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Problems of Identification and Control*

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1. INTRODUCTION

Identification problems have received a great deal of interest during recent years, [6, 8, 12, 13]. One of the motivations has been the desire to apply modern control theory to practical problems in industry and the biosciences. Almost all results in modern control theory require, however, a description of the system in terms of differential or difference equations and a description of the disturbances as stochastic processes, characterized by stochastic differential or difference equations or by second order properties such as covariance functions and spectral densities. In many practical problems in industry and the biosciences, descriptions of systems and disturbances are simply not available. The purpose of identification is to obtain the required descriptions.

In principle it should be possible to obtain the required information from first principles using basic physical laws. In many applications the fundamental results required are, however, not available. Typical examples of this are rate coefficients in pharmacokinetics and heat transfer coefficients in industrial processes. When the models required cannot be obtained from first principles it is necessary to derive the models from data obtained from experiments made on the process.

The identification problem is frequently formulated as follows:

Given a class of models, a criterion and measurements of input and output signals, find the particular model which fits the experimental data best in the sense of the given criterion. A wealth of methods for solving the identification problem have appeared, [8, 11, 12, 13]. The methods differ in the choice of models and criteria as well as mathematical techniques. Both models and criteria are frequently chosen quite arbitrarily, [11, 12]. Even if the main motivation for doing the identification is to solve a control problem this fact is frequently overlooked in the literature. A consequence of this has been that fundamental problems have been neglected e.g.:

— Is it possible to obtain rational choices of model structures and criteria if we know that the results of the identification will be used to design control strategies?

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— Is it necessary to take into account that the solution of the identification problem is not exact when solving the control problem?

— What do we mean by the "accuracy" of an identification problem? What accuracy is needed in a particular case?

The relation between identification and control has been observed in a few papers, [14, 17, 18]. Farison et al. [14] formulates a problem in such a way that the identification and control problems separate. Another class of problems for which this occurs is also discussed by Schwartz and Steiglitz [17].

In this paper we will get some insight into the questions raised above by analyzing a simple case, namely a linear system with one input and one output and a quadratic criterion. Our main purpose has been to look into the problem of optimal control of a system with constant but unknown parameters. It has turned out, however, that the mathematical machinery developed will permit us to deal with the case when the parameters are stochastic processes. We will thus be able to get some insight into the adaptive problem, i.e. a situation where identification and control are performed simultaneously.

The mathematical model is presented in Section 2. The solution to the control problem in the case of known parameters is discussed in Section 3, and the identification of the parameters is covered in Section 4. These two problems are almost trivial for the chosen example. The interrelations between control and identification are discussed in Section 5. In that section we consider the problem of controlling a system with constant but unknown parameters. It is assumed that the identification problem is first solved and that the results of the identification are then used to solve the control problem. The result gives some important aspects on the interrelations between identification and control.

Other aspects on the relationships between identification and control are given in Section 6 where we consider the combined problem of identification and control. The problem discussed differs from the problem of Section 5 in the respect that the data obtained during the operation of the system are used to update the solution to the identification problem. The formulation of the problem is discussed in Section 6. It turns out that the problem can be formulated as a nonlinear stochastic control problem. Such a problem will in general be extremely difficult to solve because of the curse of dimensionality. The state of the system will in general be the conditional probability distributions of the parameters, given the observations. For the particular problems it turns out, however that the conditional probability distributions are normal. This leads to a significant reduction of dimensionality.

A sufficient statistic for the conditional distributions is derived in Section 7. It turns out that it is not necessary to assume that the parameters are constant but that we can also generalize to the situation when the parameters of the system are stochastic processes without increasing the complexity of the problem. This means that we can consider truly adaptive systems. The stochastic optimization problem is solved in Section 8 where the fundamental functional equations are derived using Dynamic Programming. It is shown in particular that the choice of criterion is very important. The strategy which minimizes $Ey^2(t)$ in the steady state will be very different from the strategy which minimizes

$$E\frac{1}{N}\sum_{t=1}^{N} y^2(t).$$

In Section 9 we present results of numerical solutions for a simple system. This example clearly exhibits the differences between the different control strategies. In particular it is shown that it is possible to obtain strategies where the control and identification problems separate simply by choosing suitable loss functions. It is also shown that the control strategies obtained for the problems where identification and control separate can be significantly inferior to the case when a dual control is used.

2. A Simple Control Problem

In this section we will formulate a simple control problem for a linear system. The problem is chosen in such a way that the solution of the control problem is almost trivial if the parameters are known. The pure identification problem is also easy to solve for the particular model.

A Mathematical Model of the System

Consider a linear discrete time dynamical system with one input and one output characterized by the input-output relation

$$\begin{aligned} y(t) + a_1(t)\,y(t-1) + \cdots + a_n(t)\,y(t-n) \\ = b_1(t)\,u(t-1) + \cdots + b_n(t)\,u(t-n). \end{aligned} \tag{2.1}$$

The Eq. (2.1) thus represents the dynamics of a linear system of n-th order. As we want to formulate a control problem we would also like to introduce some disturbances. A simple way of doing this is to replace (2.1) by

$$\begin{aligned} y(t) + a_1(t)\,y(t-1) + \cdots + a_n(t)\,y(t-n) \\ = b_1(t)\,u(t-1) + \cdots + b_n(t)\,u(t-n) + e(t), \end{aligned} \tag{2.2}$$

where $\{e(t),\ t = \ldots -1, 0, 1, \ldots\}$ is a sequence of independent equally distributed normal $(0, \sigma)$ random variables. It is also assumed that $e(t)$ is independent of $y(t-1), y(t-2), \ldots, u(t-1), u(t-2), \ldots$.

The coefficients a_i and b_i will be assumed constant throughout Sections 3, 4 and 5. In such a case the model (2.2) represents an autoregressive process if $u(t) = 0$ and a general linear dynamics if $e = 0$. It can be shown that any linear system with a disturbance which is a stationary stochastic process can be approximated arbitrarily close by a model of type (2.2) if the order n is taken sufficiently large. Hence even if the model (2.2) is simple it can frequently be used as an approximation to a large class of realistic problems.

The Criterion

We will assume that the purpose of the control is to keep the output of the system as close as possible to a prescribed value which we arbitrarily take to be equal to 1. The deviation is specified by the criterion

$$E\ell_1 = E[y(t) - 1]^2 \tag{2.3}$$

or

$$E\ell_2 = E\frac{1}{N}\sum_{t=1}^{N}[y(t) - 1]^2 \tag{2.4}$$

where E denotes mathematical expectation.

The criterion (2.3) is referred to as the one stage control and (2.4) as the N-stage control. If the process $\{y(t)\}$ is ergodic the criteria (2.3) and (2.4) appear to be identical as $N \to \infty$. As will be seen later the control processes obtained by minimizing (2.3) and (2.4) can be widely different.

