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Published in:
Elteknik

1965

Document Version:
Publisher's PDF, also known as Version of record

Reprint from Elteknik 8 (1965) no. 4 (April)

Vertical Indication with a Physical Pendulum Based on Electromechanical Synthesis of a High Moment of Inertia

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621-56

Artikeln beskriver ett nytt sätt att syntetisera tröghetsnavigeringssystem. Systemet är baserat på en princip för vertikalindikering, som uppfanns av författarna 1959. Ett komplett navigeringssystem enligt den nya principen har utvecklats och byggts av Philips Teleindustri i Stockholm. Detta system har genomgått en omfattande testning i flygplan. De första testflygningarna utfördes med ett litet tvåmotorigt propellerplan. Ett stort antal testflygningar har nyligen genomförts med ett jetplan med flygtider upp till tio timmar. Syftet med testflygningarna har varit, att prova de nya systemprinciperna i praktiken samt att undersöka om navigeringsnoggrannheten är i överensstämmelse med vad som erhållits ur systemanalysen. Testflygningarna har genomförts planenligt och de nya systemprincipernas hållbarhet har alltså praktiskt demonstrerats för flygplan med hög hastighet.

Denna artikel är baserad på den första rapport, som skrivits om systemet. Denna rapport utgavs av TTN-gruppen (Teoretiska Tröghetsnavigeringsgruppen under professor B J Andersson vid KTH) i augusti 1959, och finns även tillgänglig som en FOA-rapport. Artikeln nedan behandlar en förenklad enaxlig version av systemet.

The paper describes a new principle for synthesizing inertial navigation systems. The idea behind the principle was conceived jointly by the authors early in 1959. A complete navigation system based on the idea has been developed and built by Philips Teleindustri AB in Stockholm. Successful test flights have been performed with this system. In the first series of test flights a small twin engine airplane was used. The duration of each flight was up to a few hours. Recently a large number of test flights have also been performed using a long distance jet plane with flight time up to ten hours. The purpose of these tests was to establish the validity of the system principles and to investigate whether the actual accuracy of the system was in agreement with the predesign prediction. Both these tasks are now accomplished and the feasibility of the new principle for inertial guidance has thus been established in practice for high speed aircraft.

This paper is a slightly revised version of the first report written on the subject. The report appeared in August 1959 as a report from the TTN group[1] with limited distribution. The paper deals only with a simplified single axis version of the system. In spite of the fact that much more detailed studies now are available, the authors believe that it is worth while to publish the original contribution to the subject.

Nomenclature

$a(t)$	Acceleration
a_0	Constant acceleration
a'	Ratio of the moment of intertia of gyrofloat with respect to the output axis to the moment of inertia of the gyro rotor with respect to its spin axis
$D=\frac{d}{dt}$	Differential operator
$\boldsymbol{F}$	Force acting on the pivot of the pendulum Eq. (1)
$\boldsymbol{g}$	Acceleration of gravity
g	Magnitude of $\boldsymbol{g}$
$\boldsymbol{h}$	Vector from pivot to centre of mass of pendulum
h	Magnitude of $\boldsymbol{h}$
$\boldsymbol{H}$	Angular momentum
$\boldsymbol{H}_{CM}$	Angular momentum of the pendulum with respect to its centre of mass
$\boldsymbol{H}_P$	Angular momentum of the pendulum with respect to its pivot point
J	Moment of intertia of the gyro rotor with respect to its spin axis
J_P	Moment of intertia of the pendulum with respect to its pivot point
m	Mass of the pendulum
mh	Mass unbalance of the pendulum
$m(t)$	Disturbing torque acting on the gyrofloat in normalized units ($\sec^{-2}$), (the disturbing torque equals $Jm(t)$ [Nm])
$M(t)$	Disturbing torque acting on the pendulum

