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On the Hausdorff Dimension of Fat Generalised Hyperbolic Attractors

Tomas Persson

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With 3 figures.

Abstract

We study non-invertible piecewise hyperbolic maps in the plane. The Hausdorff dimension of the attractor is estimated from below in terms of subshifts of finite type contained in the shift space. Some explicit esimates are done for a specific class of maps.

1 Introduction

A general class of piecewise hyperbolic maps was studied by Pesin in [8]. Pesin proved the existence of SRB-measures and investigated their ergodic properties. Results from Pesin's article and Sataev's article [9] are described in Section 2. The assumptions in [8] and [9] did not allow overlaps of the images. Schmeling and Troubetzkoy extended in [10] the theory in [8] to allow maps with overlaping images.

Using the results of Pesin and techniques from Solomyak's paper [11], the author of this paper proved in [6] and [7] that for two classes of piecewise affine hyperbolic maps, there exists, for almost all parameters, an invariant measure that is absolutely continuous with respect to Lebesgue measure, provided that the map expands area. The main difficulty that arises for these classes of maps is that in difference from the fat baker's transformation the symbolic space associated to the systems, changes with the parameters, and also the SRB-measure changes in a way that is hard to control. By embedding all symbolic spaces into a larger space it was possible get sufficient control to prove the result.

Solomyak's proof in [11] uses a transversality property of power series. The proofs in [6] and [7] uses that iterates of points under the maps can be written as power series with such a transversality property. For the possibility of writing iterates as power series, it is important that the directions of contraction is maped onto each other throughout the manifold. The method in [6] and [7] is therefore not good for proving similar results for more general maps. It should also be noted that this method only gives results that holds for almost every map, with respect to some parameter.

Tsujii studied in [12] a class of area-expanding solenoidal attractors and proved that generically these systems has an invariant measure that is absolutely continuous with respect to Lebesgue measure. Tsujii also used a transversality condition, but in a different way. Instead of transversality of power series, Tsujii used transversality of intersections of iterates of curves. This technique makes it possible to show the existence of an absolutely continuous invariant measure for a fixed system, provided that the appropriate transversality condition is satisfied. Tsujii proved that this transversality condition is generically satisfied.

In this paper we will use the method from Tsujii's article [12] to estimate the dimension of the attractor from below for some piecewise hyperbolic maps and show how this estimate can be applied to a particular class of systems.

In Section 2 we present the general theory of piecewise hyperbolic maps. In Section 3 we introduce a transversality condition. Under the assumtion that this transversality condition holds, a theorem that estimates the dimension from below is stated in Section 4. This theorem is proved in Section 6 and Section 5 contains explicit examples of maps that satisfy the asumptions of this theorem.

2 Piecewise Hyperbolic Maps

The first systematical study of piecewise hyperbolic maps was Pesin's article [8]. He studied maps of the following form.

Let M be a smooth Riemannian manifold with metic d, let $K \subset M$ be an open, bounded and connected set and let $N \subset K$ be a closed set in K. The set N is called the discontinuity set. Let $f \colon K \setminus N \to K$.

Put

$$K^+ = \{\, x \in K : f^n(x) \notin N \cup \partial K, n = 0, 1, 2, \dots \,\},$$
$$D = \bigcap_{n \in \mathbb{N}} f^n(K^+).$$

The attractor of f is the set $\Lambda = \overline{D}$.

The maps studied in [8] were assumed to satisfy the following conditions.

$$f \colon K \setminus N \to f(K \setminus N) \text{ is a } C^2\text{-diffeomorphism.} \tag{A1}$$

There exists $C > 0$ and $\alpha \geq 0$ such that (A2)

$$\|\mathrm{d}_x^2 f\| \leq C d(x, N^+)^{-\alpha}, \qquad \forall x \in K \setminus N,$$
$$\|\mathrm{d}_x^2 (f^{-1})\| \leq C d(x, N^-)^{-\alpha}, \qquad \forall x \in f(K \setminus N),$$

where $N^+ = N \cup \partial K$ and

$$N^- = \{\, y \in K : \exists z_n, z \in N^+ \colon z_n \to z,\ f(z_n) \to y \,\}.$$

One might want to think of N^- as the image of N^+ although f is not defined on N^+.

For $\varepsilon > 0$ and $l = 1, 2, \ldots$, let (A3)

$$D^+_{\varepsilon,l} = \{\, x \in K^+ : d(f^n(x), N^+) \geq l^{-1}e^{-\varepsilon n},\ n \in \mathbb{N}\,\},$$
$$D^-_{\varepsilon,l} = \{\, x \in \Lambda : d(f^{-n}(x), N^-) \geq l^{-1}e^{-\varepsilon n},\ n \in \mathbb{N}\,\},$$
$$D^0_\varepsilon = \bigcup_{l\geq 1}(D^+_{\varepsilon,l} \cap D^-_{\varepsilon,l}).$$

The set D^0_ε is not empty for sufficiently small $\varepsilon > 0$.

The attractor is called regular if (A3) is satisfied. For a given map, it is usually not apperent whether the condition (A3) is satisfied or not. There exist however conditions that implies (A3) and are such that it easily can be checked if they hold true. These conditions are given in the end of this section.

