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The Optimal Sampling Pattern for Linear Control Systems

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***Abstract*—In digital control systems the state is sampled at given sampling instants and the input is kept constant between two consecutive instants. By *optimal sampling problem* we mean the selection of sampling instants and control inputs, such that a given function of the state and input is minimized.**

In this paper we formulate the optimal sampling problem and we derive a necessary condition of the LQR optimality of a set of sampling instants. As the numerical solution of the optimal sampling problem is very time consuming, we also propose a new *quantization-based* sampling strategy that is computationally tractable and capable to achieve a near-optimal cost.

Finally, and probably most interesting of all, we prove that the quantization-based sampling is optimal in first-order systems for large number of samples. Experiments demonstrate that quantization-based sampling has near-optimal performance even when the system has higher order. However, it is still an open question whether or not quantization-based sampling is asymptotically optimal in any case.

***Index Terms*—Control design, Least squares approximations, Linear feedback control systems, Linear systems, Optimal control, Processor scheduling, Real-time systems.**

I. Introduction

Reducing the number of sampling instants in digital controllers may have an beneficial impact on many system features: the computing power required by the controller, the amount of needed communication bandwidth, the energy consumed by the controller, etc. In this paper, we investigate the effect of sampling on the optimal LQR. We formulate the problem as follows

$$\begin{aligned} &\mathsf{minimize}_{\bar u} \quad \int_0^T (x'Qx + \bar u' R \bar u)\, dt + x(T)'Sx(T) \\ &\mathsf{s.t.} \quad \begin{cases} \dot x = Ax + B\bar u \\ x(0) = x_0, \end{cases} \end{aligned} \tag{1}$$

where x and $\bar u$ are the state and input signals (moving over $\mathbb{R}^n$ and $\mathbb{R}^m$, resp.), $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{n\times n}$ are matrices, with Q and S positive semi-definite, R positive definite (to denote the transpose of any matrix M we use the compact Matlab-like notation M'). The control input signal $\bar u$ is constrained to be piecewise constant:

$$\bar u(t) = u_k \qquad \forall t \in [t_{k-1}, t_k)$$

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with $0 = t_0 < t_1 < \cdots < t_N = T$. The sequence $\{t_0, t_1, \dots, t_{N-1}, t_N\}$ is called *sampling pattern*, while t_k are called *sampling instants*. Often, we represent a sampling pattern by the values that separates two consecutive instants that are called *interarrivals* τ_k. The sampling instants and the interarrivals are related to one another through to the relations

$$\begin{cases} t_0 = 0 \\ t_k = \sum_{i=0}^{k-1} \tau_i \quad k \geq 1, \end{cases} \qquad \tau_k = t_{k+1} - t_k.$$

In periodic sampling we have $\tau_k = \tau$ for all k, with $\tau = T/N$ the period of the sampling.

In our formulation, we intentionally ignore disturbances to the system. While accounting for disturbances would certainly make the problem more adherent to the reality, it would also prevent us from deriving the analytical results that we propose in this paper. The extension to the case with disturbances is left as future work.

In continuous-time systems, the optimal control u that minimizes the cost in (1) can be found by solving the Riccati differential equation

$$\begin{cases} \dot K = KBR^{-1}B'K - A'K - KA - Q \\ K(T) = S \end{cases} \tag{2}$$

and then setting the input u as

$$u(t) = -R^{-1}B'K(t)x(t). \tag{3}$$

In this case, the achieved cost is

$$J_\infty = x_0' K(0) x_0.$$

For *given* sampling instants, the *optimal* values u_k of the input that minimize the cost (1) can be analytically determined through the classical discretization process described below. If we set

$$\Phi(\tau) = e^{A\tau}, \qquad \bar A_k = \Phi(\tau_k), \tag{4}$$

$$\Gamma(\tau) = \int_0^\tau e^{A(\tau - t)}\, dt\, B, \qquad \bar B_k = \Gamma(\tau_k), \tag{5}$$

$$\bar Q(\tau) = \int_0^\tau \Phi'(t) Q \Phi(t)\, dt, \qquad \bar Q_k = \bar Q(\tau_k), \tag{6}$$

$$\bar R(\tau) = \tau R + \int_0^\tau \Gamma'(t) Q \Gamma(t)\, dt, \qquad \bar R_k = \bar R(\tau_k), \tag{7}$$

$$\bar P(\tau) = \int_0^\tau \Phi(t)' Q \Gamma(t)\, dt, \qquad \bar P_k = \bar P(\tau_k), \tag{8}$$

then the problem of minimizing the cost (1) can be written as a discrete time-variant problem

$$\begin{cases} x_{k+1} = \bar A_k x_k + \bar B_k u_k \\ \text{given } x_0 \end{cases}$$

with the cost

$$J = x_N' S x_N + \sum_{k=0}^{N-1} (x_k' \bar{Q}_k x_k + u_k' \bar{R}_k u_k + 2x_k' \bar{P}_k u_k).$$

This problem is then solved using dynamic programming [1], [2]. The solution requires the backward recursive definition of the sequence of matrices $\bar{K}_k$

$$\begin{cases} \bar{K}_k = \hat{Q}_k - \hat{B}_k \hat{R}_k^{-1} \hat{B}_k' \\ \bar{K}_N = S, \end{cases} \tag{9}$$

with $\hat{Q}_k$, $\hat{R}_k$, and $\hat{B}_k$, functions of $\bar{K}_{k+1}$ as well, defined by

$$\begin{aligned} \hat{Q}_k &= \bar{Q}_k + \bar{A}_k' \bar{K}_{k+1} \bar{A}_k, \\ \hat{R}_k &= \bar{R}_k + \bar{B}_k' \bar{K}_{k+1} \bar{B}_k, \\ \hat{B}_k &= \bar{P}_k + \bar{A}_k' \bar{K}_{k+1} \bar{B}_k. \end{aligned}$$

Then, the optimal input sequence u_k is determined by

$$u_k = -\hat{R}_k^{-1} \hat{B}_k' x_k, \tag{10}$$

with minimal cost equal to

$$J = x_0' \bar{K}_0 x_0. \tag{11}$$

Equation (10) allows to compute the optimal input signal u_k for *given sampling instants* $t_0, t_1, \ldots, t_N$. In fact, the optimal input sequence depends on $\bar{A}_k$, $\bar{B}_k$, $\bar{Q}_k$, $\bar{R}_k$, $\bar{P}_k$, $\hat{Q}_k$, $\hat{R}_k$, and $\hat{B}_k$ which are all function of the inter-sample separations $\tau_k = t_{k+1} - t_k$. However, to our best knowledge, the problem of determining the optimal sampling pattern is still open.

The paper is organized as follows. In Sections II-A and II-B we recall some natural sampling techniques. In Section III we formulate the problem of optimal sampling and we report some results. Since solving the optimal sampling problem is very time consuming, in Section IV we propose a new sampling method that we call *quantization-based sampling* being related to quantization theory. In Section V we prove that quantization-based sampling is optimal for first-order systems when the number N of samples tends to ∞, while in Section VI we investigate second-order systems.

A. Related works

Triggering the activation of controllers by *events*, rather than by time, is an attempt to reduce the number of sampling instants per time unit. A first example of event-based controller was proposed by Årzén [3]. In self-triggered controller [4], the control task determines the next instant when it will be activated. Wang and Lemmon addressed self-triggered linear $\mathcal{H}_\infty$ controllers [5]. Self-triggered controllers have also been analysed and proved stable also for state-dependent homogeneous systems and polynomial systems [6]. Very recently, Rabi et at. [7] described the optimal envelope around the state that should trigger a sampling instant. Similar as in this paper, they consider the constraint of N given samples over a finite time horizon. In our paper, however, we aim at establishing a connection between quantization-based sampling (properly defined later in Section IV) and optimal sampling in absence of disturbances.

The connections between "quantization" and the control has been studied deeply in the past. Often the quantization was intended as the selection of the control input over a discrete set (rather than dense). Elia and Mitter [8] computed the optimal quantizer of the input, which was proved to be logarithmic. Xu and Cao [9] proposed a method to optimally design a control law that selects among a finite set of control inputs. The input is applied when the (scalar) state reaches a threshold. The number of thresholds is finite. In a different, although very related, research area Baines [10] proposed algorithms to find the best fitting of any function u with a piecewise linear function $\bar{u}$, which minimizes the L^2 norm of $u - \bar{u}$. However, in all of these works, the instants $t_1, \ldots, t_{N-1}$ at which the approximating function changes are not optimization variables, while in this paper we explicitly investigate the optimal selection of the sampling instants. Moreover, in our method, the control inputs $u_0, \ldots, u_{N-1}$ are not determined by a quantization procedure (as in [10]), but rather by the solution of an optimal discrete time-varying LQR problem.

Finally, a problem related to the one considered here was addressed by Kowalska and von Mohrenschildt [11] who proposed the variable time control (VTC). Similarly to our approach, they also perform the cost minimization over the sampling instants as well. However, the authors perform a linearisation of the discrete-time system in a neighbourhood of every sampling instant, losing then optimality.

The contributions of this paper are:

- the determination of a necessary condition for the optimality of a sampling pattern;
- the introduction of the *quantization-based sampling*, which is capable to provide a cost very close to the optimal one with a small computational effort;
- the proof that quantization-based sampling is optimal for first-order systems with a large number of samples;
- a numerical evaluation that shows that quantization-based sampling is near optimal for second and higher order systems as well.

II. SAMPLING METHODS

For any given sampling method, we evaluate the temporal distribution of the sampling instants by the sampling density, whereas the capacity to reduce the cost by the normalized cost. Both metrics are formally defined below.

