Semi-plenary at CDC08: Distributed Control using Decompositions and Games

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Building theoretical foundations for distributed control

A centralized paradigm dominates theory and curriculum today

We need methodology for

- Decentralized specifications
- Decentralized design
- Validation of global behavior

Approximating the Centralized Controller

Bellman’s equation $|x|^2 = \min\limits_{u} (|Ax + Bu|^2 + |x|^2 + |u|^2)$ gives $u = -Lx$ where

$$L = \begin{bmatrix} 0.3420 & 0.0737 & 0.0046 & 0.0002 \\ 0.1839 & 0.3448 & 0.0736 & 0.0047 \\ 0.0103 & 0.1840 & 0.3447 & 0.0726 \\ 0.0008 & 0.0104 & 0.1808 & 0.3296 \end{bmatrix}$$

Diagonal dominance of $L$ suggests natural approximations:

$$L_0 = \begin{bmatrix} 0.34 & 0 & 0 & 0 \\ 0 & 0.34 & 0 & 0 \\ 0 & 0 & 0.34 & 0 \\ 0 & 0 & 0 & 0.33 \end{bmatrix} \quad L_1 = \begin{bmatrix} 0.34 & 0.07 & 0 & 0 \\ 0.18 & 0.34 & 0.07 & 0 \\ 0 & 0.18 & 0.34 & 0.07 \\ 0 & 0 & 0.18 & 0.33 \end{bmatrix}$$

Today’s challenges:
- Distributed controller validation
- Distributed control synthesis

Outline

- Introduction
- Game theory and dual decomposition
  - Dynamic dual decomposition
  - Distributed validation for wind farm example
  - Distributed synthesis

Needs for distributed control theory

Three major challenges:

- Rapidly increasing complexity
- Dynamic interaction
- Information is decentralized

A “Wind Farm” Case Study

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 & 0 \\ 0.3 & \ddots & \ddots \\ \vdots & \ddots & 0.1 \\ 0 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} u_1(t) + u_1(t) \\ u_2(t) + u_2(t) \\ \vdots \\ u_n(t) + u_n(t) \end{bmatrix}$$

Minimize $V = E \sum_{t=1}^{n} (|x|^2 + |u|^2)$

Inspiration from other fields

- Congestion control in networks
- Collective motion in biology
- Oscillator synchronization in physics
- Parallelization in optimization theory
- Saddle points and equilibria in economics
- Cooperative and non-cooperative game theory

Much focus on convergence to equilibria, less on dynamic performance.

50 years old idea: Dual decomposition

$$\min_{x,y,z,w} [V_1(x,y) + V_2(x,z) + V_3(x,w)]$$

$$= \max_{p,q} \min_{x,y,z,w} [V_1(x,y) + V_2(x,z) + V_3(x,w) + p(x_1 - x) + q(x_2 - x_3)]$$

The optimum is a Nash equilibrium of the following game:

The three computers try to minimize their respective costs

- Computer 1: $\min_{x_1} [V_1(x_1, y) + px_1]$
- Computer 2: $\min_{x_2} [V_2(x_2, z) - px_2 + qx_2]$
- Computer 3: $\min_{x_3} [V_3(x_3, w) - qx_3]$

while the “market makers” try to maximize their payoffs

- Between computer 1 and 2: $\max_{p} [p(x_1 - x_2)]$
- Between computer 2 and 3: $\max_{q} [q(x_2 - x_3)]$
Potential game

The three computers try to minimize the potential function
\[
V_1(x_1,y) + V_2(x_2,z) + V_3(x_3,u) + p(x_1 - x_2) + q(x_2 - x_3)
\]
while the market makers try to maximize it.
Finding a Nash equilibrium (where no player has an incentive to change strategy) is greatly simplified by existence of a potential function.

