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On the Hausdorff Dimension of Fat Generalised Hyperbolic Attractors

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Tomas Persson

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With 3 figures.

Abstract

We study non-invertible piecewise hyperbolic maps in the plane. The Hausdorff dimension of the attractor is estimated from below in terms of subshifts of finite type contained in the shift space. Some explicit estimates are done for a specific class of maps.

1 Introduction

A general class of piecewise hyperbolic maps was studied by Pesin in [8]. Pesin proved the existence of srb-measures and investigated their ergodic properties. Results from Pesin’s article and Sataev’s article [9] are described in Section 2. The assumptions in [8] and [9] did not allow overlaps of the images. Schmeling and Troubetzkoy extended in [10] the theory in [8] to allow maps with overlapping images.

Using the results of Pesin and techniques from Solomyak’s paper [11], the author of this paper proved in [6] and [7] that for two classes of piecewise affine hyperbolic maps, there exists, for almost all parameters, an invariant measure that is absolutely continuous with respect to Lebesgue measure, provided that the map expands area. The main difficulty that arises for these classes of maps is that in difference from the fat baker’s transformation the symbolic space associated to the systems, changes with the parameters, and also the srb-measure changes in a way that is hard to control. By embedding all symbolic spaces into a larger space it was possible get sufficient control to prove the result.

Solomyak’s proof in [11] uses a transversality property of power series. The proofs in [6] and [7] uses that iterates of points under the maps can be written as power series with such a transversality property. For the possibility of writing iterates as power series, it is important that the directions of contraction is mapped onto each other throughout the manifold. The method in [6] and [7] is therefore not good for proving similar results for more general maps. It should also be noted that this method only gives results that holds for almost every map, with respect to some parameter.
Tsujii studied in [12] a class of area-expanding solenoidal attractors and proved that generically these systems has an invariant measure that is absolutely continuous with respect to Lebesgue measure. Tsujii also used a transversality condition, but in a different way. Instead of transversality of power series, Tsujii used transversality of intersections of iterates of curves. This technique makes it possible to show the existence of an absolutely continuous invariant measure for a fixed system, provided that the appropriate transversality condition is satisfied. Tsujii proved that this transversality condition is generically satisfied.

In this paper we will use the method from Tsujii’s article [12] to estimate the dimension of the attractor from below for some piecewise hyperbolic maps and show how this estimate can be applied to a particular class of systems.

In Section 2 we present the general theory of piecewise hyperbolic maps. In Section 3 we introduce a transversality condition. Under the assumption that this transversality condition holds, a theorem that estimates the dimension from below is stated in Section 4. This theorem is proved in Section 6 and Section 5 contains explicit examples of maps that satisfy the assumptions of this theorem.

## 2 Piecewise Hyperbolic Maps

The first systematical study of piecewise hyperbolic maps was Pesin’s article [8]. He studied maps of the following form.

Let $M$ be a smooth Riemannian manifold with metic $d$, let $K \subset M$ be an open, bounded and connected set and let $N \subset K$ be a closed set in $K$. The set $N$ is called the discontinuity set. Let $f : K \setminus N \rightarrow K$.

Put

$$K^+ = \{ x \in K : f^n(x) \notin N \cup \partial K, n = 0, 1, 2, \ldots \},$$

$$D = \bigcap_{n \in \mathbb{N}} f^n(K^+).$$

The attractor of $f$ is the set $\Lambda = \overline{D}$.

The maps studied in [8] were assumed to satisfy the following conditions.

$$f : K \setminus N \rightarrow f(K \setminus N) \text{ is a } C^2 \text{-diffeomorphism.} \quad (A1)$$

There exists $C > 0$ and $\sigma \geq 0$ such that

$$\|d^2f\| \leq Cd(x, N^+)^{-\sigma}, \quad \forall x \in K \setminus N,$$

$$\|d^2(f^{-1})\| \leq Cd(x, N^-)^{-\sigma}, \quad \forall x \in f(K \setminus N), \quad (A2)$$

where $N^+ = N \cup \partial K$ and

$$N^- = \{ y \in K : \exists z_n, z \in N^+ : z_n \rightarrow z, f(z_n) \rightarrow y \}.$$
One might want to think of \( N^- \) as the image of \( N^+ \) although \( f \) is not defined on \( N^+ \).

For \( \varepsilon > 0 \) and \( l = 1, 2, \ldots \), let

\[
D_{\varepsilon,l}^+ = \{ x \in K^+ : d(f^n(x), N^+) \geq l^{-1}e^{-\varepsilon n}, \; n \in \mathbb{N} \}, \quad D_{\varepsilon,l}^- = \{ x \in A : d(f^{-n}(x), N^-) \geq l^{-1}e^{-\varepsilon n}, \; n \in \mathbb{N} \},
\]

\[
D_0 = \bigcup_{l \geq 1}(D_{\varepsilon,l}^+ \cap D_{\varepsilon,l}^-).
\]

The set \( D_0 \) is not empty for sufficiently small \( \varepsilon > 0 \).

