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Englund, Jonas

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PO Box 117
221 00 Lund
+46 46-222 00 00

Large Scale Computations for Cracks with Corners

Jonas Englund[1]

[1]Numerical Analysis, Centre for Mathematical Sciences, Lund University, Box 118, SE-221 00, Lund, Sweden. E-mail: jonase@maths.lth.se

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Abstract. The problem of computing the stress field around cracks with many corners or branches is considered for a planar linearly elastic material. One can formulate the problem as an integral equation of Fredholm second kind. To resolve the stress field around corners, implementations with basis functions based on a Williams expansion have been tested. Large scale numerical examples are presented for cracks with up to a thousand corners. Typically, the number of points needed for a given accuracy is decreased by 80% and the number of iterations by 30% when the present algorithm is compared to an algorithm without basis functions. Convergence tests indicate that the implementation with basis functions achieves relative errors for stress intensity factors of less than 10^{-7} even for the largest geometries.

Introduction

The present paper is concerned with computing stress intensity factors for cracks with corners. Crack geometries with corners are common in engineering structures. In the literature many different techniques to solve the problem have been investigated, see references in [1, 2].

A common approach in linear elasticity is to use an integral equation formulation. In the present paper a Fredholm integral equation of the second kind is used. Such an equation can be very stable. It is possible to achieve linear complexity both in memory and time with the help of the fast multipole method. Since some of the involved integrals in the integral equation must interpreted in a Cauchy principal value sense, care must be taken. See Ref. [1].

It has been known for a long time how the stress field behaves asymptotically around a corner of a linearly elastic specimen. Due to the difficulty of implementation, this fact has only been used with partial success by earlier authors. In the present paper Williams' theory in combination with analytic function theory is used successfully. For a detailed description of how to use Williams' results, see [2].

Theory

The Equation. Consider a linearly elastic infinite region, D, containing a single crack L. In the elastic region, Airy's stress function $W(x,y)$ can be represented as $W(x,y) = \Re\mathrm{e}\{\bar{z}\varphi(z) + \chi(z)\}$, where $\varphi(z)$ and $\chi(z)$ are analytic functions of the complex variable $z = x + \mathrm{i}y$. Introduce $\Phi(z) = \varphi'(z)$ and $\Psi(z) = \chi''(z)$ and represent them as the Cauchy type integrals

$$\begin{aligned}
\Phi(z) &= \frac{1}{2\pi\mathrm{i}} \int_L \frac{\Omega(\tau)\rho(\tau)\,\mathrm{d}\tau}{\tau - z} + \frac{a}{2}, \quad z \in D \setminus L, \\
\Psi(z) &= -\frac{1}{2\pi\mathrm{i}} \int_L \frac{\overline{\Omega(\tau)\rho(\tau)}\,\mathrm{d}\bar{\tau}}{\tau - z} - \frac{1}{2\pi\mathrm{i}} \int_L \frac{\bar{\tau}\Omega(\tau)\rho(\tau)\,\mathrm{d}\tau}{(\tau - z)^2} + b, \quad z \in D \setminus L,
\end{aligned}$$

where a is a real constant and b a complex constant determined by the applied external forces. The function $\rho(z)$ is a weight, on L given by $\rho(z) = (\sqrt{(z-\gamma_s)(z-\gamma_e)})^{-1/2}$, where γ_s is the starting point and γ_e the endpoint of L. With the representation used above it is possible to rewrite the elastostatic differential equation as the integral equation

$$\begin{cases} (M_1\rho - M_3)\Omega(z) = \frac{\bar{n}}{n}\bar{b} - a \quad z \in L, \\ Q\Omega(z) = 0, \end{cases} \tag{1}$$

where $n = n(z)$ is the unit normal to L. Zero traction along the crack, and continuity of traction across L is implied by (1). The operator M_1 is defined by

$$M_1\left(\Omega\rho\right)(z) = \frac{1}{\pi\mathrm{i}} \int_L \frac{\Omega(\tau)\rho(\tau)\,\mathrm{d}\tau}{\tau - z}, \quad z \in L. \tag{2}$$

The operator M_3 is a compact integral operator. For a definition see [1]. The second equation of (1) contains the closure condition and says that the integral of $\Omega(z)\rho(z)$ along L should be equal to zero. The function $\Omega(z)\rho(z)$ has the physical interpretation $\Omega(z)\rho(z) = -\mathrm{d}/\mathrm{d}z(\delta u + \mathrm{i}\delta v)$. The system in (1) is not optimal from a numerical point of view. Fortunately it is easy to reduce the original system of equations into a single equation which is of Fredholm second kind [1]

$$(I - M_1\rho^{-1}M_3)\Omega(z) = M_1\rho^{-1}\left(\frac{\bar{n}}{n}\bar{b} - a\right), \quad z \in L. \tag{3}$$

From the solution of (3) it is easy to compute the normalized stress intensity factors $F = F_{\mathrm{I}} + \mathrm{i}F_{\mathrm{II}}$, see [1].