There exists $C > 0$ and $0 < \lambda < 1$ such that for every $x \in K \setminus N^+$ there exists cones $C^{\mathrm{s}}(x), C^{\mathrm{u}}(x) \subset T_xM$ such that the angle between $C^{\mathrm{s}}(x)$ and $C^{\mathrm{u}}(x)$ is uniformly bounded away from zero, (A4)

$$\mathrm{d}_x f(C^{\mathrm{u}}(x)) \subset C^{\mathrm{u}}(f(x)) \qquad \forall x \in K \setminus N^+,$$
$$\mathrm{d}_x(f^{-1})(C^{\mathrm{s}}(x)) \subset C^{\mathrm{s}}(f^{-1}(x)) \qquad \forall x \in f(K \setminus N^+),$$

and for any $n > 0$

$$\|\mathrm{d}_x f^n(v)\| \geq C\lambda^{-n}\|v\|, \qquad \forall x \in K^+,\ \forall v \in C^{\mathrm{u}}(x),$$
$$\|\mathrm{d}_x f^{-n}(v)\| \geq C\lambda^{-n}\|v\|, \qquad \forall x \in f^n(K^+),\ \forall v \in C^{\mathrm{s}}(x).$$

The last assumption makes it possible to define stable and unstable manifolds, $W^{\mathrm{s}}(x)$ and $W^{\mathrm{u}}(x)$ as well as local ones for any $x \in D^0_\varepsilon$.

The condition

There exists a point $x \in D^0_\varepsilon$ and $C, t, \delta_0 > 0$ such that for any $0 < \delta < \delta_0$ and any $n \geq 0$ (A3′)

$$\nu^{\mathrm{u}}(f^{-n}(U(\delta, N^+))) < C\delta^t,$$

where ν^{u} is the measure on the local unstable manifold of x, induced by the Riemannian measure, and $U(\delta, N^+)$ is an open δ-neigbourhood of N^+.

implies condition (A3). Pesin proved the following theorem.

Theorem 2.1 (Pesin [8]). *Assume that f satisfies the assumptions* (A1)–(A4) *and* (A3′). *Then there exists an f-invariant measure μ such that Λ can be decomposed $\Lambda = \bigcup_{i\in\mathbb{N}} \Lambda_i$ where*

- $\Lambda_i \cap \Lambda_j = \emptyset$, *if $i \neq j$,*
- $\mu(\Lambda_0) = 0$, $\mu(\Lambda_i) > 0$ *if $i > 0$,*
- $f(\Lambda_i) = \Lambda_i$, $f|_{\Lambda_i}$ *is ergodic,*
- *for $i > 0$ there exists $n_i > 0$ such that $(f^{n_i}|_{\Lambda_i}, \mu)$ is isomorphic to a Bernoulli shift.*

The metric entropy satisfy

$$h_\mu(f) = \int \sum \chi_i(x)\, \mathrm{d}\mu(x),$$

where the sum is over the positive Lyapunov exponents $\chi_i(x)$.

The measure μ in Theorem 2.1 is called SRB-measure (or Gibbs u-measure). For piecewise hyperbolic maps the SRB-measures are characterised by the property that their conditional measures on unstable manifolds are absolutely continuous with respect to Lebesgue measure and the set of typical points has positive Lebesgue measure.

For a somewhat smaller class of maps Sataev proved in [9] that the ergodic components of the SRB-measure (the sets Λ_i in Theorem 2.1) are finitely many.

The maps studied by Pesin and Sataev are all invertible on their images. Schmeling and Troubetzkoy generalised in [10] the results of Pesin to non-invertible maps: If

> the set $K \setminus N$ can be decomposed into finitely many sets K_i such that $f\colon K_i \to f(K_i)$ can be extended to a diffeomorphism from $\overline{K}_i$ to $\overline{f(K_i)}$ (A5)

and f satisfies the assumptions (A2)–(A4) and (A3′), then the statement of Theorem 2.1 is still valid. Note that $f(K_i) \cap f(K_j)$ is allowed to be non-empty so that $f\colon K \setminus N \to f(K \setminus N)$ is not a diffeomorphism. Schmeling and Troubetzkoy proved their result by lifting the map and the set K to a higher dimension; Let $\hat{K} = K \times [0, 1]$, $\hat{K}_i = K_i \times [0, 1]$ and

$$\hat{f}|_{K_i} : (x, t) \mapsto (f(x), \tau t + i/p), \quad i = 0, 1, \ldots, p-1,$$

where $\tau < 1$ and p is the number of sets K_i. The map $\hat{f}$ is then invertible if τ is sufficiently small and then $\hat{f}$ satisfies the assumptions of Theorem 2.1, in particular there is an SRB-measure $\hat{\mu}$ on the lifted set $\hat{K}$. The projection of this measure to the set K was shown to be an SRB-measure of the original map f, in the sence that the set of typical points with respect to the projected measure has positive Lebesgue measure.

It is often hard to check whether (A3′) holds. It is proved in [10] that if f satisfies (A2), (A4), (A5) and the asumptions (A6)–(A8) below, then f satisfies condition (A3′), and hence also (A3).

