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Covariance Analysis, Positivity and the Yakubovich-Kalman-Popov Lemma

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Abstract

This paper presents theory and algorithms for covariance analysis and stochastic realization without any minimality condition imposed. Also without any minimality conditions, we show that several properties of covariance factorization and positive realness hold. The results are significant for validation in system identification of state-space models from finite input-output sequences. Using the Riccati equation, we have designed a procedure to provide a reduced-order stochastic model that is minimal with respect to system order as well as the number of stochastic inputs.

Introduction

System identification deals with the problem of fitting mathematical models to time series of input-output data [14]. Important subproblems are the extraction both of a 'deterministic' subsystem—*i.e.*, computation of an input-output model—and a 'stochastic' subsystem which is usually modeled as a linear time-invariant system with white-noise inputs and outputs which represent the misfit between model and data. Evaluation of model misfit is often determined as an innovations sequence of a Kalman filter model which, in turn, also permits covariance-matrix factorization [26]. The related problem of stochastic realization has been approached by Ho and Kalman [13], Faurre [7],[8], Akaike [1], Desai and Pal [5], Larimore [17], Lindquist and Picci [18], [19]. Reasons for elaboration on stochastic models of the misfit are at least two-fold: Firstly, the stochastic model is needed to compute an appropriate Kalman filter which, in turn, is useful to compute state estimates. Secondly, residual analysis is used for statistical model validation [14].

An important observation pointed out in [25] is that state-space identification algorithms based on stochastic realization algorithm often fail to provide a positive definite solution of the Riccati equation which, in turn, brings attention to the open problem of "positive real sequences", their bias and variance [25, p. 85 ff.]. This problem refers to the partial realization problem with properties of the spectrum

$$S_{yy}(z) = \Lambda(z) = \Lambda_+(z) + \Lambda_+^T(z^{-1}) \quad (1)$$

$$\Lambda_+(z) = \frac{1}{2}\Lambda_0 + \Lambda_1 z^{-1} + \Lambda_2 z^{-2} + \cdots \quad (2)$$

A condition for $\Lambda(z)$ to be spectral density is that $\Lambda_+(z)$ be positive real on the unit circle—*i.e.*, for $z \in \{C : |z| = 1\}$—and if this condition is not fulfilled, the stochastic realization algorithm will fail [8], [19]. Rank-deficient output covariance might be found in cases of redundant measurement—*e.g.*, in sensor-array measurement. Hence, the rank-deficiency property among stochastic inputs or outputs is a generic case that requires theoretical attention. The purpose of this paper to provide the link between stochastic realization and statistical validation methodology for the framework of state-space model identification. The main results deal with the problem of rank-deficient covariance matrix factorization. The novel approach taken is to show that stochastic realization needs to address not only the state-space order determination but also that of the number of stochastic inputs.

Problems of Singular Covariance Matrices

Consider a discrete-time time-invariant system $\Sigma_n(A, B, C, D)$ with the state-space equations

$$\begin{pmatrix} x_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + \begin{pmatrix} v_k \\ e_k \end{pmatrix} \quad (3)$$

with input $u_k \in \mathbb{R}^m$, output $y_k \in \mathbb{R}^p$, state vector $x_k \in \mathbb{R}^n$ and zero-mean stochastic input sequences $v_k \in \mathbb{R}^n$, $e_k \in \mathbb{R}^p$ acting on the state dynamics and the output, respectively. The state-space identification problem is to fit system matrices A, B, C, D to data records $\{u_k\}_{k=1}^N$ and $\{y_k\}_{k=1}^N$ such that the state-space model reproduce the input-output behavior of data. The important remaining problem is to determine the model-misfit sequences of independent random variables $\{v_k\}_{k=1}^N$, $\{e_k\}_{k=1}^N$ such that they are uncorrelated with any state estimation error or input. Thus, it is assumed that u_k and y_k only are available to measurement and that the zero-mean stochastic processes $\{v_k\}, \{e_k\}$ have the covariance Q

with $q = \text{rank}(Q) \leq n + p$

$$\mathcal{E}\{\begin{pmatrix} v_k \\ e_k \end{pmatrix} \begin{pmatrix} v_j \\ e_j \end{pmatrix}^T\} = Q\delta_{kj}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \tag{4}$$