	in normalized units (sec^{-2}), (the disturbing torque equals $J_P M(t)$ [Nm])	
$\boldsymbol{M}'$	Torque acting on the pendulum	
p	Argument of Laplace transforms (the Laplace transform corresponding to the time function $f(t)$ is denoted by $f(p)$)	
$\boldsymbol{r}$	Vector from centre of earth to pivot point of pendulum	
r	Distance from centre of earth to pivot point of pendulum	
R	Radius of earth	
$R_x(t)$	Covariance function of second order random process $x(t)$	
t	Time variable	
v	Velocity	
$Y(p)$	Transfer function	
$Y_o(p)$	Transfer function	Eq. (12)
$Y_1(p)$	Transfer function	Eq. (9)
$Y_2(p)$	Transfer function	Eq. (10)
$Y_3(p)$	Transfer function	Eq. (11)
α	Parameter. See $\sigma(p)$	
γ	Parameter. See $\sigma(p)$	
δ	Average component accuracy in Schuler-tuning circuit	
ϑ	Angle. Fig. 3	
Θ	Angular orientation of pendulum with respect to intertial space. Fig. 3	
$\varkappa$	Parameter. See $\sigma(p)$	
$\lambda = \frac{mhr}{J_P}$	Parameter	
$\sigma(p)$	Transfer function from output signal of gyro to torque (in normalized units) applied to the torque generator of gyro. A proportional gyro has $\sigma(p) = \alpha p$, a rate gyro has $\sigma(p) = \alpha p + \varkappa$, an acceleration gyro has $\sigma(p) = \alpha p + \varkappa + \gamma / p$	
$\tau(p)$	Transfer function from output signal of gyro to torque (in normalized units) acting on the pendulum	
$\varphi(t)$	Output signal of gyro	
$\Phi_x(\omega)$	Power spectrum of second order random process $X(t)$	
$\psi(t)$	Vertical indication error	
ω_o	Angular velocity of gyro rotor.	
$\omega_s = \sqrt{g/R}$	Schuler frequency	

Introduction

Vertical indication is one of the key problems when designing inertial navigators for vehicles with velocities considerably less than the escape velocity.

One of the essential problems in the field of vertical indication is to obtain a pendulous system with a period of 84 minutes. As was pointed out by Schuler[2], this cannot be accomplished with a physical pendulum of reasonable size.

One method of synthesizing the system is to substitute the simple pendulum by a force-measuring and a torque-producing sub-system. By introducing a suitable coupling between these, the complete system will, in all essential respects, behave in the desired way. Systems of these types have been successfully designed by Draper et al. [3, 4, 5, 6, 15]

Another method of synthesizing the desired pendulous system is by using an ordinary physical pendulum, whose apparent moment of inertia is made very high by electromechanical aids. A single axis loop of a system based on that principle is analysed in this article. An analysis of a three-axis system has also been performed[12].

The system is described below. The article proceeds with the equations of motion of the system and with an analysis of the equations. Some practical aspects on the system are given and also a comparison of the proposed system with the classical ways of implementing the system. In an appendix the position indication loop is discussed, thereby completing the analysis of the single axis navigation system.

Description of the system

The main features of the system are illustrated in fig. 1. It consists of a physical pendulum supported in the vehicle. The high apparent moment of inertia of the pendulum is obtained by applying a torque to the pendulum, whose magnitude is proportional to the time derivative of the angular velocity of the pendulum with respect to inertial space. The torque is applied by a torque motor and the control signal is obtained from a single axis gyro attached to the pendulum with the input axis parallel to the pivot axis of the pendulum. The gyro can be a proportional gyro, a rate gyro, an acceleration gyro (i. e. a gyro with an integrator feedback from the signal generator to the torque motor), etc. With a proportional gyro two differentiations are required in the loop, one differentiation is sufficient if a rate gyro is used and an acceleration gyro requires only a constant gain in the loop.

A salient feature of the system is that there are signals in the system, proportional to the velocity of the vehicle. Position indication is thus easily obtained by integration.

The equation of motion of the pendulum

Introduce a right-handed orthogonal coordinate set $O\,x\,y\,z$ fixed to inertial space. The force of gravity is supposed to be directed towards O. The pivot point of the pendulum is P. The vector OP is denoted by $\boldsymbol{r}$. The vehicle is restricted to move in the xy-plane with the pivot axis of the pendulum parallel to the z-axis. See fig. 2.

Newton's second law of motion gives:

$$\frac{d}{dt}\boldsymbol{H}_{CM} = \boldsymbol{M}' - \boldsymbol{h} \times \boldsymbol{F} \tag{1}$$

$$m\left(\frac{d^2\boldsymbol{r}}{dt^2} + \frac{d^2\boldsymbol{h}}{dt^2}\right) = \boldsymbol{F} + m\boldsymbol{g} \tag{2}$$

Neglecting the difference between the centre of mass and the centre of gravity of the pendulum, we obtain from equations (1) and (2)

$$\frac{d}{dt}\boldsymbol{H}_P = \boldsymbol{M}' + m\boldsymbol{h} \times \left(\boldsymbol{g} - \frac{d^2\boldsymbol{r}}{dt^2}\right) \tag{3}$$

Introduce the angles ϑ, Θ and ψ according to fig. 3.

The angle ψ is the vertical indication error. Neglecting the reaction torques of the gyros we get*)

$$\left(\frac{d}{dt}\boldsymbol{H}_P\right)_z = J_P\frac{d^2\Theta}{dt^2} \tag{4}$$

*) The influence of the reaction torques is discussed in references (8), (11), (12) and (14).