Williams Expansion. On the crack, the function $\Omega(z)\rho(z)$ can be expressed as the difference between the right and the left limiting value of the analytic function $\Phi(z)$. Elementary jump relations give

$$\begin{aligned} \Omega(z)\rho(z) &= \Phi^+(z) - \Phi^-(z), \quad z \in L, & (4) \\ M_1(\Omega\rho)(z) &= \Phi^+(z) + \Phi^-(z) - a, \quad z \in L, & (5) \end{aligned}$$

where $\Phi^+(z)$ is the limit from the left and $\Phi^-(z)$ is the limit from the right relative to the orientation of the crack. Using (1), and (5) one can see that

$$M_3\Omega(z) + \frac{\bar{n}}{n}\bar{b} = \Phi^+(z) + \Phi^-(z), \quad z \in L, \tag{6}$$

and that solving Eq. (3) is formally equivalent to solving

$$\left(\Omega(z) - M_1\rho^{-1}\left(M_3\Omega(z) + \frac{\bar{n}}{n}\bar{b}\right)\right) = -M_1\rho^{-1}a, \quad z \in L. \tag{7}$$

In the work by Helsing and Jonsson [2], $\Phi^+(z)$ is expanded with the help of variable separation (which might be called a Williams expansion) around the wedge of a notched specimen. This makes it possible to represent $\Phi^+(z)$ with certain basis functions. In the present paper the idea of [2] is utilized and extended to geometries with cracks. A crack with a corner can be seen as a combination of an inward and an outward notch. One should therefore expand both $\Phi^+(z)$ and $\Phi^-(z)$. In what follows we have disregarded the limit of $\Phi(z)$ belonging to the smaller of the angles between two consecutive crack legs. For the geometries treated here this limitation will only have a small impact.

Now, consider the action of M_3. From (4) it is clear that $\Omega(z)\rho(z)$ can be expanded using a Williams expansion. This has the consequence that we can use exactly the same approach here as in [2] when applying M_3. From (6) one sees that the quantities that $M_1\rho^{-1}$ operates on in (7), left-hand side, also can be expanded using that same Williams expansion. From the right-hand side of Eq. (7) it is clear that one also has to include a constant basis function.

Placement of Panels. The integral equation (3) is discretized using a Nyström scheme. In this scheme composite quadrature is used. In order to improve accuracy, an adaptive approach is taken. The corner quadrature panels extend across the corner, and the adaptive refinement is shown in Fig. 1. The panels closest to a corner panel are split into new, shorter panels. Let N_{sp} be the total number of subdivisions of the panels closest to a corner. Denote the length of a panel not close to a corner by k. Each leg of a corner panel is placed so that it has length $k/2^{N_{sp}-2}$. Panels closest to a corner panel are subdivided N_{sp} times. This procedure makes the corner panel small in

comparison to the whole crack. It also makes sure that the distance from the corner to a non-corner panel is at least as long as the length of the non-corner panel.

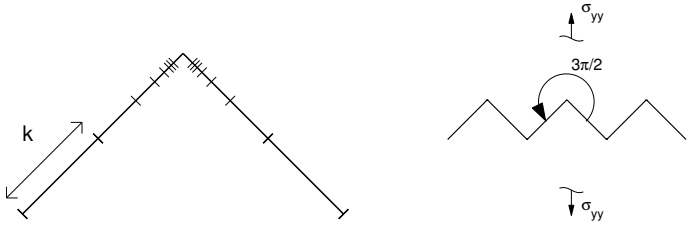

Figure 1: The left image shows panel placement close to a corner when $N_{sp} = 5$. The right image shows an example of the general geometry investigated in this paper. The crack in the figure has five corners.

Action of $M_1\rho^{-1}$. The operator $M_1\rho^{-1}$ operating on basis functions is precomputed using an adaptive algorithm. The integrals are computed in a local coordinate system. Once these integrals have been computed, a mapping from the local coordinate system to the global coordinate system gives the values of $M_1\rho^{-1}(M_3\Omega(z) + \overline{nb}/n)$. For details concerning the mapping and the local coordinate system see [2]. Using the parameter s for z and parameter t for τ, we have the following relations in the local coordinate system: $z = \mathrm{i}sn_z$ and $\tau = \mathrm{i}tn_\tau$. In the local coordinate system, n_z denotes the unit normal at z and n_τ the unit normal at τ. Integrals of the following type are needed for all basis functions $z^{\lambda_n - 1}$