As only input-output data are available, it is sufficient to consider state-space models with the same statistics as the original model of Eq. (3). Thus, by replacement of Eq. (3) by an *innovations model*—see [2, p. 230]—with $\{v_k\}_{k=1}^N$, $\{e_k\}_{k=1}^N$ replaced by a sequence of zero-mean independent identically distributed (i.i.d.) stochastic variables $w_k \in \mathbb{R}^p$ so that

$$\begin{pmatrix} x_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + \begin{pmatrix} K \\ I_p \end{pmatrix} w_k \tag{5}$$

A residual sequence computed as the estimate $\{\widehat{w}_k\}$ of the innovations sequence $\{w_k\}$ may be obtained by means of the model inverse of Eq. (5), that is

$$\begin{pmatrix} \widehat{x}_{k+1} \\ \widehat{w}_k \end{pmatrix} = \begin{pmatrix} A - KC & B - KD \\ -C & -D \end{pmatrix} \begin{pmatrix} \widehat{x}_k \\ u_k \end{pmatrix} + \begin{pmatrix} K \\ I_p \end{pmatrix} y_k$$

Both the formulation of an innovations model for and of optimal reconstruction of $\{x_k\}$ of Eq. (3)—*i.e.*, the stationary Kalman filter problem

$$\begin{aligned} \mathcal{J}^0 &= \inf_K \lim_{k\to\infty} \mathcal{J}_k(K), \quad \mathcal{J}_k(K) = \text{tr}(S_k) && (6) \\ S_k &= \mathcal{E}\{(\widehat{x}_k - x_k)(\widehat{x}_k - x_k)^T\}, \quad S = \lim_{k\to\infty} S_k && (7) \\ \widehat{x}_{k+1} &= A\widehat{x}_k + Bu_k + K(y_k - C\widehat{x}_k - Du_k) && (8) \end{aligned}$$

proceed by solving the Riccati equation

$$\begin{aligned} S &= ASA^T - KRK^T + Q_{11} \\ R &= CSC^T + Q_{22} \\ KR &= ASC^T + Q_{12} \end{aligned} \tag{9}$$

with the infimal loss

$$\mathcal{J}^0 = \lim_{k\to\infty} \text{tr}\ \mathcal{E}\{(\widehat{x}_k - x_k)(\widehat{x}_k - x_k)^T\} = \text{tr}\ (S)$$

For full-rank covariance matrices the problem of residual computation can be approached by solving the Riccati equation involving estimated system matrices A and C and covariance matrix Q. It is a standard result that a positive definite matrix S (with interpretation of variance) and a matrix K solving Riccati equation (9) exist provided that Q_{22} is positive definite and that the resulting matrix $A - KC$ is stable [2]. By the invertibility properties of R, some cases of rank-deficient Q_{22} still permit a solution—see [2, Sec. 11.3] for an approach to state estimation instrumented by a reduced-order Kalman filter and by-passing of state variables.

Example: A system that fails to exhibit positivity is

$$\begin{aligned} x_{k+1} &= Ax_k + \beta w_k = \begin{pmatrix} 1.5 & -0.9 \\ 1 & 0 \end{pmatrix} x_k + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w_k \\ y_k &= Cx_k + \delta w_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x_k + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w_k \end{aligned} \tag{10}$$

For $\{w_k\}$ such that $\mathcal{E}\{w_k\} = 0$ and $\mathcal{E}\{w_k\} = I_2$, there is an indefinite matrix

$$\begin{aligned} Q &= \begin{pmatrix} S - ASA^T & \beta - ASC^T \\ \beta^T - CSA^T & R - CSC^T \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -10.03 & -3.97 \\ 0 & 0 & -13.97 & -11.03 \\ -10.03 & -13.97 & 1 & 0 \\ -3.97 & -11.03 & 0 & 0 \end{pmatrix} \end{aligned} \tag{11}$$

which has no interpretation as a covariance matrix.