Definition 1: Given a problem, specified by x_0, A, B, Q, R, and S, an interval length T, and a number of samples N, we define the *sampling density* $\sigma_{N,\mathsf{m}} : [0, T] \to \mathbb{R}^+$ of any sampling method m as

$$\sigma_{N,\mathsf{m}}(t) = \frac{1}{N\tau_k} \qquad \forall t \in [t_k, t_{k+1}).$$

Notice that the sampling density is normalized since:

$$\int_0^T \sigma_{N,\mathsf{m}}(t)\, dt = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \sigma_{N,\mathsf{m}}(t)\, dt = \sum_{k=0}^{N-1} \frac{1}{N\tau_k} \tau_k = 1.$$

To remove the dependency on N we also define the following density.

Definition 2: Given a problem, specified by x_0, A, B, Q, R, and S, and an interval length T, we define the *asymptotic sampling density* $\sigma_{\mathsf{m}} : [0,T] \to \mathbb{R}^+$ of any sampling method m as

$$\sigma_{\mathsf{m}}(t) = \lim_{N\to\infty} \sigma_{N,\mathsf{m}}(t).$$

The density σ_{m} provides significant information only when N is large compared to the size of the interval $[0,T]$, while in reality it is often more desirable to have a small N to reduce number of execution of the controller. Nonetheless, the asymptotic density can still provide informative results that can guide the design of more efficient sampling techniques, even when N is not large.

While the density σ_{m} provides an indication of how samples are distributed over time, the quantity defined below returns a measure of the cost associated to any sampling method.

Definition 3: Given a problem, specified by x_0, A, B, Q, R, S, an interval length T, a number of samples N, we define the *normalized cost* of any sampling method m as

$$c_{N,\mathsf{m}} = \frac{N^2}{T^2}\frac{J_{N,\mathsf{m}} - J_\infty}{J_\infty}$$

where $J_{N,\mathsf{m}}$ is the minimal cost of the sampling method m with N samples and J_∞ is the minimal cost of the continuous-time systems.

The scaling factor N^2/T^2 is motivated by the observation (proved by Melzer and Kuo [12]) that the cost $J(\tau)$ of periodic sampling with period τ can be approximated by $J_\infty + k\tau^2 + o(\tau^2)$ for small values of the period τ (the Taylor expansion in a neighbourhood of $\tau \to 0$ has not first order term).

To remove the dependency on N, we also compute the limit of the normalized cost.

Definition 4: Given a problem, specified by x_0, A, B, Q, R, and S, an interval length T, we define the *asymptotic normalized cost* of any sampling method m as

$$c_{\mathsf{m}} = \lim_{N\to\infty} c_{N,\mathsf{m}}. \tag{12}$$

The asymptotic normalized cost (12) is also very convenient from an “engineering” point of view. In fact, it can be readily used to estimate the number of samples to achieve a bounded cost increase with respect to the continuous-time case. If for a given sampling method m we can tolerate at most a (small) factor ϵ of cost increase w.r.t. the continuous-time optimal controller, then

$$(1+\epsilon)J_\infty \geq J_{N,\mathsf{m}} = T^2\frac{c_{N,\mathsf{m}}J_\infty}{N^2} + J_\infty,$$

from which we deduce

$$N \geq T\sqrt{\frac{c_{N,\mathsf{m}}}{\epsilon}} \approx T\sqrt{\frac{c_{\mathsf{m}}}{\epsilon}}. \tag{13}$$

Relation (13) constitutes a good hint for assigning the number of samples in a given interval.

Notice that the cost $J_{N,\mathsf{m}}$ of any method m with N samples, can be written by Taylor expansion as

$$J_{N,\mathsf{m}} = J_\infty\big(1+T^2N^{-2}c_{N,\mathsf{m}}\big) = J_\infty\big(1+T^2N^{-2}c_{\mathsf{m}}\big) + o(N^{-2})$$

with the remainder $o(N^{-2})$ such that $\lim_{N\to\infty} N^2 o(N^{-2}) = 0$. Hence, for a small value of N, the approximation of (13) is tight as long as the remainder $o(N^{-2})$ is small. While we do not provide any analytical result in this sense, later in the experiments of Section VII, we show that this approximation is quite tight for all the considered examples.

Below, we recall the characteristics of the existing sampling methods.

A. Periodic sampling

The simplest (and almost universally used) sampling method is the one obtained by dividing the interval $[0,T]$ in N intervals of equal size; it corresponds to the choice $t_k = kT/N$ and it is called *periodic sampling* (abbreviated per). We then have that all the inter-sampling periods are equal: $\tau_k = \tau = T/N$ for all k, and the sampling density is, obviously, constant, with

$$\sigma_{\mathsf{per},N}(t) = 1/T, \qquad \forall N.$$

For the periodic case, it is possible to determine analytically the asymptotic normalized cost c_{per}. In 1971, Melzer and Kuo [12] approximated the solution $\bar{K}(\tau)$ of the Discrete Algebraic Riccati Equation to the second order of the sampling period τ, in a neighbourhood of $\tau = 0$. They showed that

$$\bar{K}(\tau) = K_\infty + X\frac{\tau^2}{2} + o(\tau^2),$$

being K_∞ the solution of the ARE of the continuous-time problem (2) and X the second order derivative of $\bar{K}(\tau)$ in 0, that is the solution of the following Lyapunov equation

$$\mathcal{A}'X + X\mathcal{A} + \frac{1}{6}\mathcal{A}'K_\infty BR^{-1}B'K_\infty\mathcal{A} = 0. \tag{14}$$

with

$$\mathcal{A} = A - BR^{-1}B'K_\infty.$$

Melzer and Kuo [12] also proved that such a solution is positive semidefinite. Hence the normalized asymptotic cost in the periodic case is

$$\begin{aligned} c_{\mathsf{per}} &= \lim_{N\to\infty}\frac{N^2}{T^2}\frac{x_0'(K_\infty + X\frac{T^2}{2N^2} + o(N^{-2}))x_0 - x_0'K_\infty x_0}{x_0'K_\infty x_0} \\ &= \frac{x_0'Xx_0}{2x_0'K_\infty x_0}. \end{aligned}$$

In the case of a first-order system ($n = 1$), assuming without loss of generality $B = R = 1$, the Lyapunov equation (14) has the solution $X = \frac{1}{12}(K_\infty - A)K_\infty^2$ and the ARE has the solution $K_\infty = A + \sqrt{A^2+Q}$. Hence the asymptotic cost becomes

$$c_{\mathsf{per}} = \frac{1}{24}(A\sqrt{A^2+Q} + A^2 + Q). \tag{15}$$

B. Deterministic Lebesgue sampling

As the number of samples $N \to \infty$, the optimal sampled-time control input $\bar{u}$ tends to the optimal continuous-time input u. It is then natural to set the sampling instants so that $\bar{u}$ approximates u as close as possible.

A tentative sampling method, that we describe here for the only purpose of a comparison with our proposed sampling method which will be described later in Section IV, is to set a threshold Δ on the optimal input u, so that after any sampling instant t_k, the next one t_{k+1} is determined such that

$$\|u(t_{k+1}) - u(t_k)\| = \Delta.$$

with u being the optimal continuous-time input. Through this sampling rule, however, we cannot establish a clear relationship between Δ and the number N of sampling instants in $[0, T]$. If we assume that the dimension of the input space is $m = 1$ (which allows us to replace the notation of the norm $\|\cdot\|$, with the notation of the absolute value $|\cdot|$), we can enforce both a constant $|u(t_{k+1}) - u(t_k)|$ (except when $\dot{u}$ changes its sign in (t_k, t_{k+1})) and a given number N of sampling instants in $[0, T]$ by the following rule

$$\forall k = 0, \ldots, N-1, \qquad \int_{t_k}^{t_{k+1}} |\dot{u}(t)|\, dt = \frac{1}{N} \int_0^T |\dot{u}(t)|\, dt, \tag{16}$$

where u is the optimal continuous-time control input. We call this sampling method *deterministic Lebesgue sampling* (abbreviated as dls), because of its similarity to the (stochastic) Lebesgue sampling proposed by Åström and Bernhardsson [13], which applied an impulsive input at any instant when the state, affected by disturbances, was hitting a given threshold.

Following this sampling rule, by construction, the asymptotic density is

$$\sigma_{\mathsf{dls}}(t) = \frac{|\dot{u}(t)|}{\int_0^T |\dot{u}(s)|\, ds}.$$

After the sampling instants $t_1, t_2, \ldots, t_{N-1}$ are determined according to Eq. (16), the values of the control input u_k are optimally assigned according to (10).

For the dls method we are unable to determine the normalized cost c_{dls}, in general. However, in Section V we analytically compute c_{dls} for first-order systems ($n = 1$).