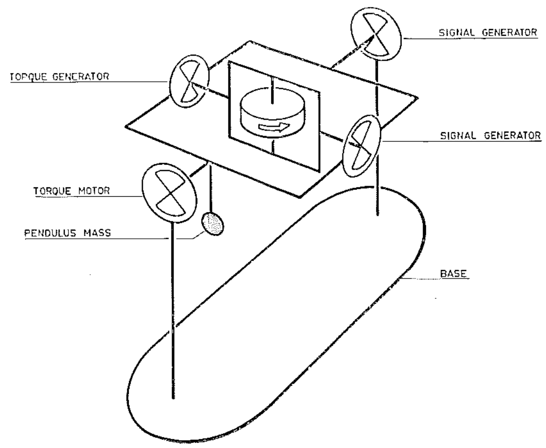


Fig. 1. Schematic diagram of the single axis system.

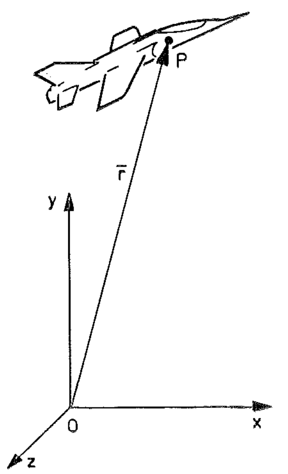


Fig. 2. Illustration of the coordinate system.

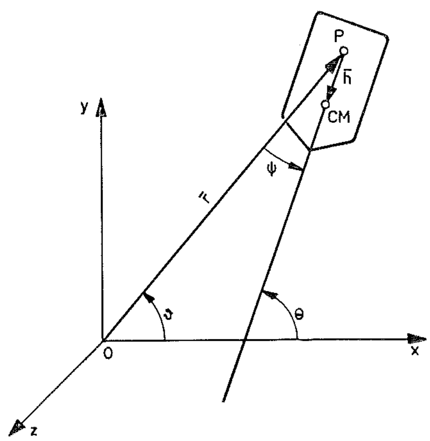


Fig. 3. Relationship of the angles defining the orientation of the pendulum and the vertical indication error.

where J_P is the moment of inertia of the pendulum with respect to its pivot axis.

The angular velocity of the pendulum is measured by a single axis gyro. Let φ be the output signal of the gyro, then

$$\varphi(t) = \frac{\omega_o D}{a'[D^2+\sigma(D)]}\Theta(t) + \frac{1}{a'[D^2+\sigma(D)]}m(t) \qquad (5)$$

where $J\sigma(D)$ is the feedback operator from the output signal of the gyro to the torque generator of the gyro. A proportional gyro has $\sigma(D) = \alpha D$, a rate gyro has $\sigma(D) = \alpha D + \varkappa$, an acceleration gyro has $\sigma(D) = \alpha D + \varkappa + \gamma D^{-1}$.

Further, let $J_P \tau(D)$ be the coupling operator from the output signal of the gyro to the torque acting on the pendulum, and $J_P M(t)$ the disturbing torque acting on the pendulum. Then

$$(M)_z = -J_p \tau(D)\varphi(t) + J_p M(t). \qquad (6)$$

Suppose it is possible to choose the operators $\tau(D)$ and $\sigma(D)$ in such a way that the tracking error $\psi(t)$ is small for all disturbing torques, eq. (3) can then be linearized in $\psi(t)$. The z-component of the linearized equation combined with the equations (5) and (6) gives

$$\left[D^2 + \frac{\omega_o D\,\tau(D)}{a'[D^2+\sigma(D)]} + \lambda\,\omega_s^2\right]\psi(t) =$$

$$\left[\lambda - 1 - \frac{\omega_o\,\tau(D)}{a'[D^2+\sigma(D)]}\right]D^2\,\vartheta(t) + M(t) + \frac{\tau(D)}{a'[D^2+\sigma(D)]}m(t)$$

$$+\lambda\left[2\frac{\dot r}{r}\dot\vartheta - \left(\frac{\ddot r}{r} - \dot\vartheta^2\right)\psi(t)\right] \qquad (7)$$

where $$\lambda = \frac{mhr}{J_P}$$

and $$\omega_s^2 = \frac{g}{r}$$

Assume that the vehicle moves at constant height, i. e. $r = R$, the radius of the earth, and that the velocity of the vehicle is considerably less than the escape velocity, i. e.

$$v \ll \sqrt{Rg} \approx 8000 \text{ m/sec}$$

The last term of the eq. (7) can then be neglected. For an analysis of the error due to variations in r, see reference 7. The rest of the equation is then linear with constant coefficients. Applying the Laplace transform, we get

$$\psi(p) = Y_1(p)\,a(p) + Y_2(p)\,M(p) + Y_3(p)\,m(p) \qquad (8)$$

where $a(p)$ is the Laplace transform of the acceleration of the vehicle, i. e.