$$M_1\rho^{-1}\tau^{\lambda_n-1}(z) = D_{1n}\int_{-1}^{0}\frac{t^{\lambda_n-1} - s^{\lambda_n-1}\,\mathrm{d}t}{\rho(\mathrm{i}tn_\tau)(t-s)} + \int_0^1 \frac{D_{2n}t^{\lambda_n-1} - D_{1n}s^{\lambda_n-1}\,\mathrm{d}t}{\rho(\mathrm{i}tn_\tau)(t - e^{-i\beta}s)}, \tag{8}$$

where $D_{1n} = \exp\{i(\beta/2 - \pi)(\lambda_n - 1)\}$ and $D_{2n} = \exp\{-i\beta(\lambda_n - 1)/2\}$. By β is meant the angle on the left of the crack (relative to the orientation) between two consecutive crack parts.

In the integrals above $\rho(z)$ appears in the integrand. Because of the adaptive algorithm used for the computation of the integrals we need to evaluate $\rho(z)$ at many different points of a corner panel. This is done by polynomial interpolation of degree eight.

Computing the integrals of Eq. (8) is costly. Therefore we will take another approach when z is far away from the corner, where the integrand is smoother. For such points we use temporary interpolation, and integration is performed

on the temporary mesh. With the exception that we have to include $\rho(z)$, this procedure is identical to the one used in [2].

Numerical Results

All numerical experiments have been performed an a SunBlade 100 workstation. All codes were written in Fortran and compiled using Sun's f77 compiler. The system of equations from the Nyström scheme was solved using the iterative solver GMRES with the residual tolerance set to $8 \cdot 10^{-16}$. On all quadrature panels, except for the corner panels and the panels at the endpoints of the crack, 8-point Gauss-Legendre quadrature has been used. On the end-panels Gauss-Jacobi quadrature with exponent $-1/2$ was used. In all experiments load of magnitude unity was applied in the direction indicated in Fig. 1. Eight basis functions from the expansion of the limit of $\Phi(z)$ were used in the calculations.

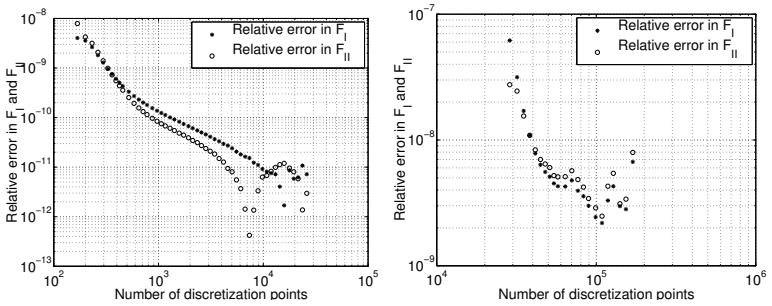

Figure 2: Left: a convergence test for a crack with one corner. Right: for a crack with 199 corners.

Convergence tests were made for cracks with different numbers of corners, see Fig. 2 and Table 1. The geometry used is the same as shown in Fig. 1. The geometry dependent factor of the normalized stress intensity factor was chosen as half the length of a line connecting the endpoints of the crack. The slope of the line between 500 and 3000 discretization points in the left part of Fig. 2 is close to one. The reason for this is that the absolute value of λ_n for the first term of the Williams expansion not included in the algorithm is equal to one.

Two typical examples of the improvements with the present algorithm compared to an algorithm without basis functions [1] will now be given. For one corner the algorithm in [1] needed about 30 iterations for convergence compared to 20 now. The present algorithm converges to eight digits for one corner using only 168 discretizations points. The same accuracy using the algorithm without basis functions needed about 1000 points. Furthermore,

the present algorithm increases the achievable accuracy for one corner with at least two digits. Another example is for a crack with 1999 corners. The present algorithm converged to five digits in 25 iterations, using $2.5 \cdot 10^7$ discretization points. The algorithm without basis functions converged to five digits in 40 iterations using $8.5 \cdot 10^5$ discretization points.

Table 1: Stress intensity factors for the left crack tip of a crack with the shape shown in Fig. 1

Number of corners	F_{I}	F_{II}	Number of iterations
1	0.5207675522	0.6411159455	20
19	0.697413839	0.417375537	34
199	0.76539692	0.34303978	34
1999	0.7883016	0.3188449	31

Conclusions

The use of a Williams expansion has been shown to improve a simpler algorithm without basis functions [1]. To achieve a certain accuracy the needed number of discretization points and iterations is decreased. The maximum achievable accuracy is increased.

The fact that the approach with basis functions around corners works for kinked cracks indicates that it also works for branched cracks. A corner of a branched crack is then simply viewed as a combination of three notches. An investigation of branched cracks is in progress.

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