Strategy-Proof and Fair Wages

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2004

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This version: March 9, 2004

Abstract

A fair division problem with indivisible objects, e.g. jobs, and one divisible good (money) is considered. The individuals consume one object and money. The class of strategy-proof and fair allocation rules is characterized. The allocation rules in the class are like a Vickrey auction bossy and like the Clark-Groves mechanisms in general not "budget balanced". The efficiency loss due to fairness and strategy-proofness becomes measurable in monetary terms.

Two interpretations of the formal model is discussed. First, it is a situation where a given sum of money has to be distributed as wages and fair wages are to be implemented. Second, it is as an auction model where a number of objects are simultaneously traded.

JEL Classification: C6, C68, C71, C78, D61, D63, D71, D78.
Keywords: Indivisibilities, fairness, strategy-proofness, wages, Vickrey-auction.

1 Introduction

This study analyses the implementation of fairness. In the economy there is a finite number of large indivisible objects and there is a limited quantity of a perfectly divisible private good. The individuals consume one object each and a quantity of the divisible good. The problem is to characterize the entire class of allocation rules[1], the outcome of which are strategy-proof

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[1] "Allocation rule" is used here synonymously with "allocation mechanism". We do not exclude, however, that the outcome of the "allocation rule" may be multi-valued.

and fair, and where available resources are never exceeded. Fairness is defined as a situation where no individual envy the consumption of any other individual, and strategy-proofness entails that no individual has incentive to misrepresent his preferences.

The model has a number of interpretations. A common one is that the objects are houses, and the classical Shapley-Scarf model of house allocation is an example (Shapley and Scarf (1974)). In the present study implementation of fair allocations is the problem and our main interpretation is that the objects are jobs or positions that the individuals are assigned (cf. Hylland and Zeckhauser (1979), Crawford and Knoer (1981), Leonard (1983)). The divisible good is called "money" and used to compensate for differences in the various objects. When the objects are jobs, the monetary compensations allocated to the various objects are called wages.

Alternatively the model may be regarded as an auction model where multiple objects are simultaneously traded. In that case fairness is simply equilibrium prices in the market and money is paid for the objects. With this interpretation, all strategy-proof auction rules satisfying a weak efficiency condition are characterized and a generalized form of a Vickrey auction is the result.

A requirement for fairness in the model is no-envy. The no-envy criterion cannot, however, in general be used as the sole test of fairness. For instance, envy-free allocations are in many cases not unique. A simple example illustrates this. Suppose there are two jobs, 1 and 2, and there are two individuals A and B. Both individuals are qualified to do the jobs and a given quantity x_0 of money has to be allocated to the jobs as wages. Let $x_0 = x_1 + x_2$ be the distribution of money. Individual A is indifferent between the two jobs if $x_1 = x_0/3$ while B is indifferent if $x_1 = x_0/2$. Obviously an envy-free allocation of jobs and wages prevails if A is assigned job 1, B is assigned job 2, and $x_0/3 \leq x_1 \leq x_0/2$.

As we can see the envy-free criterion alone cannot select a unique value of x_1, so an additional criterion is needed. Then a second problem with implementing fair wages becomes obvious. The indifference points, $x_1 = x_0/3$ for individual A and $x_1 = x_0/2$ for individual B, are private information and an implemented wage structure may depend on this information. Hence, depending on by what type of rule the choice of the wage structure is made, the individuals may have incentive not to reveal their preferences truthfully.

The class of strategy-proof allocation rules found in this study solves both these problems. Strategy-proofness means that the individuals have incentive to reveal their true preferences, and strategy-proofness is consistent with fairness for precisely one of the various distributions of money satisfying the no-envy criterion. Of course, the resource constraint concerning available quantity of money has also to be satisfied.

The fair and strategy-proof allocation rule can roughly be described in the following way. If $x = (x_1, \ldots, x_n)$ is a distribution of the divisible good across the objects (x is the wage structure), then x has to be chosen so that there is an envy-free assignment of the individuals to the objects (jobs). Moreover, the sum $\Sigma_j x_j$ has to be chosen as large as possible, but satisfying an inequality $x \leq \bar{x}$. The upper bound $\bar{x}$ is implicit in every fair and strategy-proof allocation rule and independent of which preferences are reported by the individuals. If x_0 is the available quantity of money in the economy, budget balance $x_0 = \Sigma_j x_j$ is achieved only for some individual preferences, but the inequality $\Sigma_j x_j \leq x_0$ is always satisfied. Hence, the outcome of the allocation rule is in general inefficient. This, however, reflects the well-known conflict between efficiency and strategy-proofness. Assuming efficiency and strategy-proofness often leads to dictatorial allocation rules and variations of the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975))[2]. The impossibility of budget balance for the present allocation rule is similar to the result for the Clark-Groves mechanism for public goods (Clark (1971), Groves (1973)).

A closer analysis of this allocation rule also shows that the wage an individual receives to a large extent depends on the preferences of other individuals. This means that the allocation rule is "bossy"[3] and behaves very much like a "Vickrey auction" (Vickrey (1961)), where the price the highest bidding individual pays for an object equals the second highest bid.

Various forms of the allocation rule described above have earlier been analyzed with respect to strategy-proofness in a number of studies. In most studies equilibrium prices, which are equivalent to fair distributions of money, are considered and minimal equilibrium prices are proved to be incentive compatible. That is the case e.g. in Leonard (1983) for quasi-linear preferences and in Demange and Gale (1985) for general preferences. With quasi-linear preferences, prices are similar to the incentive compatible prices in the Clark-Groves mechanism for revealing preferences for public goods. One also obtains generalization of the Vickrey auction for a single object to the case with multi objects. Recently, Sun and Yang (2003) defined a fair allocation mechanism as the one described here and proved it to be strategy-proof for general preferences.

[2]However, in the classical Shapley-Scarf model for house allocation, strategy-proofness, efficiency and individual rationality are three consistent conditions. See e.g. Ma (1994) or Svensson (1999).

[3]Bossiness means that an individual can change the allocation without changing the bundle he receives himself. Non-bossiness was introduced by Satterthwaite and Sonnenschein (1981).

The mentioned studies show the existence of a fair and strategy-proof allocation rule in this context (Theorem 1 here). Assuming a weak form of efficiency, we find in the present study that the types of allocation rules described above are in fact the only ones which are fair and strategy-proof, and also satisfy a given resource constraint concerning the divisible good (Theorem 2 here).

The model with large indivisible objects and money used in the present paper has previously and in various studies been analyzed also with respect to strategy-proofness, with respect to fairness, or with respect to Walrasian equilibrium.