The sets ∂K and N are unions of finitely many smooth curves such that the angle between these curves and the unstable cones are bounded away from zero. (A6)

The cone families $C^{\mathrm{u}}(x)$ and $C^{\mathrm{s}}(x)$ depends continuously on $x \in K_i$ and they can be extend continuously to the boundary. (A7)

There is a natural number q such that at most L singularity curves of f^q meet at any point, and $a^q > L + 1$ where (A8)

$$a = \inf_{x \in K \setminus N} \inf_{v \in C^{\mathrm{u}}(x)} \frac{|\mathrm{d}_x f(v)|}{|v|}.$$

3 A Transversality Condition

Let $\varepsilon > 0$ and $0 < \delta < 1$. We will say that an intersection of two smooth curves γ_1 and γ_2 is (ε, δ)-transversal if for any balls B_1 and B_2 of radius ε and centre in γ_1 and γ_2 respectively, there exist points $x_1 \in B_1 \cap \gamma_1$ and $x_2 \in B_2 \cap \gamma_2$ such that the following holds true. If d_1 and d_2 are the induced metrics on γ_1 and γ_2 respectively, then the intersection of the open sets

$$\bigcup_{y \in \gamma_i \cap B(x_i, \varepsilon)} B(y, \delta d_i(x_i, y)), \quad i = 1, 2,$$

is empty. The symbols $B(x, r)$ denotes the open ball of radius r around x. Note that if γ_1 and γ_2 intersect (ε, δ)-transversal then the intersection $\gamma_1 \cap \gamma_2$ can be empty.

Definition 3.1. *We will say that a piecewise hyperbolic system $f\colon K \setminus N \to K$ satisfies condition* (T) *if*

there exists numbers $\varepsilon, \delta > 0$ such that if γ_1 and γ_2 are two smooth curves such that every tangent lies in the unstable cone field, and $\gamma_1 \cap \gamma_2 = \emptyset$ then the curves $f(\gamma_1)$ and $f(\gamma_2)$ intersect (ε, δ)-transversal. (T)

4 Dimension of the Attractor

Consider a map $f\colon K \setminus N \to K \subset \mathbb{R}^2$ that satisfies the conditions (A2), (A4) and (A5)–(A8). We denote by $\chi^{\mathrm{s}}(x) < 1 < \chi^{\mathrm{u}}(x)$ the two Lyapunov exponents at the point x if they exist. If Λ_1 is and ergodic component of the attractor, then the Lyapunov exponents are constant almost everywhere and we write $\chi^{\mathrm{s}}(x) = \chi^{\mathrm{s}}$ and $\chi^{\mathrm{s}}(x) = \chi^{\mathrm{s}}$ for almost every x.

Let Λ_l be an ergodic component of the attractor. We introduce a coding of the system $\hat{f}\colon \hat{\Lambda}_l \to \hat{\Lambda}_l$. If $\hat{x} \in \hat{\Lambda}_l$ then there is a unique sequence $\hat{\underline{s}}(\hat{x}) = \{i_k\}_{k \in \mathbb{Z}}$ such that $\hat{f}^k(\hat{x}) \in K_{i_k}$ for every $k \in \mathbb{Z}$. We let $\Sigma(\hat{\Lambda}_l)$ be the set of all such sequences, that is $\Sigma(\hat{\Lambda}_l) = \hat{\underline{s}}(\hat{\Lambda}_l)$.

Theorem 4.1. *Suppose that* $f : K \setminus N \to K \subset \mathbb{R}^2$ *is a piecewise hyperbolic map that satisfies the conditions* (T), (A2), (A4) *and* (A5)–(A8). *Let* Λ_1 *be an ergodic component of the attractor. Then the Hausdorff dimension of* Λ_1 *satisfies*

$$\dim_{\mathrm{H}} \Lambda_1 \geq 1 + \frac{h_{\mathrm{top}}(\Sigma_{\mathrm{finite}})}{D_{\mathrm{u}} - D_{\mathrm{s}}},$$

where

$$D_{\mathrm{u}} = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in\Lambda_1} \sup_{|v|=1} |\mathrm{d}_x(f^n)(v)|,$$

$$D_{\mathrm{s}} = \liminf_{n\to\infty} \frac{1}{n} \log \inf_{x\in\Lambda_1} \inf_{|v|=1} |\mathrm{d}_x(f^n)(v)|,$$

and $\Sigma_{\mathrm{finite}} \subseteq \Sigma(\hat{\Lambda}_1)$ *is a subset of finite type and* $h_{\mathrm{top}}(\Sigma_{\mathrm{finite}})$ *denotes the topological entropy of* Σ_{finite}.

Theorem 4.1 is proved in Section 6.

Note that in [10], it is proved that $\dim_{\mathrm{H}} \Lambda_1 \leq 1 - \chi_{\mathrm{u}}/\chi_{\mathrm{s}}$. Hence, under the assumptions of Theorem 4.1, $\dim_{\mathrm{H}} \Lambda_1$ satisfies

$$1 + \frac{\sup h_{\mathrm{top}}(\Sigma_{\mathrm{finite}})}{D_{\mathrm{u}} - D_{\mathrm{s}}} \leq \dim_{\mathrm{H}} \Lambda_1 \leq 1 - \frac{\chi_{\mathrm{u}}}{\chi_{\mathrm{s}}}, \tag{1}$$

where the supremum is over all subshifts of finite type contained in $\Sigma(\hat{\Lambda}_1)$.

