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A NEW FAMILY OF SMOOTH STRATEGIES FOR SWINGING UP A PENDULUM

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Abstract: The paper presents a new family of strategies for swinging up a pendulum. They are derived from physical arguments based on two ideas: shaping the Hamiltonian for a system without damping; and providing damping or energy pumping in relevant regions. A two-parameter family of simple strategies without switches with nice properties is obtained. The main result is that all solutions that do not start at a zero Lebesgue measure set will converge to the upright position for a wide range of the parameters in the control law. Thus, the swing-up and the stabilization problems are simultaneously solved with a single, smooth law.

Keywords: Pendulum, Shaping Hamiltonians, Swing up.

1. INTRODUCTION

The family of the inverted pendula has attracted the attention of control researchers in recent decades as a benchmark for testing and evaluating a wide range of classical and contemporary nonlinear control methods (see (Angeli, 2001; Bloch *et al.*, 1999; Furuta, 2003; Lozano *et al.*, 2000; Utkin *et al.*, 2000; Srinivasan *et al.*, 2002; Wiklund *et al.*, 1993), to mention only a few references). The inverted pendulum displays two main problems: swinging up the pendulum to the upright position (Åström and Furuta, 2000; Gordillo *et al.*, 2003; Lozano *et al.*, 2000; Shiriaev *et al.*, 2001) and stabilizing it in this position once it is reached. These problems have traditionally been treated as two separate ones. The first one is usually solved by energy considerations (Åström and Furuta, 2000). However, this kind of controller leads to the stabilization of an homoclinic orbit. In this way, the system will eventually approach the desired point but without achieving local stability since, due to small disturbances, the system will go away from this point. One explanation of this behavior is that the desired point is a saddle point and the (attractive) homoclinic orbit is its stable manifold. On the other hand there are laws that make this point asymptotically stable (Bloch *et al.*, 1999) but the domain of attraction is limited and never reaches the horizontal plane ($|x_1| < \pi/2$). Then, the full problem is usually solved by switching between different laws: first, a law that performs the swing-up is used and, once the pendulum is near the vertical position, the controller switches to a local law (Wiklund *et al.*, 1993). In (Aracil and Gordillo, 2004; Gordillo *et al.*, 2004) a new strategy was proposed that solves both problems without commutation between different laws, but by commutation of a controller parameter. In (Srinivasan *et al.*, 2002) a single controller is proposed but it requires a strategy for commutation of the reference value.

In this paper, we consider the simplest version of the pendulum (Åström and Furuta, 2000): the control action is the acceleration of the pivot and, thus, a 2-dimensional model is used. Here, we return to the idea of (Aracil and Gordillo, 2004; Gordillo *et al.*, 2004). First, an energy shaping control law is designed in such a way that: 1) the closed-loop energy presents a minimum at the desired position; and 2) the energy shaping controller is globally defined. Since the resultant energy has other minima, a pumping and damping strategy is needed in order to carry the system into the desired basin. The main contribution of this paper is that the resultant law is smooth, and no commutations are needed, and the origin of the final closed-loop system is almost-globally asymptotically stable. The stability proof is included. The final control law has two parameters that are easy to tune.

The paper is organized as follows. In Section 2 energy shaping is used to obtain a Hamiltonian system that has a center at the desired upright position. In Section 3 a pumping-damping strategy is introduced that makes the upright position the only stable equilibrium point. Section 4 is devoted to the stability analysis of the closed loop system. The paper ends with a Section of conclusions.

2. ENERGY SHAPING

The model of the pendulum system is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - u \cos x_1, \end{aligned} \tag{1}$$

where x_1 is the angular position of the pendulum with the origin at the upright position, and x_2 is the velocity of the pendulum. Therefore, the system is defined on a cylindrical state space $S \times \mathbb{R}$.