Decentralized Bounds on Suboptimality

Given any \(p, q, \bar{x}, \bar{y}, \bar{z}, \bar{w}\), the distributed test
\[
\begin{align*}
V_1(\bar{x}, \bar{y}) + p\bar{x} &\leq \alpha \min_{x,y} [V_1(x, y) + px] \\
V_2(\bar{x}, \bar{z}) - p\bar{x} + q\bar{z} &\leq \alpha \min_{x,z} [V_2(x, z) - px + qx] \\
V_3(\bar{x}, \bar{w}) - q\bar{z} &\leq \alpha \min_{x,w} [V_3(x, w) - qx]
\end{align*}
\]
implies that the globally optimal cost \(J^*\) is bounded as
\[
V_1(\bar{x}, \bar{y}) + V_2(\bar{x}, \bar{z}) + V_3(\bar{x}, \bar{w}) \leq \alpha \min_{x,y,z,w} [V_1(x, y) + V_2(x, z) + V_3(x, w)]
\]
Proof: Add both sides up!

The saddle point algorithm

Update in gradient direction:
- Computer 1:
  \[
  \dot{x}_1 = -\partial V_1/\partial x - p
  \]
- Computer 1 and 2:
  \[
  \dot{x}_2 = -\partial V_2/\partial x + p - q
  \]
- Computer 2:
  \[
  \dot{z} = -\partial V_2/\partial z
  \]
- Computer 2 and 3:
  \[
  \dot{x}_3 = -\partial V_3/\partial x + q
  \]
- Computer 3:
  \[
  \dot{w} = -\partial V_3/\partial w
  \]
Globally convergent if \(V_1\) convex! [Arrow, Hurwicz, Usawa 1958]
Lyapunov function: \(V = x_1^2 + x_2^2 + x_3^2 + y^2 + z^2 + u^2 + p^2 + q^2\)

What do we achieve?

- Performance criteria for individual nodes
- Suboptimality bounds indicate where things went wrong
- Prices show the relative importance of different terms
- Sparsity structure useful for efficient computations

Outline

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  - Distributed synthesis

Decomposing the Cost Function

\[
\begin{align*}
\max p \min u_i \sum_i E \left[ \ell_i(x_i(t), u_i(t)) + 2\sum_j (p_{ij})^T(x_j - v_{ij}) \right] \\
= \max p \min u_i \sum_i E \left[ \ell_i(x_i(t), u_i(t)) - 2\sum_j (p_{ij})^T v_{ij} + 2\sum_j (p_{ji})^T x_i \right]
\end{align*}
\]
so agent \(i\) should minimize the stationary value of what he expects others to pay him
\[
E \left[ \ell_i(x_i(t), u_i(t)) - 2\sum_j (p_{ij})^T v_{ij}(t) + 2\sum_j (p_{ji})^T x_i(t) \right]
\]
his own cost
\[
2\sum_j (p_{ij})^T v_{ij}(t)
\]
what he pays others

A General Optimal Control Problem

Minimize \(V(u) = E \sum_i \ell_i(x_i(t), u_i(t))\)
subject to
\[
\begin{align*}
x_1(t + 1) &= f_1(x_1, v_{1j}, u_1, w_1) \\
&\vdots \\
x_n(t + 1) &= f_n(x_n, v_{nj}, u_n, w_n)
\end{align*}
\]
where
\[
v_{ij} = x_j
\]
holds for all \(i, j\).

Distributed Verification

\[
\max p \min u_i \sum_i E \left[ \ell_i(x_i(t), u_i(t)) - 2\sum_j (p_{ij})^T v_{ij} + 2\sum_j (p_{ji})^T x_i \right]
\]
Each agent \(i\) makes the comparison
\[
E f_i(\bar{x}_i, \bar{u}_i, \bar{x}_j, \bar{p}) \leq \alpha \min_{u_i, v_{ij}, w_i} E f_i(x_i, u_i, v_{ij}, \bar{p})
\]
Actual cost in node \(i\)\n
If no actual cost exceeds the expected cost by more than 10%, then the global cost is within 10% from optimal.
Theorem on Verification

Consider control laws \(\dot{u}_i = \mu_i(x)\) and stationary solutions to
\[\begin{align*}
\dot{x}_i(t + 1) &= f_i(x_i, x_j, \mu_j(x), w_i)
\end{align*}\]
where \(w_i(t)\) is stationary white noise. If \(a \geq 0\), then \(I\) implies \(II\):