$$a(t) = R\,\ddot\vartheta(t)$$

and $$Y_1(p) = \frac{1}{R}\,\frac{\lambda - 1 - Y_o(p)}{p^2[1+Y_o(p)] + \omega_s^2} \qquad (9)$$

$$Y_2(p) = \frac{1}{p^2[1+Y_o(p)] + \omega_s^2} \qquad (10)$$

$$Y_3(p) = \frac{p\,Y_o(p)}{p^2[1+Y_o(p)] + \omega_s^2} \qquad (11)$$

where $$Y_o(p) = \frac{\omega_o\,\tau(p)}{a'p\,[p^2+\sigma(p)]} \qquad (12)$$

The transformed functions are denoted by writing p for the argument. A block diagram of the system represented by the eq. (8) is shown in fig. 4.

Analysis of the system

In the linearization process above we assumed that it was possible to choose the feedback operators in such a way that the indication error is small in spite of disturbances.

The permissible error and the disturbances are supposed to be given by the specifications of the system. The synthesis problem is then to choose the operators $\sigma(p)$ and $\tau(p)$ to satisfy the specifications.

Suppose for instance that the acceleration of the vehicle and the disturbing torques are stationary random processes with the power spectra $\Phi_{aa}(\omega)$, $\Phi_{MM}(\omega)$ and $\Phi_{mm}(\omega)$. If the transfer functions $Y_1(p)$, $Y_2(p)$ and $Y_3(p)$ are strictly stable, the vertical indication error is also a stationary process whose power spectrum is $\Phi_{\psi\psi}(\omega)$. Eq. (8) gives

$$\Phi_{\psi\psi}(\omega) = Y_1(j\omega)\,Y_1(-j\omega)\,\Phi_{aa}(\omega) + Y_2(j\omega)\,Y_2(-j\omega)\,\Phi_{mm}(\omega) + Y_3(j\omega)\,Y_3(-j\omega)\,\Phi_{mm}(\omega). \tag{13}$$

The covariance function of the indication error is then

$$R_{\psi\psi}(\tau) = \int_{-\infty}^{\infty} \Phi_{\psi\psi}(\omega) e^{j\omega\tau}\, d\omega \tag{14}$$

and the r.m.s. error

$$E[\psi(t)^2] = R_{\psi\psi}(O) = \int_{-\infty}^{\infty} \Phi_{\psi\psi}(\omega)\, d\omega \tag{15}$$

Given the power spectra $\Phi_{aa}(\omega)$, $\Phi_{MM}(\omega)$ and $\Phi_{mm}(\omega)$, the problem is to choose the transfer functions $Y_1(p)$, $Y_2(p)$ and $Y_3(p)$ to yield a permissible r. m. s. error.

An obvious conclusion from the eq. (9), (10) and (11) is that all transfer functions cannot be arbitrarily small. Hence the stated problem does not necessarily have a solution.

In order to judge the possibilities of this system, we will analyse the error obtained for some deterministic disturbances.

If we choose

$$Y_o(p) = \lambda - 1 \tag{16}$$

i. e.

$$\tau(p) = \frac{\lambda - 1}{\omega_o} a' p [p^2 + \sigma(p)] \tag{16^1}$$

the acceleration of the vehicle will cause no indication error. Eq. (16[1]) is the Schuler-tuning condition of the system. Introduce eq. (16) into the eq. (9), (10) and (11) and we get

$$Y_1(p) = O \tag{17}$$

$$Y_2(p) = \frac{1}{\lambda(p^2 + \omega_s^2)} \tag{18}$$

$$Y_3(p) = \frac{\lambda - 1}{\lambda} \frac{p}{\omega(p^2 + \omega_s^2)} \tag{19}$$

Notice that the numerator of $Y_3(p)$ is proportional to p, an unbalance torque on the gyro gimbal with constant magnitude will thus give no steady state vertical indication error.

As the accuracy of the available components is limited, condition (16) cannot be accurately satisfied. Suppose for example that

$$\tau(p) = (1 - \delta) \frac{\lambda - 1}{\omega_o} p a' [p^2 + \sigma(p)] \tag{20}$$

where δ is an overall figure of the accuracy of the components. Suppose that $\delta \ll 1$. Eq. (5) is then replaced by

$$Y_1(p) = \frac{\lambda - 1}{\lambda} \frac{\delta}{R(p^2 + \omega_s^2)} \tag{21}$$

Example 1

Suppose that the disturbances have constant magnitudes a_o, M_o and m_o, respectively. Neglecting δ and $1/\lambda$ compared with 1, we obtain the following expression for the vertical indication error.