III. Optimal sampling

We now investigate the optimal solution of the problem (1). Let us introduce a notation that is useful in the context of this section. For any vector $x \in \mathbb{R}^n$, let us denote by $\boldsymbol{x}$ the following vector in $\mathbb{R}^{\boldsymbol{n}}$, with $\boldsymbol{n} = \frac{n(n+1)}{2}$,

$$\boldsymbol{x} = [x_1^2,\ 2x_1x_2, \ldots,\ 2x_1x_n,\\ x_2^2,\ 2x_2x_3, \ldots,\ 2x_2x_n, \ldots,\ x_{n-1}^2,\ 2x_{n-1}x_n,\ x_n^2]',$$

and for any matrix $M \in \mathbb{R}^{n\times n}$, let us denote by $\boldsymbol{M} \in \mathbb{R}^{\boldsymbol{n}}$ the vector

$$\boldsymbol{M} = [M_{1,1},\ M_{1,2}, \ldots,\ M_{1,n},\ M_{2,2},\ M_{2,3}, \ldots,\ M_{2,n}, \ldots,\\ \ldots,\ M_{n-1,n-1},\ M_{n-1,n},\ M_{n,n}]'.$$

This notation allows writing the cost (11) as $J = \boldsymbol{x}_0'\bar{\boldsymbol{K}}_0$, and the Riccati recursive equation (9), as

$$\begin{cases} \bar{\boldsymbol{K}}_k = r(\tau_k, \bar{\boldsymbol{K}}_{k+1}) \\ \bar{\boldsymbol{K}}_N = \boldsymbol{S}, \end{cases} \tag{17}$$

with $r : \mathbb{R} \times \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^{\boldsymbol{n}}$ properly defined from (9).

Since we search for stationary point of J, let us investigate the partial derivatives $\frac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_h}$. Firstly

$$h < k \qquad \Rightarrow \qquad \frac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_h} = 0,$$

because $\bar{K}_k$ depends only on the current and the future sampling intervals $\{\tau_k, \tau_{k+1}, \ldots, \tau_{N-1}\}$. Then we have

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_k} = \dfrac{\partial r}{\partial \tau}(\tau_k, \bar{\boldsymbol{K}}_{k+1}) \\ \dfrac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_h} = \dfrac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_k, \bar{\boldsymbol{K}}_{k+1}) \dfrac{\partial \bar{\boldsymbol{K}}_{k+1}}{\partial \tau_h} \qquad h > k \end{cases}$$

from which it follows

$$\frac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_h} = \left[\prod_{i=k}^{h-1} \frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_i, \bar{\boldsymbol{K}}_{i+1})\right] \frac{\partial r}{\partial \tau}(\tau_h, \bar{\boldsymbol{K}}_{h+1}) \qquad h \geq k \tag{18}$$

Notice that $\frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_i, \bar{\boldsymbol{K}}_{i+1}) \in \mathbb{R}^{p\times p}$.

Since the problem (1) is constrained by $\sum_{k=0}^{N-1} \tau_k = T$, from the KKT conditions it follows that at the optimal point the gradient ∇J must proportional to $[1, 1, \ldots, 1]$, meaning that all components of ∇J have to be equal to each other. Hence, a necessary condition for the optimum is that for every $h = 0, \ldots, N-2$

$$\frac{\partial J}{\partial \tau_h} = \frac{\partial J}{\partial \tau_{h+1}} \qquad \Longleftrightarrow \qquad \boldsymbol{x}_0' \frac{\partial \bar{\boldsymbol{K}}_0}{\partial \tau_h} = \boldsymbol{x}_0' \frac{\partial \bar{\boldsymbol{K}}_0}{\partial \tau_{h+1}},$$

which can be rewritten as

$$\boldsymbol{x}_0' \left[\prod_{i=0}^{h-1} \frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_i, \bar{\boldsymbol{K}}_{i+1})\right] \left[\frac{\partial r}{\partial \tau}(\tau_h, \bar{\boldsymbol{K}}_{h+1}) \right.\\ \left. - \frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_h, \bar{\boldsymbol{K}}_{h+1}) \frac{\partial r}{\partial \tau}(\tau_{h+1}, \bar{\boldsymbol{K}}_{h+2})\right] = 0. \tag{19}$$

Finding the analytical solution of (19) is an overwhelming task. Hence below we propose some special cases that provide some insights on how the general solution should be. In Section III-C we describe a numerical algorithm to find the solution.

A. Two sampling instants: optimality of periodic sampling

If $N = 2$, then (19) must be verified only for $h = 0$. In this special case condition (19) becomes

$$\boldsymbol{x}_0' \left[\frac{\partial r}{\partial \tau}(T - \tau_1, r(\tau_1, \boldsymbol{S})) \right.\\ \left. - \frac{\partial r}{\partial \bar{\boldsymbol{K}}}(T - \tau_1, r(\tau_1, \boldsymbol{S})) \frac{\partial r}{\partial \tau}(\tau_1, \boldsymbol{S})\right] = 0 \tag{20}$$

in which the only unknown is τ_1.

Eq. (20) also allows checking whether periodic sampling can be optimal or not. Let $\bar{K}_\tau$ be the solution of the DARE associated to the discretised system with period τ. Then a necessary condition for the optimality of the periodic sampling with period τ is

$$\boldsymbol{x}_0' \left[I_p - \frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau, \bar{\boldsymbol{K}}_\tau)\right] \frac{\partial r}{\partial \tau}(\tau, \bar{\boldsymbol{K}}_\tau) = 0 \tag{21}$$

where $I_{\boldsymbol{n}}$ denotes the identity matrix in $\mathbb{R}^{\boldsymbol{n}\times\boldsymbol{n}}$. If (21) is false, then we are certain that when the system state is at x_0, periodic sampling with period τ is not optimal.

B. First-order systems

In the case of a first-order system ($n = 1$ and then $\boldsymbol{n} = 1$) we can avoid using the **bold face** notation introduced at the beginning of Section III, since both M and $\boldsymbol{M}$ are the same scalar value.

The necessary condition for optimality (19) requires to compute the Riccati recurrence function $r(\tau, \bar{K})$ and its partial derivatives. For first-order order, the Riccati recurrence function r is

$$r(\tau, \bar{K}) = \frac{\bar{Q}_k \bar{R}_k - \bar{P}_k^2 + (\bar{A}_k^2 \bar{R}_k - 2\bar{A}_k \bar{B}_k \bar{P}_k + \bar{B}_k^2 \bar{Q}_k)\bar{K}}{\bar{R}_k + \bar{B}_k^2 \bar{K}} \tag{22}$$

with the following partial derivatives

$$\begin{aligned}
&\frac{\partial r}{\partial \bar{K}}(\tau, \bar{K}) = \frac{(\bar{A}_k \bar{R}_k - \bar{B}_k \bar{P}_k)^2}{(\bar{R}_k + \bar{B}_k^2 \bar{K})^2} \\
&\frac{\partial r}{\partial \tau}(\tau, \bar{K}) = \frac{1}{(\bar{R}_k + \bar{B}_k^2 \bar{K})^2}\Big(R(\bar{P}_k + \bar{A}_k \bar{B}_k \bar{K})^2 \\
&\quad + (\bar{A}_k \bar{R}_k - \bar{B}_k \bar{P}_k)(Q(\bar{A}_k \bar{R}_k - \bar{B}_k \bar{P}_k) \\
&\qquad + 2\bar{A}_k(A\bar{R}_k - B\bar{P}_k)K + 2\bar{A}_k \bar{B}_k(A\bar{B}_k - B\bar{A}_k)\bar{K}^2)\Big).
\end{aligned}$$

Let us now investigate the condition on $\tau_0, \ldots, \tau_{N-1}$ to satisfy (19). Since we assume $x_0 \neq 0$, we have that at least one of the two factors in (19) is equal to zero.

Remark 5: First, we observe that if τ_k is such that $\bar{A}_k \bar{R}_k = \bar{B}_k \bar{P}_k$, then $\frac{\partial r}{\partial \bar{K}}(\tau_k, \bar{K}) = 0$ for any possible $\bar{K}$. Let us set k^* as the minimum indices among the k such that $\bar{A}_k \bar{R}_k = \bar{B}_k \bar{P}_k$. From (18), it follows that $\frac{\partial \bar{K}_0}{\partial \tau_h} = 0$, for all $h \geq k^* + 1$. In fact, for such special τ_{k^*}, the value of $\bar{K}_{k^*}$ is

$$\bar{K}_{k^*} = \frac{\bar{Q}_{k^*} \bar{R}_{k^*} - \bar{P}_{k^*}^2}{\bar{R}_{k^*}}$$

that is independent of $\bar{K}_{k^*+1}$ and then independent of any $\tau_{k^*+1}, \ldots, \tau_{N-1}$. These are all potential critical points that need to be explicitly tested.

If instead all intersample separations are such that $\bar{A}_k \bar{R}_k$ never equals $\bar{B}_k \bar{P}_k$ (this happens if the minimum τ_k such that $\bar{A}_k \bar{R}_k = \bar{B}_k \bar{P}_k$ is larger than T, or when τ_k is small enough since $\bar{A}_k \bar{R}_k = \tau_k R + o(\tau_k)$ and $\bar{B}_k \bar{P}_k = \frac{B^2 Q}{2}\tau_k^3 + o(\tau_k^3)$), then from (19) it follows that an optimal sampling pattern must satisfy the condition

$$\begin{aligned}
&\frac{\partial r}{\partial \tau}(\tau_h, r(\tau_{h+1}, \bar{K}_{h+2})) - \\
&\qquad \frac{\partial r}{\partial \bar{K}}(\tau_h, r(\tau_{h+1}, \bar{K}_{h+2}))\frac{\partial r}{\partial \tau}(\tau_{h+1}, \bar{K}_{h+2}) = 0.
\end{aligned} \tag{23}$$

This relationship allows finding all intersample separations $\tau_0, \ldots, \tau_{N-2}$ starting from any τ_{N-1} using the backward recursive equations (23) and (17). In order to fulfil the equality $\sum_{k=0}^{N-1} \tau_k = T$ we have then to choose τ_{N-1} appropriately; this is made through an iterative procedure that we implemented to scale the value τ_{N-1} until the constraint $\sum_{k=0}^{N-1} \tau_k = T$ is verified. In Section V, this condition will be exploited to find the asymptotic behaviour of optimal sampling.