Residual Model Structures

When R is rank-deficient, however, then no K can be determined from the solution to the Riccati equation (9). In turn, determination of the residual sequence and statistical model validation are hampered. A remedy requires: *i.* a suitable residual model structure that includes the innovations model as the full-rank special case; *ii.* solution of the Riccati equation for the state-space model tested, finite data records and the possibly rank-deficient covariance matrix to find a residual realization model while preserving stability and minimum variance; *iii.* computation of the residual sequence from input-output data applied to the residual realization model.

Theorem 1 (Rank-deficient model inverse) *Let the rank-deficient innovations model be given as the state-space model*

$$\begin{pmatrix} x_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + \begin{pmatrix} \beta \\ \delta \end{pmatrix} v_k \tag{12}$$

with $u_k \in \mathbb{R}^m, v_k \in \mathbb{R}^q, x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^p, p \geq q$ *and with* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $\beta \in \mathbb{R}^{n\times q}$, $\delta \in \mathbb{R}^{p\times q}$. *Then, a left inverse of the innovations model of Eq. (12) is*

$$\begin{pmatrix} \widehat{x}_{k+1} \\ \widehat{v}_k \end{pmatrix} = \begin{pmatrix} A - K_\delta C & B - K_\delta D & K_\delta \\ -\delta^\dagger C & -\delta^\dagger D & \delta^\dagger \end{pmatrix} \begin{pmatrix} \widehat{x}_k \\ u_k \\ y_k \end{pmatrix} \tag{13}$$

with left inverse $\delta^\dagger$ *to* δ *and*

$$K_\delta = \beta\delta^\dagger + K_\perp(I_p - \delta\delta^\dagger), \quad \delta^\dagger\delta = I_q, \quad q \leq p \tag{14}$$

for some arbitrary $K_\perp$ *preserving stability of the error* $\widetilde{x}_k = x_k - \widehat{x}_k$

$$\begin{pmatrix} \widetilde{x}_{k+1} \\ \widehat{v}_k \end{pmatrix} = \begin{pmatrix} A - K_\delta C & 0_{n\times q} \\ \delta^\dagger C & I_q \end{pmatrix} \begin{pmatrix} \widetilde{x}_k \\ v_k \end{pmatrix} \tag{15}$$

and for covariance matrices $S = \mathcal{E}\{\widetilde{x}_k \widetilde{x}_k^T\}$ *and* $R = \mathcal{E}\{\varepsilon_k \varepsilon_k^T\}$ *obeying the Riccati equation*

$$\begin{aligned} \begin{pmatrix} S & 0 \\ 0 & R \end{pmatrix} &= \mathcal{T}\mathcal{S}\begin{pmatrix} S & 0 \\ 0 & I_q \end{pmatrix}\mathcal{S}^T\mathcal{T}^T && (16) \\ \mathcal{T} &= \begin{pmatrix} I_n & -K_\delta \\ 0 & I_p \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} A & \beta \\ C & \delta \end{pmatrix} \end{aligned}$$

Proof (sketch): By direct calculation based on the factorization

$$Q = \mathcal{E}\{v_k v_k^T\} = \begin{pmatrix}\beta\\ \delta\end{pmatrix}\begin{pmatrix}\beta\\ \delta\end{pmatrix}^T \tag{17}$$

and the observation that the Riccati Eq. (9) may be reformulated as (*cf.* [11], [15])

$$\begin{bmatrix}I_n & K\\ 0 & I_p\end{bmatrix}\begin{bmatrix}S & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}I_n & K\\ 0 & I_p\end{bmatrix}^T = \begin{bmatrix}A & \beta\\ C & \delta\end{bmatrix}\begin{bmatrix}S & 0\\ 0 & I_q\end{bmatrix}\begin{bmatrix}A & \beta\\ C & \delta\end{bmatrix}^T$$