$$\psi(t) = \left[\frac{a\,\delta}{R\,\omega_s^2} + \frac{M_o}{\lambda \omega_s^2}\right](1 - \cos \omega_s t) + \frac{m_o}{\omega_o \omega_s} \sin \omega_s t \tag{22}$$

Supposing the disturbances to act only during the time interval $(0,\tau)$, where $\omega_s \tau \ll 1$, we get for $t > 1/\omega_s$

$$\psi(t) = \omega_s \tau \left[\frac{a\,\delta}{R\,\omega_s^2} + \frac{M_o}{\lambda \omega_s^2}\right] \sin \omega_s t + \omega_s \tau \frac{m_o}{\omega_o \omega_s} \cos \omega_s t \tag{23}$$

To obtain the order of magnitude of the error, we assume

$$\begin{aligned} m &= 0.1 \text{ kg} \\ J &= 4 \times 10^{-6} \text{ kg m}^2 \\ J_P &= 1.6 \times 10^{-4} \text{ kg m}^2 \\ h &= 0.02 \text{ m} \\ \omega_o &= 2\,500 \text{ rad sec}^{-1} \\ \delta &= 10^{-3} \end{aligned}$$

Eq. (22) and (23) then gives

$$\psi(t) = (10^{-4} a_o + 8 \times 10^{-3} M_o)(1 - \cos \omega_s t) + 0.32\, m_o \sin \omega_s t$$

and

$$\psi(t) = (1.25 \times 10^{-7} a_o \tau + 10^{-5} M_o \tau) \sin \omega_s t + 4 \times 10^{-4} m_o t \cos \omega_s t$$

respectively.

If the disturbances are assumed to be constant and if their magnitudes are adjusted so as to give equal contributions to the maximum error, the maximum error is 3×10^{-4} rad ($\approx$ 1 minute of arc) if

$$\begin{aligned} a_o &= 0.6 \text{ m sec}^{-2} \\ J_p M_o &= 1.2 \times 10^{-5} \text{ Nm} = 120 \text{ dyncm} \\ J m_o &= 2 \times 10^{-9} \text{ Nm} = 0.02 \text{ dyncm} \end{aligned}$$

When the vehicle is accelerated to the velocity V_o in a time, much shorter than $\sqrt{R/g} \approx$ 800 sec, eq. (23) implies that the maximum vertical indication error is

$$\hat{\psi} = \delta \frac{V_o}{\sqrt{Rg}}$$

Hence with V_o = 400 m sec^{-1} and $\delta = 10^{-3}$, we get

$$\hat{\psi} = 5 \times 10^{-5} \text{ rad} = 10 \text{ seconds of arc}$$

Notice that the error due to the acceleration of the vehicle is proportional to the overall merit figure of the components and the error due to disturbing torques is inversely proportional to the mass-unbalance, the upper limit of the mass-unbalance is determined by the maximum acceleration of the vehicle and the torque capacity of the torque motors.

Compare the second part of the chapter "An approach to a practical solution" below.

Example 2

Suppose $\sigma(p) = \alpha p + \varkappa$

Eq. (16) gives

$$\tau(p) = \frac{\lambda - 1}{\omega_o} p a' [p^2 + \alpha p + \varkappa]$$

We will now analyse the error obtained if we choose

$$\tau(p) = \frac{\lambda - 1}{\omega_o} p a' \varkappa \approx \frac{mRh}{J\omega_o} a' \varkappa\, p = \frac{mRh}{H} kp \tag{24}$$

which means a considerable simplification of the instrumentation.

Eq. (9) gives

$$Y_1(p) = \frac{\lambda - 1}{R} \frac{p(p+\alpha)}{p^4 + \alpha p^3 + p^2(\lambda\varkappa + \omega_s^2) + p\,\lambda\alpha\omega_s^2 + \varkappa\lambda\omega_s^2} \quad (25)$$

Hence the system is no longer insensitive to the acceleration of the vehicle. In order to analyse the magnitude of the error obtained, we suppose that the acceleration of the vehicle is a step-function whose amplitude is a_o.

