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# Observer-Based Strictly Positive Real (SPR) Variable Structure Output Feedback Control

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**Abstract:** This paper considers switching output feedback control of linear systems and variable-structure systems. Theory for stability analysis and design for a class of observer-based feedback control systems is presented. A circle-criterion approach can be used to design an observer-based state feedback control which yields a closed-loop system with specified robustness characteristics. The approach is relevant for variable structure system design with preservation of stability when switching feedback control or sliding mode control is introduced in the feedback loop. It is shown that there exists a Lyapunov function valid over the total operating range and this Lyapunov function has also interpretation as a storage function of passivity-based control. The Lyapunov function can be found by solving a Lyapunov equation, which also generates variable structure switching surfaces. Important applications are to be found in variable structure systems with high robustness requirements.

## Introduction

For switching output feedback control in variable structure systems [1], [2], the high-gain feedback implies a challenge to stability and a variety of techniques have been considered—*e.g.*, high-gain observers [3], [4], state observer [2], [5], or other dynamic feedback [6], [7], [8], [9]. Some research on static output feedback control with geometric conditions on sliding mode and control design were presented by Żak and Hui [11], Yan and Dai [12] and Edwards *et al.* [13], [14]. Dynamic output feedback was employed in the contribution of Kwan *et al.* [15], Yan *et al.* [16] who also pointed out limitations with the static feedback approaches.

Outside the field of variable-structure systems, the qualitative analysis of transfer-function properties and its relationship to stability analysis has a long history back to [17]. As for the absolute stability problem of nonlinear feedback systems, the starting point is the Lur'e problem described by [17], [18], [19], [20], [21], [22]. Kalman demonstrated that linear-quadratic regulators satisfy a certain frequency domain inequality with a certain degree of robustness [20]. Molander and Willems introduced a synthesis of state feedback control laws with a specified gain and phase margin [23], later extended to dynamic output feedback [24], [25]. In the context of hybrid control, a related result was presented [26].

From a stability point of view, there are important differences among static output feedback and dynamic output feedback. A non-trivial problem in variable structure system is to apply the inherently high-gain feedback to systems without access to full state measurement. A drawback of solutions based on state estimation is that the separation principle is not longer true for nonlinear systems [27]. Variable structure systems using sliding-mode control are prone to stability problems related to the usage of high-gain feedback and limitations of strictly positive realness. A theoretical challenge is to single out classes of systems for which the separation principle holds—*i.e.*, where substitution of state feedback by feedback of estimated states can be done without impending instability [27]. This paper deals with application and extension of such observer-based control to variable structure systems with high-gain feedback.

*Problem Formulation*

Consider a linear time-invariant finite-dimensional system and a time-variant nonlinear feedback

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{1}$$

$$z = Sx, \quad u = -\psi(z,t), \quad z \in \mathbb{R}^m \tag{2}$$

with $(A, B)$ controllable. These authors showed that the closed-loop system is stable for certain combinations of the matrix $S$ and a condition of a cone-bounded function $\psi(\cdot,\cdot)$. Kokotović and Sussman [28] introduced the notion of feedback positive real (FPR) transfer functions with properties similar to those of Molander and Willems [23] with a global stabilizability condition formulated for $(A, B)$ controllable

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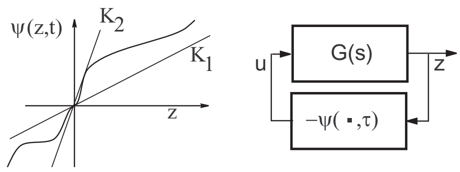


Fig. 1. Sector-bounded nonlinearity and feedback interconnection in Molander-Willems approach.

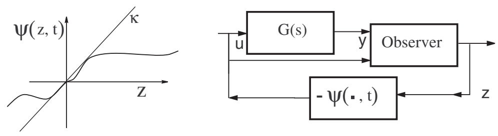


Fig. 2. Sector-bounded nonlinearity and feedback interconnection after loop transformation to sector $[0,\kappa]$ and with observer feedback included.

and $\psi(\cdot,\cdot)$ smooth. Molander and Willems provided a design procedure for $S$—*i.e.*, design for nonlinear state-feedback control—with specified gain margin [23]. They made a characterization of the conditions for stability with a high gain margin of feedback systems of the structure

$$\dot{x} = Ax + Bu, \quad z = Sx, \quad u = -\psi(Sx,t) \tag{3}$$

with $\psi(\cdot,\cdot)$ enclosed in a sector $[K_1, K_2]$—see Fig. 1. The following procedure was suggested to find a state-feedback vector $S$ such that the closed-loop system will tolerate any $\psi(\cdot,\cdot)$ enclosed in a sector $[K_1,\infty)$:

- Pick a matrix $Q = Q^T > 0$ such that $(A,Q)$ is observable;
- Solve the Riccati equation $PA + A^TP - 2K_1PBB^TP + Q = 0$ for $P$. Take $S = B^TP$ and formulate a Lyapunov function $V(x) = x^TPx$.