Covariance Analysis and Positive Realness
Conditions to find a state-space system $\Sigma\{A,B,C,D\}$ reproducing a given output autocovariance sequence $\{C_{yy}(k)\}$ is usually approached by factorizing $\{C_{yy}(k)\}$ as $\{CA^{k-1}B\}$ for $k\geq 1$ and $C_{yy}(0)=DD^T$. Such a linear system with stochastic input $\{w_k\}$ is

$$\begin{pmatrix}x_{k+1}\\ y_k\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x_k\\ w_k\end{pmatrix}, \quad \begin{matrix}\mathcal{E}\{w_k w_k^T\}=Q\\ \mathcal{E}\{x_k x_k^T\}=S\\ \mathcal{E}\{y_k y_k^T\}=R\end{matrix} \tag{18}$$

Then, the output covariance $R = C_{yy}(0)$, state covariance $P\geq 0$ and the input covariance $Q\geq 0$ must satisfy the 'positive realness' condition.

Lemma 1 (Positive Real Lyapunov Equation)
The equation of positive realness

$$0 \leq \begin{pmatrix}Q_{11} & Q_{12}\\ Q_{12}^T & Q_{22}\end{pmatrix} = \begin{pmatrix}S-ASA^T & B-ASC^T\\ B^T-CSA^T & R-CSC^T\end{pmatrix} \tag{19}$$

is equivalent to the Lyapunov equation

$$\mathcal{A}\mathcal{P}\mathcal{A}^T - \mathcal{P} = -\mathcal{Q} \tag{20}$$

where $\mathcal{Q} = \mathcal{Q}^{1/2}\mathcal{Q}^{T/2}$ *and*

$$\begin{aligned}\mathcal{P} &= \begin{pmatrix}S & 0\\ 0 & R\end{pmatrix}, \ \mathcal{A} = \begin{pmatrix}A-\kappa C & \kappa\\ C & 0\end{pmatrix}, \ \kappa = BR^\dagger\\ \mathcal{T} &= \begin{pmatrix}I_n & -\kappa\\ 0 & I_p\end{pmatrix}, \ \mathcal{Q} = \mathcal{T}Q\mathcal{T}^T\end{aligned} \tag{21}$$

The solution $\mathcal{P}=\mathcal{P}^T>0$ *exists if* $\mathcal{A}$ *has its eigenvalues within the unit circle and if* $(\mathcal{A},\mathcal{Q}^{1/2})$ *is controllable.*

Proof is made by direct calculation and by Lyapunov equation properties and generalization of Lemma 1 for rank-deficient $R=\rho\rho^T$ holds for $\kappa = B(\rho^\dagger)^T\rho^\dagger$ and

$$\mathcal{T} = \begin{pmatrix}I_n & -\kappa\\ 0 & \rho^\dagger\end{pmatrix}, \ \mathcal{A} = \begin{pmatrix}A-\kappa C & \kappa\\ \rho^\dagger C & 0\end{pmatrix} \tag{22}$$

Lemma 1 offers constructive means to find SPR transfer functions that are not necessarily minimal and is very useful for stability analysis and covariance analysis. Solutions P suitable for Lyapunov function design can be found under the relaxed condition that $(\mathcal{A},\mathcal{Q}^{1/2})$ be controllable. This Lyapunov equation and its Riccati-equation companion are also useful in the form

$$\mathcal{P} = \begin{pmatrix}A-\kappa C & \kappa\\ C & 0\end{pmatrix}\mathcal{P}\begin{pmatrix}A-\kappa C & \kappa\\ C & 0\end{pmatrix}^T + \mathcal{Q} \tag{23}$$

and as the recursive equation $\mathcal{P}_{k+1} = \mathcal{A}\mathcal{P}_k\mathcal{A}^T + \mathcal{Q}$.