5 An Example

Theorem 4.1 is not of explicit nature. In this section we give an example of maps satifying the assumptions of Theorem 4.1, and estimate the supremum in (1).

Let $K = (-1,1) \times (-1,1)$ be a square. Take $-1 < k < 1$ and let $N = \{ (x_1, x_2) \in K : x_2 = kx_1 \}$ be the singularity set. Take $\rho \neq 0$ and let ψ_1 and ψ_2 be two C^2 functions, such that $|\psi_1'|, |\psi_2'| < \rho_\psi < |\rho|/2$. We take parameters $\frac{1}{2} < \lambda < 1$, $1 < \gamma < 2$, a_1, a_2, b_1 and b_2 such that the map f defined by

$$f(x_1, x_2) = \begin{cases} (\lambda x_1 + a_1 + \rho x_2 + \psi_1(x_2), & \gamma x_2 + b_1) \quad \text{if } x_2 > kx_1 \\ (\lambda x_1 + a_2 + \psi_2(x_2), & \gamma x_2 + b_2) \quad \text{if } x_2 > kx_1 \end{cases} \tag{2}$$

maps $K \setminus N$ into K. The case $\rho \neq 0$, $\psi_1 = \psi_2 = 0$ and $\gamma = 2$ is threated in [4]. There is a picture of f in Figure 1.

We will use Theorem 4.1 to prove the following two theorems.

Theorem 5.1. *If* $a_1, a_2, -b_1 = b_2 = (\gamma - 1)$ *and*

$$(\gamma, \lambda, k, \rho) \in \{ (\gamma, \lambda, k, \rho) : \gamma > 2\lambda,\ \rho \neq 0 \}$$

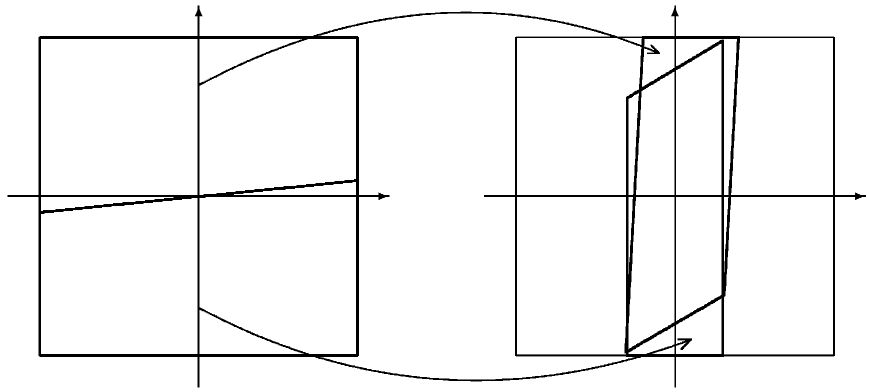

Figure 1: A picture of f with $\rho = 0.1$, $\psi_1 = \psi_2 = 0$, $\gamma = 1.8$, $\lambda = 0.3$, $k = 0.1$, $a_1 = a_2 = 0$ and $-b_1 = b_2 = 0.8$

are numbers such that $f\colon K \setminus N \to K$*, then* $f\colon K \setminus N \to K$ *defined by* (2) *has an attractor* Λ *with dimension*

$$1 + \frac{\log\gamma - \varphi(\gamma, k)}{\log\gamma - \log\lambda} \leq \dim_{\mathrm{H}} \Lambda \leq 1 - \frac{\log\gamma}{\log\lambda}, \tag{3}$$

where $\varphi(\gamma, k)$ *is continuous and* $\varphi(\gamma, k) \to 0$ *as* $k \to 0$.

Let $\psi_1 = \psi_2 = 0$, $1 < \gamma < 2$, $0 < \lambda < 1$, $a_1 = a_2 = 0$ and $b_1 = -b_2 = 1 - \gamma$. Then if $\rho = 0$, the attractor is $\Lambda = \{\, (x_1, x_2) : x_1 = 0,\ |x_2| \leq \gamma - 1 \,\}$, and so $\dim_{\mathrm{H}} \Lambda = 1$. If $\rho \neq 0$ and $\gamma > 2\lambda$ then the dimension $\dim_{\mathrm{H}} \Lambda$ satisfies the inequalities in (3). The dimension can be made arbitrarily close to 2 by choosing λ close to 1.

Proof of Theorem 5.1. We claim that if $\gamma > 2\lambda$ and $\rho \neq 0$ then f satisfies condition (T). Let us prove this claim. It is clear that the cone spanned by the vectors

$$\Big(\frac{-\rho\psi}{\gamma - \lambda}, 1\Big) \quad \text{and} \quad \Big(\frac{\rho + \rho\psi}{\gamma - \lambda}, 1\Big)$$

defines an unstable cone family at any point of $K \setminus N$. Denote this cone by C^{u}.

