $\eta = |\xi|^{\mu-1}\,\mathrm{grad}_\xi\, q$ (or rather the covector corresponding to $\mathrm{grad}_\xi q$ with respect to a Riemannian metric). We denote by V the set of all η satisfying (4.3.6). We also define $V^{\pm}$ as the set of all η satisfying

$$(4.3.7) \qquad \langle \mathrm{grad}_\xi\, q, \eta\rangle \gtrless 0 \text{ when } q(x,y,\xi) = 0,\ \xi \neq 0.$$

Note that V^+ and V^- are both convex.

If $\eta \in V$, the Taylor expansion

$$q(x,y,\xi+i\eta) = q(x,y,\xi) + i\langle \mathrm{grad}_\xi q, \eta\rangle + O(|\xi|^{\mu-2}),$$

implies that in a conical neighborhood of the zeros of q we have

$$|q(x,y,\xi+i\eta)| \geqq |q(x,y,\xi)|/2+c(x,y)|\xi|^{\mu-1} - C(x,y)|\xi|^{\mu-2}$$

$$\geqq (|q(x,y,\xi)|+c(x,y)|\xi|^{\mu-1})/2 \text{ if } |\xi| > C'(x,y).$$

Since $|\xi|^\mu$ can be bounded by a constant times $|q(x,y,\xi)|$ outside a conical neighborhood of the zeros an estimate of the same type is valid there. Hence we have for suitable continuous positive functions c and C

(4.3.8) $\quad |q(x,y,\xi+i\eta)| \geqq (|q(x,y,\xi)|+c(x,y)|\xi|^{\mu-1})/2, \quad |\xi| > C(x,y).$

The somewhat formal arguments in the beginning of section 4.1 can now be made precise as follows.

<u>Proposition 4.3.1.</u> Let q, a and η satisfy (4.3.2),(4.3.5), (4.3.6), and assume that $a(x,y,\xi) = 0$ when $|\xi| < C(x,y)$, the constant in (4.3.8). Let I be the ideal of first order differential operators defined in an example after Proposition 4.1.2. With the notation $f^\eta(x,y,\xi) = f(x,y,\xi+i\eta(x,y,\xi))$ we have then $a^\eta/q^\eta \in {}^I S_\rho^{m-\mu+1}(\Omega_1\times\Omega_2, \mathbb{R}^N)$.

<u>Proof.</u> Write $b(x,y,\xi) = a^\eta(x,y,\xi)/q^\eta(x,y,\xi)$ so that

(4.3.9) $\quad a^\eta(x,y,\xi) = b(x,y,\xi)\, q^\eta(x,y,\xi).$

The function a^η is in $S^m_{\rho,1-\rho}$. This statement is closely related to Proposition 1 .1.3 but not contained in it. However, the proof is a simple exercise using (4.2.7) and a Taylor expansion of a^η with respect to η, so it may be left for the reader.

The first step in the proof is to show that $b\in S^{m-\mu+1}_{0,1}$, that is,

(4.3.10) $\quad |D_{x,y}^{\alpha} D_\xi^\beta b(x,y,\xi)| \leqq C(1+|\xi|)^{m-\mu+1+|\alpha|}\ ; \ x,y \in K \subset\subset \Omega.$

This follows from (4.3.8)' when $\alpha = \beta = 0$ so we may use induction with respect to increasing $|\alpha+\beta|$. Application of $D_{xy}^{\alpha}D_\xi^\beta$ to (4.3.9) gives

$$D_{xy}^{\alpha} D_{\xi}^{\beta} a^{\eta} = \sum_{\alpha'+\alpha''=\alpha,} \sum_{\beta'+\beta''=\beta} c_{\alpha',\alpha'',\beta',\beta''}(D_{xy}^{\alpha'} D_{\xi}^{\beta'} b) D_{xy}^{\alpha''} D_{\xi}^{\beta''} q^{\eta}.$$

The left hand side is bounded by $C(1+|\xi|)^{m+|\alpha|}$ since $a^{\eta} \in S^{m}_{0,1}$. Since $q^{\eta} \in S_{1,0}$ for $\xi \neq 0$ by the preceding discussion of a^{η}, the terms on the right with $\alpha''+\beta'' \neq 0$ can be estimated by a constant times

$$(1+|\xi|)^{m-\mu+1+|\alpha'|+\mu-|\beta''|} = (1+|\xi|)^{m+|\alpha|+1-|\alpha''+\beta''|} \leq (1+|\xi|)^{m+|\alpha|}.$$

Using (4.3.8) we conclude that (4.3.10) is valid. Note that the proof only used tha t $a^{\eta} \in S^{m}_{0,1}$.

Next let L be a first order operator $\in I$ with homogeneous coefficients and no constant term, that is,

$$L = \sum a_j \partial/\partial\xi_j + \sum b_j \partial/\partial x_j$$

where a_j is homogeneous of degree 0 and b_j is homogeneous of degree -1 with respect to ξ,

(4.3.11) $\quad \sum a_j \partial q/\partial \xi_j + \sum b_j \partial q/\partial x_j = Lq = 0$ when $q = 0$.

If we apply L to (4.3.9) we obtain

$$(Lb)\, q^{\eta} = La^{\eta} - bLq^{\eta}.$$

Now Taylor's formula gives that $q^{\eta} - q \in S^{-1}_{1,0}$. Hence

$$Lq^{\eta} - Lq \in S^{-2}_{1,0}.$$

In view of (4.3.11) we have $Lq = cq$ with c of degree -1, so it follows that

(4.3.12) $\quad Lq^{\eta} = cq^{\eta} + d$ where $c \in S^{-1}_{1,0}$ and $d \in S^{\mu-2}_{1,0}$.

Thus $(Lb)\, q^{\eta} = La^{\eta} - b(cq^{\eta}+d)$, that is,

(4.3.13) $\quad (Lb + bc)\, q^{\eta} = La^{\eta} - bd.$

Here $La^{\eta} \in S^{m-\rho}_{\rho,1-\rho}$ and $bd \in S^{m-1}_{0,1}$ so the right hand side is in $S^{m-\rho}_{0,1}$. From the first part of the proof we now obtain

$$Lb + bc \in S^{m-\mu+1-\rho}_{0,1}$$

and since $bc \in S^{m-\mu}_{0,1}$ we conclude that

$$Lb \in S_{0,1}^{m-\mu+1-\rho}.$$

Note that the only properties of a^η used so far are that $a^\eta \in S_{0,1}^m$ and that $La^\eta \in S_{0,1}^{m-\rho}$.