Residual variance properties
By Theorem 1, the mismatch of residuals $\{\widehat{\nu}_k\}$ as compared to $\{\nu_k\}$ depends on the variance properties of the non-standard Kalman filter embodied in the residual reconstruction.

$$\begin{aligned}\begin{pmatrix}\widetilde{X}_{k+1}\\ \widehat{\nu}_k\end{pmatrix} &= \begin{pmatrix}A_0-K_0\delta^\dagger C_0 & 0\\ \delta^\dagger C_0 & I_q\end{pmatrix}\begin{pmatrix}\widetilde{X}_k\\ \nu_k\end{pmatrix} \\ &= \begin{pmatrix}I & -K_0\\ 0 & I_q\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & \delta^\dagger\end{pmatrix}\begin{pmatrix}A_0 & K_0\\ C_0 & \delta\end{pmatrix}\begin{pmatrix}\widetilde{X}_k\\ \nu_k\end{pmatrix}\end{aligned} \tag{24}$$

Let $P_k = \mathcal{E}\{\varepsilon_k\varepsilon_k^T\}$, $R_\nu = \mathcal{E}\{\nu_k\nu_k^T\}$, $R_k = \mathcal{E}\{\widehat{\nu}_k\widehat{\nu}_k^T\}$, $S_k = \mathcal{E}\{\widetilde{X}_k\widetilde{X}_k^T\}$. Variance properties can be summarized in the following Riccati-equation reformulation:

Theorem 2 (Kalman filter for rank-def. input)
Consider for given matrices A, B, C, D and zero-mean independent identically distributed stochastic processes $\{v_k\}$, $\{e_k\}$ *with covariance*

$$\mathcal{E}\{\begin{pmatrix}v_k\\ e_k\end{pmatrix}\begin{pmatrix}v_k\\ e_k\end{pmatrix}^T\} = Q = \begin{pmatrix}Q_{11} & Q_{12}\\ Q_{12}^T & Q_{22}\end{pmatrix} = \begin{pmatrix}Q_v\\ Q_e\end{pmatrix}\begin{pmatrix}Q_v\\ Q_e\end{pmatrix}^T$$

the observer of $x_k\in\mathbb{R}^n$ *based on* $u_k\in\mathbb{R}^m$, $y_k\in\mathbb{R}^p$ *as*

$$\begin{pmatrix}\widehat{x}_{k+1}\\ \varepsilon_k\end{pmatrix} = \begin{pmatrix}A-KC & B-KD\\ -C & -D\end{pmatrix}\begin{pmatrix}\widehat{x}_k\\ u_k\end{pmatrix} + \begin{pmatrix}K\\ I_p\end{pmatrix}y_k$$

and error dynamics represented by

$$\begin{pmatrix}\widetilde{x}_{k+1}\\ \varepsilon_k\end{pmatrix} = \begin{pmatrix}A-KC & I_n & -K\\ C & 0 & I_p\end{pmatrix}\begin{pmatrix}\widetilde{x}_k\\ v_k\\ e_k\end{pmatrix}$$

Let the covariance variables

$$\begin{aligned}S_k &= \mathcal{E}\{\widetilde{x}_k\widetilde{x}_k^T|\mathcal{F}_{k-1}\} \qquad (25)\\ P_k &= \mathcal{E}\{\varepsilon_k\varepsilon_k^T|\mathcal{F}_{k-1}\} = CS_kC^T + Q_{22} = \rho_k\rho_k^T\\ T_k &= \mathcal{E}\{\widetilde{x}_{k+1}\varepsilon_k^T|\mathcal{F}_{k-1}\} = AS_kC^T + Q_{12} - KP_k\end{aligned}$$

denote mathematical expectation given information up to time $k-1$. *Then,* S_k *is the solution to the Riccati matrix equation*

$$\begin{aligned}0 &= Z\begin{pmatrix}S_{k+1}-T_k(\rho_k^\dagger)^T\rho_k^\dagger T_k^T & 0\\ 0 & I_q\end{pmatrix}Z^T \qquad (26)\\ &- \begin{pmatrix}A & Q_v\\ \rho_k^\dagger C & \rho_k^\dagger Q_e\end{pmatrix}\begin{pmatrix}S_k & 0\\ 0 & I_q\end{pmatrix}\begin{pmatrix}A & Q_v\\ \rho_k^\dagger C & \rho_k^\dagger Q_e\end{pmatrix}^T\\ Z &= \begin{pmatrix}I_n & -(T_k+KR_k)(\rho_k^\dagger)^T\\ 0 & I_q\end{pmatrix} \qquad (27)\end{aligned}$$

and $tr(S_k)$ achieves its minimum for

$$K = (AS_kC^T + Q_{12})(\rho_k^\dagger)^T\rho_k^\dagger + K_\perp(I_p - \rho_k\rho_k^\dagger)$$

for arbitrary $K_\perp \in \mathbb{R}^{n\times p}$ preserving stability of $A - KC$.