Admissible Control Strategies

To specify the control problem completely it is also necessary to define the admissible control strategies. A control strategy is admissible if the value of the control signal at time t, $u(t)$, is a function of all the outputs observed up to time t i.e. $y(t), y(t-1), y(t-2), \ldots$ all previously applied control signals $u(t-1), u(t-2), \ldots$ and the a priori data, e.g. the values of the coefficients or the estimates of the coefficients and their accuracies.

3. Solution of the Control Problem in the Case of Constant Known Parameters

We now assume that the parameters of the model (2.2) are constant and known. The a priori data is thus the parameters $n, a_1, \ldots, a_n, b_1, b_2, \ldots, b_n$ and σ. The control problem is then easy to solve. We will first determine a control strategy such that the criterion (2.3) is minimal and we will then show that this strategy also minimizes the criterion (2.4). We have

$$\begin{aligned} E[y(t) - 1]^2 = E[&- a_1 y(t-1) - \cdots - a_n y(t-n) + b_1 u(t-1) + \cdots \\ &+ b_n u(t-n) - 1]^2 + 2E\{e(t)[- a_1 y(t-1) - \cdots \\ &- a_n y(t-n) + b_1 u(t-1) + \cdots + b_n u(t-n) - 1]\} \\ &+ Ee^2(t). \end{aligned} \tag{3.1}$$

Since $e(t)$ has zero mean and is independent of $y(t-1)$, $y(t-2)$, ..., $u(t-1)$, $u(t-2)$ the second term of the right member vanishes and we get:

$$\begin{aligned} E[y(t) - 1]^2 = E[&- a_1 y(t-1) - \cdots - a_n y(t-n) + b_1 u(t-1) \\ &+ \cdots + b_n u(t-n) - 1]^2 + \sigma^2 \geqslant \sigma^2, \end{aligned} \tag{3.2}$$

where equality is obtained for the control strategy

$$\begin{aligned} u(t) = \frac{1}{b_1} [&1 + a_1 y(t) + a_2 y(t-1) + \cdots + a_n y(t-n+1) \\ &- b_2 u(t-1) - \cdots - b_n u(t-n+1)]. \end{aligned} \tag{3.3}$$

This is an admissible strategy because $u(t)$ is a function of $y(t)$, $y(t-1)$,..., $u(t-1)$,... and the a priori data.

The problem is thus solved for the criterion (2.3) and the criterion (2.4) will now be considered. We would thus like to find a control strategy which minimizes

$$\sum_{t=1}^{N} [y(t) - 1]^2. \tag{3.4}$$

Consider the situation at time $N - 1$. The outputs $y(N-1)$, $y(N-2)$,... have been observed and the problem is to determine the control signal $u(N-1)$. Since $u(N-1)$ only influences the last term of the loss function, i.e.

$$[y(N) - 1]^2$$

it is apparent that the strategy (3.3) is optimal for $t = N - 1$. We also find

$$\min[y(N) - 1]^2 = \sigma^2. \tag{3.5}$$

Now consider the situation at time $N - 2$. The output signals $y(N-2)$, $y(N-3)$,... have been observed and the problem is to determine $u(N-2)$.

As $u(N-2)$ only influences the last two terms of the loss function it should be chosen so as to minimize

$$E\{[y(N)-1]^2+[y(N-1)-1]^2\}.$$

If an optimal strategy is used at the last stage we find

$$E\{[y(N)-1]^2+[y(N-1)-1]^2\}=\sigma^2+E[y(N-1)-1]^2. \tag{3.6}$$

As σ is a constant, we now find that the strategy (3.3) is optimal for all t. We also find

$$\min \sum_{t=1}^{N} [y(t)-1]^2 = N\sigma^2. \tag{3.7}$$

Summing up we get:

THEOREM 1. *Assume that the parameters of the model* (2.2) *are constant and known. Then the admissible control strategy* (3.3) *is optimal with respect to both the criterion* (2.3) *and the criterion* (2.4). *The minimal value of the expected loss is* σ^2 *in both cases.*

Remark 1. Notice that Theorem 1 still holds if the parameters of the system are time-varying but known.

Remark 2. It is well known that optimal strategies might sometimes be very sensitive to parameter variations. It has been shown in [7] that the control strategy (3.3) is not sensitive to parameter variations if the polynomial

$$b_1 z^{n-1}+b_2 z^{n-2}+\cdots+b_n=0 \tag{3.8}$$

has all its zeros inside the unit circle.

4. IDENTIFICATION OF THE PARAMETERS OF THE MODEL

If the parameters of the model (2.2) are not known they can be determined from experimental data by several methods. The least squares method is one of the simplest techniques available. In this section we will briefly review the application of the least squares method to the determination of the parameters of the model (2.2). For additional details see [6]. Using the least squares method the parameters a_i and b_i of the model (2.2) are simply determined in such a way that the criterion

$$\begin{aligned} V(a_1,\ldots,a_n,b_1,\ldots,b_n) = \sum_{t=n}^{N+n} [y(t)+a_1 y(t-1)+\cdots \\ +a_n y(t-n)-b_1 u(t-1)-\cdots-b_n u(t-n)]^2 \end{aligned} \tag{4.1}$$

is as small as possible. In (4.1) u denotes the actual values of the control signal used during the experiment and y denotes the observed outputs.

As the criterion (4.1) is quadratic in a_i and b_i it is easy to minimize V analytically. Let a and b denote vectors whose components are a_i and b_i. Introduce the column vector x defined by

$$x = \mathrm{col}[a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n] \tag{4.2}$$

and the row vector $\theta(t)$ whose components are defined by

$$\theta(t) = [-y(t-1), -y(t-2), \ldots, -y(t-n), u(t-1), u(t-2), \ldots, u(t-n)]. \tag{4.3}$$

The loss function V can then be written as:

$$\begin{aligned} V(x) &= \sum_{t=n}^{n+N} [y(t) - \theta(t)\, x]^2 = \sum_{t=n}^{n+N} y^2(t) - 2\left(\sum_{t=n}^{N+n} y(t)\, \theta(t)\right) x \\ &\quad + x^T \left(\sum_{t=n}^{N+n} \theta^T(t)\, \theta(t)\right) x. \end{aligned} \tag{4.4}$$

Assuming that the matrix $\Sigma[\theta^T(t)\, \theta(t)]$ is positive definite we find that the minimum of V with respect to x is obtained for

$$x = \hat{x} = \left[\sum_{t=n}^{N+n} \theta^T(t)\, \theta(t)\right]^{-1} \sum_{t=n}^{N+n} \theta^T(t)\, y(t). \tag{4.5}$$

The minimal value is:

$$\begin{aligned} \min V &= \sum_{t=n}^{n+N} y^2(t) - \sum_{t=n}^{n+N} y(t)\, \theta(t) \left[\sum_{t=n}^{N+n} \theta^T(t)\, \theta(t)\right]^{-1} \sum_{t=n}^{N+n} \theta^T(t)\, y(t) \\ &= \sum_{t=n}^{n+N} [y(t) - \theta(t)\, \hat{x}]^2. \end{aligned} \tag{4.6}$$