The attractor is called regular if (A3) is satisfied. For a given map, it is usually not apparent whether the condition (A3) is satisfied or not. There exist however conditions that implies (A3) and are such that it easily can be checked if they hold true. These conditions are given in the end of this section.

There exists \( C > 0 \) and \( 0 < \lambda < 1 \) such that for every \( x \in K \setminus N^+ \) there exists cones \( C^+(x), C^u(x) \subset T_xM \) such that the angle between \( C^+(x) \) and \( C^u(x) \) is uniformly bounded away from zero,

\[
d_xf(C^+(x)) \subset C^u(f(x)) \quad \forall x \in K \setminus N^+, \]

\[
d_x(f^{-1})(C^+(x)) \subset C^u(f^{-1}(x)) \quad \forall x \in f(K \setminus N^+),
\]

and for any \( n > 0 \)

\[
\|d_x f^n(v)\| \geq C\lambda^{-n}\|v\|, \quad \forall x \in K^+, \; \forall v \in C^u(x),
\]

\[
\|d_x f^{-n}(v)\| \geq C\lambda^{-n}\|v\|, \quad \forall x \in f^n(K^+), \; \forall v \in C^u(x).
\]

The last assumption makes it possible to define stable and unstable manifolds, \( W^s(x) \) and \( W^u(x) \) as well as local ones for any \( x \in D_0^0 \).

The condition

There exists a point \( x \in D_0^0 \) and \( C, t, \delta_0 > 0 \) such that for any \( 0 < \delta < \delta_0 \) and any \( n \geq 0 \)

\[
\nu^n(f^{-n}(U(\delta, N^+))) < C\delta^t,
\]

where \( \nu^n \) is the measure on the local unstable manifold of \( x \), induced by the Riemannian measure, and \( U(\delta, N^+) \) is an open \( \delta \)-neighbourhood of \( N^+ \).

implies condition (A3). Pesin proved the following theorem.
Theorem 2.1 (Pesin [8]). Assume that \( f \) satisfies the assumptions (A1)–(A4) and (A3'). Then there exists an \( f \)-invariant measure \( \mu \) such that \( \Lambda \) can be decomposed \( \Lambda = \bigcup_{i \in \mathbb{N}} A_i \) where

- \( A_i \cap A_j = \emptyset \), if \( i \neq j \),
- \( \mu(A_0) = 0 \), \( \mu(A_i) > 0 \) if \( i > 0 \),
- \( f(A_i) = A_i \), \( f|A_i \) is ergodic,
- for \( i > 0 \) there exists \( n_i > 0 \) such that \( (f^{n_i}|A_i, \mu) \) is isomorphic to a Bernoulli shift.

The metric entropy satisfy

\[
  h_{\mu}(f) = \int \sum \chi_i(x) \, d\mu(x),
\]

where the sum is over the positive Lyapunov exponents \( \chi_i(x) \).

The measure \( \mu \) in Theorem 2.1 is called srb-measure (or Gibbs u-measure). For piecewise hyperbolic maps the srb-measures are characterised by the property that their conditional measures on unstable manifolds are absolutely continuous with respect to Lebesgue measure and the set of typical points has positive Lebesgue measure.

For a somewhat smaller class of maps Sataev proved in [9] that the ergodic components of the srb-measure (the sets \( A_i \) in Theorem 2.1) are finitely many.

The maps studied by Pesin and Sataev are all invertible on their images. Schmeling and Troubetzkoy generalised in [10] the results of Pesin to non-invertible maps: If

the set \( K \setminus N \) can be decomposed into finitely many sets \( K_i \) such that \( f : K_i \to f(K_i) \) can be extended to a diffeomorphism from \( K_i \)

\( f(K_i) \), then the statement of Theorem 2.1 is still valid. Note that \( f(K_i) \cap f(K_j) \) is allowed to be non-empty so that \( f : K \setminus N \to f(K \setminus N) \) is not a diffeomorphism. Schmeling and Troubetzkoy proved their result by lifting the map and the set \( K \) to a higher dimension; Let \( \hat{K} = K \times [0, 1] \), \( \hat{K}_i = K_i \times [0, 1] \) and

\[
  \hat{f}|K_i : (x, t) \mapsto (f(x), \tau t + i/p), \quad i = 0, 1, \ldots, p - 1,
\]

where \( \tau < 1 \) and \( p \) is the number of sets \( K_i \). The map \( \hat{f} \) is then invertible if \( \tau \) is sufficiently small and then \( \hat{f} \) satisfies the assumptions of Theorem 2.1, in particular there is an srb-measure \( \hat{\mu} \) on the lifted set \( \hat{K} \). The projection of this measure to the set \( \hat{K} \) was shown to be an srb-measure of the original map \( f \), in the sense that the set of typical points with respect to the projected measure has positive Lebesgue measure.
It is often hard to check whether (A3') holds. It is proved in [12] that if \( f \) satisfies (A2), (A4), (A5) and the assumptions (A6)--(A8) below, then \( f \) satisfies condition (A3'), and hence also (A3).