Proof is made by Lemma 1 [15]. For the case non-stationary and finite-duration time series, covariance relationships of the innovations model of Eq. (24) obey the recursive equation

$$\begin{pmatrix} S_{k+1} & 0 \\ 0 & R_k \end{pmatrix} = \mathcal{F}\begin{pmatrix} S_k & 0 \\ 0 & R_v \end{pmatrix}\mathcal{F}^T \quad (28)$$

$$\mathcal{F} = \begin{pmatrix} A_0 - K_0\delta^\dagger C_0 & 0 \\ \delta^\dagger C_0 & I_q \end{pmatrix} \quad (29)$$

With $A_0 - K_0\delta^\dagger C_0$ being a stability matrix, the asymptotic covariance relationships of Eq. (24) as $k \to \infty$ are

$$\begin{aligned} S_{k+1} &= (A_0 - K_0\delta^\dagger C_0)S_k(A_0 - K_0\delta^\dagger C_0)^T \\ R_k &= \mathcal{E}\{\widehat{v}_k\widehat{v}_k^T\} = \delta^\dagger C_0 S_k C_0^T(\delta^\dagger)^T + R_v \quad (30) \\ P_k &= \mathcal{E}\{\varepsilon_k\varepsilon_k^T\} = C_0S_kC_0^T + \delta R_v\delta^T \to \delta R_v \delta^T \end{aligned}$$

Positive Realness and Factorization

We make the following constructive proof of positive realness to hold for the relaxed conditions. Let

$$\begin{aligned} R &= CSC^T + Q_{22} & (31) \\ K &= BR^\dagger = (Q_{12} + ASC^T)R^\dagger & (32) \\ H(z) &= I_m + C(zI_n - A)^{-1}K & (33) \\ &= I_m + C(zI_n - A)^{-1}(Q_{12} + ASC^T)R^\dagger \\ \Lambda_+(z) &= C(zI_n - A)^{-1}KR + \frac{1}{2}R & (34) \end{aligned}$$

and

$$\mathcal{R}_0 = \begin{pmatrix} -S + ASA^T + Q_{11} & Q_{12} + ASC^T \\ Q_{12}^T + CSA^T & Q_{22} + CSC^T \end{pmatrix} \quad (35)$$

$$\mathcal{L}_0 = \begin{pmatrix} 0 & KR \\ RK^T & R \end{pmatrix} \quad (36)$$

and the feedback transformation matrix

$$\mathcal{T} = \begin{pmatrix} I_n & -K \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} I_n & -(Q_{12} + ASC^T)R^\dagger \\ 0 & I_m \end{pmatrix} \quad (37)$$

Then the Riccati equations of Eq. (19) and Eq. (9) are reproduced as

$$\begin{aligned} 0 &= -\mathcal{L}_0 + \mathcal{R}_0 & (38) \\ &= \begin{pmatrix} -S + ASA^T + Q_{11} & Q_{12} + ASC^T - KR \\ Q_{12}^T + CSA^T - RK^T & Q_{22} + CSC^T - R \end{pmatrix} \end{aligned}$$

and $0 = \mathcal{T}(-\mathcal{L}_0 + \mathcal{R}_0)\mathcal{T}^T$, respectively. Multiplication by

$$L_2(z) = \begin{pmatrix} C(zI_n - A)^{-1} & I_m \end{pmatrix} \quad (39)$$

of $\mathcal{L}_0$, $\mathcal{R}_0$ gives

$$\begin{aligned} \mathcal{L}_1(z) &= L_2(z)\mathcal{L}_0L_2^T(z^{-1}) = \Lambda_+(z) + \Lambda_+^T(z^{-1}) & (40) \\ \mathcal{R}_1(z) &= L_2(z)\mathcal{R}_0L_2^T(z^{-1}) = H(z)RH^T(z^{-1}) & (41) \end{aligned}$$

As $\mathcal{L}_1(z) = \mathcal{R}_1(z)$, positive realness holds for $z = e^{i\theta}$, $\theta \in [0, 2\pi]$ so that

$$\Lambda_+(z) + \Lambda_+^T(z^{-1})|_{z=e^{i\theta}} = H(z)RH^T(z^{-1})|_{z=e^{i\theta}} \geq 0 \quad (42)$$

Thus, the positive realness condition is fulfilled also for the rank-deficient case for which a modified Riccati equation is to be solved.