If $\sigma_1 \subset K \cap \{x_2 > kx_1\}$ and $\sigma_2 \subset K \cap \{x_2 < kx_1\}$ are two curves such that if v_1 and v_2 are two tangent vectors of the curves, then $v_1, v_2 \in C^{\mathrm{u}}$. The vectors v_1 and v_2 are mapped by $\mathrm{d}_x f$ to

$$u_1 = \begin{bmatrix} \lambda & \rho + \psi_1(x_2) \\ 0 & \gamma \end{bmatrix} v_1 \quad \text{and} \quad u_2 = \begin{bmatrix} \lambda & \psi_2(x_2) \\ 0 & \gamma \end{bmatrix} v_2$$

respectively. One checks that u_1 is contained in the cone spanned by

$$\Big(-\rho_\psi \frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho-\rho_\psi}{\gamma}, 1\Big) \quad \text{and} \quad \Big((\rho+\rho_\psi)\frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho+\rho_\psi}{\gamma}, 1\Big)$$

and u_2 is contained in the cone spanned by

$$\Big(-\rho_\psi \frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{-\rho_\psi}{\gamma}, 1\Big) \quad \text{and} \quad \Big((\rho+\rho_\psi)\frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho_\psi}{\gamma}, 1\Big)$$

The intersection of these two cones is trivial if

$$-\rho_\psi \frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho-\rho_\psi}{\gamma} > (\rho+\rho_\psi)\frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho_\psi}{\gamma} \quad \Leftrightarrow \quad \gamma > 2\lambda.$$

This proves the claim.

By Theorem 4.1 it now follows that

$$1 + \frac{\sup h_{\mathrm{top}}(\Sigma_{\mathrm{finite}})}{\log\gamma - \log\lambda} \leq \dim_{\mathrm{H}} \Lambda \leq 1 - \frac{\log\gamma}{\log\lambda}.$$

It remains to estimate the supremum of $h_{\mathrm{top}}(\Sigma_{\mathrm{finite}})$ where Σ_{finite} is a subshift of finite type contained in the shift Σ generated by the map.

Fix all parameters except for k. The map defined by (2) with parameter k will be denoted f_k. For each k we let Σ_k denote the shift generated by the map f_k. Let $k_0 > 0$ be fixed. For any k such that $|k| < k_0$ the maps f_k and f_{k_0} coincide on the set $K \setminus \{ (x_1, x_2) : |x_2| > k_0 \}$. Let Γ_{k_0} be the set of points in Λ such that the orbit has empty intersection with the set $\{ (x_1, x_2) : |x_2| \leq k_0 \}$.

We will describe the subshifts of finite type that lie inside Γ_{k_0}. For this purpose we can consider the map f_{k_0} instead of f_k since they coinside on Γ_{k_0}.

We note that $x \in \{ (x_1, x_2) : |x_2| \leq k_0 \}$ if and only if

$$f_{k_0}(x) \notin K_{k_0} = [-1, 1] \times [-1 + k_0\gamma, 1 - k_0\gamma].$$

Hence

$$\Gamma_{k_0} = \bigcap_{n=1}^{\infty} f_{k_0}^n\big(\{ (x_1, x_2) \in \Lambda : f_{k_0}^m(x_1, x_2) \in K_{k_0},\ \forall m \geq 0 \}\big).$$

Since the dynamics of (f_{k_0}, Γ_{k_0}) is determined by the second coordinate, we are led to study the map $g \colon I \to I$ where $I = [-(\gamma-1), \gamma-1]$ and

$$g \colon x \mapsto \begin{cases} \gamma x - (\gamma - 1), & \text{if } x > 0, \\ \gamma x + (\gamma - 1), & \text{if } x \leq 0. \end{cases}$$

Hence g is the restriction of f_0 to the second coordinate, and Γ_{k_0} corresponds to the set

$$\Delta_{k_0} = \{ x \in I : -(\gamma-1) + \gamma k_0 \leq g^n(x) \leq (\gamma-1) - \gamma k_0 \}.$$

We let $I_1 = I \cap \{ x > 0 \}$ and $I_{-1} = I \cap \{ x < 0 \}$. For $x \in I$ we let $\underline{s}(x)$ denote the sequence $\{s_n\}_{n=0}^{\infty}$, where $g^n(x) \in I_{s_n}$ for all $n \geq 0$. It is easy to see that

$$S(\gamma) := \underline{s}(I) = \{ \{s_n\}_{n=0}^{\infty} : -\underline{s}(\gamma - 1) < \{s_n\}_{n=0}^{\infty} \leq \underline{s}(\gamma - 1) \},$$

where the inequalities are in the sence of the lexicographic order of $\{-1, 1\}_0^{\infty}$ with 1 larger than -1. Moreover

$$S_{k_0}(\gamma) := \underline{s}(\Delta_{k_0}) = \{ \{s_n\}_{n=0}^{\infty} : -\underline{s}(\gamma - 1 - \gamma k_0) < \{s_n\}_{n=0}^{\infty} \leq \underline{s}(\gamma - 1 - \gamma k_0) \}.$$

We note that the natural extension of $S(\gamma)$ to a two-sided infinity shift is the shift Σ_0, and the extension of $S_{k_0}(\gamma)$ is contained in ....

The shift $S(\gamma)$ is of finite type if and only if $\underline{s}(\gamma - 1)$ is periodic. Moreover $h_{\text{top}}(S(\gamma)) = \log \gamma$ for any $\gamma > 1$.