Several questions now arise. Is the estimate unbiased? What is the variance of the estimate? What conditions are required if the matrix $\Sigma\theta^T(t)\, \theta(t)$ should be nonsingular. Answers of these questions are given by:

THEOREM 2. *Let all the roots of the equation*

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0 \tag{4.7}$$

have magnitudes less than one. Assume that the limits

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t) \qquad \textit{and} \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)\, u(t+\tau) = R_u(\tau)$$

exist and that the matrix A whose elements are defined by

$$a_{ij} = R_u(i-j), \qquad i, j = 1, 2,..., n \tag{4.8}$$

is positive definite. Then the least squares estimates $\hat{x}$ converges to the true parameter value x as the number of observations N tend to infinity. For large N the estimate $\hat{x}$ is asymptotically normal with mean value x and covariance matrix $P = R^{-1}/N$ where R is a positive definite matrix defined by

$$R = \lim \frac{1}{N} \sum_{t=n}^{N+n} \theta^T(t)\, \theta(t). \tag{4.9}$$

This theorem is an extension of Mann and Wald's theorem on the consistency of the least squares estimate for an autoregressive process. An outline of the proof is given in [6]. We also have the following result.

THEOREM 3. *Let the matrix*

$$\sum_{t=n}^{N+n} \theta^T(t)\, \theta(t)$$

be definite. Then the conditional distribution of the parameters a_i and b_i of the model (2.2) *given*

$$\begin{aligned}\mathscr{Y}_{N+n} = [&y(N+n), y(N+n-1),..., y(0), u(N+n-1),\\ &\times u(N+n-2),..., u(0)]\end{aligned}$$

is normal with the mean value

$$\hat{x} = \left[\sum_{t=n}^{N+n} \theta^T(t)\, \theta(t)\right]^{-1} \sum_{t=n}^{N+n} \theta^T(t)\, y(t), \tag{4.5}$$

and the covariance matrix P defined by

$$P = \left[\sum_{t=n}^{N+n} \theta^T(t)\, \theta(t)\right]^{-1} \sigma^2. \tag{4.10}$$

A proof of this theorem is found in [6]. A slightly more general version of this theorem is also proven in Section 7 of this paper. Summing up we thus find that in the particular case the identification problem can be solved easily using the method of least squares and that the least squares estimate has several desirable properties such as asymptotic unbiasedness and asymptotic efficiency. In the next section we will investigate the relevance of these properties in relation to the solution of the control problem.

5. Separate Identification and Control

In Section 3 the control problem was solved in the case of known parameters, and in Section 4 it was shown how the parameters of the model can be identified using the method of least squares. We will now discuss the interaction between identification and control in connection with the problem of controlling a system with constant but unknown coefficients. To be specific we will consider a system governed by Eq. (2.2) where it is assumed that the parameters a_i and b_i are constant but that their numerical values are not known. It is also assumed that the object of control is to minimize the criterion (2.3) or (2.4). Throughout the section it is assumed that we first make an experiment on the system, that the outcome of this experiment is used to identify the parameters and to design a control law. This control law is then used to control the system throughout its operating period. The data obtained during the phase when the system is controlled is thus not used to improve the parameter estimates. There are many questions which arises naturally, e.g.,

- Does there exist a *separation theorem* in the sense that the optimal control law can be obtained simply by using the strategy obtained in the case of known parameters and substituting the true parameters by their estimates? (This assumption is frequently used in practical applications.)
- If a separation theorem exists, what estimates should be used?
- How much will the expected loss increase due to the fact that the parameters are not known accurately?
- Will the criteria (2.3) and (2.4) lead to the same result as was the case when the coefficients are known?

We will approach the problem by deriving the optimal control strategy and then analyzing its properties. Let us first consider the criterion (2.3). We will thus determine a control strategy which minimizes $E[y(t) - 1]^2$. To derive such a strategy we consider the situation at time $t - 1$. The outputs $y(t-1)$, $y(t-2)$,... are observed and the previous input signals $u(t-2)$,

$u(t-3),\ldots$ are known. Let $\mathscr{Y}_{t-1}$ be a vector which contains the known data, i.e.

$$\mathscr{Y}_{t-1} = [y(t-1), y(t-2),\ldots, u(t-2), u(t-3),\ldots].$$

The problem is thus to determine $u(t-1)$ as a function of $\mathscr{Y}_{t-1}$ in such a way that $E[y(t)-1]^2$ is minimal. Using the fundamental lemma of stochastic control theory [7, Lemma 3.2 of Chap. 8] we find that:

$$\min E[y(t)-1]^2 = E \min_{u(t-1)} E\{[y(t)-1]^2 \mid \mathscr{Y}_{t-1}$$

We have

$$y(t) = \theta(t)\, x + e(t) = \Sigma' \theta_i(t)\, x_i + b_1 u(t-1) + e(t), \tag{5.1}$$

where Σ' denotes the sum over 1 to $2n$ with the value $n+1$ excluded.

In Eq. (5.1) the elements $\theta_1(t),\ldots, \theta_n(t)$, $\theta_{n+2}(t),\ldots, \theta_{2n}(t)$ are equal to $-y(t-1)$, $-y(t-2),\ldots, -y(t-n)$, $u(t-2),\ldots, u(t-n)$ respectively which are all known; $u(t-1)$ is at our disposal. The components of the vector x are the parameters of the system which are not known. Compare (4.2). The identification experiment and the computation of the least squares estimate gives, however, the conditional distribution of x given the results of the identification experiment. It follows from Theorem 3 that the conditional distribution given $\mathscr{Y}_{t-1}$ is normal with the mean value $\hat{x}$ given by (4.5) and the covariance matrix P given by (4.10). We thus find

$$\begin{aligned} E\{[y(t)-1]^2 \mid \mathscr{Y}_{t-1}\} = [\Sigma'\, \theta_i(t)\, \hat{x}_i + \hat{b}_1 u(t-1) - 1]^2 \\ + \Sigma'_{ij} p_{ij} \theta_i(t)\, \theta_j(t) + u^2(t-1)\, p_{n+1,n+1} \\ + 2u(t-1)\, \Sigma'\, p_{n+1,i} \theta_i(t) + \sigma^2, \end{aligned} \tag{5.2}$$

where $\theta_i(t)$, $p_{ij}(t)$ and $\hat{x}_i$ do not depend on $u(t-1)$. We thus find that the control strategy

$$u(t-1) = \frac{\hat{b}_1 - \Sigma'\, [\hat{b}_1 \hat{x}_i + p_{n+1,i}]\, \theta_i(t)}{\hat{b}_1{}^2 + p_{n+1,n+1}} \tag{5.3}$$

will minimize (5.2). The minimal value of the loss function is given by

$$\begin{aligned} \min E[y(t)-1]^2 = \sigma^2 + \Sigma'_{i,j} p_{ij} \theta_i(t)\, \theta_j(t) + (1 - \Sigma_i{}'\, \theta_i(t)\, x_i)^2 \\ - \frac{[\hat{b}_1 - \Sigma'\, [\hat{b}_1 \hat{x}_i + p_{n+1,i}]\, \theta_i(t)]^2}{\hat{b}_1{}^2 + p_{n+1,n+1}}. \end{aligned} \tag{5.4}$$

A comparison of (3.3) and (5.3) shows that the optimal strategy for the combined problem is *not* obtained simply by substituting the true parameter values by their least squares estimates. A comparison of (3.2) and (5.4) shows that the last three terms of (5.4) represent the increase of the loss function due to the uncertainty of the identification.