C. Numerical solution

In general, finding the $\tau_0, \ldots, \tau_{N-1}$ that solve Eq. (19) is very hard. Hence we did implement a gradient descent algorithm, which iteratively performs the following steps:

1) computes the gradient $\nabla J = \left(\frac{\partial J}{\partial \tau_0}, \ldots, \frac{\partial J}{\partial \tau_{N-1}}\right)$ at the current solution;
2) project ∇J onto the equality constraint $\sum_{k=0}^{N-1} \tau_k = T$ by removing the component that is orthogonal to the constraint;
3) performs a step along the negative projected gradient and then update the solution if the cost has been reduced or reduce the length of the step if the cost is not reduced.

As it will be later shown in Section VII this numerical optimization procedure is capable to find solutions that are much better than both periodic and dls sampling. However, this considerable cost reduction has a price. The major drawback of the numerical algorithm is certainly its complexity. Moreover, being the problem non-convex, the gradient-descent algorithm does not guarantee to reach the global minimum.

Only the computation of the gradient of the matrix $\bar{K}_0$ with respect to all sampling instants has the complexity of $O(N^2 n^3)$. This step needs to be computed over and over until numerical stopping criteria of the gradient descent algorithm are reached. While still giving interesting insights on the optimal sampling pattern problem, this considerable computational cost prevents both computing the asymptotic behaviour for large N and practical applications of this result. For this reason, we propose below another solution which demonstrated surprising properties (proved later in Section V).

IV. Quantization-based sampling

In this section we describe a sampling method that is capable to provide a near-minimal cost (considerably lower than dls sampling) without requiring to execute a heavy optimization routines. The basic idea is to approximate the optimal continuous-time control input u with a piecewise constant function.

This approach is well studied under the name of *quantization*, a discretization procedure which aims to approximate a function, in the L^p sense, by means of piecewise constant functions. Given a function $u \in L^p(\Omega)$ the goal is to find a piecewise constant function $\bar{u}$ taking only N values, which realizes the best approximation of u, in the sense that the $L^p(\Omega)$ norm

$$\int_\Omega \|u(x) - \bar{u}(x)\|_p \, dx$$

is minimal. In our case, if the dimension of the input space is $m = 1$, the quantization problem can be formulated as minimizing the quantization error E_{qnt}

$$E_{\mathsf{qnt}} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} |u(t) - u_k|^2 \, dt. \tag{24}$$

with $t_0 = 0$ and $t_N = T$.

In this problem the unknowns are the constants $\{u_0, \ldots, u_{N-1}\}$ to approximate the function u, as well as the intermediate instants $\{t_1, \ldots, t_{N-1}\}$. If we differentiate the quantization error E_{qnt} with respect to u_k we find that

$$u_k = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} u \, dt. \tag{25}$$

which, not surprisingly, states that the constant u_k that better approximates u in the interval $[t_k, t_{k+1}]$ is its average value over the interval. Thanks to (25) the quantization cost can be rewritten as

$$E_{\mathsf{qnt}} = \int_0^T |u|^2 \, \mathrm{d}t - \sum_{k=0}^{N-1} (t_{k+1} - t_k)|u_k|^2.$$

Now we differentiate the error with respect to t_k, with $k = 1, \ldots, N-1$. We find

$$\begin{aligned}
\frac{\partial E_{\mathsf{qnt}}}{\partial t_k} &= -\frac{\partial}{\partial t_k}\left(\frac{1}{t_{k+1} - t_k}\left|\int_{t_k}^{t_{k+1}} u \, dt\right|^2\right) - \\
&\qquad \frac{\partial}{\partial t_k}\left(\frac{1}{t_k - t_{k-1}}\left|\int_{t_{k-1}}^{t_k} u \, dt\right|^2\right) \\
&= -\frac{1}{(t_{k+1} - t_k)^2}\left|\int_{t_k}^{t_{k+1}} u \, dt\right|^2 + 2u'(t_k)u_k + \\
&\qquad \frac{1}{(t_k - t_{k-1})^2}\left|\int_{t_{k-1}}^{t_k} u \, dt\right|^2 - 2u'(t_k)u_{k-1} \\
&= -|u_k|^2 + 2u'(t_k)u_k + |u_{k-1}|^2 - 2u'(t_k)u_{k-1} \\
&= |u_{k-1} - u(t_k)|^2 - |u_k - u(t_k)|^2
\end{aligned}$$

from which it follows that the sampling sequence that minimizes the quantization error must be such that

$$|u_{k-1} - u(t_k)|^2 = |u_k - u(t_k)|^2. \tag{26}$$

We can then define the quantization-based sampling method (abbreviated with qnt) as follows:

1) the optimal continuous-time input u is computed;
2) the piecewise-constant function $\bar{u}$ that minimizes E_{qnt} of (24) is found by applying the gradient condition of (26);
3) for the sampling instants $t_0(= 0), t_1, \ldots, t_{N-1}, t_N(= T)$ of this solution $\bar{u}$, we compute the optimal input sequence from (10), since the inputs of (25) are not optimal for the minimization of J.

An efficient implementation and a proof of convergence of this algorithm is beyond the scope of this paper. The interested reader can find our implementation of this function at github.com/ebni/samplo. Finally, we remark that the method qnt is applicable to any linear system with dimension of the input space $m = 1$ and any dimension n of the state space.

A. Asymptotic behaviour

As shown in [14], [15] the quantization problem of a function $u \in L^p(\Omega)$, is equivalent to minimize the Wasserstein's distance $W_p(\mu, \nu)$ where μ is the image measure $u^{\#}(dx/|\Omega|)$ and ν is a sum of Dirac masses

$$\nu = \frac{1}{N} \sum_{k=1}^{N} \delta_{y_k}.$$

As $N \to \infty$, the asymptotic density of points y_k can be computed and is equal to

$$\frac{f(y)^{m/(m+p)}}{\int f(y)^{m/(m+p)} \, dy}$$

where m is the dimension of the space of values of u and f is the density of the absolutely continuous part of the measure μ.

If the input space has dimension $m = 1$, then we find $f(y) = 1/|\dot{u}|\big(u^{-1}(y)\big)$, being u the solution of the Riccati equation (3). From the asymptotic density of values y_k, which is

$$\frac{|\dot{u}|^{-1/(1+p)}\big(u^{-1}(y)\big)}{\int |\dot{u}|^{-1/(1+p)}\big(u^{-1}(y)\big) \, dy}$$

we can pass to the asymptotic sampling density, which is then

$$\sigma_{\mathsf{qnt}}(t) = \frac{|\dot{u}(t)|^{p/(1+p)}}{\int_0^T |\dot{u}(t)|^{p/(1+p)} \, dt}.$$

Taking $p = 2$, i.e. minimizing the L^2 norm $\int_0^T |u - \bar{u}|^2 \, dt$, we end up with the asymptotic sampling density

$$\sigma_{\mathsf{qnt}}(t) = \frac{|\dot{u}(t)|^{2/3}}{\int_0^T |\dot{u}(s)|^{2/3} \, ds} \tag{27}$$

Equation (27) provides a very interesting intuition, which can be used as follows to determine the sampling instants. The steps, also illustrated in Figure 1, are described below:

1) the optimal continuous-time input u is computed;
2) the sampling instants $t_1, \ldots, t_{N-1}$ are determined such that their asymptotic density is (27), by construction. That is we choose $t_1, \ldots, t_{N-1}$ such that

$$\forall k = 0, \ldots, N-1, \quad \int_{t_k}^{t_{k+1}} |\dot{u}(t)|^{2/3} \, dt = \frac{1}{N} \int_0^T |\dot{u}(t)|^{2/3} \, dt \tag{28}$$

with the usual hypothesis of $t_0 = 0$, $t_N = T$

3) for such a sampling sequence $t_0(= 0), t_1, \ldots, t_{N-1}, t_N(= T)$, we compute the optimal input sequence from (10), which guarantees to minimize the control cost J for given sampling instants.

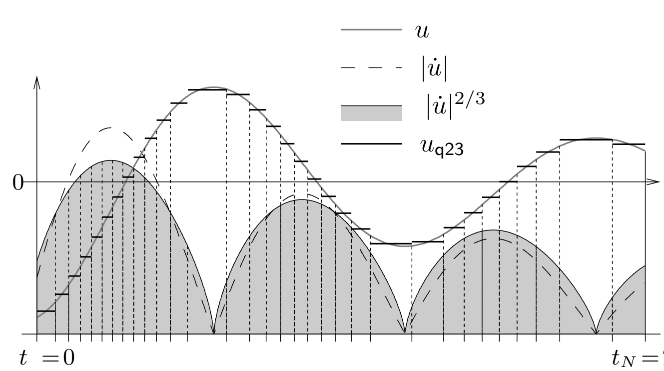

Fig. 1: Sampling according to the asymptotic density.

After the optimal continuous-time input u is computed (which has the complexity of solving a Riccati differential equation), the complexity of computing the sampling instants from u is $O(N)$. Notice that the difference between this method and the dls method (whose instants are selected according to (16)) is only in the exponent of $|\dot{u}|$.

This method is abbreviated with $\mathsf{q23}$ to remind the $2/3$ exponent in (28). Although method $\mathsf{q23}$ follows from the asymptotic sampling density (i.e. with $N \to \infty$) of method qnt, as it will be shown in Sections V and VI, this method produces sampling sequences with a near-optimal cost also for reasonably small values of N.