We now use that if $\underline{s}_\gamma(\gamma - 1 - \gamma k_0) = \underline{s}_{\gamma_0}(\gamma_0 - 1)$ for some γ_0, then $S_{k_0}(\gamma) = S(\gamma_0)$. The fact that the function $\underline{s}_\gamma$ is continuous in the product topology of $\{-1, 1\}_0^{\infty}$, now provides us with the existence of a funcion φ with the properties in the theorem. This finishes the proof. ☐

Let us end this section with an explicit estimate of the attractor of the map in Figure 1. We will use the notations from the proof of Theorem 5.1. For this map, we have

$$i_0, i_1, \ldots := \underline{s}_\gamma(\gamma - 1 - \gamma k) = 1, 1, -1, 1, -1, 1, 1, \ldots$$

If γ_0 is such that

$$j_0, j_1, \ldots := \underline{s}_{\gamma_0}(\gamma_0 - 1) = 1, (1, -1)^{\infty}.$$

then γ_0 is the unique positive root of the equation

$$\gamma = \sum_{n=0}^{\infty} \frac{j_n}{\gamma^n}. \tag{4}$$

Moreover, $\underline{s}_{\gamma_0}(\gamma_0 - 1) < \underline{s}_\gamma(\gamma - 1 - \gamma k)$, and this implies that $S(\gamma_0) \subset S_k(\gamma)$. Hence $\log \gamma - \varphi(\gamma, k) \geq \log \gamma_0 > \log 1.414$. The dimension of the attractor satisfies

$$1.193 < \dim_{\mathrm{H}} \Lambda < 1.489.$$

There is a picture of the attractor Λ in Figure 2. Similarly, if we had $k = 0$ then $\varphi(\gamma, k) = 0$ and we get the stronger estimate

$$1.328 < \dim_{\mathrm{H}} \Lambda < 1.489.$$

Since $\varphi(\gamma, k)$ does not depend on λ, we can estimate the dimension of Λ when $\gamma = 1.8$, $\lambda = 0.5$ and $k = 0.1$, by

$$1.270 < \dim_{\mathrm{H}} \Lambda < 1.848.$$

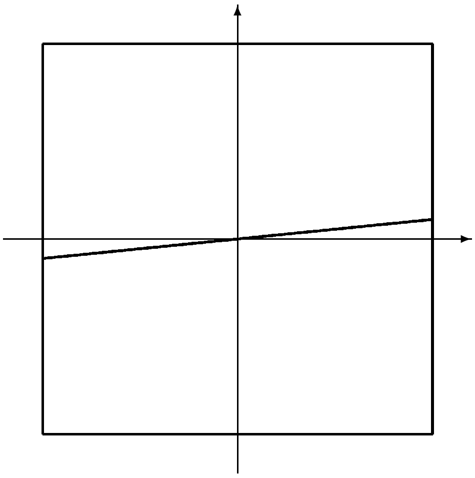

Figure 2: The attractor Λ of the map in Figure 1.

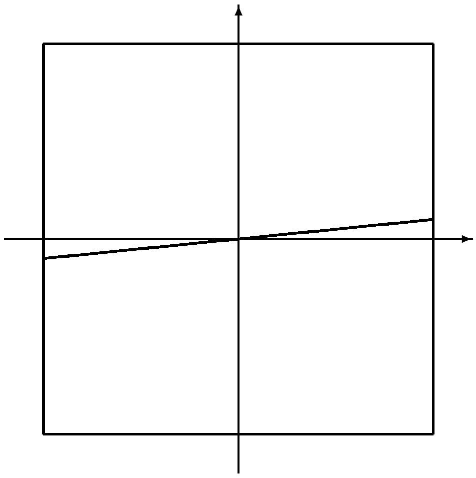

Figure 3: The attractor Λ of the map in Figure 1, with $\lambda = 0.5$.

A picture of this attractor is in Figure 3.

We can strengthen these estimates as follows. Since

$$f(K \setminus N) \subset (-\lambda - |\rho|, \lambda + |\rho|) \times (-1, 1),$$

we may consider the seqence $\underline{s}_\gamma(\gamma - 1 - \gamma(\lambda + \rho)k)$ instead of $\underline{s}_\gamma(\gamma - 1 - \gamma k)$ in the estimates above. For the attractor in Figure 2, we get

$$i_0, i_1, \ldots := \underline{s}_\gamma(\gamma - 1 - \gamma(\lambda + \rho)k) = 1, 1, 1, -1, -1, 1, -1, 1, \ldots.$$

Hence, if γ_0 is such that

$$j_0, j_1, \ldots := \underline{s}_{\gamma_0}(\gamma_0 - 1) = 1, 1, (1, -1, -1)^\infty.$$

then γ_0 is the unique positive root of the equation (4) and $\underline{s}_{\gamma_0}(\gamma_0 - 1) < \underline{s}_\gamma(\gamma - 1 - \gamma(\lambda + \rho)k)$. As above we can use this to estimate $\log \gamma - \varphi(\gamma, k) \geq \log \gamma_0 \geq \log 1.618$. This implies that the attractor in Figure 2 satisfies

$$1.269 < \dim_{\mathrm{H}} \Lambda < 1.489.$$

6 Proof of Theorem 4.1

Assume that f satisfies condition (T) with (ε_0, δ)-intersections.