\(I\) There exists \(\bar{p} = \bar{p}(x)\) satisfying
\[\begin{align*}
E_i(\dot{x}_i, \bar{u}_i, \bar{p}) &\leq a \min_{x, u, \bar{v}, \bar{w}} E_i(x, u, \bar{v}, \bar{w})
\end{align*}\]
when minimizing over stationary solutions to
\[\begin{align*}
x_i(t + 1) &= f_i(x_i, u_i, \bar{v}_i, w_i)
\end{align*}\]
\(II\) \(\sum E_i(\dot{x}_i, u_i) \leq a \min \sum E_i(x_i, u_i)\) when minimizing over stationary solutions to
\[\begin{align*}
x_i(t + 1) &= f_i(x_i, u_i, \bar{v}_i, w_i) \\
x_j(t + 1) &= f_j(x_j, u_j, \bar{v}_j, w_j)
\end{align*}\]
If dynamics is linear, \(\bar{t} \geq 0\) convex and \(a = 1\), then \(II\) implies \(I\).

A “Wind Farm” Case Study

Minimize \(V = E \sum_{i=1}^4 (|x_i|^2 + |u_i|^2)\)
\[\begin{align*}
x_1(t + 1) &= 0.6 x_1(t) + 0.1 x_2(t) + 0.1 x_4(t) + u_1(t) + w_1(t) \\
x_2(t + 1) &= 0.3 x_1(t) + 0.2 x_2(t) + 0.1 x_4(t) + u_2(t) + w_2(t) \\
x_3(t + 1) &= 0.1 x_1(t) + 0.3 x_2(t) + 0.1 x_4(t) + u_3(t) + w_3(t) \\
x_4(t + 1) &= 0.1 x_1(t) + 0.3 x_2(t) + 0.1 x_4(t) + u_4(t) + w_4(t)
\end{align*}\]

Today’s challenges: Distributed controller validation
Control synthesis

\[\begin{align*}
I_0 &= \begin{bmatrix} 0.34 & 0 & 0 & 0 \\
0 & 0.34 & 0 & 0 \\
0 & 0 & 0.34 & 0 \\
0 & 0 & 0 & 0.33 \\
\end{bmatrix} \quad I_1 = \begin{bmatrix} 0.34 & 0.07 & 0 & 0 \\
0 & 0.34 & 0.07 & 0 \\
0 & 0 & 0.34 & 0.07 \\
0 & 0 & 0 & 0.34 \\
\end{bmatrix}
\end{align*}\]

Decomposing the turbine dynamics

Minimize \(E \sum_{i=1}^4 (|x_i|^2 + |u_i|^2)\)
subject to
\[\begin{align*}
\begin{bmatrix} x_1^* \\
x_2^* \\
x_3^* \\
x_4^* \\
\end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix} x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} + \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0.1 \\
0 & 0.1 & 0 & 0 & 0.1 \\
0 & 0 & 0.1 & 0 & 0.1 \\
0 & 0 & 0 & 0.1 & 0.1 \\
\end{bmatrix} \begin{bmatrix} u_1 \\
u_2 \\
u_3 \\
u_4 \\
\end{bmatrix} \\
\begin{bmatrix} v_{12} \\
v_{21} \\
v_{23} \\
v_{24} \\
v_{41} \\
\end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
\end{align*}\]

Problem solved by the first turbine

\[\begin{align*}
x_1 &= p_{12} \\
p_{12} &= p_{21}
\end{align*}\]

Minimize \(E(|x_1|^2 + |u_1|^2 + 2p_{12}v_{12} - 2p_{21}x_1)\)
when \(x_1^* = 0.6x_1 + 0.1v_{12} + u_1 + w_1\)

Test for suboptimality:
\[\begin{align*}
E(|x_1|^2 + |u_1|^2 + 2p_{12}v_{12} - 2p_{21}x_1) &\leq a \min_{x_1, u_1} E(|x_1|^2 + |u_1|^2 + 2p_{12}v_{12} - 2p_{21}x_1)
\end{align*}\]

Validation Using Centralized Model

The variance \(E \sum_{i=1}^4 (|x_i|^2 + |u_i|^2)\) for the optimal centralized controller becomes
\[V_c = 4.9904\]
while the values for the decentralized approximations become
\[V_0 = 5.2999 \quad V_1 = 4.9917\]
These numbers were calculated using a global model.