Finally, we remark that, similarly to the method qnt, the method $\mathsf{q23}$ is applicable to any linear system with dimension of the input space $m = 1$ and any dimension n of the state space.

V. First-order systems: quantization is asymptotically optimal

These quantization-based sampling techniques (methods qnt and $\mathsf{q23}$) follow the intuitive idea that the optimal discrete-time input should mimic the optimal continuous-time input. As it will be shown later in Section VII, their excellent capability to reduce the cost appears in all the performed experiments. Unfortunately, we were unable to prove a general result that relates the costs $J_{N,\mathsf{qnt}}$ or $J_{N,\mathsf{q23}}$ to the minimal continuous-time cost J_∞ or to the cost of optimal sampling $J_{N,\mathsf{opt}}$. Nonetheless, for first-order systems ($n = 1$), we did actually prove that the asymptotic sampling density σ_{qnt} of the quantization (Equation (27)) is actually equal to the asymptotic density of the optimal sampling σ_{opt}. The only additional hypothesis we are using is that the weight S of the final state at the instant T is set equal to the continuous ARE solution K_∞, which implies $K(t) = K_\infty,\ \forall t \in [0, T]$. As it will be shown later in the proof, this assumption is needed only to simplify the expression of the optimal continuous-time control input. Proving the asymptotic optimality of quantization-based sampling in more general hypothesis required a too involved mathematical development. Moreover, we observe that assuming the weight of the final state equal to the solution of the ARE is not very stringent, since the state at time $t = T$ is going to be small anyway, especially for large T.

Also, throughout this section we assume that if $Q = 0$ then $A > 0$, otherwise the optimal input is obviously $\bar{u}(t) = 0$, which is a constant function independently of the sampling instants.

For such a first-order system we are actually able to compute analytically both the asymptotic optimal sampling density σ_{opt} and asymptotic normalized cost c_{opt}.

Lemma 6: Consider a first-order system ($n = 1$) with weight of the final state S equal to the solution of the continuous ARE. Then the optimal sampling pattern has asymptotic density

$$\sigma_{\mathsf{opt}} \propto |\dot{u}|^{2/3} \tag{29}$$

being u the optimal continuous-time input.

Proof: Up to suitable normalizations of the cost function and of the system dynamics, we can assume, without loss of generality, that $B = 1$, and $R = 1$. From (2) it follows that the solution of the ARE, and then the weight of the final state, is

$$S = A + \sqrt{A^2 + Q}.$$

This assumption enables us to have a simple expression for the optimal continuous-time input u, that is

$$u(t) = -x_0(A + \sqrt{A^2 + Q})e^{-\sqrt{A^2+Q}\,t}. \tag{30}$$

From (4)–(8), the discretised system with an intersample separation of τ_k gives the following discrete-time model

$$\begin{aligned}
\bar{A}_k &= 1 + A\tau_k + \frac{A^2}{2}\tau_k^2 + \frac{A^3}{6}\tau_k^3 + \frac{A^4}{24}\tau_k^4 + o(\tau_k^4),\\
\bar{B}_k &= \tau_k + \frac{A}{2}\tau_k^2 + \frac{A^2}{6}\tau_k^3 + \frac{A^3}{24}\tau_k^4 + o(\tau_k^4),\\
\bar{Q}_k &= Q(\tau_k + A\tau_k^2 + \frac{2A^2}{3}\tau_k^3 + \frac{A^3}{3}\tau_k^4) + o(\tau_k^4),\\
\bar{R}_k &= \tau_k + Q(\frac{1}{3}\tau_k^3 + \frac{A}{4}\tau_k^4 + \frac{7A^2}{60}\tau_k^5) + o(\tau_k^5),\\
\bar{P}_k &= Q(\frac{1}{2}\tau_k^2 + \frac{A}{2}\tau_k^3 + \frac{7A^2}{24}\tau_k^4) + o(\tau_k^4).
\end{aligned}$$

Since we investigate the asymptotic optimal sampling density ($N \to \infty$ and then $\tau_k \to 0$), we realize that $\bar{A}_k\bar{R}_k = \tau_k + o(\tau_k)$ never equals $\bar{B}_k\bar{P}_k = \frac{Q}{2}\tau_k^3 + o(\tau_k^3)$, for small τ_k. By Remark 5 the optimal solution must satisfy Eq. (23) which establishes a relationship between τ_h and τ_{h+1}. Since they both tend to zero, we write τ_h as a function of τ_{h+1}:

$$\tau_h = \alpha\tau_{h+1} + \beta\tau_{h+1}^2 + o(\tau_{h+1}^2), \tag{31}$$

with α and β suitable constants to be found from Eq. (23).

Approximating the Riccati recurrence function r of (22) to the fourth order[1] w.r.t. τ, we find

$$\begin{aligned}
r(\tau, \bar{K}) = \bar{K} &- (\bar{K}^2 - 2A\bar{K} - Q)\tau\\
&+ \big(\bar{K}^3 - 3A\bar{K}^2 + (2A^2 - Q)\bar{K} + AQ\big)\tau^2\\
&- \Big(\bar{K}^4 - 4A\bar{K}^3 + (\frac{55}{12}A^2 - \frac{4}{3}Q)\bar{K}^2\\
&\quad + (-\frac{4}{3}A^3 + \frac{5}{2}QA)\bar{K} - \frac{2}{3}QA^2 + \frac{1}{4}Q^2\Big)\tau^3\\
&+ \Big(\bar{K}^5 - 5A\bar{K}^4 + (\frac{49A^2}{6} - \frac{5}{3}Q)\bar{K}^3 + \frac{19}{4}(-A^3 + QA)\bar{K}^2\\
&\quad + (\frac{2A^4}{3} - \frac{13QA^2}{4} + \frac{7Q^2}{12})\bar{K} + \frac{QA^3}{3} - \frac{Q^2A}{2}\Big)\tau^4\\
&+ o(\tau^4)
\end{aligned} \tag{32}$$

[1] For computing this and next expressions, we made use of the symbolic manipulation tool “Maxima” (http://maxima.sourceforge.net/).

with the partial derivatives

$$\begin{aligned}\frac{\partial r}{\partial \tau} &= -(\bar{K}^2 - 2A\bar{K} - Q)\\ &\quad + 2\big(\bar{K}^3 - 3A\bar{K}^2 + (2A^2 - Q)\bar{K} + AQ\big)\tau\\ &\quad - 3\Big(\bar{K}^4 - 4A\bar{K}^3 + (\frac{55}{12}A^2 - \frac{4}{3}Q)\bar{K}^2\\ &\qquad + (-\frac{4}{3}A^3 + \frac{5}{2}QA)\bar{K} - \frac{2}{3}QA^2 + \frac{1}{4}Q^2\Big)\tau^2\\ &\quad + 4\Big(\bar{K}^5 - 5A\bar{K}^4 + (\frac{49A^2}{6} - \frac{5}{3}Q)\bar{K}^3 + \frac{19}{4}(-A^3 + QA)\bar{K}^2\\ &\qquad + (\frac{2A^4}{3} - \frac{13QA^2}{4} + \frac{7Q^2}{12})\bar{K} + \frac{QA^3}{3} - \frac{Q^2A}{2}\Big)\tau^3\\ &\quad + o(\tau^3)\\ \frac{\partial r}{\partial \bar{K}} &= 1 - 2(\bar{K} - A)\tau + (3\bar{K}^2 - 6A\bar{K} + 2A^2 - Q)\tau^2\\ &\quad - \Big(4\bar{K}^3 - 12A\bar{K}^2 + 2(\frac{55}{12}A^2 - \frac{4}{3}Q)\bar{K}\\ &\qquad - \frac{4}{3}A^3 + \frac{5}{2}QA\Big)\tau^3 + o(\tau^3).\end{aligned}$$

We now replace in the necessary condition for optimality (23) the expressions above. If we write τ_h as a function of τ_{h+1} (Eq. (31)), we find

$$\begin{aligned}&-\tau_{h+1}^2 \frac{Q + A\bar{K}_{h+2}}{12}\Big[\Big(6(\alpha-1)(\alpha+1)^2 A\bar{K}_{h+2}^2\\ &\quad + 2\big[(\alpha+1)\big((2\alpha^2 - 2\alpha - 1)Q + (-4\alpha^2 - 2\alpha + 5)A^2\big)\\ &\qquad - 3\alpha\beta A\big]\bar{K}_{h+2}\\ &\quad - 6\big((\alpha+1)^2(\alpha-1)A + \alpha\beta\big)Q\Big)\tau_{h+1}\\ &\quad - 3(\alpha^2 - 1)(Q + A\bar{K}_{h+2})\Big] + o(\tau_{h+1}^3) = 0\end{aligned}$$

from which we have

$$\begin{aligned}&\Big(6(\alpha-1)(\alpha+1)^2 A\bar{K}_{h+2}^2\\ &\qquad + 2\big[(\alpha+1)\big((2\alpha^2 - 2\alpha - 1)Q\\ &\qquad\qquad + (-4\alpha^2 - 2\alpha + 5)A^2\big) - 3\alpha\beta A\big]\bar{K}_{h+2}\\ &\qquad - 6\big((\alpha+1)^2(\alpha-1)A + \alpha\beta\big)Q\Big)\tau_{h+1}\\ &- 3(\alpha^2 - 1)(Q + A\bar{K}_{h+2}) + o(\tau_{h+1}) = 0. \qquad (33)\end{aligned}$$

From Equation (33) we have that both coefficient of the zero order term and of the first order term in τ_{h+1} are zero. By setting the constant (that is $-3(\alpha^2-1)(Q + A\bar{K}_{h+2})$) equal to zero, we find $\alpha^2 = 1$. However, from (31), we observe that $\alpha = -1$ is not feasible, since it will lead to negative intersample separations. Hence we have $\alpha = 1$. By replacing $\alpha = 1$ in the coefficient of τ_{h+1} in (33) and setting it equal to zero, we find

$$(2Q + 2A^2 + 3\beta A)\bar{K}_{h+2} + 3\beta Q = 0$$

from which we find

$$\beta = -\frac{2}{3}\frac{(Q + A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}.$$

Recalling the expression (31), we can now assert that a necessary condition for the optimality of a sampling pattern is that

$$\tau_h = \tau_{h+1} - \frac{2}{3}\frac{(Q + A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}\tau_{h+1}^2 + o(\tau_{h+1}^2). \qquad (34)$$

We are now going to exploit (34) to find the asymptotic sampling density of the optimal pattern.