Let $\hat{f}$ be the lift of f as described in Section 2. We let $\hat{\Lambda}$ denote the attractor of $\hat{f}$. Let

$$\Sigma = \{\, \underline{a} \in \{1, 2, \ldots, p\}^{\mathbb{Z}} : \exists \hat{x} \in \hat{K} \text{ such that } \hat{f}^k(\hat{x}) \in \hat{K}_{a_k},\ \forall k \in \mathbb{Z} \,\},$$

as defined in Section 4. Then there is an one-to-one correspondance $\rho \colon \Sigma \to \hat{K}$, defined in the natural way. Let $\pi \colon \hat{K} \to K$ be the projection $\pi(x, y) = x$. A cylinder is a set of the form

$$_k[\underline{a}]_l := \{\, \underline{b} \in \Sigma : b_i = a_i,\ \forall i = k, k+1, \dots, l \,\}.$$

We assume that $\hat{f}$ is ergodic with respect to the SRB-measure. (If not, we can just take an ergodic component.) We let Σ_0 be a subshift of finite type contained in $\Sigma(\hat{\Lambda})$.

Let $\underline{a} \in \Sigma_0$. Then there is a unique point $\hat{x} \in \hat{\Lambda}$ such that $\hat{x} = \rho(\underline{a})$. Let $x = \pi(\hat{x})$. Then for any sequence $\underline{b} \in \Sigma_0$, there is a point $\hat{y} = \rho(\underline{c}) \in \hat{\Lambda}$), such that

$$\begin{aligned} \underline{a} &= \quad \dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots, \\ \underline{c} &= \quad \dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots, a_M, b_0, b_1, \dots. \end{aligned}$$

That is, $\hat{y}$ is contained in the unstable manifold of $\hat{x}$. Since Σ_0 is of finite type, this can be done for any point $x \in \Lambda_0 := \Lambda \cap \pi(\rho(\Sigma_0))$, with uniform bounds on M. The uniform bound on M provides a bound from below of the length of the unstable manifold around a point $x \in \Lambda$. This proves that there exists a number $l_0 > 0$ such that there exists a local unstable manifold $W^{\mathrm{u}}_{l_0}(x)$ of length l_0 around x, for any $x \in \Lambda_\varepsilon$. We take l_0 so small that $l_0 < \varepsilon_0$.

Take $\varepsilon > 0$ and a constant C such that for any n and any $x \in \Lambda$ holds.

$$\begin{aligned} \sup_x \sup_{|v|=1} |\mathrm{d}_x(f^n)(v)| &\leq C e^{(D_{\mathrm{u}}+\varepsilon)n}, \\ \inf_x \inf_{|v|=1} |\mathrm{d}_x(f^n)(v)| &\geq C^{-1} e^{(D_{\mathrm{s}}-\varepsilon)n}. \end{aligned}$$

We estimate the dimension of Λ_0. For this purpose we define measures μ_n with support in Λ_0.

For a cylinder $_{-n}[\underline{a}]_0 \subset \Sigma_0$ take a point $x(_{-n}[\underline{a}]_0) \in \pi(\rho(_{-n}[\underline{a}]_0 \cap \Sigma_0))$. Then $W^{\mathrm{u}}_{l_0}(x(_{-n}[\underline{a}]_0))$ exists. We let $\mathcal{W}_n$ be the collection of such unstable manifolds. Let

$$\mu_n = \frac{1}{\#\mathcal{W}_n} \sum_{W^{\mathrm{u}}_{l_0} \in \mathcal{W}_n} \nu_{W^{\mathrm{u}}_{l_0}},$$

where $\nu_{W^{\mathrm{u}}_{l_0}}$ is the normalised Lebesgue measure on the set $W^{\mathrm{u}}_{l_0}$.

By taking a subsequence we can achieve that μ_n converges weakly to a measure μ with support in Λ_0.

We will use the following method, originating from Frostman [5], to estimate the dimension of Λ_0. If

$$\iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^s} < \infty,$$

then $\dim_{\mathrm{H}} \Lambda \geq \dim_{\mathrm{H}} \Lambda_0 \geq s$. For a proof of this, see Falconer's book [3].

Let M be a number. Then

$$\iint \min\left\{M, \frac{1}{|x-y|^s}\right\} \mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) \\ \to \iint \max\left\{M, \frac{1}{|x-y|^s}\right\} \mathrm{d}\mu(x)\mathrm{d}\mu(y), \quad \text{as } n \to \infty,$$

and

$$\iint \min\left\{M, \frac{1}{|x-y|^s}\right\} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \\ \to \iint \frac{1}{|x-y|^s} \mathrm{d}\mu(x)\mathrm{d}\mu(y), \quad \text{as } M \to \infty.$$

We will therefore estimate

$$E_s(n,M) = \iint \min\left\{M, \frac{1}{|x-y|^s}\right\} \mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y).$$

It is clear that $E_s(n,M) \leq M$. By the definition of the measure μ_n we immediately get that

$$E_s(n,M) \\ = \sum_{W\in\mathcal{W}_n} \sum_{V\in\mathcal{W}_n} \frac{1}{(\#\mathcal{W}_n)^2} \iint \min\left\{M, \frac{1}{|x-y|^s}\right\} \mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y). \quad (5)$$

