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The motion of electrons in the field of a homogeneously winded toroid

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The motion of electrons in the field of a homogeneously winded toroid

By SVEN GÖSTA NILSSON

With 3 figures in the text

The focusing properties of the toroid are investigated for the use of it as a beta-spectroscope or as a device for space focusing in a more complicated spectroscope.

We assume a winding carrying an electric current to be distributed homogeneously along the whole surface of a toroid. Expressed in cylindrical coordinates in accordance with Fig. 1 the field inside the toroid can be easily determined:

$$B = B_\varphi = \frac{\text{const}}{r} = \frac{B_0\, r_0}{r}.$$

The field is thus constant in the φ- and z-direction as long as we stay inside the toroid.

We further assume a β-radioactive point source to be placed inside the toroid at $r = r_0$, $\varphi = 0$, $z = 0$. The limits for the angles of injection will be determined by an entrance slit. Possible directions inside this entrance slit will be described by the parameters ξ and η, where ξ and η both measure deviations from the tangent at the point of injection, ξ in the z-direction, η in the r-direction. (Fig. 2.)

The relativistic equation of motion reads

$$\frac{d}{dt}\left(\frac{m_0\,\bar{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = -\,e\bar{v}\times\bar{B}$$

where $\bar{v}$ is the velocity vector and $-e$ the charge of the electron. From this equation it is immediately apparent that $|\bar{v}| = $ const. is a solution. (The Lorentz force performs no work.) The relativistic mass $\frac{m_0}{\sqrt{1-v^2/c^2}} = m$ can thus be brought outside the differential operator $\frac{d}{dt}$, and the classical formulae are formally regained.

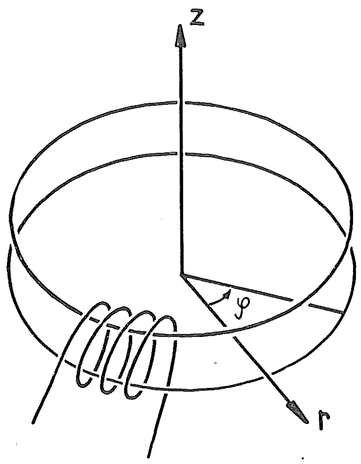

Fig. 1. The orientation of the cylindrical coordinate system in relation to the toroid.

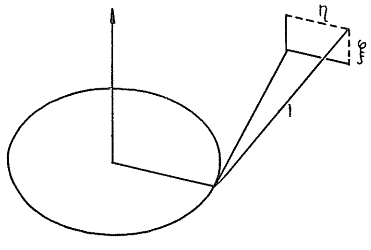

Fig. 2. The significance of the parameters ξ and η as determining the direction of the initial ray.

Introducing new variables

$$\varrho = \frac{r}{r_0}, \quad x = \frac{z}{z_0}, \quad \tau = \frac{e\,B_0}{m} \cdot t, \quad \alpha = \frac{m\,v_0}{e\,B_0\,r_0}$$

the equation (resolved into its components) can be rewritten

$$\ddot{\varrho} - \varrho\,\dot{\varphi}^2 = \frac{1}{\varrho}\dot{x} \tag{1}$$

$$\frac{1}{\varrho}\frac{d}{d\tau}(\varrho^2\,\dot{\varphi}) = 0 \tag{2}$$

$$\ddot{x} = -\frac{\dot{\varrho}}{\varrho} \qquad \text{where } \dot{\varrho} = \frac{d\varrho}{d\tau} \text{ etc.} \tag{3}$$

The initial conditions on those equations are

$$\begin{aligned} \varrho(0) &= 1 & \dot{\varrho}(0) &= \alpha\,\eta \\ \varphi(0) &= 0 & \dot{\varphi}(0) &= \beta = \alpha\sqrt{1 - \xi^2 - \eta^2} \\ x(0) &= 0 & \dot{x}(0) &= \alpha\,\xi. \end{aligned}$$

An equation containing only ϱ and its derivatives with respect to τ, can be obtained by integrating (2) and (3) once and substituting into (1)

$$\ddot{\varrho} = \frac{\beta^2}{\varrho^2} + \frac{\xi\,\alpha - \log\varrho}{\varrho}. \tag{4}$$

As the right hand side of this equation is the derivative of a function having a single maximum and tending monotonously towards $-\infty$ on both sides of this maximum, it is easily found that the solutions of this equation are *periodic* in τ.

The approximate solution of the differential equation (4).

The periodicity of the solutions proved, we now proceed to look for approximate solutions of the equation. We put

$$\varrho = 1 + f, \tag{5}$$

and assume the field to be strong enough so that $|f| \ll 1$. This condition is equivalent to α being sufficiently small.

We substitute (5) into (4), expand in powers of f, eliminate the constant term in the expansion by a new substitution $f = y + a$, and obtain, admitting terms up to the order y^3 only:

$$\ddot{y} + w^2 y = b y^2 + d y^3 \tag{6}$$

where w^2, b and d are all of order 1 if $|\xi|_{\max} |\eta|_{\max}$ and α are assumed small of order 0.1.

This equation (6) is now tackled by a modification of the Lindstedt-Poincaré method.