Let us compute the derivative of the asymptotic density σ_{opt} of the optimal sampling at a generic instant t_{h+1}. By Definitions 1 and 2, we have

$$\begin{aligned}\dot{\sigma}_{\mathsf{opt}}(t_{h+1}) &= \lim_{N\to\infty}\frac{\sigma_{\mathsf{opt}}(t_{h+2}) - \sigma_{\mathsf{opt}}(t_{h+1})}{\tau_{h+1}}\\ &= \lim_{N\to\infty}\frac{\frac{1}{N\tau_{h+1}} - \frac{1}{N\tau_h}}{\tau_{h+1}}\\ &= \lim_{N\to\infty}\frac{1 - \frac{1}{1 - \frac{2}{3}\frac{(Q+A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}\tau_{h+1}}}{N\tau_{h+1}^2}\\ &= \lim_{N\to\infty}\frac{-\frac{2}{3}\frac{(Q+A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}\tau_{h+1}}{N\tau_{h+1}^2}\\ &= -\frac{2}{3}\frac{(Q + A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}\lim_{N\to\infty}\frac{1}{N\tau_{h+1}}\\ &= -\frac{2}{3}\frac{(Q + A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}\sigma_{\mathsf{opt}}(t_{h+2})\end{aligned}$$

from which we obtain the differential equation

$$\dot{\sigma}_{\mathsf{opt}}(t) = -\frac{2}{3}\frac{(Q + A^2)K(t)}{Q + AK(t)}\sigma_{\mathsf{opt}}(t) \qquad (35)$$

being $K(t)$ the solution of the Riccati differential equation (2). If $S = A + \sqrt{Q + A^2}$, then $K(t)$ is constantly equal to S. Then in such special case, Eq. (35) becomes

$$\begin{aligned}\dot{\sigma}_{\mathsf{opt}}(t) &= -\frac{2}{3}\frac{(Q + A^2)(A + \sqrt{Q + A^2})}{Q + A(A + \sqrt{Q + A^2})}\sigma_{\mathsf{opt}}(t)\\ &= -\frac{2}{3}\sqrt{Q + A^2}\,\sigma_{\mathsf{opt}}(t)\end{aligned}$$

which is solved by

$$\sigma_{\mathsf{opt}}(t) = c\,e^{-\frac{2}{3}\sqrt{Q + A^2}\,t}$$

with c suitable constant such that $\int_0^T \sigma_{\mathsf{opt}}(t)\,dt = 1$.

From the expression (30) of the optimal continuous-time input we obtain that $\sigma_{\mathsf{opt}} \propto |\dot{u}|^{2/3}$. The Lemma is then proved. ■

Basically Lemma 6 states that, by tolerating the weak assumption that the weight S of the final state $x(T)$ is equal to the solution of the continuous ARE, the asymptotic density σ_{opt} of the optimal sampling is the same as the asymptotic density σ_{q23} of the quantization-based sampling. In addition to this result, the next Lemma also provides an exact computation of the asymptotic normalized cost c_{opt} (see Definition 4) of the optimal sampling. This result allows quantifying the benefit of optimal sampling.

The following Lemma provides a more general result from which the asymptotic normalized cost c_{opt} of the optimal sampling is derived later in Corollary 8.

Lemma 7: Consider a first-order system ($n = 1$). Let us assume, up to suitable normalizations, that $B = 1$ and $R = 1$

and that the weight of the final state is equal to the solution of the continuous ARE, $S = A+\sqrt{A^2+Q}$. Then, the asymptotic normalized cost of the sampling method mα with asymptotic sampling density

$$\sigma_{\mathsf{m}\alpha}(t) = \frac{\alpha(S-A)}{1-e^{-\alpha(S-A)T}}e^{-\alpha(S-A)t} \propto |\dot{u}(t)|^{\alpha} \qquad (36)$$

is

$$c_{\mathsf{m}\alpha} = \frac{S}{12(S-A)T^2}\frac{1-e^{-2(1-\alpha)(S-A)T}}{2(1-\alpha)}\left(\frac{1-e^{-\alpha(S-A)T}}{\alpha}\right)^2 \qquad (37)$$

Proof: Under our hypotheses, the solution if the Riccati differential equation is $K(t) = S = A + \sqrt{Q + A^2}$, for all $t \in [0, T]$. Hence the optimal continuous-time cost is

$$J_\infty = x_0^2 S.$$

Since we are investigating the normalized cost c_{opt} (see Definition 3), we consider the sequence:

$$\xi_k = \frac{N^2}{T^2}\left(\frac{\bar{K}_k}{S} - 1\right)$$

so that $c_{\mathsf{opt}} = \lim_{N\to\infty}\xi_0$. From the definition of ξ_k it follows that

$$\bar{K}_k = S(\frac{T^2}{N^2}\xi_k + 1).$$

From (32), by approximating $\bar{K}_k$ to the third order of τ_k, we have

$$\begin{aligned}\bar{K}_k &= \bar{K}_{k+1} - (\bar{K}_{k+1}-S)(\bar{K}_{k+1}-2A+S)\tau_k \\ &\quad + (\bar{K}_{k+1}-A)(\bar{K}_{k+1}-S)(\bar{K}_{k+1}-2A+S)\tau_k^2 \\ &\quad - \Big(\bar{K}_{k+1}^4 - 4A\bar{K}_{k+1}^3 + (\frac{55}{12}A^2 - \frac{4}{3}Q)\bar{K}_{k+1}^2 \\ &\quad + (-\frac{4}{3}A^3 + \frac{5}{2}QA)\bar{K}_{k+1} - \frac{2}{3}QA^2 + \frac{1}{4}Q^2\Big)\tau_k^3 + o(\tau_k^3)\end{aligned}$$

which allows to find a recurrent relationship that defines ξ_k

$$\begin{cases}\xi_k = \xi_{k+1} - \xi_{k+1}(S\frac{T^2}{N^2}\xi_{k+1} + 2(S-A))\tau_k \\ \quad + \xi_{k+1}(S\frac{T^2}{N^2}\xi_{k+1} + 2(S-A))(S\frac{T^2}{N^2}\xi_{k+1} + S - A)\tau_k^2 \\ \quad + \frac{S(S-A)^2}{12}\frac{N^2}{T^2}\tau_k^3 + o(\tau_k^3) \\ \xi_N = 0.\end{cases} \qquad (38)$$

From (38), it follows that the discrete derivative of ξ_k is

$$\begin{aligned}\frac{\xi_{k+1}-\xi_k}{\tau_k} &= \xi_{k+1}(S\frac{T^2}{N^2}\xi_{k+1} + 2(S-A)) \\ &\quad - \xi_{k+1}(S\frac{T^2}{N^2}\xi_{k+1} + 2(S-A))(S\frac{T^2}{N^2}\xi_{k+1} + S - A)\tau_k \\ &\quad - \frac{S(S-A)^2}{12}\frac{N^2}{T^2}\tau_k^2 + o(\tau_k^2).\end{aligned}$$

By definition of asymptotic sampling density (see Definitions 1 and 2), as $N \to \infty$, the intersample separation τ_k tends to zero with

$$\tau_k = \frac{1}{N\sigma_{\mathsf{m}\alpha}(t_k)} + o\Big(\frac{1}{N}\Big).$$

From this observation, as $N \to \infty$, the discrete derivative of ξ_k becomes the differential equation

$$\dot{\delta}(t) = 2(S-A)\delta(t) - \frac{S(S-A)^2}{12\,T^2}\sigma_{\mathsf{m}\alpha}^{-2}(t),$$

where $\delta(t)$ is the limit of ξ_k. With the sampling density $\sigma_{\mathsf{m}\alpha}(t)$ of (36), the differential equation above becomes

$$\begin{cases}\dot{\delta}(t) = 2(S-A)\delta(t) - \frac{S(1-e^{-\alpha(S-A)T})^2}{12\alpha^2T^2}e^{2\alpha(S-A)t} \\ \delta(T) = 0,\end{cases}$$

which is a first-order linear non-homogeneous differential equation, whose explicit solution is

$$\begin{aligned}\delta(t) = \frac{S(1-e^{-\alpha(S-A)T})^2}{24(S-A)(1-\alpha)\alpha^2T^2}\Big(&e^{2\alpha(S-A)t} \\ &- e^{-2(1-\alpha)(S-A)T}e^{2(S-A)t}\Big).\end{aligned}$$

Since the asymptotic normalized cost $c_{\mathsf{m}\alpha}$ coincides with $\delta(0)$, we obtain (37) and the Lemma is proved. ■

The reason for assuming a sampling density as in (36) is quite simple: periodic, dls, and optimal sampling (q23) are all special cases of the asymptotic density (36). In fact:

- the periodic sampling has constant sampling density, hence it corresponds to the case $\alpha = 0$;
- the dls sampling corresponds, by construction, to the case $\alpha = 1$;
- from Lemma 6, the optimal sampling corresponds to the case $\alpha = 2/3$.