The sets \( \partial K \) and \( N \) are unions of finitely many smooth curves such that the angle between these curves and the unstable cones are bounded away from zero.

The cone families \( C_u(x) \) and \( C_s(x) \) depends continuously on \( x \in K \) and they can be extend continuously to the boundary.

There is a natural number \( q \) such that at most \( L \) singularity curves of \( f^q \) meet at any point, and \( a^q > L + 1 \) where

\[
a = \inf_{x \in K \setminus N} \inf_{v \in C^i(x)} \frac{|d_x f(v)|}{|v|}.
\]

3 A Transversality Condition

Let \( \varepsilon > 0 \) and \( 0 < \delta < 1 \). We will say that an intersection of two smooth curves \( \gamma_1 \) and \( \gamma_2 \) is \((\varepsilon, \delta)\)-transversal if for any balls \( B_1 \) and \( B_2 \) of radius \( \varepsilon \) and centre in \( \gamma_1 \) and \( \gamma_2 \) respectively, there exist points \( x_1 \in B_1 \cap \gamma_1 \) and \( x_2 \in B_2 \cap \gamma_2 \) such that the following holds true. If \( d_1 \) and \( d_2 \) are the induced metrics on \( \gamma_1 \) and \( \gamma_2 \) respectively, then the intersection of the open sets

\[
\bigcup_{y \in \gamma_i \cap B(x_i, \varepsilon)} B(y, \delta d_i(x_i, y)), \quad i = 1, 2,
\]

is empty. The symbols \( B(x, r) \) denotes the open ball of radius \( r \) around \( x \). Note that if \( \gamma_1 \) and \( \gamma_2 \) intersect \((\varepsilon, \delta)\)-transversal then the intersection \( \gamma_1 \cap \gamma_2 \) can be empty.

Definition 3.1. We will say that a piecewise hyperbolic system \( f : K \setminus N \to K \) satisfies condition \((T)\) if

there exists numbers \( \varepsilon, \delta > 0 \) such that if \( \gamma_1 \) and \( \gamma_2 \) are two smooth curves such that every tangent lies in the unstable cone field, and \( \gamma_1 \cap \gamma_2 = \emptyset \) then the curves \( f(\gamma_1) \) and \( f(\gamma_2) \) intersect \((\varepsilon, \delta)\)-transversal.

4 Dimension of the Attractor

Consider a map \( f : K \setminus N \to K \subset \mathbb{R}^2 \) that satisfies the conditions (A2), (A4) and (A5)--(A8). We denote by \( \chi^u(x) < 1 < \chi^s(x) \) the two Lyapunov exponents at the point \( x \) if they exist. If \( \Lambda_1 \) is an ergodic component of the attractor, then the Lyapunov exponents are constant almost everywhere and we write \( \chi^u(x) = \chi^u \) and \( \chi^s(x) = \chi^s \) for almost every \( x \).

Let \( \Lambda_I \) be an ergodic component of the attractor. We introduce a coding of the system \( \hat{f} : \hat{\Lambda}_I \to \Lambda_I \). If \( \hat{x} \in \hat{\Lambda}_I \) then there is a unique sequence \( \hat{z}(\hat{x}) = \{i_k\}_{k \in \mathbb{Z}} \) such
that $f^k(\bar{s}) \in K_0$ for every $k \in \mathbb{Z}$. We let $\Sigma(\hat{\lambda})$ be the set of all such sequences, that is $\Sigma(\hat{\lambda}) = \hat{\lambda}(\hat{\lambda})$.

**Theorem 4.1.** Suppose that $f: K \setminus N \rightarrow K \subset \mathbb{R}^2$ is a piecewise hyperbolic map that satisfies the conditions $(T)$, $(A2)$, $(A4)$ and $(A5)$–$(A8)$. Let $\Lambda_1$ be an ergodic component of the attractor. Then the Hausdorff dimension of $\Lambda_1$ satisfies

$$\dim_H \Lambda_1 \geq 1 + \frac{h_{\text{top}}(\Sigma_{\text{finite}})}{D_a - D_s},$$

where

$$D_a = \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in \hat{\lambda}_1} |d_x(f^n)(v)|,$$

$$D_s = \liminf_{n \to \infty} \frac{1}{n} \log \inf_{x \in \hat{\lambda}_1} |d_x(f^n)(v)|,$$

and $\Sigma_{\text{finite}} \subseteq \Sigma(\hat{\lambda}_1)$ is a subset of finite type and $h_{\text{top}}(\Sigma_{\text{finite}})$ denotes the topological entropy of $\Sigma_{\text{finite}}$.