We will next use dual decomposition to see that the control laws can be both validated and synthesized in a distributed way.

Problem solved by the first turbine

\[\begin{align*}
x_1 &= p_{12} \\
p_{12} &= p_{21}
\end{align*}\]

Minimize \(E(|x_1|^2 + |u_1|^2 + 2p_{12}v_{12} - 2p_{21}x_1)\)
when \(x_1^* = 0.6x_1 + 0.1v_{12} + u_1 + w_1\)

using measurements of \(x\) and knowledge of the joint spectral density of \(x, v, p_{13}\) and \(p_{21}\).

Notice: Once the price sequences \(p_{12}(t), p_{21}(t)\) are given, no other knowledge of the outside world is relevant. However, since future prices are usually not available, knowledge of other states can be useful for price prediction.

Performance degradation due to decentralization

\[\begin{align*}
\begin{bmatrix} 0.24 & 0.34 & 0.34 & 0.34 \\
0 & 0.24 & 0.34 & 0.34 \\
0.36 & 0.34 & 0.24 & 0.34 \\
0.36 & 0.34 & 0.34 & 0.24 \\
\end{bmatrix} L_s = \begin{bmatrix} 0.26 & 0.28 & 0.28 & 0.28 \\
0.26 & 0.28 & 0.28 & 0.28 \\
0.26 & 0.28 & 0.28 & 0.28 \\
0.26 & 0.28 & 0.28 & 0.28 \\
\end{bmatrix} L_c
\end{align*}\]

Compare expected and actual costs for the two control laws:
\[u = -L_s x \quad \bar{p} = \bar{L}_x:\]
\[\begin{align*}
1.5647 &\leq 1.5350 \quad \alpha \\
0.9132 &\leq 0.8585 \quad \alpha \\
0.9132 &\leq 0.8585 \\
0.9132 &\leq 0.8585 \\
1.5647 &\leq 1.5350 \quad \alpha \\
1.5741 &\leq 1.5740 \quad \alpha \\
1.062 &\leq \alpha = 1.27 \\
V_0 &\leq \alpha = 1.0094
\end{align*}\]
This allows us to modify the control law in the gradient direction using correlation estimates from the
By the maximum principle, optimal solutions to
\[ E_{x^t} = \max_{p \in \bar{p}} \left[ Ax + \dot{Av} + Bu + w \right] \text{ with } v = Sx \]
Dynamic programming gives control law as well as prices:
\[ |x|^2 = \max_{p \in \bar{p}} \left[ Ax + \dot{Av} + Bu + |x|^2 + \frac{|u|^2}{2} - 2pt(v - Sx) \right] \]
\[ p(t) = \begin{bmatrix} p_{11}(t) \\ p_{21}(t) \\ p_{31}(t) \\ p_{41}(t) \end{bmatrix} = \begin{bmatrix} 0.0342 & 0.2574 & 0.0010 & 0.0002 \\ 0.5545 & 0.1013 & 0.0382 & 0.0038 \\ 0.0025 & 0.2755 & 0.0676 & 0.0364 \\ 0.0038 & 0.0382 & 0.1913 & 0.5545 \end{bmatrix} \]
\[ M = \begin{bmatrix} \sum \bar{m}_{11} & \bar{m}_{12} & \bar{m}_{13} & \bar{m}_{14} \\ \bar{m}_{21} & \bar{m}_{22} & \bar{m}_{23} & \bar{m}_{24} \\ \bar{m}_{31} & \bar{m}_{32} & \bar{m}_{33} & \bar{m}_{34} \\ \bar{m}_{41} & \bar{m}_{42} & \bar{m}_{43} & \bar{m}_{44} \end{bmatrix} \]
The same \( p \) and \( u(t) = -Lx(t) \) as in classical solution.