In Figure 2 we plot the asymptotic normalized cost as α varies. The system in the plot has $A = 1$ and $Q = 8$ (and $B = R = 1$,

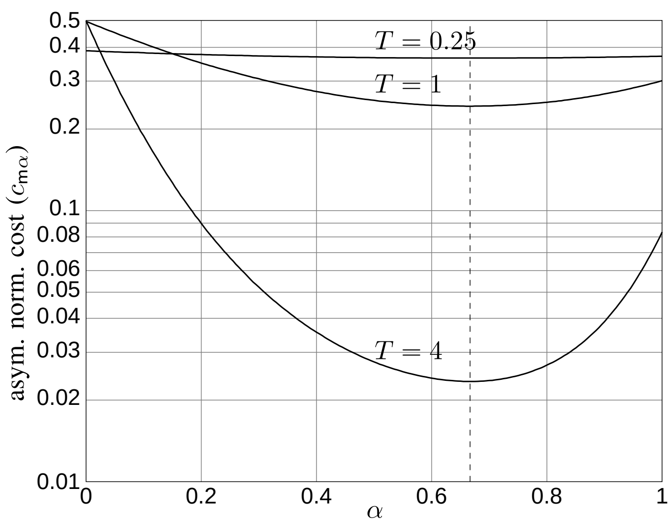

Fig. 2: Asymptotic normalized cost as function of α.

$S = A + \sqrt{A^2+Q} = 4$). We plot the cost $c_{\mathsf{m}\alpha}$ for three different values of T: 0.25, 1, and 4. This experiment confirms the validity of Lemma 6: the minimal cost occurs when $\alpha = \frac{2}{3}$ (denoted in the figure by a dashed vertical line).

By evaluating the cost of (37) with α equal to 0, 1, and $\frac{2}{3}$ we can then find the explicit cost expression of the asymptotic normalized cost for the periodic, the dls, and the optimal sampling respectively, as stated in the following Corollary.

Corollary 8: Consider a first-order system ($n = 1$). Let us assume, up to suitable normalizations, that $B = 1$ and $R = 1$ and that the weight of the final state is equal to the solution of the continuous ARE, $S = A + \sqrt{A^2 + Q}$. Then, the asymptotic normalized costs of the periodic, dls, and optimal sampling are, respectively

$$c_{\mathsf{per}} = c_{\mathsf{m}0} = \frac{A\sqrt{A^2+Q} + A^2 + Q}{24}(1 - e^{-2\sqrt{A^2+Q}\,T}), \tag{39}$$

$$c_{\mathsf{dls}} = c_{\mathsf{m}1} = \frac{A + \sqrt{A^2+Q}}{12\,T}(1 - e^{-\sqrt{A^2+Q}\,T})^2, \tag{40}$$

$$c_{\mathsf{opt}} = c_{\mathsf{m}\frac{2}{3}} = \frac{9}{32T^2}\left(\frac{A}{\sqrt{A^2+Q}} + 1\right)(1 - e^{-\frac{2}{3}\sqrt{A^2+Q}\,T})^3. \tag{41}$$

Notice that as $T \to \infty$, the cost c_{per} coincides with the one derived earlier in (15), which was a consequence of the second order approximation of the cost of periodic sampling (14) derived by Melzer and Kuo[12].

In Figure 3, we draw the asymptotic normalized costs for the three sampling methods: periodic, dls, and optimal (method q23). As expected, the cost of optimal sampling is always

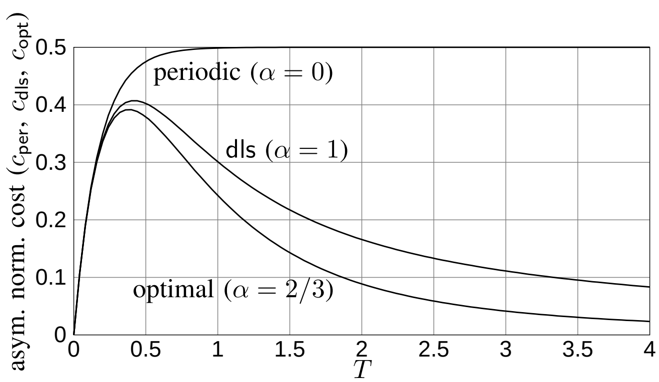

Fig. 3: Asymptotic normalized cost as function of T.

lower than the other two methods. As $T \to \infty$, the cost c_{per} tends to the constant of (15), that is $\frac{1}{2}$ in this case. The cost c_{dls} tends to zero as $\frac{1}{4T}$, while c_{opt} tends faster to zero with $\frac{3}{8T^2}$.

VI. Second-order systems

In Section V, we show that the quantization-based sampling is optimal for first-order systems. First-order systems, however, are quite special cases. For example, they never exhibits oscillations in the optimal control input. Hence, the second natural investigation that we perform, is on systems that can oscillate. For this purpose we assume

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad Q = qI, \quad R = 1, \tag{42}$$

Notice that we must assume $q > 0$, otherwise the optimal input is always $u(t) = 0$. The system of (42) has the following solution of the ARE

$$K_\infty = \begin{bmatrix} \omega\sqrt{\rho^2 + 2\rho - 3} & \omega(1-\rho) \\ \omega(1-\rho) & \rho\omega\sqrt{\rho^2+2\rho-3} \end{bmatrix}$$

with ρ defined as

$$\rho = \sqrt{1 + \frac{q}{\omega^2}} > 1.$$

For such a system, the characteristic polynomial $\chi(s)$ of the closed loop system with optimal state-feedback, is:

$$\chi(s) = \det(sI - (A - BR^{-1}B'K_\infty)) = s^2 + 2\omega_n\zeta s + \omega_n^2 \tag{43}$$

with the following damping ratio ζ and natural frequency ω_n:

$$\zeta = \frac{1}{2}\sqrt{(\rho - 1)\left(1 + 3\frac{1}{\rho}\right)},$$
$$\omega_n = \omega\sqrt{\rho}.$$

Hence, by properly choosing the problem parameters q and ω, we can construct overdamped, critically damped, and under-damped systems with any natural frequency.

If we assume $x_0 = [1\ 0]'$, then the cost of the optimal continuous-time input is

$$J_\infty = x_0' K_\infty x_0 = \omega\sqrt{\rho^2 + 2\rho - 3}.$$

With this initial condition, if the closed-loop system is over-damped (that is when $\rho > 3$), then the optimal input is

$$u(t) = 2\sqrt{2}\omega\frac{\sqrt{\rho-1}}{\sqrt{\rho-3}}\sinh\Big(\omega_n t\sqrt{\zeta^2-1} - \log\frac{\sqrt{\rho^2-1}+\sqrt{\rho^2-9}}{\sqrt{8}}\Big)e^{-\omega_n t\zeta},$$

if the system is underdamped ($1 < \rho < 3$), the optimal input is

$$u(t) = 2\sqrt{2}\omega\sqrt{\frac{\rho-1}{3-\rho}}\sin\Big(\omega_n t\sqrt{1-\zeta^2} - \arctan\sqrt{\frac{9-\rho^2}{\rho^2-1}}\Big)e^{-\omega_n t\zeta},$$

and finally, if the system is critically damped ($\rho = 3$), then the optimal input simply is

$$u(t) = (4\omega^2 t - 2\omega\sqrt{3})e^{-\omega\sqrt{3}t}.$$

For such a second order systems we are not capable to demonstrate that the asymptotic (with the number of samples $N \to \infty$) sampling density of the quantization problem (that, we remind, is proportional to $|\dot{u}|^{\frac{2}{3}}$) is the same as the asymptotic sampling density of the optimal LQR problem (1). Instead, we propose a numerical evaluation suggesting that the two asymptotic densities may coincide, even in the second order case. More precisely, let us define a sampling method mα with the sampling instants $t_0(=0), t_1, \ldots, t_{N-1}, t_N(=T)$ such that

$$\forall k = 0, \ldots, N-1,$$
$$\int_{t_k}^{t_{k+1}} |\dot{u}(t)|^\alpha\,dt = \frac{1}{N}\int_0^T |\dot{u}(t)|^\alpha\,dt.$$

As observed in Section V, such a method is of our interest, because periodic, dls, and q23 sampling methods are all special cases for α equal to 0, 1, and 2/3, respectively.

N	$c_{N,\mathsf{per}}$	$c_{N,\mathsf{dls}}$	$c_{N,\mathsf{q23}}$	$c_{N,\mathsf{qnt}}$	$c_{N,\mathsf{num}}$
4	0.4958	0.3200	0.2541	0.2539	0.2536
8	0.4980	0.3082	0.2454	0.2454	0.2454
16	0.4986	0.3033	0.2432	0.2432	0.2432
32	0.4987	0.3017	0.2426	0.2426	0.2426
64	0.4987	0.3012	0.2425	0.2425	0.2425
128	0.4988	0.3011	0.2424	0.2425	0.2424
256	0.4988	0.3010	0.2424	0.2427	0.2424
512	0.4988	0.3010	0.2424	0.2435	0.2424
∞	$\frac{1}{2}(1-e^{-6})$	$\frac{1}{3}(1-e^{-3})^2$	$\frac{3}{8}(1-e^{-2})^3$	—	$\frac{3}{8}(1-e^{-2})^3$
$\epsilon = 2\%$	$N \geq 4.99$	$N \geq 3.88$	$N \geq 3.48$	$N \geq 3.49$	$N \geq 3.48$

TABLE I: First-order system ($A = B = R = 1,\ Q = 8$): normalized costs $c_{N,\mathsf{m}}$, with varying N.