Continuous-time Systems: Now consider the following continuous-time reformulation of the Yakubovich-Kalman-Popov (YKP) matrix equation

$$-Q = -\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} = \begin{pmatrix} PA + A^TP & PB - C^T \\ B^TP - C & -(D + D^T) \end{pmatrix}$$

Then, this YKP matrix equation may be reformulated as the special Lyapunov equation

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} = -Q \quad (43)$$

Theorem 3 *Assume that $Q = Q^T > 0$ and a LTI state-space system $\{A, B, C, D\}$ be given. Let*

$$\mathcal{A} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix}, \mathcal{P} = \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)} \quad (44)$$

If $(\mathcal{A}, Q^{1/2})$ is observable and if all eigenvalues of $\mathcal{A}$ are in the open left-half plane, then the Lyapunov equation

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} = -Q^{T/2}Q^{1/2} = -Q \quad (45)$$

provides a unique positive definite solution $P = P^T > 0$ to the YKP matrix equation for $\{A, B, C, D\}$.

Moreover, there are rational functions $G(s) = C(sI - A)^{-1}B + D$ and $\Gamma(s) = (Q_1 \quad Q_1(sI - A)^{-1}B + Q_2)$, Q_1, Q_2 matrices, satisfying

$$\begin{aligned} \Gamma_2^T(-s)\Gamma_2(s) &= G(s) + G^T(-s) & (46) \\ \Gamma_2^T(-i\omega)\Gamma_2(i\omega) &= G(i\omega) + G^T(-i\omega) \geq 0 & (47) \end{aligned}$$

Proof: We make the following constructive proof of positive realness: Let

$$L(s) = \begin{pmatrix} I_n & 0 \\ C(sI_n - A)^{-1} & I_m \end{pmatrix}, \quad E = \begin{pmatrix} I_n & 0 \\ 0 & 0_m \end{pmatrix}$$

$$R(s) = \begin{pmatrix} I_n & -(sI_n - A)^{-1}B \\ 0 & I_m \end{pmatrix} \quad (48)$$

First, the transmission zeros of the system $\{A, B, C, D\}$ are found from the generalized eigenvalue problem of

$$sE - \mathcal{A} = L(s)\begin{pmatrix} sI_n - A & 0 \\ 0 & G(s) \end{pmatrix} R(s) \quad (49)$$

Let $Q^{1/2}$ denote a matrix factor of $Q = Q^T \geq 0$ so that

$$Q = Q^{T/2}Q^{1/2} = \begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \quad (50)$$

Assume that for $(\mathcal{A}, Q^{1/2})$ observable and eigenvalues of $\mathcal{A}$ with negative real part, the Lyapunov equations

$$\begin{aligned} \mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} &= -Q \quad (51) \\ PA + A^TP &= -Q_1^TQ_1 \stackrel{\text{def}}{=} -Q_{11} \end{aligned}$$

has provided the positive definite solutions $\mathcal{P}$ and P and that the solution obtained has been brought to block-diagonal form of Eq. (44). Then, expand the Lyapunov equation (51) into

$$Q = \mathcal{P}(sE - \mathcal{A}) + (-sE - \mathcal{A}^T)\mathcal{P} \quad (52)$$

By Eq. (49), it follows that

$$\mathcal{P}(sE - \mathcal{A}) = \begin{pmatrix} P(sI_n - A) & 0 \\ C & G(s) \end{pmatrix} R(s)$$