Let us now prove by induction that if $L_1 \dots L_j a \in S_{0,1}^{m-\rho j}$ for $j \leq k$ and arbitrary homogeneous $L_1, \dots, L_j \in I$ as above, then $L_1 \dots L_k b \in S_{0,1}^{m-\mu+1-k\rho}$. We know this already when $k = 0,1$ and assume that the statement is proved with k replaced by $k-1$. Set $L_k = L$. By hypothesis $L_1 \dots L_{k-1} L_k a^\eta \in S_{0,1}^{m-\rho k}$, and the inductive hypothesis gives that $L_1 \dots L_{k-1} bd \in S_{0,1}^{m-\mu+1+\mu-2-\rho(k-1)}$. Hence $L_1 \dots L_{k-1}(La^\eta - bd) \in S_{0,1}^{m-k\rho}$, also with any other operators of the same type instead of $L_1, \dots, L_{k-1}$. In view of (4.3.13) and the inductive hypothesis it follows that $L_1 \dots L_{k-1}(L_k b + bc) \in S_{0,1}^{m-k\rho-\mu+1}$. The inductive hypothesis also gives $L_1 \dots L_{k-1}(bc) \in S_{0,1}^{m-\mu-(k-1)\rho} \subset S_{0,1}^{m-\mu+1-k\rho}$, which completes the proof that $L_1 \dots L_k b \in S_{0,1}^{m-\mu+1-k\rho}$ and so proves the proposition.

Remark. The proof only used the fact that $a \in {}^I S^m$ and might be more natural if the proposition were stated in that way. However, this more general statement is an immediate consequence of the one which we have proved in view of the last statement in Proposition 4.1.2.

To complete the definition of (4.3.1) we note that

$$\varphi^\eta(x,y,\xi) = \varphi(x,y,\xi+i\eta) = \varphi(x,y,\xi) + \psi(x,y,\xi), \quad \psi \in S_{1,0}^0.$$

Hence $e^{i\psi} \in S_{1,0}^0$ so that $e^{i\psi} a^\eta / q^\eta \in {}^I S_\rho^{m-\mu+1}$. For a given function $\eta \in V$ we choose an almost analytic extension a_1 of a function which is equal to a for large $|\xi|$ but vanishes for $|\xi| < C(x,y)$ so that (4.3.8) becomes applicable with a replaced by a_1. Using Theorem 4.1.3 we can now define

$$A^\eta u(x) = \iint e^{i\varphi^\eta(x,y,\xi)} a_1^\eta(x,y,\xi)/q^\eta(x,y,\xi) u(y)dy \; d(\xi_1+i\eta_1) \wedge \dots \wedge d(\xi_N+i\eta_N). \tag{4.3.13}$$

Here we have used the nota tions of exterior calculus to express the integration element briefly. Note that $d(\xi_1+i\eta_1)\wedge\ldots\wedge d(\xi_n+i\eta_n)$ (where differentials are taken with x,y regarded as parameters) is equal to $D(x,y,\xi)$ $d\xi_1\wedge\ldots\wedge d\xi_n$ where $D(x,y,\xi)\in S^0_{1,0}$, in fact $D(x,y,\xi) = 1 +$ an asymptotic sum of terms homogeneous of degree -1, -2, ... Thus we could have written $D(x,y,\xi)d\xi$ instead. If a is sufficiently rapidly decreasing at infinity it is perhaps best to write

$$A^\eta u(x) = \iint_{\gamma^\eta_{xy}} e^{i\varphi(x,y,\zeta)}a_1(x,y,\zeta)/q(x,y,\zeta)\, u(y)\, dy\, d\zeta_1\wedge\ldots\wedge d\zeta_n$$

where γ^η_{xy} is the cycle $\xi \to \xi+i\eta(x,y,\xi)$.

We shall postpone for a moment the discussion of the singularities of the kernel of A^η but observe that this operator has all the properties listed in Theorem 4.1.3 with m replaced by $m-\mu+1$. Our next purpose is to show that modulo operators with C^∞ kernel the operator A^η is independent of all the choices made provided that $\eta\in V^+$ (or $\eta\in V^-$); representatives for these classes of operators will be denoted by A^+ and A^- respectively.

First of all, for a fixed choice of η it is clear in view of (4.2.10) tha t different choices of a_1 and the almost analytic continuations will only change $e^{i\varphi^\eta(x,y,\xi)}a_1^\eta(x,y,\xi)/q^\eta(x,y,\xi)$ by a term which is rapidly decreasing at infinity. The same is true for the derivatives. It remains to discuss the dependence on η. Assume for example that $\eta^0, \eta^1\in V^+$. Then we have $(1-t)\eta^0+t\eta^1\in V^+$ if $0\leq t\leq 1$, and for sufficiently large $C(x,y)$ we have (4.3.8) for all these vector fields. Let Γ_{xy} be the chain

$$\mathbb{R}^{N+1} \ni (t,\xi) \to \xi + (1-t)\eta^0(x,y,\xi) + t\eta^1(x,y,\xi) \in \mathbb{C}^N, \quad 0\leq t\leq 1.$$

If f is rapidly decreasing at infinity we have by Stokes' formula

$$\int_{\Gamma_{xy}} \bar\partial f \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = \sum_0^1 (-1)^{j-1} \int_{\gamma_{xy}^{\eta^j}} f\, d\zeta_1 \wedge \ldots \wedge d\zeta_n.$$

Hence we obtain for every $\varepsilon > 0$

$$\iint_{\Gamma_{xy}} \bar\partial\, (e^{i\varphi(x,y,\zeta)} a_1(x,y,\zeta)/q(x,y,\zeta))u(y)dyd\zeta_1 \wedge \ldots \wedge d\zeta_n e^{-\varepsilon(\zeta,\zeta)}$$

$$= \sum_0^1 (-1)^{j-1} \iint_{\gamma_{xy}^{\eta^j}} e^{i\varphi(x,y,\zeta)} a_1(x,y,\zeta)/q(x,y,\zeta)\; u(y)dyd\zeta_1 \wedge \ldots \wedge d\zeta_n.$$