In Figure 4 we illustrate the normalized cost for $\omega \in \{5, 25\}$ and $q \in \{1, 10, 100\}$, with $N = 500$ sampling instants, as α varies. Surprisingly, we have that in all cases the normalized

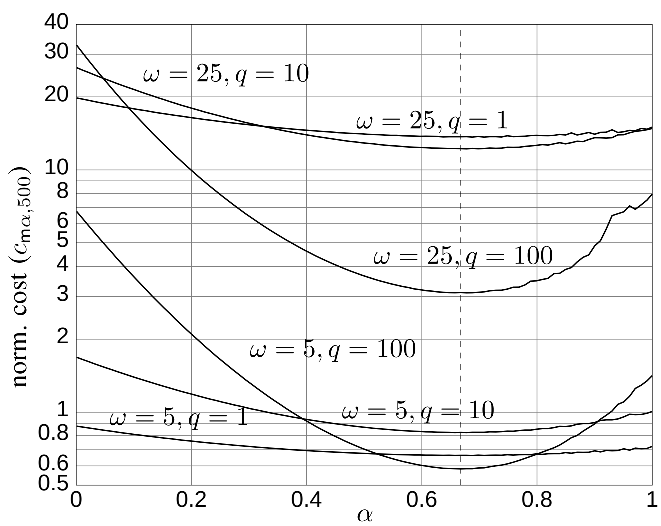

Fig. 4: Normalized cost $c_{\mathsf{m}\alpha,500}$ for second order systems.

cost reaches the minimum at $\alpha = \frac{2}{3}$. This observation suggests that a result analogous to Lemma 6 may hold also for second-order systems. In addition, we observe that the normalized cost is higher in all cases when the optimal input has larger variations ($w = 25$). When the cost of the state (represented by q) is large compared to the cost of the input, then the choice of the sampling method has a stronger impact on the overall cost.

VII. Numerical evaluation

In this section we investigate how the normalized cost varies with the number of samples N. We compare the following sampling methods:

- periodic sampling (per),
- deterministic Lebesgue sampling (dls), with sampling instants determined according to (16);
- quantization based on the theoretical asymptotic density of (q23), with sampling instants determined according to (28);
- quantization based on the exact condition of gradient equal to zero of Equation (26) (abbreviated with qnt). For large N this method tends to q23;
- optimal numerical solution (num), computed by the gradient-descent algorithm described in Section III-C.

In all experiments of this section the length of the interval is $T = 1$. Also noticed that in all cases, the optimal input signals $u_0, \dots, u_{N-1}$ are selected according to (10), while the sampling sequence depends on the chosen method.

In the first experiment we tested a first-order system, with $A = 1$ and $Q = 8$. In Table I we report the normalized costs as N grows. In the row corresponding to $N = \infty$ we report the theoretical values, as computed from (39), (40), and (41). We observe that, in this case, the convergence to the asymptotic limit is quite fast. This supports the approximation made in (13) and, more in general, the adoption of the asymptotic normalized cost as a metric to judge sampling methods, even with low N. Also, in the last row, we report the bound on the number of samples, for each sampling method, if a cost increase of at most $\epsilon = 2\%$ is tolerated w.r.t. the continuous-time case.

In Table II, we report similar data for a second order system of the kind described in (42), with $\omega = 5$ and $q = 100$. Such a choice makes the closed-loop system underdamped. We observe again that the convergence to the limit is fast.

In the final experiment, we tested the following third-order system

$$A = \begin{bmatrix} 1 & 12 & 0 \\ -12 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{44}$$

with initial condition $x_0 = [1\ 0\ 0]'$.

In Table III, we report the computed normalized costs corresponding to $Q = 0$ (upper part of the table), $Q = 10\,I$ (middle portion), and $Q = 100\,I$ (bottom of the table). The weights to input u and to the final state $x(T)$ were always assumed constant ($R = I$ and $S = K_\infty$).

In the last row of each case tables, we report again the estimate of the needed number of samples in $[0, 1]$, if it is tolerated a cost increase of $r = 2\%$ w.r.t. the continuous control input (as computed from (13)). The interested reader can find the code for performing these experiments at github.com/ebni/samplo.

Below we provide some comments on the data reported in this section.

- The run-time of the experiments of Table III took one day on a 2.40 GHz laptop. The weight of this simulation prevented us to perform it on a higher dimension systems or with larger number of samples.

N	$c_{N,\mathsf{per}}$	$c_{N,\mathsf{dls}}$	$c_{N,\mathsf{q23}}$	$c_{N,\mathsf{qnt}}$	$c_{N,\mathsf{num}}$
4	8.9956	1.7499	1.9642	1.8199	1.6878
8	7.2440	5.7915	0.7064	0.5578	0.5497
16	6.8638	4.1775	0.6356	0.5638	0.5596
32	6.7721	1.9258	0.5936	0.5704	0.5656
64	6.7494	1.5956	0.5818	0.5747	0.5724
128	6.7437	1.4830	0.5828	0.5799	0.5774
256	6.7423	1.4355	0.5826	0.5845	0.5805
512	6.7419	1.4332	0.5839	0.5951	0.5823
$\epsilon = 2\%$	$N \geq 18.36$	$N \geq 8.47$	$N \geq 5.40$	$N \geq 5.45$	$N \geq 5.39$

TABLE II: Second-order system ($\omega = 5,\ q = 100$): normalized costs $c_{N,\mathsf{m}}$, with varying N.

N	$c_{N,\mathsf{per}}$	$c_{N,\mathsf{dls}}$	$c_{N,\mathsf{q23}}$	$c_{N,\mathsf{qnt}}$	$c_{N,\mathsf{num}}$
10	14.488	9.4078	8.2823	5.9003	5.8911
20	13.382	8.0191	8.1115	6.6710	6.5765
40	13.136	8.4132	7.8431	7.2797	7.1273
$\epsilon = 2\%$	$N \geq 25.63$	$N \geq 20.51$	$N \geq 19.80$	$N \geq 19.08$	$N \geq 18.88$
10	16.615	7.2274	4.9684	2.9200	2.9139
20	14.640	6.3719	3.7374	3.2204	3.1786
40	14.221	4.8349	3.6936	3.6597	3.4165
$\epsilon = 2\%$	$N \geq 26.66$	$N \geq 15.55$	$N \geq 13.59$	$N \geq 13.53$	$N \geq 13.07$
10	30.959	10.161	1.7244	0.96569	0.94476
20	25.509	24.900	1.1534	1.1478	0.99627
40	24.352	4.5791	1.2179	1.2095	1.1631
$\epsilon = 2\%$	$N \geq 34.89$	$N \geq 15.13$	$N \geq 7.80$	$N \geq 7.78$	$N \geq 7.63$

TABLE III: Third-order system (of Eq. (44) with, from top to bottom, $Q = 0$, $Q = 10\, I$, and $Q = 100\, I$): normalized costs $c_{N,\mathsf{m}}$, with $N \in \{10, 20, 40\}$.

- The experiments confirm the validity of the asymptotic density of the quantization-based sampling of (28), since the cost achieved by q23 tends to the cost of the numerical quantization qnt as N grows.
- The capacity of both quantization-based sampling and dls sampling to reduce the cost w.r.t. periodic sampling is much higher in all those circumstances with high variation of the optimal continuous-time input u (such as when Q is larger compared to R). This behaviour is actually proved for first-order systems. In fact, from Equations (39)–(41), if $Q \to \infty$, we have $c_{\mathsf{per}} \approx Q$, $c_{\mathsf{dls}} \approx \sqrt{Q}$, and $c_{\mathsf{opt}} \approx 1/\sqrt{Q}$.
- The cost achieved by the quantization-based sampling (qnt and, q23 for larger N) appears to be very close to the optimal one, even for higher order systems. However, it is still an open question whether Lemmas 6 and 7 can be proved in general or not.

VIII. Conclusions and future works

In this paper we investigated the effect of the sampling sequence over the LQR cost. We formulate the problem for determining the optimal sampling sequence and we derive a necessary optimality condition based on the study of the gradient of the cost w.r.t. the sampling instants. Hence, following a different path of investigation, we proposed a *quantization-based* sampling, which selects the sampling instants (but not control sequence) in the way that better approximates the optimal control input. Surprisingly, this sampling method is demonstrated to be optimal for first-order systems and large number of samples per time unit. For second-order systems, such an asymptotic optimality is apparent from our numerical experiments, although it is not formally proved.

Being this research quite new, there are more open issues than questions with answers. Among the open problems we mention:

- proving the asymptotic optimality of the quantization-based sampling even in general (higher order) cases;
- the application of the proposed methods to closed-loop feedback where the state is also affected by disturbances;
- possible more efficient implementation of the gradient optimization procedure;
- the investigation of global minimization procedures which could lead to a higher cost reduction (gradient descent algorithms could indeed fall into local minima);
- the investigation of different approaches to approximate the optimal control input.

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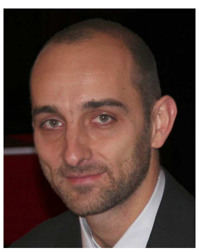

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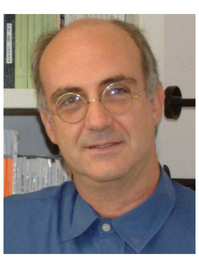

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