Miyagawa (2001) and Svensson and Larsson (2002) also consider a model with indivisible objects and money and characterize the class of strategy-proof allocation rules under some additional conditions. A basic assumption there is "non-bossiness" and as a consequence of this assumption, there is only a *finite number* of distributions of money. This property excludes fair allocation rules. Instead the allocation rules are serially dictatorial, or if a primary distribution of property rights is assumed (individual rationality), the outcomes of the allocation rules are fixed-price allocations, where Gale's top trading cycle procedure determines the final allocation of the objects and money.

For proofs of existence and analysis of fair allocations (with general preferences), see e.g. Svensson (1983), Maskin (1987) or Alkan, Demange, and Gale (1991). For existence of a Walrasian equilibrium see e.g. Gale (1984), Quinzii (1984) or Svensson (1984) and for Nash implementation of price equilibrium in this type of model, see Svensson (1991).

The paper is organized in the following way. In Sections 2 and 3 the formal model is introduced and some basic results derived. Section 4 introduces the concept of an allocation rule, while Section 5 contains the main result. In Section 6 the interpretation of the model as a multi object auction is to some extent discussed. The proofs can be found in an Appendix.

2 The model

There is a finite number of *individuals,* $N = \{1, 2, ..., n\}$, and the same number of indivisible *objects* (jobs for instance) in the economy. The index set N is used to label the objects as well, so $j \in N$ denotes object j. There is also a divisible good (called money), available in a finite quantity $x_0 \in \mathbb{R}$.

An individual consumes one object and an amount of money. *Preferences* over consumption bundles $(j, x_j) \in N \times \mathbb{R}$ are *quasi-linear,* and represented by total utility functions, $u_{ij} + x_j$ for $i \in N$. The number u_{ij} denotes the utility individual i derives from object j. A list $u = (u_1, u_2, \ldots, u_n)$ of individual utility functions is a *preference profile,* or for short a *profile.* A profile is an element in $\mathcal{U} = \mathbb{R}^{n^2}$. A profile $u = (u_1, u_2, \ldots, u_n)$ can also be denoted (u_i, u_{-i}) for $i \in N$.

An *allocation* is a list of consumption bundles. It is a pair (a, x), where $a : N \to N$ is a bijective mapping assigning the object a_i to individual i, and where $x \in \mathbb{R}^n$ distributes the quantity x_j of money to object j, and hence also x_j to the individual i with $a_i = j$. We call a the *assignment function* and x the *distribution function.* The set of all conceivable allocations is denoted $\mathcal{A}$. An allocation (a, x) is *feasible* if the resource constraint $\Sigma_j x_j \leq x_0$ is respected. If $\Sigma_j x_j = x_0$ the allocation is *budget balanced.*

3 The fair set

For a given profile $u \in \mathcal{U}$, an allocation (a, x) is defined to be *fair* if it is envy-free[4], i.e. if $u_{ia_i} + x_{a_i} \geq u_{ij} + x_j$ for all i and j. If (a, x) is a fair allocation, a is the *fair assignment* and x the *fair distribution.* For a given profile $u \in \mathcal{U}$, the set of *fair allocations* is denoted $\Phi(u)$, and the corresponding set of fair distributions, called the *fair set,* is denoted $F(u)$. Formally,

$$F(u) = \{x \in \mathbb{R}^n;\ (a, x) \in \Phi(u) \text{ for some assignment } a\}.$$

If the fair set is non-empty it contains also several distributions and a fair selection may require some additional criterion. In this study distributions called fair and optimal are of particular interest. Let $\bar{x} \in \mathbb{R}^n$ be a fixed distribution of money. Given this as an upper bound on fair distributions, an element in the fair set is called *optimal* if it maximizes the total sum $S(x) = \Sigma_j x_j$ of money. Formally, for a given vector $\bar{x}$ of quantity restrictions and a preference profile $u \in \mathcal{U}$, a distribution x is *fair and optimal*[5] with respect to $\bar{x}$ if $x \in F(u)$ and

$$S(x) = \max S(x') \text{ s.t. } x' \leq \bar{x} \text{ and } x' \in F(u).$$

Correspondingly, a fair allocation (a, x) is optimal with respect to $\bar{x}$ if the distribution x is optimal with respect to $\bar{x}$. The following proposition collects some essential properties of the fair set.

[4] In some cases the term fair is reserved for allocations that are envy-free and efficient, see e.g. Varian (1974) or Svensson (1983). However, the term fair meaning envy-free was introduced by Foley (1967).

[5] The concept of fair and optimal allocations was introduced by Sun and Yang (2003).

Proposition 1 *(i) For each profile $u \in \mathcal{U}$, the fair set $F(u)$ is a non-empty and closed subset of $\mathbb{R}^n$, (ii) If $x, y \in F(u)$ and (a, x) is a fair allocation, then (a, y) is also a fair allocation. (iii) If $x, y \in F(u)$ then $z \in F(u)$, where $z_j = \max[x_j, y_j]$. (iv) A fair and optimal distribution is unique.*

The second point in the proposition says that even if the fair set contains more than one fair distribution, and hence also several utility distributions, one and the same assignment is fair for all fair distributions. If there is an interior point in the fair set the statement immediately follows from continuity of preferences, but in other cases some more arguments are required. This point will simplify the analysis to come.

4 Fair allocation rules

By an *allocation rule (AR)* we mean a non-empty correspondence φ which, for each profile $u \in \mathcal{U}$, selects a set of allocations, $\varphi(u) \subset \mathcal{A}$. Below some conceivable properties of an allocation rule are defined.

An AR φ is *essentially single-valued (ESV)* if all elements selected by φ are identical from a utility point of view for all individuals. Hence, for each profile u, the outcome of an essentially single-valued AR is a unique utility distribution. Formally, for all $u \in \mathcal{U}$, for all $(a, x), (b, y) \in \varphi(u)$, and for all $i \in N$, it is true that $u_{ib_i} + y_{b_i} = u_{ia_i} + x_{a_i}$.

An AR φ is *Pareto-indifferent (PI)* if for any two allocations, which all individuals are indifferent between, either both or none belong to the outcome set of the AR. Formally, φ is PI if for all $u \in \mathcal{U}$ and $(b, y) \in \mathcal{A}$,

$$(b, y) \in \varphi(u) \text{ if } (a, x) \in \varphi(u) \text{ and } u_{ib_i} + y_{b_i} = u_{ia_i} + x_{a_i} \text{ for all } i \in N.$$

An AR φ is *fair* if all allocations in the set $\varphi(u)$ are fair, i.e. $\varphi(u) \subset \Phi(u)$ for all $u \in \mathcal{U}$.

We also define strategy-proof allocation rules. An AR φ is *manipulable* if there is an individual $i \in N$, two profiles $u, v \in \mathcal{U}$, and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(v_i, u_{-i})$, such that $u_{ib_i} + y_{b_i} > u_{ia_i} + x_{a_i}$. This definition of manipulation means that some element in $\varphi(v_i, u_{-i})$ is strictly better than some element in $\varphi(u)$, according to i's preferences u_i. Note that if φ is an ESV allocation rule then manipulation means that some element in $\varphi(v_i, u_{-i})$ is strictly better than each element in $\varphi(u)$ for some individual[6]. If the AR is not manipulable, it is *strategy-proof (SP)*.