When $\varepsilon \to 0$ the right hand side converges to the difference between the two two oscillatory integrals $A^{\eta^1}u(x)$ and $A^{\eta^0}u(x)$. On the left hand side the limit of the integrand is rapidly decreasing in view of (4.2.9), so we conclude that $(A^{\eta^1} - A^{\eta^0})$ is an integral operator with the kernel

$$\int_{\Gamma_{xy}} \bar\partial\, (e^{i\varphi(x,y,\zeta)} a_1(x,y,\zeta)/q(x,y,\zeta)) \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n.$$

This is a C^∞ function since the integrand remains rapidly decreasing after any number of differentiations with respect to x or y (this follows by combining (4.2.9) and (4.2.7) using standard convexity properties of derivatives, or we could otherwise ha ve stated (4.2.9) in a stronger form involving also differentiations with respect to x and y). The results obtained so far are summed up in the following

<u>Theorem 4.3.2.</u> Assume that (4.3.2) - (4.3.5) are fulfilled. Then the integral (4.3.13) with $\eta \in V^+$ (V^-) defines a class of operators A^+ (or A^-) modulo operators with C^∞ kernel, such that

(i) $A^\pm$ is a continuous map of $C_0^k(\Omega_2)$ into $C^j(\Omega_1)$ if $m-\mu+1+N+j < k\rho$;

(ii) $A^\pm$ is a continuous map of $\mathcal{E}'^j(\Omega_2)$ into $\mathcal{D}'^k(\Omega_2)$ if $m-\mu+1+N+j<k$.

We shall now examine the location of the singularities of the kernel of $A^\pm$. In part the results could be obtained from (iv) in Theorem 4.1.3 but we get more precise information by studying what can be achieved by a suitable choice of $\eta \in V^\pm$ in (4.3.13).

We shall prove that if $(x,y) \in \Omega_1 \times \Omega_2$ and for some $\eta \in V^+$

$$\langle \mathrm{grad}_\xi \varphi(x,y,\xi), \eta(x,y,\xi) \rangle > 0, \quad \xi \neq 0, \tag{4.3.14}$$

then the kernel of A^+ is C^∞ in a neighborhood of (x,y). In doing so we may assume that (4.3.14) is valid in all of $\Omega_1 \times \Omega_2$ for we can otherwise replace these open sets by smaller ones. We shall then push the integration contour in (4.3.1 3) further out in the direction η. This can be done without any difficulties from the exponential function for

$$\begin{aligned} \mathrm{Re}\, i\, \varphi(x,y,\xi+it\eta(x,y,\xi)) &= -t\langle \mathrm{grad}_\xi \varphi, \eta \rangle + O(|\xi|^{-1}t^2) < \\ &< -tc(x,y) \text{ if } 1 < t < |\xi|/C(x,y), \ |\xi| > C(x,y) \end{aligned}$$

where c, C are positive continuous functions. We can now repeat the discussion preceding Theorem 4.3.2 with $\eta^0 = \eta$ and $\eta^1 = t\eta$, where $t \to +\infty$. Assuming as we may that $\rho < 1$ we recall that by (4.2.8) we have $|\eta| = o(|\xi|)$ in supp $a(x,\xi+i\eta)$ so in the limit we obtain the nicely convergent integral

$$A^\eta u(x) = -\int_{\Gamma_{xy}} \bar\partial \left(e^{i\varphi(x,y,\zeta)} a_1(x,y,\zeta)/q(x,y,\zeta)\right) \wedge u(y)\, dy\, d\zeta_1 \wedge \ldots \wedge d\zeta_n.$$

Here Γ_{xy} is now the chain

$$\mathbb{R}^{N+1} \ni (t,\xi) \to \xi + it\eta(x,y,\xi) \in \mathbb{C}^N$$

restricted to $t > 1$. As before it follows that A^η has a C^∞ kernel given by the integral

$$-\int_{\Gamma_{xy}} \bar\partial \left(e^{i\varphi(x,y,\zeta)} a_1(x,y,\zeta)/q(x,y,\zeta)\right) \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n.$$

Now one can find $\eta \in V^+$ satisfying (4.3.14) unless for some ξ with $q(x,y,\xi) = 0$ we have $\mathrm{grad}_\xi \varphi(x,y,\xi) = \lambda\, \mathrm{grad}_\xi q(x,y,\xi)$ where $\lambda \leq 0$. In fact, we can then satisfy (4.3.) and (4.3.7) (with the upper inequality) at the same time locally and using a partition of unity we find that η also exists globally. Hence we have proved

Theorem 4.3.3. The kernel of the operators A^+ (A^-) defined in Theorem 4.3.2 are in C^∞ excepts at points (x,y) where for some $\xi \in \mathbb{R}^N$, $\xi \neq 0$, with $q(x,y,\xi) = 0$ we have $\mathrm{grad}_\xi \varphi = \lambda\, \mathrm{grad}_\xi\, q$ for some $\lambda \leq 0$ (≥ 0).

Note that for each of the operators A^+ and A^- we obtain only "one half" of the singularities possible by (iv) in Theorem 4.1.3. A further improvement can be made as follows. Let us take a partition of $\Sigma = \{(x,y,\xi); |\xi| = 1\}$ as in Proposition 4.2.3, thus $\Sigma = F_+ \cup F_-$ where F_+ and F_- are closed. Choose ψ_j according to Proposition 4.2.3 with $\rho < 1$. We now define A as the sum of the + operator corresponding to $\psi_1 a$ and the - operator corresponding to $_2 a$. Them the kernel of A can only be singular at (x,y) if one can find $\xi \in \mathbb{R}^N$ with $q(x,y,\xi) = 0$ such that $(x,y,\xi) \in F_+$ (F_-) and $\mathrm{grad}_\xi\, \varphi(x,y,\xi) = \lambda\, \mathrm{grad}_\xi q(x,y,\xi)$ with $\lambda \leq 0$ $(\lambda \geq 0)$. This will be useful in Chapter V where any one of these operators A will be a fundamental solution and a favorable choice of F_+ and F_- gives precise results on the location of singularities of the solutions.

