A new upper bound on the first-event error probability for maximum-likelihood decoding of fixed binary convolutional codes

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In this correspondence we derive tighter upper bounds of both Viterbi- and Van de Meeberg-type for binary convolutional codes on the binary symmetric channel (BSC). Our bound is significantly better than Van de Meeberg's bound and generally also better than Post's bound for rates above the computational cut-off rate $R_{\text{comp}}$.

Viterbi used the union bound in his derivation. While the union bound is quite tight when there are few channel symbols in error, it is rather loose when we have many errors among the channel symbols. Thus, let us separate the error event into two disjoint events corresponding to few ($F$) and many ($M$) errors, respectively. If we let $E$ denote the event that the first information symbol is erroneously decoded by maximum-likelihood decoding on the binary symmetric channel (BSC), we have


In Section II we will use a random-walk argument to upper bound the probability that we have many channel errors, $P[M]$. The union bound is used in Section III to obtain a good upper bound on the probability that the first information symbol is erroneously decoded and that we have few channel errors, $P[E,F]$. These two bounds are combined in Section IV to a tighter Viterbi-type bound, and in Section V we give an improved Van de Meeberg-type bound. Finally, in Section VI we discuss some numerical results.

II. Many Channel Errors—Random Walk

To obtain an upper bound on the probability that we have many channel errors $P[M]$, we use a random-walk argument that is based on a lemma given by Gallager [6, p. 312].

Let $z_1, z_2, \cdots$ be a sequence of statistically independent identically distributed random variables. Then for any $\lambda < 0$ such that

$$E[2^{\lambda z_i}] \leq 1,$$

and for any $f$,

$$P\left[ \min_i \sum_{i=1}^n z_i \leq -f \right] \leq 2^M.$$  

Let us accrue a metric $s_i$ when we have a channel error and a metric $s_i$ when the channel symbol is correctly received. Then the cumulative metric along the correct path is a random walk with $P[z_i = s_i] = p$ and $P[z_i = -s_i] = q = 1 - p$, where $p$ is the crossover probability of the BSC.

Suppose we choose metrics

$$s_e = \log \frac{p}{a},$$

and

$$s_\varepsilon = \log \frac{q}{1 - a},$$

where $p < a < 1$ is a parameter to be chosen later. Then condition (2) will be satisfied for those $\lambda$ such that

$$P\left( \frac{p}{a} \right)^\lambda + q (1 - a)^\lambda \leq 1.$$

Noting that $\lambda = -1$ causes (6) to be satisfied, we have from (3) that

$$P\left[ \min_i \sum_{i=1}^n z_i \leq -f \right] \leq 2^{-f}.$$
Let \( S_k \) denote the cumulative metric for the first \( k \) channel symbols, and suppose we have \( j_k \) errors among them. Then
\[
S_k = j_k s_e + (k - j_k) s_c.
\]  
(8)

Now we can more precisely state what we mean by "few" and "many" errors. Those error patterns for which \( S_k \) stays above the barrier at \( -f \) contain few errors, and those error patterns for which the cumulative metric hits or crosses the barrier contain many errors. Few errors, i.e., \( \min_k S_k > -f \), is equivalent to
\[
j_k s_e + (k - j_k) s_c > -f,
\]
(9)
or
\[
j_k < \frac{f}{s_e} + \frac{k s_c}{s_e} \leq r_k,
\]
(10)
In Fig. 1 we illustrate inequality (10).

From inequality (7) we have
\[
P[M] = P \left[ \min S_k \leq -f \right] \leq 2^{-f},
\]
(11)
where \( f \) is a parameter to be chosen later. The bound (11) will be exploited in Section IV.

It is interesting to notice that our choice of the metrics \( s_e \) and \( s_c \) is closely related to the Fano metric [7]. To see this, let \( \varphi_0 \) be the solution of
\[
\varphi R = E_0(\varphi),
\]
(12)
where \( E_0(\varphi) \) is the Gallager function. If we choose a particular value of \( \varphi \), viz.
\[
\varphi = p^4/(1+q_0) = q^4/(1+q_0)
\]
(c.f. [6, p. 146]) we obtain from (4) and (5)
\[
s_e = \varphi_0(1 + q_0)(1 + q_0) \log 2p - R
\]
(13)
and
\[
s_c = \varphi_0(1 + q_0) \log 2q - R,
\]
(14)
which differ by a factor exactly \( \varphi_0/(1 + q_0) \) from the corresponding Fano metrics.