[6]In fact, under the ESV assumption manipulation means not only that some element in $\varphi(v_i, u_{-i})$ is strictly better than each element in $\varphi(u)$ for i but also that no element in $\varphi(u)$ is strictly better than an element in $\varphi(v_i, u_{-i})$ according to preferences u_i. For a proof of this result, see Bogomolnaia, Debt, and Ehlers (2002).

Throughout the paper we will consider fair allocation rules that are ESV and PI. The following proposition shows that such an AR not only selects a unique utility distribution but also a unique distribution of money for each preference profile. Moreover, the PI property implies that all fair assignments corresponding to the (unique) fair distribution also belongs to the outcome set of the AR.

Proposition 2 *Let φ be a fair, ESV and PI allocation rule and let $u \in \mathcal{U}$. If $(a,x),(b,y) \in \varphi(u)$ then $x = y$. If $(a,x) \in \varphi(u)$ and $(b,x) \in \Phi(u)$, then $(b,x) \in \varphi(u)$.*

Hence for fair and ESV allocation rules φ there is a unique distribution of money for each profile $u \in \mathcal{U}$. Denote this relationship by f, i.e. $(a,x) \in \varphi(u)$ if and only if $x = f(u)$. The function f is called a *fair distribution rule (FDR).* For allocation rules φ that are fair, ESV and PI, the corresponding FDR f and the AR φ will be used interchangeably.

Finally, we define efficiency for fair distribution rules. Call a fair distribution rule f *efficient*[7] at a profile $u \in \mathcal{U}$ if $f(u) = x$ and $y < x$[8] imply that $f(v) \neq y$ for all $v \in \mathcal{U}$. Hence if the distribution x is the outcome at some profile and x strictly dominates a distribution y then y is never the outcome at any profile. The following is an example of a strategy-proof AR that is fair and optimal w.r.t. the vector $(1,1)$. With minor modifications, the example will also be used later in the paper to illustrate various properties of allocation rules.

Example 1 *A strategy-proof, efficient, fair and optimal allocation rule.*

Consider a case with two individuals. Normalize the utility functions so $u_{i2} = 0$, $u_{11} = 1 - \alpha$ and $u_{21} = 1 - \beta$ represent the preferences of individual 1 and 2 respectively. Let φ be a fair, ESV and PI allocation rule where the corresponding distribution rule $f(u) = x$ is defined as follows. For $\alpha \leq \beta$:

$$\begin{array}{ll} x_1 = \beta \text{ if } \beta \leq 1, & x_2 = 1 \text{ if } \alpha \leq 1, \\ x_1 = 1 \text{ if } \beta \geq 1, & x_2 = 2 - \alpha \text{ if } \alpha \geq 1. \end{array}$$

When $\alpha \leq \beta$ the assignment is $a_1 = 1$. Let the allocation rule be symmetric, so when $\alpha \geq \beta$, the assignment is $a_1 = 2$ and the distribution above is changed so α becomes β and vice versa. One easily sees that φ is strategy-proof – the money one individual receives depends only on the preferences of

[7] Note that the outcome of an efficient FDR is not necessarily Pareto optimal – for many profiles the outcome is not budget balanced.

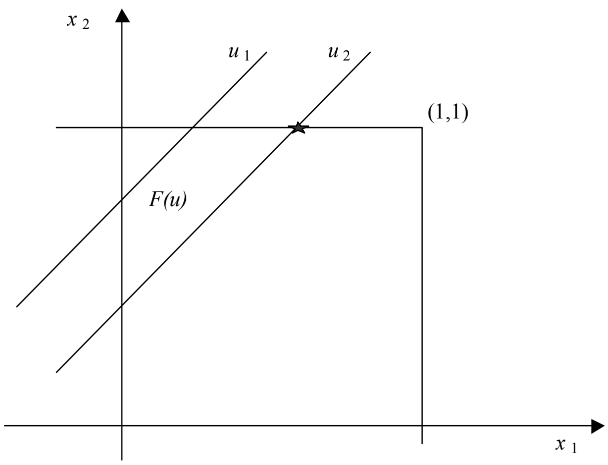
[8] $y < x$ iff $y_j < x_j$ for all j.

the other individual as far as the assignment does not change. The outcome is also envy-free. In addition, the AR is obviously efficient at all profiles – the range of f is the set

$$S = \left\{x \in \mathbb{R}^2;\ x_j \leq 1,\ j = 1, 2,\ x_2 = 1 \text{ when } x_1 \leq 1,\ x_1 = 1 \text{ when } x_2 \leq 1\right\}.$$

■

The following figure illustrates the situation in Example 1 for the case $\alpha < \beta < 1$. The line u_1 (u_2) indicates the distributions where individual 1 (2) is indifferent between the two bundles $(1, x_1)$ and $(2, x_2)$. The fair set is the area $F(u)$ between the lines u_1 and u_2, and the star denotes the fair and optimal distribution $(\beta, 1)$. The FDR is efficient at all profiles but budget balance does not prevail; $1 + \beta < 2$. So the allocation is not efficient.



The main objective of this study is to characterize the set of strategy-proof and fair distribution rules. The theorem below shows that this set is non-empty and in next section we give the full characterization of the set. Various forms of the theorem have been proved for different preference domains. Leonard (1983) prove the theorem assuming preferences to be quasi-linear and $\bar{x} = 0$. Proofs with general preferences can be found in Demange and Gale (1985) and in Sun and Yang (2003).

Theorem 1 *Let $\bar{x} \in \mathbb{R}^n$ and let $f(u)$ be the distribution rule which is fair and optimal w.r.t. $\bar{x}$. Then f is strategy-proof and efficient at all profiles.*

5 The main result

Now consider the class of fair and strategy-proof allocation rules that are essentially single-valued and Pareto-indifferent. Denote this class $\mathcal{F}$. Theorem 1 shows that $\mathcal{F}$ is non-empty, and the question is now whether all elements in the class are of this type, i.e. fair and optimal w.r.t. some quantity restrictions. This is in fact the case if some "inefficient" allocation rules are excluded. The example below shows an AR in $\mathcal{F}$ that is not efficient.