Multiplication of the operators $A^\pm$ by pseudo-differential operators to the left or to the right gives new operators of the same type, that is, with the same φ and q but with a new a. To prove this we first note that by a substitution

$$(x,y,\xi) \longrightarrow (x,y,\Phi(x,y,\xi))$$

one can locally reduce q to a function independent of x and y, multiplied by a homogeneous function without zeros. In establishing this fact we may assume that the degree of homogeneity μ of q is $\neq 0$. Then the vector $v(\xi) = \mathrm{grad}_\xi q(x_0,y_0,\xi)$ is then always $\neq 0$, for grad $q = 0$ implies $q = 0$

by Euler's identity for homogeneous functions, so $v(\xi) \neq 0$ in view of (4.3.2). Now we define $\Phi(x,y,\xi) = \xi + t\,v(\xi)$ where t shall be chosen so tha t

$$q(x,y,\ \xi+tv(\xi)) = q(x_0,y_0,\xi) \quad \text{and} \quad t = 0 \text{ when } x=x_0, y=y_0.$$

Since $\partial/\partial t$ of the left hand side is $\neq 0$ at x_0,y_0 when $t = 0$, the implicit function theorem implies that there is a unique solution t, homogeneous of degree $\frac{2-\mu}{\,}$ with respect to ξ, defined and C^∞ for (x,y) in a neighborhood of (x_0,y_0), and Φ then gives a substitution with the desired property. But as soon as q is independent of x (or y) we can use the proofs of the multiplicative properties of Fourier integral operators given im section 3.1 to discuss left (or right) multiplication by properly supported pseudo-differential operators. Since we shall not actually use these results the detailed proofs will not be given but we keep in mind until Chapter V that q should be chosen independent of x (y) if one wants to operate to the left (right) by a pseudo-differential operator.

Chapter V

Operators of principal type

5.1. Construction of a right parametrix. Let $P = P_m(x,D) + \ldots + P_0(x,D)$ be a differential operator of principal type in the open set $\Omega \subset \mathbb{R}^n$. Thus $P_j(x,\xi)$ is homogeneous of degree j with respect to ξ and

$$P_m(x,\xi) = 0 \implies \operatorname{grad}_\xi P_m(x,\xi) \neq 0. \tag{5.1.1}$$

We also make the essential assumptiom that P_m has real coefficients. Our methods are also applicable if Re $\operatorname{grad}_\xi P_m(x,\xi)$ and Im $\operatorname{grad}_\xi P_m(x,\xi)$ are linearly independent whenever $P_m(x,\xi) = 0$ but so far they are not sufficient for the general "principally normal" case which has been studied before by means of energy integral methods. On the other hand, the results obtained by the use of Fourier integral operators are more precise. (Using the results of section 3.1 it would cause no difficulties to study pseudo-differential operators P in the same way but we shall refrain from doing so in the hope that this will make the exposition easier to follow.)

A right parametrix in Ω is a continuous map $E: C_0^\infty(\Omega) \to C^\infty(\Omega)$ such that $PEu - u = Ku$, $u \in C_0^\infty$, where K has a C^∞ kernel. In the case where P has constant coefficients, one can construct Eu be interpreting the integral

$$(2\pi)^{-n} \iint e^{i\langle x,\xi\rangle} P(\xi)^{-1}\hat{u}(\xi)\, d\xi$$

as a suitable contour integral. We shall make an analogous approach here using the results of Chapter IV. Thus we write, at first formally

$$Eu(x) = (2\pi)^{-n} \iint e^{i\varphi(x,y,\xi)} a(x,y,\xi)/q(y,\xi)\, u(y)\, dy d\xi$$

where φ, a, q should be as in section 4.3. Note that q is required to be independent of x so that we can opera te with $P(x,D)$ under the integral sign without introducing worse singularities. (See the remarks at the end of section 4.3.) The function φ should satisfy the hypotheses of Theorem 2.1.2 so that $L^{m'}_{1,0}(\Omega,\Omega,\varphi) = L^{m'}_{1,0}(\Omega)$ is a standard space of pseudo-differential operators. More precisely, we shall choose φ so that

$$(5.1.2) \qquad \varphi(x,y,\xi) = \langle x-y,\xi\rangle + O(|x-y|^2|\xi|).$$

Then the identity operator can be represented in the form

$$u(x) = (2\pi)^{-n} \iint e^{i\varphi(x,y,\xi)} I(x,y,\xi)\, u(y)\, dy d\xi$$

where $I(x,x,\xi) - 1 \in S^{-1}_{1,0}(\Omega, \mathbb{R}^n)$.

Formally the equa tion $PEu = u + Ku$ should be satisfied with an operator K having a smooth kernel provided that apart from an error which decreases rapidly at infinity

$$(5.1.3) \qquad e^{-i\varphi(x,y,\xi)}\, P(x,D)(e^{i\varphi(x,y,\xi)} a(x,y,\xi)/q(y,\xi)) = I(x,y,\xi).$$

Actually we shall at first only achieve somewhat less, namely that this equation is satisfied to a high degree of accuracy on the diagonal with $I(x,x,\xi)$ replaced by 1. The leading term in the left hand side of (5.1.3) is $P_m(x, \operatorname{grad}_x \varphi)a/q$. We would like to have $P_m(x, \operatorname{grad}_x \varphi(x,y,\xi)) = q(y,\xi)$ so that our equation (5.1.3) in the first approximation becomes $a = 1$. Since $\operatorname{grad}_x \varphi(x,y,\xi) = \xi$ when $x = y$ in view of the requirement (5.1.2), this means that we must have

$$(5.1.4) \qquad q(y, \xi) = P_m(y, \xi)$$

$$(5.1.5) \qquad P_m(x, \operatorname{grad}_x \varphi) = P_m(y, \xi),\ \varphi(x,x,\xi) = 0,\ \operatorname{grad}_x \varphi(x,y,\xi) = \xi \text{ when } x=y.$$