Since the last three terms of the right-side of (5.4) depend on $u(t-2)$, $u(t-3)$ etc., we find that the strategy (5.3) will not minimize the criterion (2.4). The solution of the problem for the criterion (2.4) is given in Section 8.

Summing up we thus find that it is not sufficient just to compute the least squares estimate in order to obtain the optimal control but that the knowledge of the conditional probability distribution of the parameters is required. We summarize the result as:

THEOREM 4. *Consider the system* (5.1) *with constant but unknown parameters. The control law* (5.3) *then minimizes the criterion* (2.3).

6. Combined Identification and Control

The solution discussed in Section 5 has the apparent drawback that the data obtained during the control phase are not used to improve the parameter estimates. This possibility was excluded already in the problem formulation. In this section we will investigate a combined estimation and control problem. At each step of the process all the available information is used both to determine a suitable parameter estimate and a suitable value of the control signal.

It turns out that the problem obtained can be solved using the theory of optimal control of Markov processes [1, 2, 3, 20]. We will consider the system (2.2) with the criteria (2.3) and (2.4). The coefficients of (2.2) are assumed to be unknown. It turns out that the problem can be generalized slightly without introducing extra mathematical complications. It is thus possible to solve the problem in the case that the parameters a_i and b_i are Gauss-Markov processes with the same mathematical machinery. We can thus consider a truly adaptive problem.

Formulation

Consider the model (2.2). Let the parameters a_i and b_i be timevarying

$$
\begin{aligned}
x_1(t) &= a_1(t);\\
x_2(t) &= a_2(t);\\
&\vdots\\
x_n(t) &= a_n(t);\\
x_{n+1}(t) &= b_1(t);\\
x_{n+2}(t) &= b_2(t);\\
&\vdots\\
x_{2n}(t) &= b_n(t).
\end{aligned}
\tag{6.1}
$$

Assume that x is a Gauss-Markov process which satisfies the stochastic difference equation

$$x(t + 1) = \Phi x(t) + v(t), \tag{6.2}$$

where Φ is a known constant $2n \times 2n$ matrix and $\{v(t), t = t_0, t_0 + 1, ...\}$ is a sequence of independent equally distributed normal vectors with zero mean value and covariance R_1. The initial state of the system (6.2) is assumed normal with mean value

$$Ex(t_0) = m \tag{6.3}$$

and covariance

$$\operatorname{cov}[x(t_0), x(t_0)] = R_0 . \tag{6.4}$$

It is assumed that $e(t)$ is independent of $x(t_0)$. The case of constant coefficients is included in (6.2) because we can always choose $\Phi = I$ (the identity matrix) and $R_0 = R_1 = 0$. The input-output relation of the system (2.2) can be written in the compact form

$$y(t) = \theta(t)\, x(t) + e(t) \tag{6.5}$$

where the vector θ is defined by (4.3). It is also assumed that $e(t)$ and $v(s)$ are independent for all t and s.

We are thus considering a linear time-varying system whose parameters are Gauss–Markov processes. Notice that the control signal appears as components of the vector θ.

The criterion is taken so as to minimize the expected loss given by (2.3) or (2.4). It turns out that these two criteria will give results which are mostly different.

The admissible control strategies are the ones defined in Section 2.

Outline of Solution

Before entering the details we will first outline the main steps in the solution. The problem we have formulated is an optimal control problem for a Markov process with incomplete state information. It is known [1, 2, 20] that such problems can be solved using Dynamic Programming if a suitable hyperstate is chosen. The hyperstate is in general infinite dimensional. It is in essence the conditional distribution of the original state of the original Markov process given the observed outputs. In this particular case the hyperstate is the conditional distribution of x given all the observed outputs. This conditional distribution will be derived in Section 7. It turns out that the conditional distribution is Gaussian which gives a considerable reduction of dimension.

7. The Hyperstate of the Problem

In this section we will consider the model (6.2) described in the previous section. We will derive a convenient form for the conditional distribution of $x(t)$ given all applied inputs and all observed outputs. For this purpose we introduce $\mathcal{Y}_t = [y(t), y(t-1),..., u(t-1), u(t-2),...]$ as the vector whose components are all inputs applied and all outputs observed up to time t. The conditional distribution of $x(t)$ given $\mathcal{Y}_t$ is given by the following.

THEOREM 5. *Consider the model* (6.2) *with the output defined by* (6.5). *The conditional distribution of $x(t)$ given $\mathcal{Y}_{t-1}$ is normal with mean $\hat{x}(t)$ and covariance $P(t)$, where $\hat{x}$ and P satisfies the difference equations*

$$\hat{x}(t+1) = \Phi\hat{x}(t) + K(t)[y(t) - \theta(t)\hat{x}(t)], \tag{7.1}$$

$$P(t+1) = [\Phi - K(t)\theta(t)] P(t)\Phi^T + R_1, \tag{7.2}$$

where

$$K(t) = \Phi P(t)\theta^T(t)[\theta(t)P(t)\theta^T(t) + R_2]^{-1}, \tag{7.3}$$

and the initial conditions are

$$\hat{x}(t_0) = m, \tag{7.4}$$

$$P(t_0) = R_0. \tag{7.5}$$

Proof. If $\theta(t)$ was known a priori then the theorem would be identical to the Kalman filtering theorem. Going through the details of Kalman's proof we find, however, that the arguments used in the proof still hold because $\theta(t)$ is a function of $\mathcal{Y}_{t-1}$. Compare Eq. (4.3).

Remark 1. Notice that Theorem 5 includes Theorem 3 as a special case!

Remark 2. Notice that Theorem 3 can be easily generalized to the case when the parameters a_i and b_i are stochastic processes given by arbitrary linear stochastic differential equations.