Example 2 *A strategy-proof and fair allocation rule that is not efficient.*

Consider the following modification of the allocation rule in Example 1. Individual preferences are the same – $u_{i2} = 0$, $u_{11} = 1 - \alpha$ and $u_{21} = 1 - \beta$ – but the distribution rule $f(u) = x$ corresponding to the fair, ESV and PI allocation rule φ is now defined as follows. For $\alpha \leq \beta$:

$$\begin{array}{ll} x_1 = \beta \text{ if } \beta \leq 0, & x_2 = 1 \text{ if } \alpha \leq 0, \\ x_1 = \beta/2 \text{ if } 0 \leq \beta \leq 2, & x_2 = (2-\alpha)/2 \text{ if } 0 \leq \alpha \leq 2, \\ x_1 = 1 \text{ if } \beta \geq 2, & x_2 = 2 - \alpha \text{ if } \alpha \geq 2. \end{array}$$

When $\alpha \leq \beta$ then the assignment is $a_1 = 1$. Let the allocation rule be symmetric, so when $\alpha \geq \beta$, the assignment is $a_1 = 2$ and the distribution above is changed so α becomes β and vice versa. One easily sees that φ is strategy-proof – the money one individual receives depends only on the preferences of the other individual as far as the assignment does not change. The AR is, however, not efficient. For instance, let u be given by $\alpha = 0$ and $\beta = 2$. Then $f(u) = (1, 1)$. On the other hand, if v is defined by $\alpha = \beta = 1$, then $f(v) = (1, 1)/2 < (1, 1) = f(u)$, and hence, f is not efficient at the profile v. In Example 1 the range of the distribution rule f is the set S, while the range is now $S \cup T$, where T is the triangle

$$T = \left\{ x \in \mathbb{R}^2;\ x_j \leq 1,\ j = 1, 2 \text{ and } x_1 + x_2 \geq 1 \right\}.$$

Clearly all distributions in T, except when $x_1 = 1$ or $x_2 = 1$, are inefficient. ■

The example points at a potential inefficiency for allocation rules in $\mathcal{F}$. Another type of inefficiency is budget imbalance, i.e. if not all available money is used. It is, however, not possible to achieve budget balance for AR in $\mathcal{F}$ for all profiles, but only for some. In the characterization theorem below, those allocation rules in $\mathcal{F}$ which achieve budget balance at least at *some* profile and is also efficient at that particular profile, are considered. The main result is then the following.

Theorem 2 *Let $\varphi \in \mathcal{F}$ be such that for some $\bar{u} \in \mathcal{U}$, $\Sigma_j f_j(\bar{u}) = x_0$, and f is efficient at the profile $\bar{u}$. Then for all $u \in \mathcal{U}$, $f(u)$ is the unique distribution that is fair and optimal w.r.t. $\bar{x}$,* where $\bar{x} = f(\bar{u})$.

As Theorem 2 shows, budget balance is not consistent with strategy-proofness and fairness for all profiles. A measure of the magnitude of inefficiency the fair and optimal allocation rule contains is the difference between available money and the money actually used by the AR at various profiles, i.e. the nonnegative number $x_0 - \Sigma_j f_j(u)$.

A question is how severe this lack of budget balance is. To illuminate this issue we consider the class of strategy-proof and budget balanced, but not necessarily fair, allocation rules in a two individual economy. Denote by $W(u) = u_{1a_1} + u_{2a_2} + x_1 + x_2$ the (utilitarian) welfare sum an allocation rule yields. In the example below we calculate $W(u)$ for various allocation rules and various profiles u. We find that there is no allocation rule with a dominating welfare sum for all profiles.

Allocation rules in the two individual economy which are budget balanced are also non-bossy. Then strategy-proofness implies that there is only a finite (at most two) number of allocations in the range of the allocation rule. Let (a, x) and (b, y), with $x_0 = x_1 + x_2 = y_1 + y_2$, be the two possible allocations in the range. There is also only two types of allocation rules. In the first one the outcome is (a, x) unless not both individuals prefer (b, y). The second one is dictatorial, e.g. individual 1 always makes the choice between the two allocations. This class of strategy-proof and budget balanced allocation rules in the two individual economy was derived by Schummer (2000). In the example below we let $\bar{x} = (1, 1)$ be the quantity restriction in the fair and optimal AR, and hence budget balance means that $x_1 + x_2 = 2$.

Example 3 *Aggregate welfare for different allocation rules.*

Once more consider the case with two individuals and preferences according to: $u_{i2} = 0$ while $u_{11} = 1 - \alpha$ and $u_{21} = 1 - \beta$. Let φ be the AR that is fair and optimal w.r.t. $\bar{x} = (1, 1)$. Moreover, let φ' and φ'' be two allocation rules, both having a range $\{(a, \bar{x}), (b, \bar{x})\}$ where $a_1 = 1$ while $b_1 = 2$. In the allocation rule φ', $(a, \bar{x})$ is the outcome unless not both individuals prefer (one strictly) $(b, \bar{x})$ to $(a, \bar{x})$. The allocation rule φ'' is dictatorial and individual 1 makes the choice. Individual 2 makes the choice if individual 1 is indifferent between the two allocations.

Now let us calculate the aggregate welfare for these three allocation rules. Denote by $W_\varphi(u) = u_{1a_1} + u_{2a_2} + x_1 + x_2$ the welfare sum, where φ is the optimal and fair AR. $W_{\varphi'}$ and $W_{\varphi''}$ denotes the welfare sum for the two other allocation rules. One easily finds

α, β	W_{φ}	$W_{\varphi'}$	$W_{\varphi''}$	$W_{\varphi} - W_{\varphi'}$	$W_{\varphi} - W_{\varphi''}$
$\alpha < \beta < 1$	$2 - \alpha + \beta$	$3 - \alpha$	$3 - \alpha$	< 0	< 0
$\alpha < 1 < \beta$	$3 - \alpha$	$3 - \alpha$	$3 - \alpha$	$= 0$	$= 0$
$1 < \alpha < \beta$	$4 - 2\alpha$	$3 - \alpha$	$3 - \beta$	< 0	< 0
$\beta < \alpha < 1$	$2 - \beta + \alpha$	$3 - \alpha$	$3 - \alpha$	$2\alpha - \beta - 1$	$2\alpha - \beta - 1$
$\beta < 1 < \alpha$	$3 - \beta$	$3 - \beta$	$3 - \beta$	$= 0$	$= 0$
$1 < \beta < \alpha$	$4 - 2\beta$	$3 - \alpha$	$3 - \beta$	$1 + \alpha - 2\beta$	< 0

Let

$$A = \{(\alpha, \beta) \in \mathbb{R}^2;\ \beta < \alpha < 1 \text{ and } 2\alpha > 1 + \beta\},$$

$$B = \{(\alpha, \beta) \in \mathbb{R}^2;\ 1 < \beta < \alpha \text{ and } 2\beta < 1 + \alpha\},$$

$$C = \{(\alpha, \beta) \in \mathbb{R}^2;\ \alpha < 1 < \beta\} \text{ and } D = \{(\alpha, \beta) \in \mathbb{R}^2;\ \beta < 1 < \alpha\}.$$