III. FEW CHANNEL ERRORS UNION BOUND

To upper bound the probability that we make an error when we decode the first information symbol and have few channel symbol errors, \( P[E,F] \), we use the union bound and obtain
\[
P[E,F] = \sum_k \sum_i P[E_{ki}, F],
\]
(16)
where \( E_{ki} \) is the event that a path of length \( k \) and weight \( i \) is causing a decoding error.

A path has few channel errors only if it stays below the barrier in Fig. 1 for all \( k \). If we take all paths of length \( k \) with \( j_k < r_k \) channel errors we will get all paths with few channel errors together with the paths with many channel errors that take one or more detours above the barrier. Hence we have
\[
P[E_{ki}, F] \leq P[E_{ki}, j_k < r_k],
\]
(17)
where
\[
P[E_{ki}, j_k < r_k] = \sum_{j_k < r_k} \left( \frac{k}{j_k} \right) p^{j_k} q^{k-j_k} P[E_{ki}, j_k].
\]
(18)
If we multiply the sum in (18) by \( \eta^{-(\alpha - j_k)} \), \( 0 < \eta < 1 \), we obtain the inequality
\[
P[E_{ki}, j_k < r_k] \leq \eta^{-\alpha} \sum_{j_k < r_k} \left( \frac{k}{j_k} \right) \eta p^{j_k} q^{k-j_k} P[E_{ki}, j_k].
\]
(19)
Let us introduce
\[
p_0 \triangleq \frac{\eta p}{\eta p + q} \leq p
\]
(20)
and
\[
q_0 \triangleq 1 - p_0 = \frac{q}{\eta p + q} > q.
\]
(21)
Substituting (20) and (21) into (19) and rearranging (19) we get
\[
P[E_{ki}, j_k < r_k] \leq \left( \frac{p_0 q_k}{p_0 q} \right)^k \sum_{j_k < r_k} \left( \frac{k}{j_k} \right) p_0^{j_k} q_0^{k-j_k} P[E_{ki}, j_k].
\]
(22)

Overbounding by summing over all \( 0 \leq j_k \leq k \), we have
\[
P[E_{ki}, j_k < r_k] \leq \left( \frac{p_0 q_k}{p_0 q^0} \right)^k \left( \frac{q}{q_0} \right) p_0^k D, (23)
\]
where \( P[E_{ki}, p_0] \) is the probability that a decoding error is caused by a path of length \( k \) and weight \( i \) on an improved BSC with crossover probability \( p_0 \).

Using the Bhattacharyya bound [8] we obtain from (23)
\[
P[E_{ki}, j_k < r_k] \leq \left( \frac{p_0 q_k}{p_0 q^0} \right)^k \left( \frac{q}{q_0} \right) \left( \frac{2}{\sqrt{p_0 q_0}} \right)^k.
\]
(24)
Using the definition of \( r_k \) given in (14) and rearranging (24) we get
\[
P[E_{ki}, j_k < r_k] \leq \left( \frac{p_0 q_k}{p_0 q^0} \right)^{-\alpha} L^k D, (25)
\]
where
\[
L = \frac{q}{q_0} \left( \frac{p_0}{q_0} \right)^{\alpha / (\alpha - \alpha_k)}
\]
(26)
and
\[
D = 2 \sqrt{p_0 q_0}.
\]
(27)
Finally, we combine (16) and (17) with (25) and obtain the following upper bound on the probability of having few channel symbol errors and making an error when decoding the first information symbol:
\[
P[E,F] \leq \left( \frac{p_0 q_k}{p_0 q^0} \right)^{-\alpha} \sum_i a_{ki} L^k D
\]
(28)
where \( a_{ki} \) is the number of paths of length \( k \) and weight \( i \)
IV. A Tighter Viterbi-Type Bound

We now show how to combine the bounds (1), (15), and (28) to obtain a new upper bound on the first-event error probability,

\[
P[E] \leq \left( \frac{p_{q_0}}{p_0 q} \right)^{/(s_c - s_e)} T(D, L) + 2^{-f}.
\]