The conditions (5.1 .5) are of course a consequence of

(5.1.6) $P_m(x, \operatorname{grad}_x \varphi(x,y,\xi)) = P_m(y,\xi)$, $\operatorname{grad}_x \varphi(x,y,\xi) = \xi$ when $x = y$ and

$$\varphi(x,y,\xi) = \langle x-y,\xi\rangle \text{ when } \langle x-y,\ t(y,\ \xi)\rangle = 0.$$

Here $t(y,\xi)$ shall be chosen as a C^∞ function, homogeneous with respect to ξ for $\xi \neq 0$, such that $\langle \operatorname{grad}_\xi P_m(y,\ \xi),\ t(y,\xi)\rangle \neq 0$; for example $t(y,\xi)$ could be $\operatorname{grad}_\xi P_m(y,\xi)$ (or rather the corresponding covector). This is a non-characteristic initial value problem and since P_m is real it follows from the Hamilton-Jacobi integration theory in view of the homogeneity with respect to ξ that in a neighborhood Ω_1 of any point in Ω the conditions (5.1.6) define uniquely a phase functiom φ. We shrink Ω so that this is assumed to be true in the whole of Ω; we may even assume that $\operatorname{grad}_x \varphi \neq 0$, $\operatorname{gra\,d}_y \varphi \neq 0$ when $\xi \neq 0$ and that $\operatorname{grad}_\xi \varphi = 0 \Longleftrightarrow x = y$. Finally we choose Ω so small that (4.3.4) is fulfilled. This is possible in virtue of the remarks made in the example at the end of section 4.1.

By Leibniz' formula we can write (cf. (3.1.5)) using (5.1.6)

$$e^{-i\varphi(x,y,\xi)}P(x,D)(e^{i\varphi(x,y,\xi)}a(x,y,\xi)) =$$

$$= P_m(y,\ \xi)\ a(x,y,\xi) + \sum_1^n P_m^{(j)}(x,\ \operatorname{grad}_x\ )D_{x_j} a + ba + \dots$$

Here b is homogeneous of degree m-1 and the dots indicate a sumof homogeneous functions of degree $\leqq m+\mu-2$ if a is of degree $\leqq \mu$. We shall choose a so that $a(x,x,\xi) = 1$ and

(5.1.7) $$e^{-i\varphi(x,y,\xi)}P(x,D)(e^{i\varphi(x,y,\xi)}a(x,y,\xi)) - P_m(y,\xi)a(x,y,\xi) \in S^{-\infty}_{1,0}.$$

Write formally $a = \sum_0^\infty a_\nu$ where a_ν is homogeneous of degree m. Then the terms of order $m-1-\nu$ in the left hand side of (5.1.7) will cancel provided that

(5.1.8) $$\sum_1^n P_m^{(j)}(x,\ \operatorname{gra\,d}_x \varphi)\ D_{x_j} a_\nu + ba_\nu = r_\nu$$

where r_ν is a homogeneous function of degree $m-1-\nu$ which is determined by the terms $a_0, \ldots, a_{\nu-1}$ which precede a_ν. In particular, $r_0 = 0$. We solve the linear first order equations (5.1.8) with the boundary conditions

$$(5.1.9) \qquad a_0(x,x,\xi) = 1, \; a_\nu(x,x,\xi) = 0, \; \nu > 0.$$

Since it suffices to do so when $|\xi| = 1$ and then extend a_ν by homogeneity it is clear that this is possible if Ω is chosen sufficiently small.

If we now choose $a \in S^0_{1,0}$ according to Proposition 1.1.3 so that $a \sim \sum_0^\infty a_\nu$ for $|\xi| > 1$, it follows that

$$(5.1.10) \qquad a(x,x,\xi) = 1 \text{ for large } |\xi|,$$

$$(5.1.11) \quad e^{-i\varphi(x,y,\xi)}P(x,D)(e^{i\varphi(x,y,\xi)}a(x,y,\xi)) = P_m(y,\xi)a(x,y,\xi)+r(x,y,\xi)$$

where $r \in S^{-\infty}$. With a chosen in this way we introduce

$$Fu(x) = (2\pi)^{-n}\iint e^{i\varphi(x,y,\xi)}\, a(x,y,\xi)/P_m(y,\xi)\, u(y)\, dyd\xi$$

or rather the two operators F^+ and F^- corresponding to deformation of $\mathbb{R}^n$ in the directions $\pm i\, \mathrm{grad}_\xi P_m(y,\xi)$. Now we have for the almost analytic continuations of a and

$$(5.1.12) \quad e^{-i\varphi(x,y,\zeta)}P(x,D)(e^{i\varphi(x,y,\zeta)}a(x,y,\zeta)) = P_m(y,\zeta)a(x,y,\zeta) + r(x,y,\zeta)$$

where r and all of its derivatives are rapidly decreasing when $|\mathrm{Im}\,\zeta| = O(|\zeta|^\sigma)$ for some $\sigma < 1$. In fact, if we write out the left hand side of (5.1.12) explicitly so that the exponentials disappear and note that sums and products of almost analytic continuations are also almost analytic continuations, we obtain (5.1.12) using the last part of Theorem 4.2.2.

Now we may compute $PF^\pm u$ by operating under the integral sign. This is obvious if we introduce a convergence factor $e^{-\varepsilon(\zeta,\zeta)}$ which is then removed by letting $\varepsilon \to 0$. In the resulting integral we can make a change of integration back to $\mathbb{R}^n$ by using Stokes' formula as in section 4.3. This gives $P F^{\pm} = A + K^{\pm}$ where K^+ and K^- have C^∞ kernels and

$$Au(x) = (2\pi)^{-n} \iint e^{i\varphi(x,y,\xi)} a(x,y,\xi)\, u(y)\, dy d\xi.$$

Since $a \in S^0_{1,0}$ and $a(x,x,\xi) = 1$ for large $|\xi|$, we have $A = I + R_1 + R_2$ where I is the identity operator, $R_1 \in L^{-1}_{1,0}$ is properly supported and R_2 has a C^∞ kernel. It follows that the operator A has an approximate inverse. In fact, let B be a properly supported pseudo-differential operator of order 0 with the symbol

$$\sigma_B \sim \sum_0^\infty (-1)^k \sigma_{R_1^k}.$$

Then we have $AB = (I + R_1)B + R_2 = I + R_3$ where R_3 has a C^∞ kernel.