Then $W_{\varphi} - W_{\varphi'} > 0$ if and only if $(\alpha, \beta) \in A \cup B$. Moreover $W_{\varphi} - W_{\varphi'} = 0$ when $(\alpha, \beta) \in C \cup D$, and finally, $W_{\varphi} - W_{\varphi'} \leq 0$ for profiles where $(\alpha, \beta) \in \mathbb{R}^2 - (A \cup B \cup C \cup D)$. So the fair and optimal AR performs better than the alternative with budget balance for certain preference profiles while its aggregate utility sum is equal or smaller for other profiles. Hence, from an efficiency point of view there is no obvious choice between the various possible strategy-proof allocation rules. There are efficiency losses due to the strategy-proofness property of similar magnitude in all allocation rules (in this example). ■

6 Multi object auctions

One interpretation of the model with money and indivisible objects is as an auction model where multiple objects are simultaneously traded. In this section we shortly compare the fair and optimal allocation rule in this context with similar or equivalent allocation rules in some previous studies. In this context it is also natural to talk about equilibrium prices p instead of fair distributions x. The relationship is of course $p = -x$. In the analysis so far sellers' reservation prices and buyers' willingness to participate in the allocation procedure have not played any role. Now, however, those values are important. We conclude the section with an example of how the multi object auction can be designed.

In Koopmans and Beckmann (1957) (equilibrium) prices supporting an efficient allocation of the objects are derived, and in Leonard (1983) those prices are examined with respect to incentive compatibility. Leonard also provides an extensive discussion on the interpretation of the model as a multi object auction. Below we will compare the incentive compatible selection of equilibrium prices in Leonard, called Clark-Groves (CG) prices, to the outcome of the fair and optimal allocation rule. We find that the CG prices are equal to the outcome of the fair and optimal allocation rule if the implicit quantity restrictions $\bar{x}$ that characterize each such rule is chosen zero, $\bar{x} = 0$. With the multi object auction interpretation of the model there are sellers and buyers of the objects. A main difference between the two rules of selecting prices is how the sellers' reservation prices and buyers' willingness to pay for the objects are treated. With CG-prices it is individually rational for the buyers to participate in the auction while the sellers' reservation prices are not directly related to the outcome. On the other hand, with optimal and fair allocation rules, the implicit quantity restrictions $\bar{x}$ may be chosen equal to the sellers' reservation prices (except for the sign, $p = -x$). But then the outcome of the rule is not related to any participation constraints of the buyers.

In Demange and Gale (1985), however, the role of the sellers' as well as the buyers' reservation prices are explicitly analyzed. By definition their set of equilibrium price vectors, say E, contains only prices respecting all reservation prices. The set E is a lattice and has precisely one maximal element $\bar{p}$ and one minimal element $\underline{p}$. The minimal price vector $\underline{p}$ is exactly the same, except for the sign, as the outcome of the fair and optimal distribution rule; $\underline{p}= -f(u)$, where $-\bar{x}$ is the sellers reservation price vector. Demange and Gale also show that this selection of equilibrium prices is incentive compatible, i.e. our Theorem 1.

Let us more formally compare the incentive compatible selection of equilibrium prices in Leonard (1983) and in Demange and Gale (1985) with the fair and optimal distribution rule.

Suppose there are n sellers of the various objects and denote by $r \in \mathbb{R}^n$ the vector of reservation prices, r_j for object j, $j = 1, 2, \ldots, n$. Furthermore, denote by u_{i0} individual i's utility of consuming "no object", which implicitly also defines i's reservation prices for the various objects. Hence, a utility profile u is now an element in $\mathbb{R}^{n(n+1)}$. Let $E(r, u)$ be the set of equilibrium prices defined by

$$E(r, u) = \{p \in \mathbb{R}^n;\ p \geq r\} \cap \bar{F}(u),$$

where

$$\bar{F}(u) = \{p \in \mathbb{R}^n;\ u_{ia_i} - p_{a_i} \geq \max(u_{ij} - p_j, u_{i0}) \text{ for all } i, j, \text{ for some } a\}$$

is the set of "fair" prices which are individually rational ($u_{ia_i} - p_{a_i} \geq u_{i0}$).

Provided that the set $E(r,u)$ of equilibrium prices is non-empty, there is precisely one price vector $\underline{p}$ which is minimal in $E(r,u)$, i.e. $\underline{p} \leq p$ for all $p \in E(r,u)$. Demange and Gale (1985) show that the minimal price vector also defines incentive compatible prices. Hence with $\bar{x} = -\underline{p}$ this is the content in our Theorem 1.

A variant of Theorem 1 can also be found in Leonard (1983). There equilibrium prices are those which support an efficient assignment of the objects. This analysis is based on Koopmans and Beckmann (1957). Given the set of equilibrium prices, Leonard selects one price vector and shows that it is incentive compatible. The procedure is the following.

Efficiency requires that the assignment a of objects to individuals is chosen so that the utility sum $\Sigma_i u_{ia_i}$ is maximal. To obtain equilibrium prices Koopmans and Beckmann solve the following linear programming problems for nonnegative utility profiles, $u \geq 0$ (and hence assume $u_{i0} = 0$). The solution to the first problem identifies an efficient allocation of the objects:

$$\max_{\delta} \Sigma_{ij} \delta_{ij} u_{ij} \text{ s.t.}$$

$$\Sigma_j \delta_{ij} \leq 1, \text{ for all } i, \ \Sigma_i \delta_{ij} \leq 1 \text{ for all } j, \text{ and } \delta_{ij} \geq 0.$$

This problem has at least one integer solution, where $\delta_{ij} = 1$ or 0. $\delta_{ij} = 1$ means that individual i is assigned object j. Denote by $a_i = j$ this assignment. The second problem is the dual to the first one and identifies equilibrium prices p supporting the efficient allocation:

$$\min_{s,p} (\Sigma_i t_i + \Sigma_j p_j) \text{ s.t.}$$

$$t_i + p_j \geq u_{ij} \text{ for all } i, j, \text{ and } t, p \geq 0.$$

If a is the assignment for an integer solution to the primal problem and (t^*, p^*) a corresponding solution to the dual problem then $\Sigma_i u_{ia_i} = \Sigma_i t_i^* + \Sigma_j p_j^*$, and one easily sees that the allocation (a, x^*), with $x^* = -p^*$, is envy-free ($t_i^* + p_{a_i}^* = u_{ia_i}$). Typically there are several solutions and several equilibrium prices. The set of equilibrium prices obtained as solutions to the dual linear programming problem above is $E(r,u)$ if $u_{i0} = 0$ and e.g. $r = 0$. Now Leonard (1983) shows that if $p^* \in E(r,u)$ is the selection of equilibrium prices such that $\Sigma_j p_j^*$ is *minimal*, then the individuals have no incentive to misrepresent their preferences. Since $p \geq 0$ is a restriction for the choice of prices, we see that the rule for selecting p is exactly the same as the one for selecting the fair distribution x in the fair and optimal allocation rule if we let $p = -x$ and $\bar{x} = 0$. Leonard also shows that this type of prices is of the same kind as the incentive compatible Clark-Groves prices for public goods.