Summing up we thus find that the conditional distribution of $x(t)$ given $\mathcal{Y}_{t-1}$ is normal in spite of the fact that the process $\{y(t), t = t_0, t_0 + 1,...\}$ is not normal. We also find that in order to carry out the computations given by (7.1) and (7.2) it is necessary to store $\theta(t)$. We thus find that the hyperstate of the system given by Eqs. (6.2) and (6.5) is the triplet $\hat{x}$, P, θ. Based on this triplet we can then generate the conditional distribution of $x(t)$ and $y(t)$ given $\mathcal{Y}_{t-1}$.

8. Dynamic Programming

Having obtained the conditional distributions we will now solve the control problem formulated in Section 6 using Dynamic Programming.

Consider the situation at time $t-1$. The outputs $y(t-1)$, $y(t-2)$,... have been observed. The past inputs $u(t-2)$, $u(t-3)$,... are known and the problem is to determine $u(t-1)$ in such a way as to minimize the expected loss. By changing $u(t-1)$, only the part

$$\sum_{k=t}^{n} [y(k)-1]^2$$

is influenced. It follows from a fundamental result of stochastic control theory [7, Lemma 8:3.2] that

$$\min E \sum_{k=t}^{n} [y(k)-1]^2 = E_{\mathscr{Y}_{t-1}} \min E \left\{ \sum_{k=t}^{n} [y(k)-1]^2 \mid \mathscr{Y}_{t-1} \right\} \tag{8.1}$$

where $\mathscr{Y}_{t-1}$ denotes the vector $[y(t-1), y(t-2),..., u(t-2), u(t-3),...]$ and it is assumed that the minimum exists.

It was shown in Section 7 that $E[\cdot \mid \mathscr{Y}_{t-1}]$ is a function of $\hat{x}(t)$, $P(t)$, $\theta(t)$ and t. Also notice that the $n+1$-th component of $\theta(t)$ equals $u(t-1)$, the control variable which should be determined.

Introduce the vector $\tilde{\theta}(t)$ defined by

$$\tilde{\theta}(t) = [-y(t-1),..., -y(t-n), u(t-2),..., u(t-n)] \tag{8.2}$$

which equals the vector $\theta(t)$ with the component $u(t-1)$ removed and the function V defined by

$$V(\hat{x}(t), P(t), \tilde{\theta}(t), t) = \min E \left\{ \sum_{k=t}^{N} [y(k)-1]^2 \mid \mathscr{Y}_{t-1} \right\} \tag{8.3}$$

$\tilde{\theta}$ is thus a vector which contains all elements of θ except $u(t-1)$. Using Dynamic Programming we find the following functional equation for V:

$$\begin{aligned} V(\hat{x}(t), P(t), \tilde{\theta}(t), t) = \min_{u(t-1)} E\{[y(t)-1]^2 \\ + V(\hat{x}(t+1), P(t+1), \tilde{\theta}(t+1), t+1) \mid \mathscr{Y}_{t-1}\}. \end{aligned} \tag{8.4}$$

It follows from Eq. (6.5) and Theorem 3 that the conditional distribution of $y(t)$ given $\mathscr{Y}_{t-1}$ is normal with mean value

$$E[y(t) \mid \mathscr{Y}_{t-1}] = \theta(t)\, \hat{x}(t) \tag{8.5}$$

and the covariance

$$\operatorname{cov}[y(t), y(t) \mid \mathscr{Y}_{t-1}] = \theta(t)\, P(t)\, \theta^T(t) + \sigma^2. \tag{8.6}$$

Hence

$$E\{[y(t) - 1]^2 \mid \mathscr{Y}_{t-1}\} = [\theta(t)\,\hat{x}(t) - 1]^2 + \theta(t)\,P(t)\,\theta^T(t) + \sigma^2. \tag{8.7}$$

Furthermore it follows from Eq. (7.1) that the conditional distribution of $\hat{x}(t+1)$ given $\mathscr{Y}_{t-1}$ is normal with mean value

$$E\hat{x}(t+1) \mid \mathscr{Y}_{t-1}] = \Phi\hat{x}(t) \tag{8.8}$$

and the covariance

$$\operatorname{cov}[\hat{x}(t+1), \hat{x}(t+1) \mid \mathscr{Y}_{t-1}] = K(t)\,[\sigma^2 + \theta(t)\,P(t)\,\theta^T(t)]\,K^T(t). \tag{8.9}$$

It follows from Eq. (7.2) that $P(t+1)$ is simply a deterministic function of $P(t)$ and $\theta(t)$.

The Eqs. (4.3) and (8.2) imply that

$$\tilde{\theta}_i(t+1) = \begin{cases} -y(t), & i = 1; \\ \tilde{\theta}_{i-1}(t), & i = 2, 3,\ldots, n, n+2,\ldots, 2n; \\ u(t-1), & i = n+1. \end{cases} \tag{8.10}$$

The conditional distribution of $\tilde{\theta}(t+1)$ given $\mathscr{Y}_{t-1}$ is then easily obtained from (8.5) and (8.6).

Exploiting the Eqs. (8.5-10), we find that the functional Eq. (8.4) reduces to

$$\begin{aligned} V(\hat{x}(t), P(t), \tilde{\theta}(t), t) = \min_{u(t-1)} \Big\{ & [\theta(t)\,\hat{x}(t) - 1]^2 + \sigma^2 \\ & + \theta(t)\,P(t)\,\theta^T(t) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(\hat{x}(t+1), P(t+1), \\ & \times \tilde{\theta}(t+1), t+1)\, e^{-(s^2/2)}\, ds \Big\}, \end{aligned} \tag{8.11}$$

where

$$\hat{x}(t+1) = \Phi\hat{x}(t) + K(t)\sqrt{\sigma^2 + \theta(t)\,P(t)\,\theta^T(t)}\; s; \tag{8.12}$$

$$P(t+1) = [\Phi - K(t)\,\theta(t)]\,P(t)\,\Phi^T + R_1; \tag{8.13}$$

$$K(t) = \Phi P(t)\,\theta^T(t)\,[\sigma^2 + \theta(t)\,P(t)\,\theta^T(t)]^{-1}; \tag{8.14}$$

$$\begin{aligned} \tilde{\theta}_1(t+1) &= -\theta(t)\,\hat{x}(t) + \sqrt{\sigma^2 + \theta(t)\,P(t)\,\theta^T(t)}\; s; \qquad (8.15) \\ \tilde{\theta}_i(t+1) &= \tilde{\theta}_{i-1}(t), \qquad i = 2,\ldots, n, n+2,\ldots 2n-1; \\ \tilde{\theta}_{n+1}(t+1) &= u(t-1). \end{aligned}$$