At last we can now define $E^{\pm} = F^{\pm} B$. These operators are continuous from $C_0^\infty(\Omega)$ to $C^\infty(\Omega)$ and from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$, and we have

$$P E^{\pm} = (A + K^{\pm}) B = I + K_1^{\pm}$$

where $K_1^{\pm}$ has a C^∞ kernel. Thus $E^{\pm}$ is a right parametrix. In view of the pseudo-local property of B, the singularities of the kernel of $E^{\pm}$ must be the same as those of the kernel of $F^{\pm}$. The latter can be determined using Theorem 4.3.3; we only have to examine the meaning of the condition that

(5.1.9) $\qquad \operatorname{grad}_\xi \varphi = \lambda \operatorname{grad}_\xi P_m(y,\xi)$ for some $\xi \neq 0$ and $\lambda \in \mathbb{R}$, $P_m(y,\xi) = 0$.

If $\lambda = 0$ this is equivalent to $x = y$, so we assume that $\lambda \neq 0$. Note that (5.1.9) implies that $\sum s_j \partial\varphi/\partial\xi_j = 0$ if $\sum s_j \partial P_m(y,\xi)/\partial\xi_j = 0$.

To study (5.1.9) we shall use the fact that

(5.1.10) $\qquad P_m(x, \operatorname{grad}_x \varphi) = P_m(y,\xi) = 0,$

so we have to recall some of the Hamilton-Jacobi integration theory. This relates the solution of the non-linear equation $P_m(x, \operatorname{grad} \varphi) = 0$ to the Hamilton equations

$$dx_j/dt = P_m^{(j)}(x,\ \theta) = \partial P_m(x,\theta)/\partial\theta_j,\ d\theta_j/dt = -P_{m,j}(x,\theta) = -\partial P_m(x,\theta)/\partial x_j.$$

The solutions of these equations are curves in the cotangent space, called bicharacteristic strips; their projections on the x variable are called bicharacteristics. Now if we have a solution of the equation $P_m(x, \operatorname{grad}\psi) = 0$ we obtain by differentiation that $-P_{m,j}(x,\ \operatorname{grad}\psi) = \sum P_m^{(k)}(x, \operatorname{grad}\psi)\ \partial^2\psi/\partial x_j \partial x_k$. If we integrate the equations $dx_j/dt = P_m^{(j)}(x,\ \theta)$ with $\theta = \operatorname{grad}\psi$, the remaining Hamilton equations will therefore be satisfied automatically. If $\theta = \operatorname{grad}\psi$ at one point of a bicharacteristic strip this will therefore be valid along the whole strip.

In our situation we have a solution of the equations $P_m(x,\ \operatorname{grad}_x\varphi) = P_m(y,\xi)$ depending on the parameters y and ξ. Differentiation with respect to ξ gives if s_k are constants, with y and ξ fixed now,

$$\sum P_m^{(j)}(x,\ \operatorname{grad}_x\varphi)\ \partial^2\varphi/\partial x_j\partial\xi_k s_k = 0 \text{ if } \sum s_k P_m^{(k)}(y,\xi) = 0.$$

Thus $\sum s_k\partial\varphi/\partial\xi_k$ is then constant on the bicharacteristics corresponding to φ. (This expresses the fact that bicharacteristics arise as intersections of level surfaces of infinitesimally close solutions of the equation.) Provided that Ω_1 is sufficiently small, these will intersect the plane $\langle x-y,\ t(y,\xi)\rangle = 0$ used in (5.1.6) for this is transversal to the direction $\operatorname{grad}_\xi P_m(y,\xi)$ of the bicharacteristic curve through y. Since $\operatorname{grad}_\xi\varphi = x-y + O(|x-y|^2)$ cannot have the direction $\operatorname{grad}_\xi P_m(y,\xi)$ when x is in this transversal plane and sufficiently close to but different from y, we conclude that (5.1.9) implies that x lies on a bicharacteristic starting at y, that is, on the curve determined by the initial value problem

(5.1.11) $dx_j/dt = P_m^{(j)}(x,\ \theta),\ d\theta_j/dt = -P_{m,j}(x,\ \theta),\ x(0) = y,\ \theta(0) = \xi,$

where we recall that $P_m(y,\ \xi) = 0$. Since $\partial \operatorname{grad}_\xi\varphi/\partial t = \operatorname{grad}_\xi P_m(y,\xi),\ t = 0,$

and gra $d_\xi \varphi(x,y,\xi) \neq 0$ for $x \neq y$, we conclude that grad $\varphi(x,y,\xi)$ will have the same direction as $\mathrm{grad}_\xi P_m(y,\xi)$ for $t > 0$ and the opposite direction for $t < 0$.

Let $C^{\pm}(y,\xi)$ be the parts of the bicharacteristic (5.1.11) corresponding to parameter values $t \geqq 0$ and $t \leqq 0$ respectively. Then we have found that E^+ (E^-) has a C^∞ kernel at (x,y) except when $x \in C^-(y,\xi)$ ($x \in C^+(y,\xi)$) for some ξ with $P_m(y, \xi) = 0$.

A still better result, essentially due to Grusin |1a| for operators with consta nt coefficients, can be obtained from the remarks following Theorem 4.3.3. In fact, let $\sum = \{(y, \xi) \in \Omega \times \mathbb{R}^n, P_m(y,\xi) = 0, |\xi| = 1\}$ which is a closed set, and let $\sum = F_+ \cup F_-$ where F_+ and F_- are closed subsets. Then we can choose a corresponding partition of unity ψ_1, ψ_2 according to Proposition 4.2.3. Let F be the sum of the +operator corresponding to $\psi_2 a$ and the -operator corresponding to $\psi_1 a$ as explained after Theorem 4.3.3. Then E = FB is still a right parametrix, for PF can be computed as before by letting P act under the sign of integration, the factors ψ_1 and ψ_2 being independent of x. In view of the quoted remarks following Theorem 4.3.3 we obtain