Example 4 *An auction method for multiple objects*

An advantage with the fair and optimal allocation rule used as an auction method is that the sellers' reservation prices can be set to $\underline{p}$ $(= -\bar{x})$. The problem with this is that there is no guarantee that potential buyers want to participate. To overcome this problem we can simply give the buyers the option to refrain from buying the object assigned to him by the allocation rule if he prefers not to trade, i.e. for individual i, $u_{i0} > u_{ia_i} - p_{a_i}$. In that case the seller of the object keeps the object.

Hence the auction method now consists of three steps:

1. The sellers report their reservation prices, say $\underline{p} \in \mathbb{R}^n$, and the buyers their preferences $u \in \mathcal{U}$, to the auctioneer.

2. The prices and assignments according to the fair and optimal allocation rule is applied based on the complete profile u. The allocation becomes (a, p).

3. Individual i receives object a_i and pay the price p_{a_i} provided that $u_{a_i} - p_{a_i} \geq u_{i0}$. Otherwise the object assigned to him remains at the seller.

At step 1 the sellers' choices are strategic. To optimize they have to guess the preferences of the potential buyers. The buyers, however, have no incentives to misrepresent their preferences. They also have the option not to buy when the outcome of the auction is revealed, so it is individually rational to participate. If there is only one seller the auction reduces to the usual Vickrey auction where the second highest bid determines the price. This bid, however, can be the sellers reservation price. ■

Remark 1 *It is well-known that generalizations of Vickrey's sealed bid single object auction may be incentive compatible for sellers or for buyers but not for both. See e.g. Demange and Gale (1985).*

7 Appendix – the proofs

Proposition 1 *(i) For each profile $u \in \mathcal{U}$, the fair set $F(u)$ is a non-empty and closed subset of $\mathbb{R}^n$. (ii) If $x, y \in F(u)$ and (a, x) is a fair allocation, then (a, y) is also a fair allocation. (iii) If $x, y \in F(u)$ then $z \in F(u)$, where $z_j = \max[x_j, y_j]$. (iv) A fair and optimal distribution is unique.*

Proof. (i) For a proof of $F(u) \neq \varnothing$, see e.g. Svensson (1983). $F(u)$ is closed since preferences are continuous.

(ii) Let $x, y \in F(u)$ and (a, x) be a fair allocation. Without loss of generality (abbreviated: w.l.g.), we assume that $a_i = i$ for all $i \in N$ and $y_j = x_j + \delta_j$ with $\delta_j \geq \delta_{j+1}$ for all $j \in A$. If $\delta_j = \delta_{j+1}$ for all j, (a, y) is obviously a fair allocation, so assume that $\delta_j = \delta_1$ if $j \leq k$ and $\delta_k > \delta_{k+1}$. Also let (b, y) be a fair allocation. Then we can choose $b_i = i$ for $i \leq k$, since (a, x) is fair and $u_{ii} + x_i \geq u_{ij} + x_j$ if and only if $u_{ii} + x_i + \delta_i \geq u_{ij} + x_j + \delta_j$. Moreover, a fair assignment b cannot be chosen so that $b_i > k$ for $i \leq k$ because if chosen so then $u_{ib_i} + x_{b_i} + \delta_{b_i} < u_{ii} + x_i + \delta_i$. In summary, for individuals $i \leq k$, $b_i \leq k$ is necessary and $b_i = i$ possible for a fair assignment b. Now we can repeat the the arguments above for the group of individuals i with $k < i \leq k'$, where $\delta_j = \delta_{k+1}$ if $k < j \leq k'$ and $\delta_{k'} > \delta_{k'+1}$. We then obtain $b_i = i$ also for those individuals. Repeating the arguments now leads to $b_i = i$ for all individuals.

(iii) Since $x, y \in F(u)$, $(a, x), (a, y) \in \Phi(u)$ for some assignment a by (ii) above. W.l.g. let $a_i = i$ for all i. Then for all $i \in N$,

$$u_{ii} + x_i \geq u_{ij} + x_j \text{ and } u_{ii} + y_i \geq u_{ij} + y_j \text{ for all } j,$$

and hence,

$$u_{ii} + z_i = \max[u_{ii} + x_i, u_{ii} + y_i] \geq \max[u_{ij} + x_j, u_{ij} + y_j] = u_{ij} + z_j \text{ for all } j.$$

(iv) Let $u \in \mathcal{U}$ and $\bar{x} \in \mathbb{R}^n$ be given and let $y, y' \in F(u)$ be fair and optimal w.r.t. $\bar{x}$. Then $z = \max(y, y') \in F(u)$ by property (iii) above. Moreover, $z \leq \bar{x}$. If $z \neq y$ then $S(y) < S(z)$, so $z = y = y'$ must be the case. ∎

Proposition 2 *Let φ be a fair, ESV and PI allocation rule and let $u \in \mathcal{U}$. If $(a, x), (b, y) \in \varphi(u)$ then $x = y$. If $(a, x) \in \varphi(u)$ and $(b, x) \in \Phi(u)$, then $(b, x) \in \varphi(u)$.*

Proof. Let $(a, x), (b, y) \in \varphi(u)$. Then $u_{ia_i} + x_{a_i} = u_{ib_i} + y_{b_i}$ for all $i \in N$ since φ is ESV, while $u_{ia_i} + x_{a_i} \geq u_{ib_i} + x_{b_i}$ since φ is fair. But then $y_{b_i} \geq x_{b_i}$ and for symmetry reasons, $x_{b_i} \geq y_{b_i}$. This shows that $x = y$.

Now assume that $(a, x) \in \varphi(u)$ and $(b, x) \in \Phi(u)$. Then for all $i \in N$,

$$u_{ia_i} + x_{a_i} \geq u_{ij} + x_j \text{ for all } j \text{ and } u_{ib_i} + x_{b_i} \geq u_{ij} + x_j \text{ for all } j$$

and hence, $u_{ia_i} + x_{a_i} = u_{ib_i} + x_{b_i}$. Then by PI, $(b, x) \in \varphi(u)$. ∎.

Theorem 1 *Let $\bar{x} \in \mathbb{R}^n$ and let $f(u)$ be the distribution rule which is fair and optimal w.r.t. $\bar{x}$. Then f is strategy-proof and efficient at all profiles.*

Proof. Let $f(u)$ be optimal and fair w.r.t. $\bar{x}$. W.l.g. we let $\bar{x}_j = 1$ for all j. Moreover by Proposition 1 (iv), $f(u)$ is unique. Now let φ be the corresponding fair AR that is ESV and PI. Suppose that φ is not SP. Then there are profiles $u, v \in \mathcal{U}$, an individual $l \in N$, and two fair allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(v_l, u_{-l})$ such that $u_{lb_l} + y_{b_l} > u_{la_l} + x_{a_l}$. Since $u_{la_l} + x_{al} \geq u_{lb_l} + x_{b_l}$, $y_{b_l} > x_{b_l}$ must be the case.