Our goal is to design a controller that is able to swing up the pendulum from (almost) all initial conditions and to maintain the pendulum at the upright position. We will base the derivation on the potential energy shaping method, choosing as desired Hamiltonian functions of the form

$$H_d(x_1, x_2) = V_d(x_1) + \frac{x_2^2}{2}, \tag{2}$$

where the potential energy V_d should have a single minimum at the desired upright position. A generalized Hamiltonian target system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -k_a \end{bmatrix} \begin{bmatrix} D_{x_1} H_d \\ D_{x_2} H_d \end{bmatrix}, \tag{3}$$

which, with H_d as given by (2), yields

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -V_d'(x_1) - k_a x_2. \end{aligned} \tag{4}$$

One of the problems for choosing an appropriate $V_d(x_1)$ function is related to the term $\cos x_1$, affecting to the control signal, u, in the second equation of (1). For instance, the most elementary choice is $V_d = -\cos x_1$, which has an appropriate shape (a single minimum at the desired upright position), but it leads to the control law $u = 2\tan x_1$ (for the case $k_a = 0$) which cannot be implemented in the full range $|x_1| \leq \pi$ because for $x_1 = \pm\pi/2$ the feedback law is unbounded.

To solve the matching problem of the open (1) and closed (4) loop behaviors, and in order to avoid the division by $\cos x_1$, a good choice of V_d' is

$$V_d' = -\sin x_1 + \beta(x_1) \cos x_1, \tag{5}$$

and then, for $k_a = 0$ (that is, for the conservative case), $u = \beta(x_1)$ (the case $k_a \neq 0$ will be discussed later). Some additional conditions should be imposed on $\beta(\cdot)$. First, $\beta(0) = 0$ to guarantee that the origin $(0, 0)$ is an equilibrium of the closed-loop system. Just as the pendulum behaves in a cylindrical state space, the closed-loop system should display some periodicity. Then, it is reasonable to make $\beta(x_1) = \sin x_1 \bar{\beta}(\cos x_1)$. This choice facilitates the integration of (5) to get V_d. We must also impose that $V'(0) = 0$, $V''(0) > 0$, $V_d(x_1) = V_d(-x_1)$.

A family of functions V_d that fulfill these conditions is given by

$$V_d = a_0 + \cos x_1 - a_2 \cos^2 x_1 - \cdots, \tag{6}$$

which allows us to determine $\beta(x_1)$ from (5).

The simplest case of this family is obtained by taking $a_0 = -1/(4a)$, $a_2 = a$ and $a_k = 0, \forall k > 2$, which leads to

$$V_d(x_1) = \cos x_1 - a \cos^2 x_1 - \frac{1}{4a}, \tag{7}$$

where the value of a_0 has been chosen in order to achieve $\max V_d = 0$. With this V_d we obtain

$$V_d' = -\sin x_1 + 2a \cos x_1 \sin x_1,$$

and then with the feedback law

$$u = 2a \sin x_1, \tag{8}$$

the matching of the open and closed loop systems is solved. It should be noted that this law, with $a = 1$, was proposed in (Furuta, 2003) in another context.

Control law (8) gives a closed loop system with the Hamiltonian

$$H_d(x_1, x_2) = \cos x_1 - a \cos^2 x_1 + \frac{x_2^2}{2} - \frac{1}{4a} \tag{9}$$

So far we have focused our attention in the solvability of the matching problem and we have not considered the global behavior of the target system. When we plot V_d as in Fig. 1 we see that the shape of the graph has some undesirable features. As required, for $a > 0.5$, it has a minimum at the origin, but it has another minimum which makes that the desired equilibrium point is not the only possible equilibrium. This additional minimum coincides with the undesired hanging position.

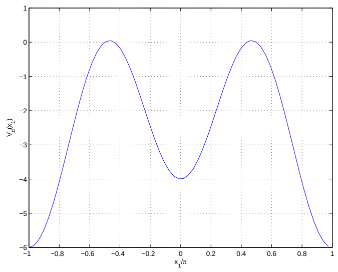

Fig. 1. Shape of V_d for $a = 5$.