Thus, for

$$\begin{aligned} \Gamma(s) &= Q^{1/2}R^{-1}(s) = \begin{pmatrix} Q_1 & Q_1(sI - A)^{-1}B + Q_2 \end{pmatrix} \\ G_{12}(s) &= C^T + (-sI_n - A^T)P(sI_n - A)^{-1}B \quad (53) \\ &= Q_{11}(sI_n - A)^{-1}B + C^T - PB \quad (54) \end{aligned}$$

one finds that

$$\begin{aligned} \Omega(s) &= \Gamma^T(-s)\Gamma(s) = R^{-T}(-s)Q^TQR^{-1}(s) \quad (55) \\ &= R^{-T}(-s)(\mathcal{P}(sE - \mathcal{A}) + (-sE - \mathcal{A}^T)\mathcal{P})R^{-1}(s) \\ &= \begin{pmatrix} -(PA + A^TP) & G_{12}(s) \\ G_{12}^T(-s) & G(s) + G^T(-s) \end{pmatrix} \end{aligned}$$

with rank deficit only at the transmission zeros of $\{A, B, C, D\}$. By the matrix equations (, 56), the simultaneous transfer-function positivity and the positive definite Lyapunov equation properties follow from the diagonal matrix equation blocks of $\Gamma^T(-s)\Gamma(s)$ that

$$\begin{aligned} -(PA + A^TP) &= \Gamma_1^T(-s)\Gamma_1(s) = Q_1^TQ_1 \quad (56) \\ G(s) + G^T(-s) &= \Gamma_2^T(-s)\Gamma_2(s) \quad (57) \end{aligned}$$

As for $s = i\omega$, it follows that

$$\Gamma_2^T(-i\omega)\Gamma_2(i\omega) = G(i\omega) + G^T(-i\omega) \geq 0 \quad (58)$$

which shows that the Nyquist curve of the transfer function $G(s)$ is situated to the right of the imaginary axis thus fulfilling the 'positive-real' condition. ∎

Remark: The converse result holds for $Q = Q^{1/2}Q^{T/2}$ and for a controllable pair $(\mathcal{A}, Q^{1/2})$. If all eigenvalues of $\mathcal{A}$ are in the open left-half plane, then the Lyapunov equation

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^T = -Q^{1/2}Q^{T/2} = -Q \quad (59)$$

provides a unique positive definite solution and positive definite solution $P = P^T > 0$ to the YKP matrix equation for $\{A, B, C, D\}$.

Discussion

Several properties of positivity and factorization remain valid for stable nonminimal realizations. As state-space model identification provides controllable state dynamics as well as uncontrollable stochastic dynamics, minimality tests may be replaced by some test of the property

$$\mathcal{E}\{\hat{v}_i\hat{v}_j^T\} = \hat{Q}\delta_{ij}, \ \mathcal{E}\{\hat{v}_iu_j^T\} = 0, \quad \forall i, j \quad (60)$$

Correlation test can be made by direct application of statistical validation methods [14, Sec. 9.4]. Although the reconstructed innovations sequence will exhibit no autocorrelation, the resulting prediction error sequence may still exhibit autocorrelation.

Conclusions

The problem of stochastic residual realization to accompany estimated input-output models in the case of state-space model identification such as multi-input multi-output state-space model identification is solved and its relationship to positive realness is shown. The case considered includes the problem of rank-deficient residual covariance matrices, a case which is encountered in applications with mixed stochastic-deterministic input-output properties as well as for cases where outputs are linearly dependent, thus extending previous results of partial realization [8], [10]. Also without any minimality conditions, we show that several properties of covariance factorization and positive realness hold. In addition, we provide a constructive method to solve the Riccati equation of covariance analysis by means of a reduction to a Lyapunov equation. The case considered includes the problem of rank-deficient residual covariance matrices, a case which is encountered in applications with mixed stochastic-deterministic input-output properties as well as for cases where outputs are linearly dependent. Our approach has been the formulation of a rank-deficient innovations model and inverse innovations model in the form of a state-space inverse model and a left transfer function inverse applicable to the rank-deficient model output. This extension is related to a rank-deficient

Riccati equation and is accompanied by a reformulation of the Riccati equation and nonminimal positive realness conditions.

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