Theorem 4.1 is proved in Section 6.

Note that in [10], it is proved that $\dim_H \Lambda_1 \leq 1 - \chi_a/\chi_s$. Hence, under the assumptions of Theorem 4.1, $\dim_H \Lambda_1$ satisfies

$$1 + \frac{\sup h_{\text{top}}(\Sigma_{\text{finite}})}{D_a - D_s} \leq \dim_H \Lambda_1 \leq 1 - \frac{\chi_a}{\chi_s}, \quad (1)$$

where the supremum is over all subshifts of finite type contained in $\Sigma(\hat{\lambda}_1)$.

### 5 An Example

Theorem 4.1 is not of explicit nature. In this section we give an example of maps satisfying the assumptions of Theorem 4.1, and estimate the supremum in (1).

Let $K = (-1,1) \times (-1,1)$ be a square. Take $-1 < k < 1$ and let $N = \{(x_1, x_2) \in K : x_2 = kx_1\}$ be the singularity set. Take $\rho \neq 0$ and let $\varphi_1$ and $\varphi_2$ be two $C^2$ functions, such that $|\varphi'_1|, |\varphi'_2| < \rho \rho' < |\rho|/2$. We take parameters $\frac{1}{2} < \lambda < 1$, $1 < \gamma < 2$, $a_1$, $a_2$, $b_1$ and $b_2$ such that the map $f$ defined by

$$f(x_1, x_2) = \begin{cases} (\lambda x_1 + a_1 + \rho x_2 + \varphi_1(x_2), \gamma x_2 + b_1) & \text{if } x_2 > kx_1 \\ (\lambda x_1 + a_2 + \varphi_2(x_2), \gamma x_2 + b_2) & \text{if } x_2 > kx_1 \end{cases} \quad (2)$$

maps $K \setminus N$ into $K$. The case $\rho \neq 0$, $\varphi_1 = \varphi_2 = 0$ and $\gamma = 2$ is threatened in [4]. There is a picture of $f$ in Figure 1.

We will use Theorem 4.1 to prove the following two theorems.

**Theorem 5.1.** If $a_1, a_2, b_1 = b_2 = (\gamma - 1)$ and

$$(\gamma, \lambda, k, \rho) \in \{(\gamma, \lambda, k, \rho) : \gamma > 2\lambda, \rho \neq 0\}$$

then $\Lambda_1$ is a subset of finite type and $h_{\text{top}}(\Sigma_{\text{finite}})$ denotes the topological entropy of $\Sigma_{\text{finite}}$.\textit{}}
are numbers such that $f: K \setminus N \to K$, then $f: K \setminus N \to K$ defined by (2) has an attractor $\Lambda$ with dimension

$$1 + \frac{\log \gamma - \varphi(\gamma, k)}{\log \gamma - \log \lambda} \leq \dim_H \Lambda \leq 1 - \frac{\log \gamma}{\log \lambda}, \tag{3}$$

where $\varphi(\gamma, k)$ is continuous and $\varphi(\gamma, k) \to 0$ as $k \to 0$.

Let $\varphi_1 = \varphi_2 = 0$, $1 < \gamma < 2$, $0 < \lambda < 1$, $a_1 = a_2 = 0$ and $b_1 = b_2 = 1 - \gamma$. Then if $\rho = 0$, the attractor is $\Lambda = \{ (x_1, x_2) : x_1 = 0, |x_2| \leq \gamma - 1 \}$, and so $\dim_H \Lambda = 1$. If $\rho \neq 0$ and $\gamma > 2\lambda$ then the dimension $\dim_H \Lambda$ satisfies the inequalities in (3). The dimension can be made arbitrarily close to 2 by choosing $\lambda$ close to 1.

Proof of Theorem 5.1. We claim that if $\gamma > 2\lambda$ and $\rho \neq 0$ then $f$ satisfies condition (T). Let us prove this claim. It is clear that the cone spanned by the vectors

$$\left( \frac{-\rho \phi}{\gamma - \lambda}, 1 \right) \text{ and } \left( \frac{\rho + \rho \phi}{\gamma - \lambda}, 1 \right)$$

defines an unstable cone family at any point of $K \setminus N$. Denote this cone by $C_u$.