The Laplace transform of the vertical indicating error is

$$\psi(p) = \frac{\lambda - 1}{R} \frac{a_o(p+\alpha)}{p^4 + \alpha p^3 + p^2(\lambda\varkappa + \omega_s^2) + p\,\lambda\alpha\omega_s^2 + \varkappa\lambda\omega_s^2}$$

Assuming $\varkappa \gg \omega_s^2 \qquad \lambda\varkappa \gg \alpha^2$

$\lambda \gg 1 \qquad \varkappa \gg \alpha\,\omega_s^2$

we get

$$\psi(t) \approx \frac{a_o}{\varkappa R}\left[e^{-\frac{\alpha}{2\varkappa}\omega_s^2 t}\left(\cos\omega_s t + \frac{\alpha}{\omega_s}\sin\omega_s t\right) - e^{-\frac{\alpha}{2}t}\left(\cos\sqrt{\lambda\varkappa}\,t + \frac{3}{2}\frac{\alpha}{\lambda\omega}\sin\sqrt{\lambda\varkappa}\,t\right)\right] \quad (26)$$

Introducing $\varkappa = 10^4 \text{ sec}^{-2}$

$\alpha = 140 \text{ sec}^{-1}$

$m = 0.1 \text{ kg}$

$J_P = 1.6 \times 10^{-4} \text{ kg m}^2$

$h = 0.02 \text{ m}$

we get $\lambda = \dfrac{mrh}{J_P} = 8 \times 10^7$

and the vertical indication error has its maximum

$$\psi \max = 1.8 \times 10^{-6}\, a_o$$

for $t = 1\,250$ sec. This means that the vertical indication error induced by having the simplified operator, eq. 24, is less than a second of an arc when the acceleration of the vehicle has a constant magnitude of 1 m sec^{-2}.

An approach to a practical solution

Condition for Schuler-tuning

Eq. (16) gives the condition for Schuler-tuning

$$\tau(p) = \frac{\lambda - 1}{\omega_o} a'p\,[p^2 + \sigma(p)]$$

This condition can be satisfied in many different ways, e. g.

I) $\sigma(p) = \alpha p, \qquad \tau(p) = \dfrac{\lambda - 1}{\omega_o} a'p\,(p^2 + \alpha p)$

II) $\sigma(p) = \alpha p + \varkappa, \qquad \tau(p) = \dfrac{\lambda - 1}{\omega_o} a'p\,(p^2 + \alpha p + \varkappa)$

III) $\sigma(p) = \alpha p + \varkappa + \dfrac{\gamma}{p}, \quad \tau(p) = \dfrac{\lambda - 1}{\omega_o} a'p\left(p^2 + \alpha p + \varkappa + \dfrac{\gamma}{p}\right)$

etc.

The first scheme means that the gyro is integrating and that $\tau\,(p)$ includes two differentiations. This method is not very attractive since the output signal of the gyro is proportional to the distance travelled, which means difficulties in keeping the output signal of the gyro within reasonable limits.

The second scheme means that a rate gyro is used and that $\tau\,(p)$ includes one differentiation. The output signal of the gyro is essentially proportional to the velocity of the vehicle. In the case III the gyro gimbal is still harder restrained to the zero position, and the output signal is zero in the steady state if the vehicle moves with constant velocity.

To find the orders of magnitudes involved we will analyse case II. The magnitude of the coefficient $\varkappa$ can be estimated by the following inequality

$$\frac{\omega_o v_{max}}{a'R\,\varphi_{max}} \leqq \varkappa \leqq \frac{\omega_o \Delta v}{a'R\,\Delta\varphi} \quad (27)$$

Here is

- Δv the allowable velocity resolution (this is essentially determined by the time of flight and the permissible position error)
- v_{max} the maximum velocity of the vehicle
- $\Delta\varphi$ the resolution of the signal generator
- $\varphi_{\max}$ the maximum allowable output signal of the gyro
- ω_o the angular velocity of the gyro
- R the radius of the earth
- a' ratio between moments of intertia of gyro gimbal and gyro rotor

The inequality (27) is consistent if

$$\frac{\varphi_{max}}{\Delta\varphi} \geqq \frac{v_{max}}{\Delta v} \quad (28)$$

If this condition is not satisfied, scheme II cannot be used for Schuler-tuning.

Assume for example the following numerical values:

$V_{max} = 400 \text{ msec}^{-1} \qquad \omega_o = 2\,500 \text{ rad sec}^{-1}$

$\varphi_{max} = 10^{-4} \text{ rad} \qquad a' = 2.5$

The inequality (27) then gives

$$\varkappa \geqq 600.$$

Let us now return to the implementation of the system. Suppose that equation (28) holds. In order to simplify the instrumentation we choose

$$\tau(p) = \frac{\lambda - 1}{\omega_o} a'p$$

The error obtained by this approximation can be estimated from the analysis given in example 2 above. The output signal of the gyro should therefore be differentiated and fed to the torque motor of the pendulum.

Magnitude of the unbalance

We will now consider the magnitude of the unbalance mh. If the disturbing torque acting on the stable element has a constant magnitude M_o, the steady state vertical indication error $\Delta\psi$ is

$$\Delta\psi = \frac{M_o}{mgh}$$

Hence, the greater the unbalance mh, the larger the allowable disturbing torque. However, a large unbalance will require a high capacity torque motor. We have

$$M_{max} = mh\, a_{max}$$

where M_{max} is the maximum torque from the torque motor and a_{max} the maximum acceleration of the vehicle.