The method referred to, is applicable to an equation of the form $\ddot{y} + w^2 y = b y^2$, where, however, $|b|$ is to be $\ll 1$. Apart from the existence of a second term on the right side of the equation the main difficulty is the condition $|b| \ll 1$. This can be fullfilled by an artifice. We can prove $|y|$ to be small of the order α^2 and substitute $y = x q$, where $q \sim \alpha^2 \ll 1$. (6) now takes the form

$$\ddot{x} + w^2 x = b q x^2 + d q^2 x^3. \tag{7}$$

We then write, expanding in powers of q

$$x = x_0 + q x_1 + q^2 x_2 + \cdots$$
$$w^2 = w_0^2 + q w_1^2 + q^2 w_2^2 + \cdots$$

The rapidity with which we obtain satisfactory values on w_0^2 and x depends on the smallness of q.

We admit terms only to the order α^2. The real power of the method, however, is apparent first when higher approximations are aquired.

Determination of the focusing angle Φ.

We use expression (2), integrate once and expand in powers of f, insert the expression found above, and finally integrate over a whole period $\tau = \dfrac{2\pi}{w_0}$. For the integrated expression, we use the notation Φ_τ.

The ray corresponding to the central ray in ordinary lenses is not the "tangential" ray with initial conditions $\xi = 0$, $\eta = 0$. Instead $\xi = -\alpha$, $\eta = 0$ gives the solution $\varrho = \text{const} = 1$, $x = \text{const} \cdot \tau$.

Here the Lorentz force compensates the "centrifugal force". We define the Φ_τ corresponding to this latter ray as the focusing angle Φ.

$$\Phi = 2\pi\alpha\left(1 - \frac{3}{2}\alpha^2\right). \tag{7}$$

The magnitude of the field or the value of α thus essentially determines the magnitude of Φ.

The form of the image in a plane through the z-axis at the angle Φ is then derived.

ϱ and x are determined by Taylor expansions from points on the respective rays corresponding to the respective Φ_τ:s

$$\varrho_\Phi = \varrho_{\Phi_\tau} + \dot{\varrho}\,\Delta\tau + \cdots$$

where $\Delta\tau$ is determined from

$$\Phi - \Phi_\tau = \dot{\varphi}_{\Phi_\tau}\,\Delta\tau + \tfrac{1}{2}\,\ddot{\varphi}_{\Phi_\tau}\,(\Delta\tau)^2$$

x is determined anlogously. The final expression for x is:

$$x_\Phi = \pi\alpha\left[-2\alpha + \alpha^3 + \xi'\left(\xi'^2 + \eta^2 - 2\alpha^2\right)\right], \quad \text{where } \xi' = \xi + \alpha. \tag{8}$$

At this point it is to be noticed that beside the sine-oscillation in x there is a constant drift, to the second order independent of the initial x-velocity. This drift is opposite in direction for electrons and positrons — a property which makes the apparatus useful for instance for the study of γ-rays via pair creation.

Form of entrance slit. The resolving power without regard to the finite dimensions of the source and the counter slit.

x is now considered as function of the parameters ξ and η. The form of the entrance slit determines the limits of variation of the latter. The slit form is chosen to make $T/\Delta x$ maximum, where T signifies the transmission and Δx the total variation in x.

The best form of the entrance slit is shown by the egg-shaped line in Fig. 3. The dotted curves are the locus of rays exactly at focus in Φ. The signs + and — signify the "landscaping" of the $x(\xi, \eta)$-surface.

Under these conditions the transmission $T = \alpha^2\,\%$, and the "base spread"

$$\mu_0 = \frac{\Delta\varrho_0}{\varrho}\left[\text{or } \frac{\Delta(H\varrho)}{H\varrho}\right] \simeq \tfrac{1}{2}\,\alpha^2.$$

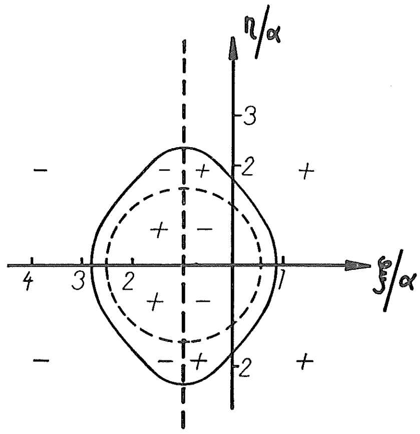

Fig. 3. The eggshaped curve shows the optimal form of the entrance slit as function of the parameters of injection. The dotted curves signify the ξ, η-locus of rays exactly confocal with the central ray. The signs + and — denote the level of x in relation to x_0 of the central ray.

The "spread" μ or the relative half width of the line is of the order $1/2\ \mu_0$, and the resolving power $R = \frac{1}{\mu}\cdot$

Table I.

a	T	μ	R	Φ
0.1	1 %	1/2 %	400	35°
0.2	4 %	2 %	100	68°
0.2[1]	1 %	1 %	200	68°

Calculations are in progress on the problem of combining the toroid with a spectroscope of the flat type for obtaining double focusing of a high order.

My thanks are due to Professor Kai Siegbahn for kindly suggesting the problem and supervising the work.

[1] Signifies another entrance slit then the optimal one.

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