If $\sigma_1 \subset K \cap \{ x_2 > kx_1 \}$ and $\sigma_2 \subset K \cap \{ x_2 < kx_1 \}$ are two curves such that if $v_1$ and $v_2$ are two tangent vectors of the curves, then $v_1, v_2 \in C_u$. The vectors $v_1$ and $v_2$ are mapped by $d_x f$ to

$$u_1 = \begin{bmatrix} \lambda & \rho + \varphi_1(x_2) \\ 0 & \gamma \end{bmatrix} v_1 \text{ and } u_2 = \begin{bmatrix} \lambda & \varphi_2(x_2) \\ 0 & \gamma \end{bmatrix} v_2$$

respectively. One checks that $u_1$ is contained in the cone spanned by
\[
\left( -\rho \frac{\lambda}{\gamma(\gamma - \lambda)} + \frac{\rho - \rho \phi}{\gamma}, 1 \right) \quad \text{and} \quad \left( (\rho + \rho \phi) \frac{\lambda}{\gamma(\gamma - \lambda)} + \frac{\rho + \rho \phi}{\gamma}, 1 \right)
\]
and $u_2$ is contained in the cone spanned by
\[
\left( -\rho \frac{\lambda}{\gamma(\gamma - \lambda)} + \frac{\rho - \rho \phi}{\gamma}, 1 \right) \quad \text{and} \quad \left( (\rho + \rho \phi) \frac{\lambda}{\gamma(\gamma - \lambda)} + \frac{\rho \phi}{\gamma}, 1 \right)
\]
The intersection of these two cones is trivial if
\[
-\rho \frac{\lambda}{\gamma(\gamma - \lambda)} + \frac{\rho - \rho \phi}{\gamma} > (\rho + \rho \phi) \frac{\lambda}{\gamma(\gamma - \lambda)} + \frac{\rho \phi}{\gamma} \iff \gamma > 2\lambda.
\]
This proves the claim.

By Theorem 4.1 it now follows that
\[
1 + \frac{\sup h_{\text{top}}(\Sigma_{\text{finite}})}{\log \gamma - \log \lambda} \leq \dim_{\text{H}} A \leq 1 - \frac{\log \gamma}{\log \lambda}.
\]
It remains to estimate the supremum of $h_{\text{top}}(\Sigma_{\text{finite}})$ where $\Sigma_{\text{finite}}$ is a subshift of finite type contained in the shift $\Sigma$ generated by the map.

Fix all parameters except for $k$. The map defined by (2) with parameter $k$ will be denoted $f_k$. For each $k$ we let $\Sigma_k$ denote the shift generated by the map $f_k$. Let $k_0 > 0$ be fixed. For any $k$ such that $|k| < k_0$ the maps $f_k$ and $f_{k_0}$ coincide on the set $K \setminus \{(x_1, x_2) : |x_2| > k_0 \}$. Let $\Gamma_{k_0}$ be the set of points in $\Lambda$ such that the orbit has empty intersection with the set $\{(x_1, x_2) : |x_2| \leq k_0 \}$.

We will describe the subshifts of finite type that lie inside $\Gamma_{k_0}$. For this purpose we can consider the map $f_{k_0}$ instead of $f_k$ since they coinide on $\Gamma_{k_0}$.

We note that $x \in \{(x_1, x_2) : |x_2| \leq k_0 \}$ if and only if
\[
f_{k_0}(x) \notin K_{k_0} = [-1, 1] \times [-1 + k_0 \gamma, 1 - k_0 \gamma].
\]
Hence
\[
\Gamma_{k_0} = \bigcap_{m=1}^{\infty} f_{k_0}^m \left( \{(x_1, x_2) \in \Lambda : f_{k_0}^m(x_1, x_2) \in K_{k_0}, \forall m \geq 0 \} \right).
\]

Since the dynamics of $(f_{k_0}, \Gamma_{k_0})$ is determined by the second coordinate, we are led to study the map $g : I \to I$ where $I = [-\gamma - 1, \gamma - 1]$ and
\[
g : x \mapsto \begin{cases} 
\gamma x - (\gamma - 1), & \text{if } x > 0, \\
\gamma x + (\gamma - 1), & \text{if } x \leq 0.
\end{cases}
\]
Hence $g$ is the restriction of $f_0$ to the second coordinate, and $\Gamma_{k_0}$ corresponds to the set
\[
\Delta_{k_0} = \{ x \in I : -(\gamma - 1) + \gamma k_0 \leq g^n(x) \leq (\gamma - 1) - \gamma k_0 \}. 
\]
We let $I_1 = I \cap \{ x > 0 \}$ and $I_{-1} = I \cap \{ x < 0 \}$. For $x \in I$ we let $g(x)$ denote the sequence $\{ s_n \}_{n=0}^{\infty}$, where $g^n(x) \in I_n$ for all $n \geq 0$. It is easy to see that

$$S(\gamma) := g(I) = \{ \{ s_n \}_{n=0}^{\infty} : -2(\gamma - 1) < \{ s_n \}_{n=0}^{\infty} \leq 2(\gamma - 1) \},$$

where the inequalities are in the sense of the lexicographic order of $\{-1, 1\}_0^\infty$ with 1 larger than $-1$. Moreover

$$S_{h_0}(\gamma) := g(\Delta h_0) = \{ \{ s_n \}_{n=0}^{\infty} : -2(\gamma - 1 - \gamma k_0) < \{ s_n \}_{n=0}^{\infty} \leq 2(\gamma - 1 - \gamma k_0) \}.$$