If instead of the potential energy V_d we consider the shape of the energy H_d, it is clear that the two minima of V_d give rise to two "wells" in the energy function H_d (Fig. 2). One is the good one, associated with the desired equilibrium, but the other is an undesirable well, because it precludes the global nature of the attraction basin of the equilibrium at the upright position.

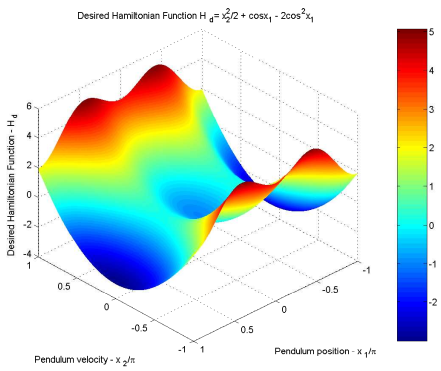

Fig. 2. Shape of H_d showing the desired well around $(0, 0)$ for the upright position and the undesirable well for the hanging position.

To overcome the difficulty with the undesirable well we propose a strategy that consists in pumping energy inside this well, to make the trajectories to leave it. This strategy is discussed in the next section.

A last remark concerning family (6). One should wonder whether there is a member of this family with a single minimum at the origin. It is straightforward to show that this is not possible as for all members of the family the slope of V_d at $\pi/2$ is negative (resp. positive for $-\pi/2$).

3. DAMPING AND PUMPING

Since for $k_a = 0$ the system is Hamiltonian all trajectories are stable but not asymptotically stable. It is then easy to influence the system significantly by changing damping $k_a \neq 0$. Even a small change can have a major impact. To do so we introduce the control law

$$u = 2a \sin x_1 + v \tag{10}$$

where v is a new control signal which have to be chosen to provide the appropriate damping or pumping. The first term of the control law gives a closed-loop system with the Hamiltonian (9). Therefore it follows that

$$\dot{H}_d = -x_2 v \cos x_1$$

To increase and decrease H_d it is natural to make v proportional to $x_2 \cos x_1$ and we will choose

$$v = x_2 F(x_1, x_2) \cos x_1$$

where function F is negative in the region of the state space where we want to pump energy into the system and positive where we would like to damp the system. The control law is

$$u = 2a \sin x_1 + x_2 F(x_1, x_2) \cos x_1 \tag{11}$$

The problem is now to find a simple function F which dissipates energy in regions A and B and which increases the energy in region C (Fig. 3).

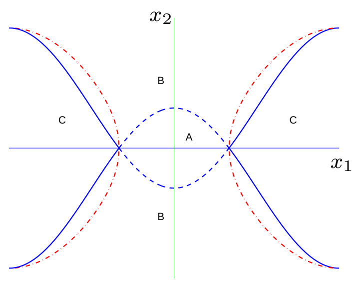

Fig. 3. The level curve for $H_d(x_1, x_2) = 0$ separates the state space into three regions. Region A, which contains the desired equilibrium where the pendulum is upright, is bounded by the dashed line. Region C, which contains the equilibrium where the pendulum hangs straight down, is bounded by the full line. $F = 0$ is the dot-dashed curve

A natural choice is to choose F negative in region C and positive elsewhere. To have a continuous control law the function should also vanish on the boundary of region C.

We could consider choosing $F = H$. This function has the nice property that it vanishes on the boundary of region C. The function is negative in region C and positive in region B but unfortunately it is also negative in region A and it vanishes on the boundary of the region A. This function gives a closed loop with a limit cycle corresponding to the dashed curve in Fig. 3.

To find an approximation to the boundary of region C we will try to find a simple function $W(x_1)$ that matches the potential energy function V for $x_1 \geq x_1^0 = \arccos 1/2a$. We have

$$V_d(x_1^0) = 0$$
$$V_d(\pi) = -1 - a - \frac{1}{4a} = -\frac{(2a+1)^2}{4a}.$$

A simple function which matches these values and which is close to V for intermediate points is

$$W(x_1) = \frac{2a+1}{4a}(2a\cos x_1 - 1).$$

Function $F(x_1, x_2) = W(x_1) + x_2^2/2$ approximates H_d quite well as is shown in Figure 3. It is positive in region A, negative in region C and positive in almost all of region B. Since $\dot{H}_d \geq 0$ in C, the strategy guarantees that all solutions starting in C will leave the set C. Similarly since $\dot{H}_d \leq 0$ in A it follows that all solutions starting in A will converge to the equilibrium at the origin.