To get the order of magnitude of the quantities involved, we assume $\Delta\psi = 1''$

$$m = 0.1 \text{ kg}$$
$$a_{max} = 50 \text{ m sec}^{-2}$$

The following numerical values are then obtained:

h cm	M_0 dyncm	M_{max} Nm
0.1	0.05	0.005
1	0.5	0.05
10	5.0	0.5
100	50	5

We will now summarize a few practical aspects on the system.

Suspension

As the system is based on an ordinary physical pendulum, it is necessary to use low friction suspension in order to obtain a small steady state vertical indication error. For systems with "seconds-of-an-arc" accuracy it is necessary to have friction levels down to the order of magnitude of dyncm. Additional requirements on the suspension are also given by the initial alignment. A fast alignment of the system is usually obtained by allowing the pendulum to oscillate with its natural frequency and only providing damping through the servo system. The accuracy of the initial alignment is essentially given by the mass-unbalance and the suspension forces. The orders of magnitude involved are given in the table above. The accepted torque disturbances will thus be very small. Ordinary bearings are excluded. Among available solutions are hydraulic or gas bearings, but even spring suspension with a compensating restraint system may be accepted.

Also notice that it is very important that there are no velocity errors when the system is Schuler-tuned. To see this we notice that the vertical indication error due to the initial alignment errors for an undamped system is given by

$$\psi(t) = \psi_o \cos \omega_s t + \frac{\dot{\psi}_o}{\omega_s} \sin \omega_s t$$

Hence, the initial velocity errors are amplified by ω_s^{-1}. To consider this quantitatively, assume that the system is oscillating with the frequency ω_n with the amplitude a and that the system is Schuler-tuned by increasing the apparent moment of inertia. The amplitude of the vertical indication error will then be between a and $a\ \omega_n/\omega_s$, depending on the timing of the Schuler-tuning. The maximum error is obtained if the system is Schuler-tuned, when the oscillation has maximal velocity $\dot{\psi} = a\omega_n$.

Gyro

If the alternative II above (i. e, a rate gyro with a differentiating network between the signal generator of the gyro and the torque motor of the pendulum) is chosen, it is necessary to have a rate gyro with very high resolution. This can be achieved by spring restraining existing floated gyros. According to equation 13 the restraint must be very weak, corresponding to a natural frequency of some periods per second.

Consequently the damping must also be very low. A typical floated gyro has

$$J = 5 \times 10^{-6} \text{ kg m}^2$$
$$D = 7.5 \times 10^{-4} \text{ Nm rad}^{-1} \text{ sec.}$$

Assuming that a rate gyro is realized by a proportional feedback around the proportional gyro and that the gain is chosen so that the system is critically damped, we find that the eigen-frequency of the system is 125 rad sec^{-1}.

Conclusions and comparison with a Reisch-Draper system

The basic idea of this approach to the problem of vertical indication is the use of an ordinary physical pendulum, whose apparent moment of inertia is made very high by electromechanical aids. This is obtained by applying a torque to the pendulum proportional to the angular acceleration of the pendulum. In one of the alternatives outlined in this article, the torque is obtained by providing the pendulum with a rate gyro whose output signal is differentiated, amplified and fed to a torque motor. A signal proportional to the angular acceleration of the pendulum can of course also be obtained in other ways, e. g. by measuring the angle between the pendulum and an inertial reference and differentiating twice. A system of this type is described in reference 9.

A salient feature of the inertial navigation systems based on the described method for vertical indication is that there is no need for conventional accelerometers. This is the main difference between the system described in this article and the Draper systems based on the space-integrator concept. The requirements on low friction suspension is another feature of the system described in this article. This has as a consequence that the gyros are very well isolated from the motions of the vehicle.

Appendix

The position indication loop

As the vehicle is restricted to move on a circle, the position indication consists of determining the angle $\vartheta(t)$. This problem can be solved in the way illustrated in the block diagram of fig. 5.

The system is supposed to be Schuler-tuned, i. e. the transfer function $\tau\ (p)$ is chosen according to eq. (4). In fig. 4 the block $\tau\ (p)$ is split up into two parts. The signal at the point F is $\dot{\vartheta}^*\ (t)$ where

$$\dot{\vartheta}^*(t) = [RD^2Y_1(D) + 1]\,\vartheta(t) + DY_2(D)M(t) + DY_3(D)\,\frac{1}{a'\omega_o}\,m(t)$$

If the disturbances are small, this signal is an estimate of $\dot{\vartheta}\ (t)$, which explains the notation. An estimate $\vartheta^*\ (t)$ of the angle $\vartheta\ (t)$ can thus be obtained by integrating $\dot{\vartheta}^*\ (t)$, hence

$$\vartheta^*(t) = [RD^2Y_1(D) + 1]\,\vartheta(t) + Y_2(D)M(t) + \\ + \left[-Y_3(D) + \frac{1}{a'\omega_o D}\right] m(t)$$

Fig. 4. Block diagram of the vertical indication loop.