Distributed gradient iteration for control law

By the maximum principle, optimal solutions to
\[ \min E(|x|^2 + |u|^2) \]
\[ x^* = Ax + \dot{Av} + Bu + w \text{ and } v = Sx \]
must minimize the Hamiltonian
\[ E(|x|^2 + |u|^2 + 2p_{12}v_{12} - 2p_{21}x_1) \]
This allows us to modify the control law
\[ u_1 = [I_{11} \ I_{12}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]
in the gradient direction using correlation estimates from the time interval \( t = 1, \ldots, T \).

Gradient iteration for the wind park

```
cost = 5.3183
L =
0.0366 0.0411 0 0 0
0.0386 0.0623 0.0546 0 0
0 0.0555 0.0686 0.0544 0
0 0 0.0554 0.0620 0.0405
0 0 0 0.0385 0.0363```

```
cost = 7.4944
L =
0.0138 0.0195 0 0 0
0.0162 0.0283 0.0294 0 0
0 0.0264 0.0333 0.0294 0
0 0 0.0264 0.0283 0.0195
0 0 0 0.0162 0.0138```

```
cost = 4.4277
L =
0.0709 0.0629 0 0 0
0.0666 0.1025 0.0749 0 0
0 0.0853 0.1070 0.0744 0
0 0 0.0851 0.1016 0.0611
0 0 0 0.0662 0.0697```
Gradient iteration for the wind park

$$\text{cost} = 3.9476$$

$$L = \begin{bmatrix} 0.1187 & 0.0812 & 0 & 0 & 0 \\ 0.0987 & 0.1494 & 0.0885 & 0 & 0 \\ 0 & 0.1146 & 0.1509 & 0.0879 & 0 \\ 0 & 0 & 0.1144 & 0.1479 & 0.0777 \\ 0 & 0 & 0 & 0.0976 & 0.1155 \end{bmatrix}$$

Gradient iteration for the wind park

$$\text{cost} = 3.6674$$

$$L = \begin{bmatrix} 0.1820 & 0.0903 & 0 & 0 & 0 \\ 0.1324 & 0.2041 & 0.0920 & 0 & 0 \\ 0 & 0.1419 & 0.2032 & 0.0917 & 0 \\ 0 & 0 & 0.1416 & 0.2023 & 0.0853 \\ 0 & 0 & 0 & 0.1296 & 0.1743 \end{bmatrix}$$

Gradient iteration for the wind park

$$\text{cost} = 3.5166$$

$$L = \begin{bmatrix} 0.2654 & 0.0777 & 0 & 0 & 0 \\ 0.1611 & 0.2684 & 0.0755 & 0 & 0 \\ 0 & 0.1607 & 0.2674 & 0.0759 & 0 \\ 0 & 0 & 0.1604 & 0.2672 & 0.0731 \\ 0 & 0 & 0 & 0.1549 & 0.2479 \end{bmatrix}$$

Gradient iteration for the wind park

$$\text{cost} = 3.4732$$

$$L = \begin{bmatrix} 0.2347 & 0.0393 & 0 & 0 & 0 \\ 0.1152 & 0.2363 & 0.0449 & 0 & 0 \\ 0 & 0.1187 & 0.2393 & 0.0444 & 0 \\ 0 & 0 & 0.1189 & 0.2369 & 0.0410 \\ 0 & 0 & 0 & 0.1103 & 0.2131 \end{bmatrix}$$

Convergence rate versus state dimension

For a fixed number of iterations and fixed sparsity structure of $L, M$, the computational cost grows linearly with $n$!

Conclusions

We have seen dynamic dual decomposition used for

- Distributed validation
- Distributed synthesis

Benefits to be obtained

- Reduced complexity
- Control structure reflects plant structure
- Flexibility and robustness

We have the tools to deal with dynamics!

Welcome to join the efforts!

Much (most) remains to be done and much is happening already at this conference!

See [Rantzer CDC07]
[Rantzer ACC09] covers much of this lecture. Working paper on www.control.lth.se/user/anders.rantzer

Lund University funds postdocs and will also hire new faculty members to complement the competence of our current staff.
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