3.1 A Family of Strategies

Summarizing we have obtained the following control strategies that are parameterized by $a \geq 0.5$,

$$u = 2a\sin x_1 + bx_2 F(x_1, x_2)\cos x_1, \qquad (12)$$

where

$$F(x_1, x_2) = \frac{2a+1}{4a}(2a\cos x_1 - 1) + \frac{x_2^2}{2}. \qquad (13)$$

This control law has good physical interpretation. The first term shapes the energy function so that the equilibrium at the origin is a center. The second term introduces energy damping in region A, energy pumping in region C and almost all of region B. The sizes of the regions are adjusted by the parameter a and the amount of damping by the parameter b.

It should be noticed that with control law (12) the system apparently preserves the generalized Hamiltonian structure of (3) with $k_a = bF\cos x_1$. However, as F changes sign it does not agree strictly with the definition proposed by Van der Schaft (van der Schaft, 1989), where the damping term is always dissipative. Furthermore, H_d is not a Lyapunov function. However in some way it fits quite well the physical meaning of Van der Schaft's generalized Hamiltonian systems.

4. STABILITY ANALYSIS

We will now analyze the closed-loop system.

4.1 Equilibria

The closed-loop system has the equilibria $(0, 0)$, $(\pm\arccos(1/2a), 0)$ and $(\pi, 0)$. If $a > 0.5$ the equilibrium at the origin is stable. The equilibria at $x_1 = \pm\arccos(1/2a)$ are saddle nodes and the equilibrium at the $x_1 = \pi$ is unstable if $a > 0$.

Since there are several equilibria the system can not be globally stable, but it may be almost-globally stable in the sense that for every initial condition, except for a set of Lebesgue measure zero, the trajectories tend to the origin. This set is formed by the saddles and the unstable equilibria together with the stable manifold of the saddles.

4.2 Stability Proof

To formulate a stability criterion for control we introduce the following functions:

$$\varphi_H(x) \triangleq \sqrt{\frac{1}{2a} + 2a\cos^2 x - 2\cos x}$$
$$\varphi_F(x) \triangleq \sqrt{\frac{1+2a}{2a}(1 - 2a\cos x)}, \qquad x_0 \leq x \leq \pi$$
$$\Phi(a) \triangleq \int_0^{x_0} \varphi_H(x)\cos^2(x)F(x, \varphi_H(x))dx$$
$$+ \int_{x_0}^{\pi} \varphi_F(x)\cos^2(x)F(x, \varphi_H(x))dx$$

where $x_0 = \arccos(1/(2a))$. The function $\varphi_H(x_1)$ is the x_2 coordinate of the upper curve defined by $H(x_1, x_2) = 0$ and the function $\varphi_F(x_2)$ is the x_2 coordinate of the curve defined by $F(x_1, x_2) = 0$. We have the following result.

Proposition 1. Consider system (1) with control law (12)–(13) and $b > 0$. The origin is almost-globally asymptotically stable for any $a > 0.5$ such that $\Phi(a) > 0$ [1].

Proof Since there are multiple equilibria we cannot use Lyapunov theory. We can however use a similar reasoning based on the energy function H_d. We will investigate how the energy changes in the different regions.