We note that the natural extension of $S(\gamma)$ to a two-sided infinity shift is the shift $\Sigma_0$, and the extension of $S_{h_0}(\gamma)$ is contained in $\Sigma_0$. The shift $S(\gamma)$ is of finite type if and only if $g(\gamma - 1)$ is periodic. Moreover $h_{\text{top}}(S(\gamma)) = \log \gamma$ for any $\gamma > 1$.

We now use that if $S_{h_0}(\gamma - 1 - \gamma k_0) = S_{h_0}(\gamma_0 - 1)$ for some $\gamma_0$, then $S_{h_0}(\gamma) = S(\gamma_0)$. The fact that the function $g_\gamma$ is continuous in the product topology of $\{-1, 1\}_0^\infty$, now provides us with the existence of a function $\varphi$ with the properties in the theorem. This finishes the proof.

Let us end this section with an explicit estimate of the attractor of the map in Figure 1. We will use the notations from the proof of Theorem 5.1. For this map, we have

$$i_0, i_1, \ldots : = g(\gamma - 1 - \gamma k) = 1, 1, -1, 1, -1, 1, 1, \ldots$$

If $\gamma_0$ is such that

$$j_0, j_1, \ldots : = g_{\gamma_0}(\gamma_0 - 1) = 1, (1, -1)^\infty,$$

then $\gamma_0$ is the unique positive root of the equation

$$\gamma = \sum_{n=0}^{\infty} \frac{j_n}{\gamma^n}.$$  \hfill (4)

Moreover, $g(\gamma_0 - 1) < g(\gamma - 1 - \gamma k)$, and this implies that $S(\gamma_0) \subset S_k(\gamma)$. Hence

$$\log \gamma - \varphi(\gamma, k) \geq \log \gamma_0 > \log 1.414.$$  \hfill (5)

The dimension of the attractor satisfies

$$1.193 < \dim_H A < 1.489.$$  \hfill (6)

There is a picture of the attractor $A$ in Figure 2. Similarly, if we had $k = 0$ then $\varphi(\gamma, k) = 0$ and we get the stronger estimate

$$1.328 < \dim_H A < 1.489.$$  \hfill (7)

Since $\varphi(\gamma, k)$ does not depend on $\lambda$, we can estimate the dimension of $A$ when $\gamma = 1.8, \lambda = 0.5$ and $k = 0.1$, by

$$1.270 < \dim_H A < 1.848.$$  \hfill (8)
A picture of this attractor is in Figure 3.

We can strengthen these estimates as follows. Since

$$f(K \setminus N) \subset (-\lambda - |\rho|, \lambda + |\rho|) \times (-1, 1),$$

we may consider the sequence $\mathcal{L}_p(\gamma - 1 - \gamma(\lambda + \rho)k)$ instead of $\mathcal{L}_p(\gamma - 1 - \gamma k)$ in the estimates above. For the attractor in Figure 2, we get

$$i_0, i_1, \ldots := \mathcal{L}_p(\gamma - 1 - \gamma(\lambda + \rho)k) = 1, 1, 1, -1, -1, 1, -1, 1, \ldots$$

Hence, if $\gamma_0$ is such that

$$j_0, j_1, \ldots := \mathcal{L}_{\gamma_0}(\gamma_0 - 1) = 1, 1, (1, -1, -1)^\infty,$$

then $\gamma_0$ is the unique positive root of the equation (4) and $\mathcal{L}_{\gamma_0}(\gamma_0 - 1) \leq \mathcal{L}_p(\gamma - 1 - \gamma(\lambda + \rho)k)$. As above we can use this to estimate $\log \gamma - \varphi(\gamma, k) \geq \log \gamma_0 \geq \log 1.618$. This implies that the attractor in Figure 2 satisfies

$$1.269 < \dim_H \Lambda < 1.489.$$

6 Proof of Theorem 4.1

Assume that $f$ satisfies condition (T) with $(\varepsilon_0, \delta)$-intersections.

Let $\hat{f}$ be the lift of $f$ as described in Section 2. We let $\Lambda$ denote the attractor of $\hat{f}$.