The first three terms of (8.11) represent the immediate loss and the last term the accumulated loss over the last steps starting with $t+1$. Notice that the control variable $u(t-1)$ equals the $n+1$-th component of $\theta(t)$. It follows from (8.12), (8.13) and (8.15) that $u(t-1)$ influences $\hat{x}(t+1)$, $P(t+1)$, and $\tilde{\theta}(t+1)$. This implies that the choice of the control signal $u(t-1)$ influences both the future parameter estimates, their accuracy and the future values of the output signal. Due to the non-linearity of (8.11) it is in general not possible to carry out the minimization explicitly except at the last step. For $t = N$ we have, however,

$$V(\hat{x}(t), P(t), \tilde{\theta}(t), t) = \min_{u(t-1)}\{[\theta(t)\,\hat{x}(t) - 1]^2 + \sigma^2 + \theta(t)\,P(t)\,\theta^T(t)\} \tag{8.16}$$

where the function to be minimized is quadratic. We thus get

$$u(t-1) = \frac{\hat{x}_{n+1}(t) - \sum_{i=1}^{2n}{}' \,[\hat{x}_{n+1}(t)\,\hat{x}_i(t) + p_{n+1,i}(t)]\,\theta_i(t)}{\hat{x}^2_{n+1}(t) + p_{n+1,n+1}(t)}, \tag{8.17}$$

where Σ' means that the term corresponding to $i = n+1$ is excluded. We also find

$$V(\hat{x}, P, \tilde{\theta}, N) = \sigma^2 + \Sigma' p_{ij}\theta_i\theta_j + (1 - \Sigma_i{}'\theta_i(t)\,x_i)^2 - \frac{[\hat{x}_{n+1} - \Sigma'(\hat{x}_{n+1}\hat{x}_i + p_{n+1,i})\,\theta_i]^2}{\hat{x}^2_{n+1} + p_{n+1,n+1}}. \tag{8.18}$$

Notice that the strategy (8.17) is optimal if the criterion (2.3) is chosen.

Summing up we now find

THEOREM 6. *Assume that the minimum of the loss function exists. The optimal strategy is then given by the functional Eq.* (8.11) *where* $\hat{x}(t+1)$, $P(t+1)$, *and* $\tilde{\theta}(t+1)$ *are given by* (8.12), (8.13) *and* (8.15). *The initial condition of* (8.11) *is given by* (8.18). *The minimal value of the expected loss is*

$$EV(m, R_0, \tilde{\theta}(t_0), t_0),$$

where E denotes mathematical expectation over the distribution of $\tilde{\theta}(t_0)$.

Remark 1. It was mentioned previously that the analysis includes the case of constant but unknown parameters as the special case, $\Phi = I$ and $R_1 = 0$. It follows from the Eqs. (8.12-15) that this is not a significant simplification. The effort required to solve the combined estimation and control problem in the case of constant but unknown parameters thus is not significantly easier than to solve the problem in the case of stochastic parameters.

Remark 2. Notice the drastic difference between the optimal strategies for the criteria (2.3) and (2.4). When the criterion is chosen (2.3) we find that the optimal strategy is given by (8.17). This strategy will only minimize the immediate loss. It does not attempt to pick the control signal in such a way that the future estimates are improved.

9. Examples

In order to illustrate the results we will now give a few examples. In order to get simple computations we will consider the following simple system:

$$y(t) = xu(t-1) + e(t) \tag{9.1}$$

where x is the unknown parameter and $\{e(t)\}$ a sequence of independent normal $(0, \sigma)$ stochastic variables. The system (9.1) thus has no dynamics and an unknown gain parameter. It is assumed that the purpose is to keep the output of the system as close to one as possible. The performance of the system is evaluated by the expected loss using the loss function (2.3) or (2.4).

Separate Identification and Control

We will first consider the case when identification and control are performed separately. Choosing an arbitrary input signal $\{u(t),\ t = 0, 1,..., N-1\}$, observing the corresponding output $\{y(t), t = 1, 2,..., N\}$ we find from Eq. (4.5) that the least squares estimate is given by

$$\hat{x} = \frac{\sum_{t=1}^{N} u(t-1)\, y(t)}{\sum_{t=1} u^2(t-1)}. \tag{9.2}$$

The variance of the estimate is

$$P = \frac{\sigma^2}{\sum_{t=1}^{N} u^2(t-1)}. \tag{9.3}$$

It also follows from Theorem 3 that the conditional distribution of x given $\{y(t),\ t = 1, 2,..., N\}$ is normal with mean $\hat{x}$ and variance P. Using the results of the identification experiment to design a control law we find from Theorem 4 that the control law

$$u(t) = \frac{\hat{x}}{\hat{x}^2 + P} = \frac{1}{\hat{x}} \frac{\hat{x}^2}{\hat{x}^2 + P} \tag{9.4}$$

will minimize the criterion (2.3). The minimal value of the expected loss is given by

$$V_1 = E[y(t) - 1]^2 = \sigma^2 + \frac{P}{\hat{x}^2 + P}. \tag{9.5}$$

As the rightside of Eq. (9.5) does not depend on $u(t - 2),...$ we find the strategy (9.4) will also minimize the criterion (2.4). If the parameter x is known the optimal strategy is simply

$$u(t) = \frac{1}{x} \tag{9.6}$$

and the minimal loss is

$$V_0 = E[y(t) - 1]^2 = \sigma^2. \tag{9.7}$$

A comparison of (9.4) and (9.6) now shows that the effect of the uncertainty is to reduce the gain of the system by the factor $\hat{x}^2/(\hat{x}^2 + P)$.

A comparison of (9.5) and (9.7) also shows that the relative increase of the loss function due to the uncertainty of the parameters is

$$\frac{V_1 - V_0}{V_0} = \frac{P}{\sigma^2[\hat{x}^2 + P]}.$$

For example if $\hat{x} = 1$, $\sigma = 0.5$ we find that the parameter must be determined with the accuracy $P = 0.0025$ ($\sigma_x = 0.05$) if the uncertainty of the parameters should not increase the loss function by more than 1 % compared to the case of known parameters. If it is assumed that $|u(t)| = 1$ during the identification experiment we find from (9.3) that this uncertainty of the parameters corresponds to an identification period of $N = 100$ samples.

If a 10 % increase of the loss function due to the uncertainty of the parameters is permitted we find that an identification period of $N = 10$ samples is sufficient.

Combined Identification and Control

We will now discuss the combined identification and control problem. It is assumed that the parameter x satisfies the stochastic difference equation

$$x(t + 1) = 0.9x(t) + v(t) \tag{9.8}$$

where $\{v(t)\}$ is a sequence of independent normal (0, 1) stochastic variables. The initial condition of (9.8) is assumed to be normal with zero mean and the variance

$$\text{Var } x_0(t) = \frac{1}{1 - 0.9^2} = 5.26.$$

This means that the gain parameter of the system is a stationary Gauss–Markov process with zero mean value. As the gain of the system changes according to (9.8) it is intuitively clear that the control problem can be very difficult. The gain parameter can change sign.