In region A, which is bounded and contains the origin, we have $H_d \leq 0$ and $\dot{H}_d \leq 0$. All trajectories entering the region converge to the origin. Convergence is faster the larger b is. In region C we have $H_d \leq 0$ and $\dot{H}_d \geq 0$. All trajectories will leave the region, faster the larger b is. In region B we have $H_d \geq 0$ but the derivative $\dot{H}_d$ can be both

[1] An equilibrium of a dynamical system is said to be almost-globally stable if all the trajectories –except for a set of initial conditions of zero Lebesgue measure– converge to it.

positive and negative. Therefore we divided the region into two subregions $B_1 = \{x \in B | F(x) > 0\}$ and $B_2 = \{x \in B | F(x) \leq 0\}$; that is, B_2 is the region where the control law "incorrectly" injects energy, see Fig. 4.

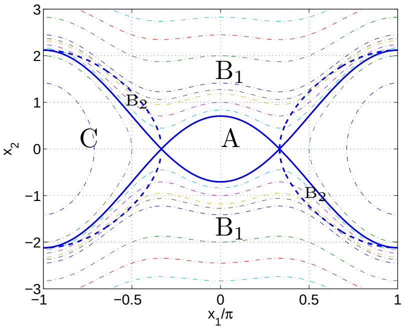

Fig. 4. Curves $F = 0$ (dashed), $H_d = 0$ (solid) and other H_d level curves (dot-dashed).

Introduce $\mathrm{D} = \mathrm{B}_2 \bigcap C$. In B_1 we have damping and the energy will decrease. It thus remains to investigate what happens with trajectories originating in D. To see this we will investigate the total change in energy along the trajectories (we will only consider the case for $x_2 > 0$. The case $x_2 < 0$ can be analyzed with a similar argument). In D we have $\dot{x}_1 > 0$ except at the saddle equilibrium. Taking into account that in D $\dot{H}_d \geq 0$, almost all trajectories in D will thus eventually leave D. We will separate two cases characterized by trajectories that enter A after leaving D and trajectories that do not. The trajectories that enter A will converge to the origin. We will show that the trajectories that do not directly enter A will encounter a net energy loss over a period (i.e. a complete pendulum revolution) if $\Phi(a) > 0$. In this way and taking into account that x_1 can be defined on the manifold S, the system will evolve towards region A after enough revolutions.

We have

$$\dot{H}_d(x_1, x_2) = -bx_2^2 \cos^2 x_1 F(x_1, x_2). \tag{14}$$

Dividing by $\dot{x}_1$ gives

$$dH_d(x_1, x_2) = -bx_2 \cos^2 x_1 F(x_1, x_2) dx_1$$

Consider the change of energy over the interval $-\pi \leq x_1 \leq 0$ for trajectories that start in D and do not enter region A, see Fig. 4.

The total energy change along the trajectory is ΔH_d where

$$\begin{aligned} -\Delta H_d &= \int_{-\pi}^{0} bx_2 \cos^2 x_1 F(x_1, x_2) dx_1 \\ &= \int_{0}^{\pi} bx_2 \cos^2 x_1 F(x_1, x_2) dx_1 \\ &= \int_{0}^{x_0} bx_2 \cos^2 x_1 F(x_1, x_2) dx_1 \\ &+ \int_{x_0}^{\pi} bx_2 \cos^2 x_1 F(x_1, x_2) dx_1 \\ &\geq \int_{0}^{x_0} b\varphi_H(x_1) \cos^2 x_1 F(x_1, \varphi_H(x_1)) dx_1 \\ &+ \int_{x_0}^{\pi} b\varphi_F(x_1) \cos^2 x_1 F(x_1, \varphi_H(x_1)) dx_1 \\ &= b\Phi(a), \end{aligned}$$

The inequality is obtained by observing that, in B_2 we have $x_2 < \varphi_F(x_1)$ and $F(x_1, \varphi_H(x_1)) \leq F(x_1, x_2)$, and in the region B_1 we have $x_2 > \varphi_H(x_1)$ and $F(x_1, x_2) > F(x_1, \varphi_H(x_1))$. Also notice that $F(x_1, x_2)$ is positive in $\mathrm{A} \cup \mathrm{B}_1$ and negative in D.