<u>Theorem 5.1.1.</u> Let P be a differential operator of principal type with real principal part, defined in an open set $\Omega \subset \mathbb{R}^n$. Every point in Ω has then an open neighborhood $\Omega_1 \subset \Omega$ where for any two closed sets $F_+, F_- \subset \sum = \{(y,\xi) \in \Omega_1 \times \mathbb{R}^n, P_m(y,\xi) = 0, |\xi| = 1\}$ with $F_+ \cup F_- = \sum$ one can find a right parametrix E of P which is a continuous map from $C_0^\infty(\Omega_1)$ to $C^\infty(\Omega_1)$ and $\mathcal{E}'(\Omega_1)$ to $\mathcal{D}'(\Omega_1)$ such that the

kernel of E is in C^∞ in $\Omega_1 \times \Omega_1$ outside

$$\{(x,y) \in \Omega_1 \times \Omega_1;\ x \in C_+(y,\xi) \text{ for some } \xi \text{ such that } (y,\xi) \in F_+ \text{ or}$$
$$x \in C_-(y,\xi) \text{ for some } \xi \text{ such that } (y,\xi) \in F_- \}.$$

Roughly speaking, this means that x shall avoid an arbitrarily chosen half of the bicharacteristic cone with vertex at y.

In section 5.2 we shall need a left parametrix E, that is, a map such that EP = I + K where K has a C^∞ kernel. Now the adjoint of a right parametrix for P is obviously a left parametrix for the adjoint of P, so we obtain from Theorem 5.1.1

Theorem 5.1.2. Let P be a differential operator of principal type with real principal part, defined in an open set $\Omega \subset \mathbb{R}^n$. Every point in Ω has then an open neighborhood $\Omega_1 \subset \Omega$ where for any two closed sets $F_+, F_- \subset \Sigma = \{(x,\xi) \in \Omega_1 \times \mathbb{R}^n,\ P_m(x,\xi) = 0,\ |\xi| = 1\}$ with $F_+ \cup F_- = \Sigma$ one can find a left parametrix E of P which is a continuous map from $C_0^\infty(\Omega_1)$ to $C^\infty(\Omega_1)$ and $\mathcal{E}'(\Omega_1)$ to $\mathcal{D}'(\Omega_1)$ such that the kernel of E is in C^∞ in $\Omega_1 \times \Omega_1$ outside

$$\{(x,y) \in \Omega_1 \times \Omega_1;\ y \in C_+(x,\xi) \text{ for some } \xi \text{ such that } (x,\xi) \in F_+ \text{ or}$$
$$y \in C_-(x,\xi) \text{ for some } \xi \text{ such that } (x,\xi) \in F_- \}.$$

5.2. The singularities of solutions of differential equations of principal type. As in section 5.1 we denote throughout by P a differential operator of principal type with real principal part in an open set $\Omega \subset \mathbb{R}^n$. Let $u \in \mathcal{D}'(\Omega)$ and

$$Pu = f \in C^\infty(\Omega). \tag{5.2.1}$$

We wish to study the singular support of u, denoted sing supp u, which is defined as the complement of the largest open subset of Ω where $u \in C^\infty$.

Theorem 5.2.1. Let $x_0 \in \Omega$ and assume that for every bicharacteristic curve through x_0 we have sing supp $u \cap C(x_0, \xi) \subset C^+(x_0, \xi)$ or $\subset C^-(x_0, \xi)$. Then $x_0 \notin$ sing supp u.

In other words, a singularity cannot propagate along half the bicharacteristic conoid only.

Proof. Since Ω can be replaced by any smaller neighborhood of x_0, we may assume that the conclusion of Theorem 5.1.2 is valid in Ω. Let $\psi \in C_0^\infty(\Omega)$ be equal to 1 in a neighborhood of x_0 and set $K = \operatorname{supp} d\psi$. This is a compact subset of $\Omega \setminus \{x_0\}$. If now E is one of the left parametrices of Theorem 5.1.2, we have $EP\psi u - \psi u \in C^\infty$, hence

$$u - EP\psi u \in C^\infty \text{ in a neighborhood of } x_0. \tag{5.2.2}$$

We wish to choose E so that $EP\psi u \in C^\infty$ in a neighborhood of x_0. The singular support of $P\psi u$ is contained in $K \cap$ sing supp u so we just have to make sure that $E(x_0, y)$ is in C^∞ when (x,y) is in a neighborhood of $\{x_0\} \times K \cap$ sing supp u. This will be true for the left parametrix of Theorem 5.1.2 provided that

$$\begin{aligned} &C^+(x_0,\xi) \cap K \cap \text{sing supp } u = \emptyset\,, \quad (x_0,\xi) \in F^+; \\ &C^-(x_0,\xi) \cap K \cap \text{sing supp } u = \emptyset\,, \quad (x_0,\xi) \in F^-. \end{aligned} \tag{5.2.3}$$

Since by hypothesis the open sets

$$O^{\pm} = \left\{ \xi;\ |\xi| = 1,\ C^{\pm}(x_0,\xi) \cap K \cap \text{sing supp } u = \emptyset \right\}$$

cover the unit ball, it is clear that we can choose F^+, F^- with the properties required in Theorem 5.1.2 so that (5.2.3) is valid. This proves the theorem.

Remark 1. The proof used only a weaker version of Theorem 5.1.2 where

F^+ and F^- are replaced by open sets. This is the result proved by Grusin |1a| when the coefficients are constant. We could have restricted the discussion in Chapter IV to $= 1$ if we had only wanted to prove this result.

Remark 2. The proof did not use the full hypotheses either for only K sing supp u played a role and not sing supp u. When P_m has constant coefficients a simple modification of the proof of Theorem 5.2.1 shows that if x sing supp u, then a bicharacteristic through x belongs to sing supp u. (This result is due to Grusin |1a|.) Is this true also when the coefficients are variable.

Remark 3. Zerner and the author have shown that for every bicharacteristic curve one can find a solution of (5.2.1) with sing supp u equal to the bicharacteristic curve. This is clearly a converse of the strong type of results discussed in Remark 2.

As an application of Theorem 5.2.1 we shall now prove a result which has previously been obtained by the author with energy integral arguments; it is clear that Theorem 5.2.1 also implies improvements of it.