(29)

The bound (29) is valid for all \( f \). By taking the derivative of the right side of (29) we find that its minimum is obtained for

\[
f_0 = \frac{(s_c - s_e) \left( \log(s_c - s_e) - \log T(D, L) - \log \frac{p_{q_0}}{p_0 q} \right)}{\log \frac{p_{q_0}}{p_0 q} + s_c - s_e}.
\]

(30)

Inserting (30) and rearranging (29) give the upper bound

\[
P[E] \leq 2^{h(y)} T(D, L)^{\gamma},
\]

where \( h(y) \) is the binary entropy function and

\[
\gamma^{-1} = 1 + \frac{\log \frac{p_{q_0}}{p_0 q}}{s_c - s_e}.
\]

(32)

Finally, we use (4) and (5) and obtain a new Viterbi-type bound,

\[
P[E] \leq \inf_{0 < p_0 \leq p < a < 1} 2^{h(y)} T(D, L)^{\gamma},
\]

(33)

where

\[
\gamma^{-1} = 1 + \frac{\log \frac{p_{q_0}}{p_0 q}}{\log \frac{q a}{p(1 - a)}}
\]

(34)

\[
D = 2\sqrt{p_0 q_0}
\]

(35)
New Van de Meeberg type bound compared with Van de Meeberg's bound and simulations for $R = 1/2$, $m = 4$ code with $G^{(1)} = 72$ and $G^{(2)} = 62$.

Fig. 6. New Van de Meeberg type bound compared with Van de Meeberg's bound and simulations for $R = 1/2$, $m = 8$, ODP code with $G^{(1)} = 557$ and $G^{(2)} = 751$.

We notice that

$$\frac{q^d - p^d}{q - p} = q^{-} \left( \frac{p}{q} \right) \left( \frac{1}{1 - 2p} \right) < \left( \frac{\sqrt{q}}{2^i} \right) \frac{1}{1 - 2p}.$$

For $p \leq 0.38$ and $d_\infty > 4$ (see Fig. 2) the bound (40) is tighter than (39), and we have (for all practical values of $p$ and $d_\infty$ slightly improved) the Van de Meeberg-type bound

$$P_{2i} < \left( \frac{2^8}{8} \right) \frac{1}{1 - 2p} \left( \frac{2^8}{1 - 2p} \right)^{2i}.$$

Since $P_{2i} = P_{2i-1}$, $i \geq 1$, we can now rewrite our Viterbi-type bound (33) as

$$P[E] < \inf_{u < p_0 < p} \inf_{p < a < 1} 2^{8\gamma} \left[ \frac{2^8}{8} \right] \left( \frac{2^8}{1 - 2p} \right)^{2i} \left( \frac{2^8}{1 - 2p_0} \right)^{2i},$$

where $\gamma$, $\delta$, $D$, and $L$ are given in (34), (38), (35), and (36), respectively.
Nonbinary Codes, Correcting Single Deletion or Insertion

GRIGORY TENENGOLTS

Abstract—In many digital communications systems, bursts of insertions or deletions are typical errors. A new class of nonbinary codes is proposed that correct a single deletion or insertion. Asymptotically, the cardinality of these codes is close to optimal. The codes can be easily implemented.

I. INTRODUCTION

In communications systems, the disturbance of synchronization can cause digits to be deleted or inserted. Binary codes that correct single deletion or insertion were introduced by Sellers [1], Levenshtein [2], and Ullman [3]. Codes that correct multiple deletions or insertions were studied by Calabi and Harnett [4] and by Tanaka and Kasai [5].

Another class of binary codes correcting synchronization errors was given by Tenengolts [6]. These codes correct bit loss and substitution error in the preceding bit. Such errors are typical, for example, in tape perforation because of a fault in the tape transport mechanism.

In many practical systems, groups of symbols are inserted or deleted (see, for example, Rainborm [7]). Therefore, a synthesis of nonbinary codes that correct deletions or insertions is of interest. This correspondence considers a class of nonbinary codes, correcting single deletion or insertion. We will show that the cardinality (number of codewords) of these codes is close to asymptotically optimal. The codes can be easily implemented.

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