Let

$$\Sigma = \{ \mathcal{A} \in \{1, 2, \ldots, p\}^Z : \exists \hat{x} \in \hat{K} \text{ such that } \hat{f}^k(\hat{x}) \in \hat{K}_{\mathcal{A}_k}, \forall k \in \mathbb{Z} \},$$
as defined in Section 4. Then there is an one-to-one correspondence \( \rho: \Sigma \to \hat{K} \), defined in the natural way. Let \( \pi: \hat{K} \to K \) be the projection \( \pi(x, y) = x \). A cylinder is a set of the form

\[
k[a] := \{ b = (b_i)_{i=1}^l : b_i = a_i, \forall i = k, k+1, \ldots, l \}.\]

We assume that \( \hat{f} \) is ergodic with respect to the srb-measure. (If not, we can just take an ergodic component.) We let \( \Sigma_0 \) be a subshift of finite type contained in \( \Sigma(A) \).

Let \( \bar{a} \in \Sigma_0 \). Then there is a unique point \( \hat{x} \in A \) such that \( \hat{x} = \rho(\bar{a}) \). Let \( x = \pi(\hat{x}) \). Then for any sequence \( b \in \Sigma_0 \), there is a point \( \hat{y} = \rho(\xi) \in A \), such that

\[
\bar{a} = \ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots, \quad \xi = \ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots, a_{M}, b_0, b_1, \ldots.
\]

That is, \( \hat{y} \) is contained in the unstable manifold of \( \hat{x} \). Since \( \Sigma_0 \) is of finite type, this can be done for any point \( x \in \Lambda_0 := A \cap \pi(\rho(\Sigma_0)) \), with uniform bounds on \( M \).

The uniform bound on \( M \) provides a bound from below of the length of the unstable manifold around a point \( x \in A \). This proves that there exists a number \( l_0 > 0 \) such that there exists a local unstable manifold \( W^u_{l_0}(x) \) of length \( l_0 \) around \( x \), for any \( x \in \Lambda_\varepsilon \).

We take \( l_0 > 0 \) and a constant \( C \) such that for any \( n \) and any \( x \in A \) holds.

\[
\sup_x \sup_{|\nu|=1} |d_x(f^n)_{\nu}| \leq Ce^{(D_\varepsilon \varepsilon)\rho},
\]

\[
\inf_x \inf_{|\nu|=1} |d_x(f^n)_{\nu}| \geq C^{-1}e^{(-D \varepsilon)\rho}.
\]

We estimate the dimension of \( \Lambda_0 \). For this purpose we define measures \( \mu_n \) with support in \( \Lambda_0 \).

For a cylinder \( \nu_n[a] \subset \Sigma_0 \) take a point \( x(-n[a]) \in \pi(\rho(-n[a] \cap \Sigma_0)) \). Then \( W_{\nu_n}^u(x(-n[a])) \) exists. We let \( W_n \) be the collection of such unstable manifolds. Let

\[
\mu_n = \frac{1}{\#W_n} \sum_{W_n^\nu \in W_n} \nu_{W_n^\nu},
\]

where \( \nu_{W_n^\nu} \) is the normalised Lebesgue measure on the set \( W_n^\nu \).

By taking a subsequence we can achieve that \( \mu_n \) converges weakly to a measure \( \mu \) with support in \( \Lambda_0 \).

We will use the following method, originating from Frostman [3], to estimate the dimension of \( \Lambda_0 \). If

\[
\int \int \frac{d\mu(x)d\mu(y)}{|x - y|^s} < \infty,
\]

then \( \dim_H \Lambda \geq \dim_H A_0 \geq s \). For a proof of this, see Falconer’s book [3].
Let $M$ be a number. Then
\[
\int \int \min\left\{ M, \frac{1}{|x-y|} \right\} d\mu(x)d\mu(y) \to \int \int \max\left\{ M, \frac{1}{|x-y|} \right\} d\mu(x)d\mu(y), \quad \text{as } n \to \infty,
\]
and
\[
\int \int \min\left\{ M, \frac{1}{|x-y|} \right\} d\nu(x)d\nu(y) \to \int \int \frac{1}{|x-y|} d\nu(x)d\nu(y), \quad \text{as } M \to \infty.
\]

We will therefore estimate
\[
E_s(n, M) = \int \int \min\left\{ M, \frac{1}{|x-y|} \right\} d\mu(x)d\mu(y).
\]

It is clear that $E_s(n, M) \leq M$. By the definition of the measure $\mu_n$ we immediately get that
\[
E_s(n, M) = \sum_{W \in W_n} \sum_{V \in W_n} \frac{1}{(\# W_n)^2} \int \int \min\left\{ M, \frac{1}{|x-y|} \right\} d\nu_V(x)d\nu_W(y). \tag{5}
\]