W.l.g. we let $a_i = i$ for all i and $l = 1$, and hence $y_{b_1} > x_{b_1}$. Also let $G = \{j \in N;\ y_j > x_j\}$. We now prove that for all $i \in N - G$, $u_{ii} + x_i > u_{ij} + x_j$ for all $j \in G$.

For $i > 1$ and $i \in G$:

$$u_{ib_i} + y_{b_i} \geq u_{ii} + y_i > u_{ii} + x_i \geq u_{ij} + x_j \geq u_{ij} + y_j \text{ for all } j \in N - G.$$

Hence by fairness, for $i > 1$, $b_i \in G$ when $i \in G$. Moreover, $y_{b_1} > x_{b_1}$ and hence $b_1 \in G$. Then by fairness, $i \in G$ if and only if $b_i \in G$. Then also $i \in N - G$ if and only if $b_i \in N - G$. Now we have for $i \in N - G$:

$$u_{ii} + x_i \geq u_{ib_i} + x_{b_i} \geq u_{ib_i} + y_{b_i} \geq u_{ij} + y_j > u_{ij} + x_j \text{ for all } j \in G.$$

Hence for $i \in N - G$, $u_{ii} + x_i > u_{ij} + x_j$ for all $j \in G$. Moreover, $x_j < y_j \leq 1$ for all $j \in G$. This shows that x is not optimal - a contradiction. Hence φ must be strategy-proof.

Finally suppose that f is not efficient at a profile $u \in \mathcal{U}$. Then there is a profile $v \in \mathcal{U}$ such that $f(u) = x < y = f(v)$. But $y_j \leq 1$ for all j so $x_j < 1$ for all j. Then obviously $z \in F(u)$ and $z_j \leq 1$ if $z_j = x_j + \epsilon$ and $\epsilon > 0$ is sufficiently small. This shows that x is not optimal – a contradiction. Hence f must be efficient at all profiles. ■

Theorem 2 *Let $\varphi \in \mathcal{F}$ be such that for some $\bar{u} \in \mathcal{U}$, $\Sigma_j f_j(\bar{u}) = x_0$, and f is efficient at the profile $\bar{u}$. Then for all $u \in \mathcal{U}$, $f(u)$ is the unique distribution that is fair and optimal w.r.t. $\bar{x}$, where $\bar{x} = f(\bar{u})$.*

To prove the theorem a number of lemmas will be useful. The presumptions for the lemmas are the same as in Theorem 2, and are not repeated. Moreover, by choice of units we let $\bar{x} = e$, where $e_j = 1$ for all j. Hence, $x_0 = n$.

Lemma 1 *Let $u, v \in \mathcal{U}$ and $f(u) = x$. Then $f(v) \leq f(u)$ if $v_{ij} = -x_j$ for all i, j.*

Proof. Let $f(u) = x$ and a be the assignment where $a_i = i$ for all i. With no loss of generality, we assume that $(a, x) \in \Phi(u)$. Let $f(v) = y$. Then obviously

$(a, y) \in \Phi(v)$. Also let $v^k = (v_1, \ldots, v_k, u'_{k+1}, \ldots, u'_n)$ and $x^k = f(v^k)$. We first prove that $(a, x^k) \in \Phi(v^k)$ for all k.

If $(a, x^k) \in \Phi(v^k)$ then also $(a, x^k) \in \Phi(v^{k-1})$, because:

$$v_{kk} + x_k^k \geq v_{kj} + x_j^k \text{ for all } j \quad \Leftrightarrow \quad x_k^k - x_k \geq x_j^k - x_j \text{ for all } j,$$

and then

$$u_{kk} + x_k^k = u_{kk} + x_k + x_k^k - x_k \geq u_{kj} + x_j + x_j^k - x_j = u_{kj} + x_j^k \text{ for all } j.$$

But then $x^k, x^{k-1} \in F(v^{k-1})$, and hence by Proposition 1(ii), $(a, x^{k-1}) \in \Phi(v^{k-1})$ since $(a, x^k) \in \Phi(v^k)$. By induction it now follows that $(a, x^k) \in \Phi(v^k)$ for all k.

By fairness we have for all $i \leq k$,

$$v_{ii} + x_i^k \geq v_{ij} + x_j^k \text{ for all } j, \text{ i.e. } x_i^k - x_i \geq x_j^k - x_j \text{ for all } j,$$

and by strategy-proofness,

$$u'_{kk} + x_k^{k-1} \geq u'_{kk} + x_k^k \text{ and } v_{kk} + x_k^k \geq v_{kk} + x_k^{k-1}, \text{ i.e. } x_k^k = x_k^{k-1}.$$

Thus

$$x_k^k - x_k = x_k^{k-1} - x_k \leq x_{k-1}^{k-1} - x_{k-1} \text{ and hence, } x_n^n - x_n \leq x_1^1 - x_1.$$

But $x_1^1 = x_1$ so $x_n^n \leq x_n$. We also have $x_n^n - x_n \geq x_j^n - x_j$ for all j, and hence, $x^n \leq x$ which means that $f(v) \leq f(u)$. ■

Let $u \in \mathcal{U}$ be a profile, $\alpha \in \mathbb{R}$ a given number and $k \in N$. Then a distribution $x \in F(u)$ is *fair and minimal w.r.t.* α if $x_k \geq \alpha$ and there is no nonempty group $G \subset N$ and $y \in F(u)$ such that $y_k \geq \alpha$ and $y_j < x_j$ for all $j \in G$.

This means that as little money as possible consistent with fairness has been distributed across the objects $j \neq k$ for a fixed minimal amount of money to object k.

Lemma 2 *If $f(u) = x$ and $y \in F(u)$ is fair and minimal w.r.t. x_k, then $f(v) = y$ if $v_{ij} = -y_j$.*

Proof. W.l.g. let $k = 1$ and $(a, y) \in \Phi(u)$ with $a_i = i$ for all i. Then $(a, x) \in \varphi(u)$. Moreover, $(a, y) \in \Phi(v_1, u_{-1})$ and hence, $y \in F(v_1, u_{-1})$. Now let $z = f(v_1, u_{-1})$. Then $(a, z) \in \varphi(v_1, u_{-1})$ by Proposition 1(ii). Then by strategy-proofness,

$$u_{11} + x_1 \geq u_{11} + z_1 \text{ and } v_{11} + z_1 \geq v_{11} + x_1 \text{ and hence, } z_1 = x_1.$$

Then by fairness, $z_1 - y_1 \geq z_j - y_j$ for all j, and hence, $z \leq y$ since $y_1 = z_1 = x_1$. Moreover, $(a, z) \in F(u)$ since $y_1 = x_1$. But then $z = y$ since y is fair and minimal.