Minimization of $E[y(t) - 1]^2$

If the criterion given by Eq. (2.3) is chosen we find that the optimal control problem can be solved analytically using Theorem 6. The optimal control law is given by the equation (8.17) which in this particular case reduces to

$$u(t) = \frac{\hat{x}(t)}{\hat{x}^2(t) + P(t)} = \frac{1}{\hat{x}(t)} \cdot \frac{\hat{x}^2(t)}{\hat{x}^2(t) + P(t)}, \tag{9.9}$$

where $\hat{x}$ and P are given by the Eqs. (8.12), (8.13), and (8.14):

$$\hat{x}(t+1) = 0.9\hat{x}(t) + K(t)\,[y(t) - u(t-1)\,\hat{x}(t)]; \tag{9.10}$$

$$P(t+1) = 0.9[0.9 - K(t)\,u(t-1)]\,P(t) + 1; \tag{9.11}$$

$$K(t) = \frac{0.9P(t)\,u(t-1)}{\sigma^2 + P(t)\,u^2(t-1)}. \tag{9.12}$$

It is not easy to analyze the properties of the system (9.1) when the parameter x is given by (9.8) and the control law (9.9) is used because the equations for the closed loop system are strongly nonlinear. In order to get some insight into the properties of the system we will therefore use simulations. In Fig. 1 we show some results of a simulation of the system. Notice in particular the strange behaviours of the control signal. There are long intervals during which the control signal is practically zero. This means in fact that the system is not controlled at all during this interval.

The graph of the parameter estimate $\hat{x}$ shows that the estimate agress reasonably well with the true parameter value except at the intervals when the control signal assumes very low values. The graph of the variance of the parameter estimate also shows that the variance is close to the steady state value

$$P_\infty = \frac{1}{1 - 0.9^2} = 5.26 \tag{9.13}$$

when the control signal is small.

Minimization of $E \sum_{t=1}^{N} [y(t) - 1]^2$

It is thus clear that the control strategy obtained by minimizing the criterion $E[y(t) - 1]^2$ has several undesirable features. We will therefore turn to the criterion (2.4). In this case we have to solve the functional Eq. (8.11).

This cannot be done analytically. The state of the system is $\hat{x}$ and P and we thus have to solve a Dynamic Programming problem with two variables. This is easily done using discretization. To simplify the computations we introduce the variables

$$z = u\hat{x}, \qquad w = \frac{P}{\hat{x}^2}$$

and the variables z and w are quantized instead of the original variables. With exact state information the optimal control is $u\hat{x} = 1$, i.e. $z = 1$. This

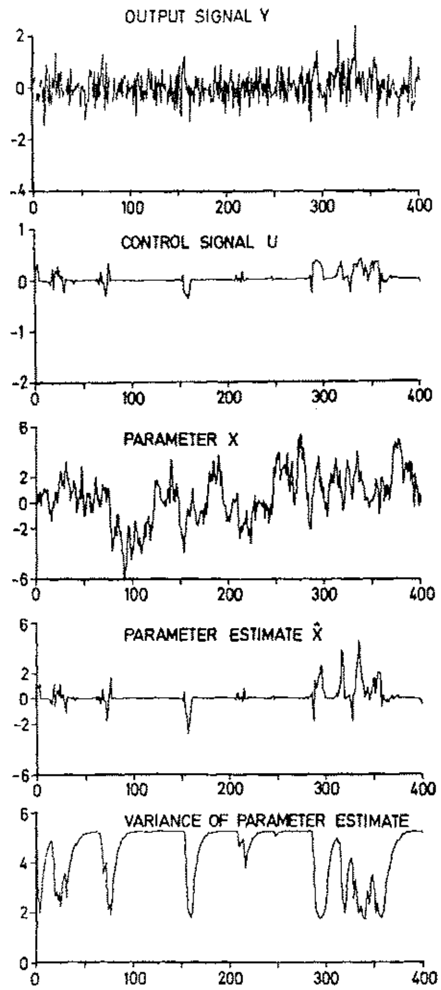

FIG. 1. Result from simulation using the control law, $u(t) = \hat{x}(t)/(\hat{x}(t)^2 + P(t))$ which minimize $E\ell_1$.

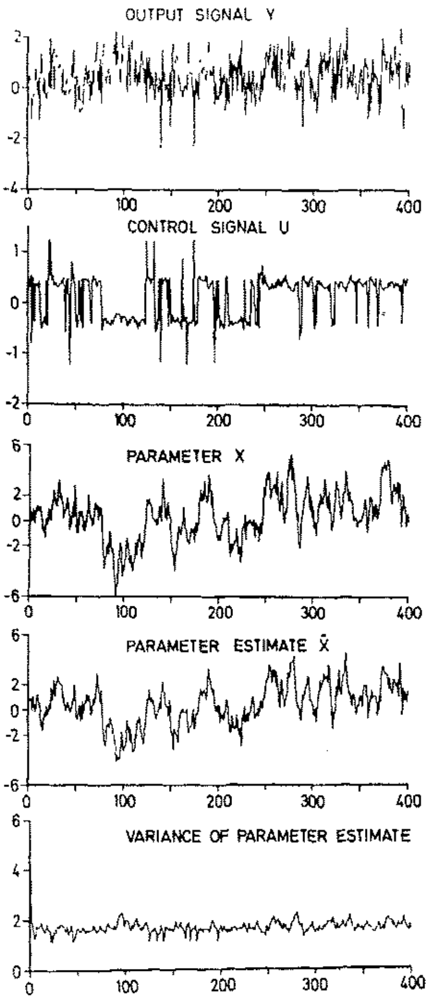

FIG. 2. Result from simulation using the control table obtained through Dynamic Programming when minimizing $E\ell_2$.

means that z can be interpreted as a weighting factor. See also (9.9). The variable w can be interpreted as the relative variance of the estimate of w. In the computations twenty levels of quantization were used for both z and w.

The integration in (8.11) was performed using Simpson's formula. It was found that the control table achieved steady state after about 20 steps at the iteration. In the simulation we only used the steady state control law. To check the results the average loss was evaluated both by integration in the loss table and by simulation of the optimal control strategies.

In Fig. 2 we show the results of a simulation. It is of interest to compare with the results of Fig. 1. By choosing the criterion (2.4) the control signal will not only eliminate the instantaneous error but it will also improve the accuracy of the estimation error. This effect is clearly seen by comparing the variances of the parameter estimates shown in Fig. 1 and Fig. 2.

In Fig. 3 we finally give a comparison between the accumulated errors obtained using strategies which minimizes (2.3) and (2.4). This graph also shows the estimated loss obtained by integration in the loss table of the Dynamic Programming problem. For further discussions of the dual control law see [19].