We rewrite (5) as

$$E_s(n) = J_1 + J_2,$$

with

$$J_1 = \sum_{W\in\mathcal{W}_n} \frac{1}{(\#\mathcal{W}_n)^2} \iint \min\left\{M, \frac{1}{|x-y|^s}\right\} \mathrm{d}\nu_W(x)\mathrm{d}\nu_W(y),$$

$$J_2 = \sum_{W\in\mathcal{W}_n} \sum_{\substack{V\in\mathcal{W}_n \\ V\neq W}} \frac{1}{(\#\mathcal{W}_n)^2} \iint \min\left\{M, \frac{1}{|x-y|^s}\right\} \mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y).$$

To estimate J_1 we note that

$$\iint \min\left\{M, \frac{1}{|x-y|^s}\right\} \mathrm{d}\nu_W(x)\mathrm{d}\nu_W(y) \leq M.$$

Hence

$$J_1 \leq \sum_{W\in\mathcal{W}_n} \frac{M}{(\#\mathcal{W}_n)^2} = \frac{M}{\#\mathcal{W}_n},$$

and so $J_1 \to 0$ as $n \to \infty$.

We will now estimate J_2. First we estimate that if $m = m(W, V)$ is the largest number such that ${}_{n-m}[\underline{a}]_n = {}_{n-m}[\underline{b}]_n$ then

$$\iint \frac{1}{|x-y|^s}\,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y) \le C_0 e^{(D_\mathrm{u}-D_\mathrm{s}+2\varepsilon)(s-1)m(W,V)}, \tag{6}$$

where C_0 does not depend on V and W. Indeed, if $m = m(W, V)$, then V and W intersect $(\varepsilon_0, C^2 e^{(D_\mathrm{u}-D_\mathrm{s}+2\varepsilon)m}\delta)$-transversal and we can estimate

$$\iint \frac{1}{|x-y|^s}\,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y) \le C_0 \int_{\gamma_1}\int_{\gamma_2} \frac{1}{|x-y|^s}\,\mathrm{d}x\mathrm{d}y,$$

where γ_1 and γ_2 are the curves

$$\gamma_1 = \{\,(x_1,x_2) : x_1 = 0, |x_2| < l_0\,\},$$
$$\gamma_2 = \{\,(x_1,x_2) : |x_2| < l_0, x_2 = C^2 e^{(D_\mathrm{u}-D_\mathrm{s}+2\varepsilon)m}\delta x_1\,\},$$

and C_0 is a constant, that depend only on the second derivative of the map. To prove (6), one easily checks that there exists a constant C_1 such that

$$C_0 \int_{\gamma_1}\int_{\gamma_2} \frac{1}{|x-y|^s}\,\mathrm{d}x\mathrm{d}y \le C_1 e^{(D_\mathrm{u}-D_\mathrm{s}+2\varepsilon)(s-1)m}.$$

Since Σ_0 is of finite type there is a constant C_2 such that for $W \in \mathcal{W}_n$

$$\#\{\,V \in \mathcal{W}_n : m(W,V) = k\,\} \le C_2 e^{h_\mathrm{top}(\Sigma_0)(n-k)}$$

This yield

$$\begin{aligned}
J_2 &= \sum_{k=0}^{n-1}\sum_{W\in\mathcal{W}_n}\sum_{\substack{V\in\mathcal{W}_n\\ m(W,V)=k}} \frac{1}{(\#\mathcal{W}_n)^2}\iint \min\left\{M, \frac{1}{|x-y|^s}\right\}\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y)\\
&\le \sum_{k=0}^{n-1}\sum_{W\in\mathcal{W}_n}\sum_{\substack{V\in\mathcal{W}_n\\ m(W,V)=k}} C_1 e^{(D_\mathrm{u}-D_\mathrm{s}+2\varepsilon)(s-1)m(W,V)}\frac{1}{(\#\mathcal{W}_n)^2}\\
&\le \sum_{k=0}^{n-1} C_1 C_2^2 e^{h_\mathrm{top}(\Sigma_0)n} e^{h_\mathrm{top}(\Sigma_0)(n-k)} e^{(D_\mathrm{u}-D_\mathrm{s}+2\varepsilon)(s-1)k}\frac{1}{e^{h_\mathrm{top}(\Sigma_0)2n}}.
\end{aligned}$$

This sum is bounded, uniformly over n, if $h_\mathrm{top}(\Sigma_0) > (D_\mathrm{u} - D_\mathrm{s} + 2\varepsilon)(s-1)$, or equivalently

$$s < 1 + \frac{h_\mathrm{top}(\Sigma_0)}{D_\mathrm{u} - D_\mathrm{s} + 2\varepsilon}. \tag{7}$$

We conclude that, if s satisfies (7) then the integral

$$\iint \min\left\{M, \frac{1}{|x-y|^s}\right\}\mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

is uniformly bounded and hence converges as $M \to \infty$. This proves that

$$\iint \frac{1}{|x-y|^s} \, d\mu(x) d\mu(y) < \infty$$

if (7) holds true. Hence

$$\dim_H \Lambda \geq \dim_H \Lambda_0 \geq 1 + \frac{h_{top}(\Sigma_0)}{D_u - D_s + 2\varepsilon}.$$

Let $\varepsilon \to 0$.

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