Finally, using LaSalle arguments it can be seen that the only invariant set with $\dot{H}_d = 0$ exclusively contains equilibria. Since the only stable equilibria is the origin the proof is completed. □

Remark 1. The result is intuitively reasonable since region A increases with increasing a.

Remark 2. The plot of the function Φ Fig. 5 shows that stable systems are obtained for all $a > 0.94$ and $b > 0$.

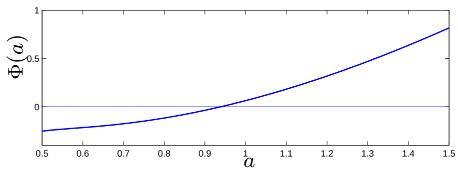

Fig. 5. Plot of function $\Phi(a)$.

Remark 3. The estimate is conservative because energy pumping in D is overestimated and damping outside D is underestimated. This means that it is possible to have stable systems for values of a smaller than 0.94 (and for any value of b). By means of simulations it can be seen that the actual value for this bound is $a = 0.81$. The system can also be stable for smaller values of a by making b sufficiently large. Precise conditions for this are not yet known.

Remark 4. The control law (12) proposed in this paper has a quite engineering and intuitive interpretation. Remember the expression (14) giving the instantaneous damping $\dot{H}_d$. It is evident that the damping is modulated by the term $\cos^2 x_1$, which acts as a weighting factor that determines the value of $\dot{H}_d$. It is clear that for values of x_1 around $\pi/2$ it takes very small values, but for x_1 close to 0 or to π these values are close to unity.

This means that for values of x_1 close to $\pi/2$, where we are "incorrectly" injecting energy, this injection is penalized with a low value for $\cos^2 x_1$. Otherwise, for x_1 around 0, the desired damping is reinforced. Furthermore, for x_1 around π damping or pumping, depending on x_2, is also correctly reinforced. Therefore, term $\cos^2 x_1$ appropriately modulates the energy injection or dissipation in such a way that dissipation is greater than injection, making the system to display the desired convergent behavior towards the upright position.

In order to clarify the ideas of the proof and the behavior of the closed-loop system, Fig. 6 shows the results of two simulations with $a = 1$. The initial condition has been chosen so that both trajectories enter in zone D. In the first simulation with $b = 0.9$ the system enters directly zone A while in the second simulation with $b = 0.25$ the system needs a complete revolution to achieve the net energy loss so it enters in zone A.

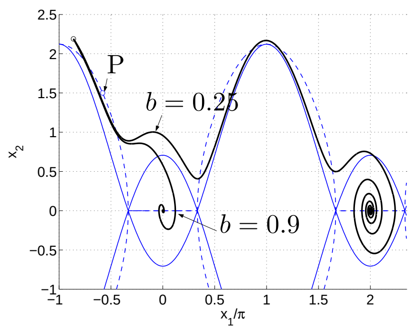

Fig. 6. Results of two simulations for $a = 1$ and with $b = 0.9$ and $b = 0.25$.

Remark 5. It is important to mention a drawback of this new smooth law. While the damping-pumping term can be made arbitrarily small by decreasing parameter b (as can be done in previous laws (Åström and Furuta, 2000)), the energy shaping term can not. The lower bound on a is 0.5 and, thus for $|x_1| = \pi/2$ the absolute value of the term $2a \sin x_1$ is greater than 1. In real systems with real parameters, this law could saturate. The effect of this saturation deserves some analysis and will be studied elsewhere.

5. CONCLUSIONS

In this paper a new approach to the problem of the inverted pendulum is proposed. First we introduce a feedback that modifies the Hamiltonian of the closed loop system so that the origin becomes a center instead of a saddle. By necessity this introduces two more equilibria of the closed loop system. This shaping of the Hamiltonian can intuitively be interpreted as shaping the force term of the system. Next we have used the damping term to introduce damping around the desired equilibrium to change it from being stable to being asymptotically stable. Then we have introduced negative damping (energy pumping) in the region around the equilibrium at $x_1 = \pi$ to make this unstable. In this way a single, smooth controller is obtained that swings up the pendulum from almost all positions.

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