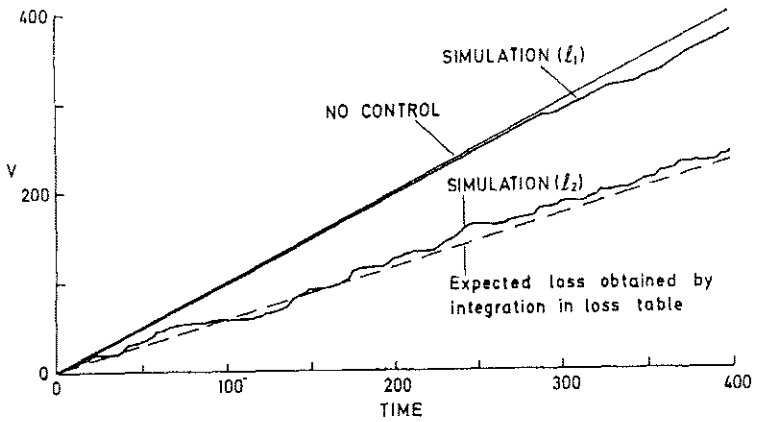

FIG. 3. Accumulated loss $V = \sum_{k=1}^{t} (y(k) - 1)^2$ for the strategies which minimize $E\ell_1$ and $E\ell_2$.

10. CONCLUSION

The relationships between identification and control have been investigated for a simple regression model. The analysis has shown that if the results of the identification problem will be used to design control strategies it is not sufficient to compute the conditional probability distribution of the parameters of the system given the outputs obtained during the identification experiment. For the simple regression model it was shown that the conditional distribution is normal with a mean value equal to the least squares estimate.

Two different problems have been pursued, called separate identification and control and combined identification and control. In the first problem the identification problem is first solved separately and the data obtained during the operation of the system are not used to improve the parameter estimates. In the combined problem estimation and control are performed simultaneously and the input-output pairs obtained at each stage are used to update the parameter estimates.

It turns out that when solving the combined estimation and control problem the case of constant but unknown parameters is not significantly easier than the problem with stochastically varying parameters. We have therefore considered the case when the coefficients are Gauss–Markov processes.

Two different criteria have also been used. The criteria are characterized by the loss functions

$$\ell_1 = [y(t) - 1]^2 \qquad \text{and} \qquad \ell_2 = \frac{1}{N} \sum_{k=1}^{N} [y(t) - 1]^2.$$

In the separate identification and control problem the determination of optimal strategies reduces to a linear quadratic control problem.

The optimal strategy for continued identification and control problem can be obtained analytically for the loss function ℓ_1. The solution of the combined problem for the loss function ℓ_2 requires Dynamic Programming. The state space of the problem is $\hat{x}$, P and $\tilde{\theta}$ a vector of n past outputs and $n - 1$ past inputs. For a system of n-th order the dimension of the problem is then $2n^2 + 5n - 1$.

Even if the optimal strategies for the combined problem with the loss function ℓ_1 can be derived analytically, it is not easy to analyze the properties of the closed loop system because the equations are nonlinear. Simulation of a simple example has shown, however, that the equations have interesting properties. It has been shown that the system exhibits a "*falling asleep*" effect in the sense that it happens that the control signal can become close to zero over long periods. Similar phenomena have been observed in other adaptive systems. There is a marked difference between the systems obtained when minimizing the loss functions (2.3) and (2.4) in this respect. The strategy which minimizes (2.4) does not exhibit the "falling asleep" effect.

This paper should be regarded as an initial attempt to investigate the relationships between identification and control. There are many questions which remain to be answered. For example it would be highly desirable to provide mathematical analysis which gives insight into the difference between the properties of the systems which minimizes (2.3) and (2.4). This is essentially a problem of analyzing nonlinear stochastic differential equations of a particular class.

It would also be highly desirable to look into the computational aspects of the functional Eq. (8.11) as well as to extend the results to more general models.

Another problem of great interest would be to investigate the Eqs. (8.11–15) in the case of constant but unknown parameters to find out if the control strategy converges to the control stategy for known parameters as $N \to \infty$.

References

1. M. Aoki, "Optimization of Stochastic Systems," Academic Press, New York, 1967.
2. K. J. Åström, Optimal control of Markov processes with incomplete state information, *J. Math. Anal. Appl.* **10** (1965), 174–205.
3. K. J. Åström, Optimal control of Markov processes with incomplete state information II—The convexity of the loss function, *J. Math. Anal. Appl.* **26** (1969), 403–406.
4. K. J. Åström and T. Bohlin, Numerical identification of linear dynamic systems from normal operating records, *in* "Proceedings of the IFAC Congress on Self-Adaptive Control Systems," Teddington, 1965, Plenum Press, New York 1966.
5. K. J. Åström, On the achievable accuracy in identification problems, *in* "Proceedings of the IFAC Symposium on Identification in Automatic Control Systems," Prague, 1967, pp. 1–8, Academia, Prague 1967.
6. K. J. Åström, "Lectures on the Identification Problem—The Least Squares Method," Report 6806, Lund Institute of Technology, Division of Automatic Control, 1968.
7. K. J. Åström, "Introduction to Stochastic Control Theory," Academic Press, New York, 1970.
8. A. V. Balachrishnan and V. Peterka, "Identification in Automatic Control Systems" (survey paper), Fourth IFAC Congress, Warsaw, 1969.
9. R. Bellman, "Adaptive Control Processes—A Guided Tour," Princeton Univ. Press, Princeton, N. J., 1961.
10. R. Bellman and R. Kalaba, "Quasilinearization and Nonlinear Boundary Value Problems," American Elsevier Company, New York, 1965.
11. M. Cuenod and A. P. Sage, "Comparison of Some Methods Used for Process Identification" (survey paper), IFAC Symposium on Identification in Automatic Control Systems, Prague, 1967, Academia, Prague 1967.
12. P. Eykhoff *et al.*, "Systems Modeling and Identification" (survey paper), Proceedings of the 3rd Congress of IFAC, London, 1967, Butterworths, London 1966.
13. P. Eykhoff, Process parameter and state estimation (survey paper), IFAC Symposium on Identification in Automatic Control Systems, Prague, 1967.
14. J. B. Farison, R. E. Graham, and Rey. C. Shelton, Identification and control of linear discrete systems, *IEEE Trans. Automatic Control* **12** (1967), 438–442.
15. A. A. Feldbaum, "Optimal Control Systems," Academic Press, New York, 1965.
16. L. Meier, Combined optimum control and estimation theory, Tech. Report NAS-2-2457, Stanford Research Institute, Menlo Park, California, October, 1955.
17. S. C. Schwartz and K. Steiglitz, The Identification and Control of Unknown Linear Discrete Systems, Technical Report No. 24, Information Sciences and Systems Laboratory, Dept. of EE., Princeton University, Princeton, New Jersey, June, 1968.
18. D. D. Sworder, A study of the relationship between identification and optimization in adaptive control problems, *J. Franklin Inst.* **281** (1966), 198–213.
19. B. Wittenmark, On adaptive control of low order systems, Report 6918, Lund Institute of Technology, Division of Automatic Control, 1969.
20. M. Wonham, Lectures on Stochastic Control Theory, Report, Centre for Dynamical Systems, Brown University.