Now let $v^k = (v_1, \ldots, v_k, u_{k+1}, \ldots, u_n)$ and $x^k = f(v^k)$. We prove by induction that $x^k = y$. The statement is true for $k = 1$ according to the first part of the proof. Assume that $x^{k-1} = y$ and $(a, y) \in \Phi(v^{k-1})$. Then obviously $(a, y) \in \Phi(v^k)$. Moreover, $x^k = f(v^k)$ so $(a, x^k) \in \Phi(v^k)$ by Proposition 1(ii) and hence, $(a, x^k) \in \varphi(v^k)$ by PI. Then by fairness for $i \leq k$, $x_i^k - y_i = \delta = \max_j(x_j^k - y_j)$. By strategy-proofness we have $u_{kk} + x_k \geq u_{kk} + x_k^k$ and $v_{kk} + x_k^k \geq v_{kk} + x_k$ and hence, $z_k = x_k^k$. But $x_1 = y_1$ so $\delta = 0$ and hence, $x_i^k - y_i = 0$ for $i \leq k$ and $x^k \leq y$. Moreover, $(a, x^k) \in F(u)$ since $x_i^k = y_i$ for $i \leq k$. But then $x^k = y$ since y is fair and minimal. Then by induction, $x^k = y$ for all k and hence $y = x^n = f(v)$. ■

Let $E = \{x \in \mathbb{R}^n;\ x_j \leq 1 \text{ for all } j \text{ and } x_j = 1 \text{ for some } j\}$.

Lemma 3 *Let $f(u) = x$ and $u_{ij} = -x_j$ for all i, j. Then $x \in E$.*

Proof. W.l.g. assume that $x_1 \geq x_j$ for all j. Moreover let $a_i = i$ for all $i \in N$ be an assignment. Let $y \in E$ and $y = x + te$ for some $t \in \mathbb{R}$ (and hence, $y_1 = 1$). Then $(a, x) \in \varphi(u)$. Let $\delta \in \mathbb{R}$ and define a profile $v \in \mathcal{U}$ according to:

$v_{11} = -y_1 + \delta$ and $v_{1j} = -y_j$ for $j > 1$,

$v_{i1} = -y_1$, $v_{ii} = -y_i$, and $v_{ij} = -\delta$ for $j \in N - \{1, i\}$, when $i > 1$.

Then $(a, y) \in \Phi(v)$ if δ is sufficiently large because,

$v_{11} + y_1 = \delta - 1 \geq v_{1j} + y_j = 0$ for $j > 1$,

$v_{ii} + y_i = v_{i1} + y_1 = 0 \geq v_{ij} + y_j = y_j - \delta$ for $j \in N - \{1, i\}$, when $i > 1$.

Moreover, $(a, e) \in \Phi(v)$ if δ is sufficiently large because,

$v_{11} + 1 = 1 - y_1 + \delta \geq v_{1j} + 1 = 1 - y_j$ ´for $j > 1$,

$v_{ii} + 1 = 1 - y_i \geq v_{i1} + 1 = 1 - y_1 = 0 \geq v_{ij} + 1 = 1 - \delta$ for $j \in N - \{1, i\}$, when $i > 1$.

But the FDR f is weakly efficient at the distribution e by assumption and hence, $(a, e) \in \varphi(v)$.

Now let $z \in F(v)$ be fair and minimal w.r.t. $z_1 = y_1 = 1$. $G = \{j \in N;\ z_j < y_j\}$. Then for $i \in G$,

$$v_{ii} + z_i < v_{ii} + y_i = v_{i1} + y_1 = v_{i1} + z_1.$$

Hence, $G = \varnothing$ and $z \geq y$. But z is minimal w.r.t. $z_1 = y_1$ and $y \in F(v)$ so $z = y$. Moreover, $f(v) = e$ and hence by Lemma 2, $f(u') = y$ if $u' \in \mathcal{U}$ and $u'_{ij} = -y_j$. But $f(u') = f(u)$ by strategy-proofness, and hence, $f(u) \in E$. ■

Also let $g(u) \in E \cap F(u)$ be the optimal and fair distributions for profiles $u \in \mathcal{U}$. Then,

Lemma 4 *Let $f(u) \in E$ for all $u \in \mathcal{U}$. Then $f(u) = g(u)$ for all $u \in \mathcal{U}$.*

Proof. The lemma is proved by induction. Let

Id: $f(u) = g(u)$ if $u_{ij} = -y_j$ for at least k individuals i, where $y = g(u)$.

Id is trivially true when $k = n$. Now assume that Id is true for k. Let $u \in \mathcal{U}$, $y = g(u)$ and $u_i = v_i$ for $k-1$ individuals i, where $v_{ij} = -y_j$. Also let $(a, x) \in \varphi(u)$ and assume that $x \neq y$. W.l.g. we assume that $a_i = i$ and $x_1 < y_1$.

Now $u_1 \neq v_1$ must be the case, because if $u_1 = v_1$ then $u_{11} + x_1 = x_1 - y_1 < 0$ and $u_{1j} + x_j = x_j - y_j = 0$ for some j. But then $(a, x) \notin \varphi(u)$, which is a contradiction.

Let $z = f(v_1, u_{-1})$. According to Id, $z = g(v_1, u_{-1})$. But then

$$(a, x) \in \varphi(u) \Rightarrow (a, y) \in \Phi(u) \Rightarrow (a, y) \in \Phi(v_1, u_{-1}) \Rightarrow (a, z) \in \varphi(v_1, u_{-1}).$$

Hence, $y \leq z$. Moreover by strategy-proofness,

$$u_{11} + x_1 \geq u_{11} + z_1 \text{ and } v_{11} + z_1 \geq v_{11} + x_1 \text{ and hence, } z_1 = x_1.$$

But $y_1 \leq z_1$, $z_1 = x_1$ and $x_1 < y_1$ is a contradiction. Thus the assumption $x_1 < y_1$ is false and $x = y$ must be the case. ■

Proof of Theorem 2. Let $u \in \mathcal{U}$ and $f(u) = x$. If $x_j > 1$ for some j then let $y \in F(u)$ minimal w.r.t. $y_j = x_j$. Also let $v \in \mathcal{U}$ with $v_{ij} = -y_j$. Then $f(v) = y$ by Lemma 2, and hence, $y \notin E$. But this is a contradiction to Lemma 3. Hence $x \leq e$. If $x < e$ then $f(v) < e$ by Lemma 1 if $v \in \mathcal{U}$ with $v_{ij} = -x_j$. This is also a contradiction to Lemma 3. Then by Lemma 4, f is optimal w.r.t. $f(u) \leq e$. ■

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