We rewrite (5) as
\[
E_s(n) = J_1 + J_2,
\]
with
\[
J_1 = \sum_{W \in W_n} \frac{1}{(\# W_n)^2} \int \int \min\left\{ M, \frac{1}{|x-y|} \right\} d\nu_W(x)d\nu_W(y),
\]
\[
J_2 = \sum_{W \in W_n} \sum_{V \in W_n} \frac{1}{(\# W_n)^2} \int \int \min\left\{ M, \frac{1}{|x-y|} \right\} d\nu_V(x)d\nu_W(y).
\]

To estimate $J_1$ we note that
\[
\int \int \min\left\{ M, \frac{1}{|x-y|} \right\} d\nu_W(x)d\nu_W(y) \leq M.
\]

Hence
\[
J_1 \leq \sum_{W \in W_n} \frac{M}{(\# W_n)^2} = \frac{M}{\# W_n},
\]
and so $J_1 \to 0$ as $n \to \infty$. 
We will now estimate $J_2$. First we estimate that if $m = m(W, V)$ is the largest number such that $n - m \leq n - mC_1$, then

$$
\int \int \frac{1}{|x - y|^t} \, d\nu_{V}(x) d\nu_{W}(y) \leq C_0 e^{(D_u - D_s + 2\varepsilon)(s - 1)m(W, V)},
$$

(6)

where $C_0$ does not depend on $V$ and $W$. Indeed, if $m = m(W, V)$, then $V$ and $W$ intersect $(\varepsilon_0, C^2 e^{(D_u - D_s + 2\varepsilon)m_0}\delta_{\varepsilon})$-transversal and we can estimate

$$
\int \int \frac{1}{|x - y|^t} \, d\nu_{V}(x) d\nu_{W}(y) \leq C_0 \int_{\gamma_1} \int_{\gamma_2} \frac{1}{|x - y|^t} \, dx \, dy,
$$

where $\gamma_1$ and $\gamma_2$ are the curves

$$
\gamma_1 = \{ (x_1, x_2) : x_1 = 0, |x_2| < l_0 \},
$$

$$
\gamma_2 = \{ (x_1, x_2) : |x_2| < l_0, x_2 = C^2 e^{(D_u - D_s + 2\varepsilon)m_0} s x_1 \},
$$

and $C_0$ is a constant, that depend only on the second derivative of the map. To prove (6), one easily checks that there exists a constant $C_1$ such that

$$
C_0 \int_{\gamma_1} \int_{\gamma_2} \frac{1}{|x - y|^t} \, dx \, dy \leq C_1 e^{(D_u - D_s + 2\varepsilon)(s - 1)m}.
$$

Since $\Sigma_0$ is of finite type there is a constant $C_2$ such that for $W \in \mathcal{W}_n$

$$
\# \{ V \in \mathcal{W}_n : m(W, V) = k \} \leq C_2 e^{h_{\top}(\Sigma_0)(n-k)}
$$

This yield

$$
J_2 = \sum_{k=0}^{n-1} \sum_{W \in \mathcal{W}_n} \sum_{V \in \mathcal{W}_n} \frac{1}{(\# \mathcal{W}_n)^2} \int \int \min \left\{ M, \frac{1}{|x - y|^t} \right\} \, d\nu_{V}(x) d\nu_{W}(y)
$$

$$
\leq \sum_{k=0}^{n-1} \sum_{W \in \mathcal{W}_n} \sum_{V \in \mathcal{W}_n} C_1 e^{(D_u - D_s + 2\varepsilon)(s - 1)m(W, V)} \frac{1}{(\# \mathcal{W}_n)^2}
$$

$$
\leq \sum_{k=0}^{n-1} C_1 C_2 e^{h_{\top}(\Sigma_0)n} e^{h_{\top}(\Sigma_0)(n-k)} e^{(D_u - D_s + 2\varepsilon)(s - 1)k} \frac{1}{e^{h_{\top}(\Sigma_0)2n}}.
$$

This sum is bounded, uniformly over $n$, if $h_{\top}(\Sigma_0) > (D_u - D_s + 2\varepsilon)(s - 1)$, or equivalently

$$
s < 1 + \frac{h_{\top}(\Sigma_0)}{D_u - D_s + 2\varepsilon}.
$$

(7)

We conclude that, if $s$ satisfies (7) then the integral

$$
\int \int \min \left\{ M, \frac{1}{|x - y|^t} \right\} \, d\mu(x) d\mu(y)
$$
is uniformly bounded and hence converges as $M \to \infty$. This proves that

$$\int \int \frac{1}{|x-y|} d\mu(x) d\mu(y) < \infty$$

if (7) holds true. Hence

$$\dim H A \geq \dim H A_0 \geq 1 + \frac{h_{\text{top}}(\Sigma_0)}{D_u - D_s + 2\varepsilon}.$$ 

Let $\varepsilon \to 0$. 
On the Hausdorff Dimension of Fat Generalised Hyperbolic Attractors

References


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