Theorem 5.2.2. Let P be of principal type with real principal part in Ω, and let $\psi \in C^\infty(\Omega)$ be a function such that $\{x;\ \psi(x) \leq c\}$ is a compact subset of Ω for every c, and ψ is pseudo-convex in the sense that

$$(5.2.4)\qquad \sum_{j,k=1}^{n} \partial^2\psi/\partial x_j\partial x_k\, P_m^{(j)}(x,\xi)P_m^{(k)}(x,\xi) + \sum_{j,k=1}^{n}\big(P_{m,k}^{(j)}(x,\xi)P_m^{(k)}(x,\xi) - P_{m,k}(x,\xi)P_m^{(kj)}(x,\xi)\big)\partial\psi/\partial x_j > 0,\ 0 \neq \xi,\ x \in \Omega,$$

$$P_m(x,\xi) = 0, \quad \sum P_m^{(j)}(x,\xi)\partial\psi/\partial x_j = 0.$$

If $u \in \mathcal{E}'(\Omega)$ it follows that the supremum of ψ in sing supp u is equal to the supremum of ψ in sing supp Pu.

It is well known (see e.g. the author's lecture at the Stockholm Congress 1962) that this implies existence theorems for P in Ω.

Proof. The meaning of (5.2.4) is that on a bicharacteristic curve the second derivative of ψ is positive where the first derivative vanishes. Let c_1, c_2 be the smallest constants such that $\psi(x) \leqq c_1$ (resp. c_2) when $x \in$ sing supp u (resp. sing supp Pu). Then $c_2 \leqq c_1$. Assume that $c_2 < c_1$. For every x_0 with $\psi(x_0) = c_1$ we can then apply Theorem 5.2.1 since near x_0 one half of every bicharacteristic curve through x_0 must lie in the set $\{x; \psi(x) > c_1\}$ where u is known to be in C^∞. It follows that no point x_0 with $\psi(x_0) = c_1$ belongs to sing supp u, which contradicts the definition of c_1 and shows that $c_1 = c_2$.

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C o r r e c t i o n s

page 1.4 line 3 the subscript ρ is missing

page 1.10, line 12 $\in$ should be $\subset$

page 2.4, line 10 the subscript ρ is missing.

page 2.5, line 8 from below $\alpha!$ should be $\gamma!$.

page 2.11 line 11 two integral sigms are missing.

page 2.12 line 13 should be ... that χ has compact

line 17 Ω missing

page 2.17, formula (2.2.2); one integral sign missing

page 2.18 , formula (2.2.4). An integral sign is missing.

line 2 from below. Theorem 2.1.1 should be Theorem 2.2.1.

page 3.3, line 4 the exponent should end with $-(\rho - \delta)\nu$

line 6 replace $\varphi_x^!$ by $\lambda\varphi_x^!$

line 7 insert λ in the exponential function; replace $|\beta|$ by ν on the right hand side

line 2 from below insert a missing φ.

page 4.1. line 6 from below replace φ by q.

page 4.2, line 13 replace $m-\mu$ by $\mu - m$.

page 4.5, line 1. Insert φ on the left hand side

page 4.8, line 10. Replace t_α by $t_{|\alpha|}$.

formula (4.2.6): right hand side should be $(1+|\xi|)^{m-\rho(|\beta+\gamma|+t)}|\eta|^t$

line 5 from below χ is missing

page 4.11, line 6 from below should be ... take $\eta = |\xi|^{1-\mu}\,\mathrm{grad}_\xi\, q$

page 4.13, line 10 from below. should be ... $S^{\mu-1}_{1,0}$. Hence

line 9 from below, μ is missing there too.

page 4.18, line 10. ψ is missing

page 5.3, line 10 from below. φ is missing

page 5.4, line 4. should be ... extend a_ν by homogeneity

line 5. Ω is missing

line 12 from below ... a and φ

page 5.5, line 3 Since $a \in S^0_{1,0}$ ...

page 5.7, line 1 $\mathrm{grad}\,\varphi(x,y,\xi)$ should be $\mathrm{grad}_\xi\,\varphi(x,y,\xi)$.

page 1.6, line 11 should be $M = \dots + \chi$

Some comments to the text by **Lars Hörmander***:* **Fourier integral operators,** *Lectures at the Nordic Summer School in Mathematics, 1969. [Ho69a]*

This text is a very interesting document from a time of intense development of microlocal analysis. Pseudodifferential operators had already been around for a few years, and with the present text the author took the first steps towards a systematic treatment of Fourier integral operators. See [Ho83-85] for an account of the modern theory. The text was not intended for publication by its author, but I think it has a considerable historical value.

The second part of the notes deals with singular Fourier integral operators and the motivation here is to invert pseudodifferential operators which are not elliptic. The author approaches this problem in a very direct way by leaving the real domain and reaching a region where the symbol is invertible, somewhat in the spirit of the treatment of hyperbolic equations. To be able to do so without analyticity assumptions, Hörmander introduces the notion of almost analytic functions, which fulfill the Cauchy-Riemann equations to infinite order on the real domain.

In a parallel work, Nirenberg [Ni69] introduced almost analytic functions in his proof of the Malgrange preparation theorem. Mather [Ma69] found an alternative construction. Dynkin [Dy70, Dy72, Dy74, Dy80, Dy93] has also introduced such functions with motivations from complex and harmonic analysis with different function spaces.

The commentator has had a great benefit from [Ho69a] in his works with A. Melin [MeSj75], L. Boutet de Monvel [BoSj76] and others on the theory of Fourier integral operators with complex valued phase functions. Almost analytic functions here permit to give the right geometric descriptions of many quantities in complexified phase space and they are useful in the analysis as well.

Dynkin [Dy70, Dy72] has used almost analytic functions to develop functional calculus for classes of operators. Unaware of these works, B. Helffer and the commentator [HeSj89] reintroduced the simple functional formula for spectral problems in mathematical physics. See also [Da95, JeNa94, DiSj99]. The construction of almost analytic extensions was announced by Hörmander in [Ho69b].

We thank Gerd Grubb for supplying a well preserved copy of the lecture notes [Ho69a] and for pointing out two typos in this text which are now corrected.

Chissey en Morvan, July, 2018
Johannes Sjöstrand

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