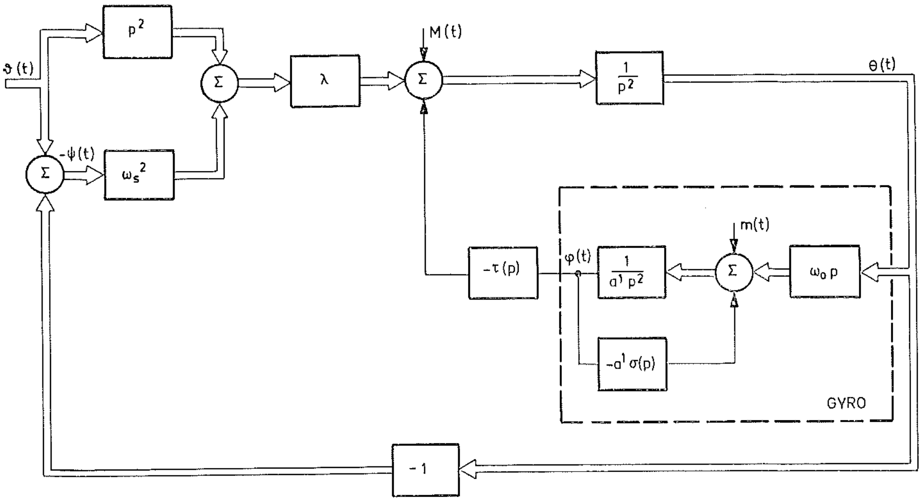
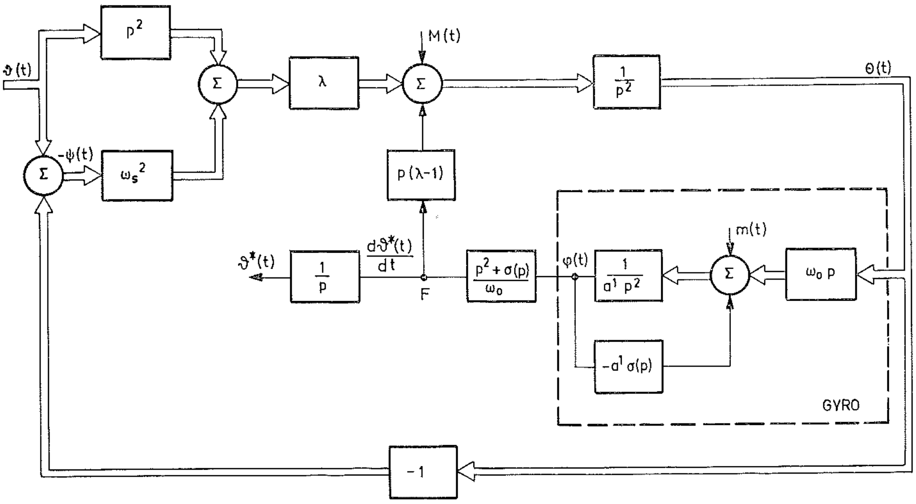


Fig. 5. Block diagram of the complete system with position indication.



Defining the position indication error† $\varepsilon\ (t)$ as

$$\varepsilon(t) = \vartheta^*(t) - \vartheta(t)$$

we get
$$\varepsilon(t) = \psi(t) + \frac{1}{a'\omega_o D}\, m(t)$$

The position indication error is thus the sum of the vertical indication error and the time integral of the disturbing torque acting on the gyrofloat. The vertical indication error is already discussed in "Analysis of the system". For a discussion of the last term of the error we refer to reference 8. Assume that the acceleration of the vehicle and the disturbing torques are stationary random processes with the power spectra $\Phi_{aa}\ (\omega)$, $\Phi_{MM}\ (\omega)$ and $\Phi_{mm}\ (\omega)$ respectively. Applying Lemma 1 of reference 8 we get the following expression for the ensemble average of the squared error

$$E\,[\varepsilon(t)^2] = \frac{t}{a'\omega_o}\int_{-\infty}^{\infty} R_{mm}(\tau)\, d\tau + \underset{t\longrightarrow\infty}{0(1)}$$

where
$$R_{mm}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau}\Phi_{mm}(\omega)\, d\omega$$

The long-time position error is thus entirely determined by the random drift of the gyro.

† In order to get the position indication error in metres we have to multiply $\varepsilon(t)$ by R (radius of earth).

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