The algorithm provides a robustness result which fulfills an FPR condition—*i.e.*, the stability condition will be that of an SPR condition on $S(sI - A + K_1BS)^{-1}B$, the design procedure being based on a circle-criterion proof and involving a solution of a Riccati equation. The Molander-Willems equations may be summarized as a Yakubovich-Kalman-Popov matrix equation

$$\mathcal{P}=\begin{bmatrix} P & 0\\ 0 & I_m\end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A-K_1BS & B\\ -S & 0\end{bmatrix}, \quad P>0$$

$$\mathcal{Q}=\begin{bmatrix} Q & 0\\ 0 & 0\end{bmatrix}, \quad Q>0, \quad -\mathcal{Q}=\mathcal{P}\mathcal{A}+\mathcal{A}^T\mathcal{P}, \tag{4}$$

$$\frac{dV}{dt}=\begin{bmatrix} x\\ u+Sx\end{bmatrix}^T \left[\mathcal{P}\mathcal{A}+\mathcal{A}^T\mathcal{P}\right]\begin{bmatrix} x\\ u+Sx\end{bmatrix} \tag{5}$$

$$=-\begin{bmatrix} x\\ u+Sx\end{bmatrix}^T \mathcal{Q}\begin{bmatrix} x\\ u+Sx\end{bmatrix} < 0, \|x\| \neq 0$$

which is actually a special Lyapunov equation with properties described elsewhere [24], [25]. In the context of observer-based state feedback control, however, the controllability condition presents a problem of application, prompting extension of strictly positive realness to cases including state estimation [25], [29], [30].

The purpose of this paper is to generalize the application of SPR/FPR design with application to circle criterion to the case of switching output feedback control using observer-based state feedback control.

## Problem Formulation

Assume a problem formulation with a linear system and nonlinear feedback of cone-bounded nonlinear variation described by the function $\psi(\cdot,t)$

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x\in\mathbb{R}^n, \quad u,\sigma\in\mathbb{R}^m && (6)\\ \sigma &= Sx, \quad u = -\psi(\sigma,t), && (7)\\ 0 &\geq \psi^T(\sigma,t)(\psi(\sigma,t)-\kappa\sigma), \quad 0<\kappa\in\mathbb{R}^{m\times m} && (8)\end{aligned}$$

As a Lyapunov function candidate, the circle criterion applies the Lyapunov function candidate

$$V(x) = x^TPx \tag{9}$$

which for $P = P^T > 0$ satisfies requirements on 'positivity', 'radial growth', 'continuity' and 'differentiability'. Suppose that $\dot{x} = Ax + Bu, y = Cx$ and $P = P^T > 0$ and $A^TP + PA = -S^TS - \epsilon P$, $PB = C^T\kappa - S^TR$ and define $V(x) = x^TPx$. Then, if $u = -\psi(\sigma,t)$ where $\psi(\cdot,\cdot)$ fulfills the cone condition

$$\psi^T(\sigma,t)(\psi(\sigma,t)-\kappa\sigma) \leq 0 \tag{10}$$

we have for $V(x) = x^TPx$ that for $\|x\| \neq 0$

$$\begin{aligned} \frac{dV}{dt} &= x^TP\dot{x} + \dot{x}^TPx && (11)\\ &\leq x^T(A^TP+PA)x + 2x^TPBu\\ &- 2\psi^T(\sigma,t)(\psi(\sigma,t)-\kappa\sigma)\\ &\leq -(W_1x - W_2\psi(\sigma,t))^T(W_1x - W_2\psi(\sigma,t))\\ &- \epsilon x^TPx \leq -\epsilon x^TPx < 0\end{aligned}$$

The circle criterion predicts asymptotic stability of the closed-loop system if the derivative $dV/dt$ along the system trajectories

$$\frac{dV}{dt} = \dot{x}^TPx + x^TP\dot{x} \leq -\begin{bmatrix} x\\ \psi(\sigma,t)\end{bmatrix}^T W \begin{bmatrix} x\\ \psi(\sigma,t)\end{bmatrix} \tag{12}$$

with

$$\begin{aligned} W &= \begin{bmatrix} W_1^T\\ W_2^T\end{bmatrix}\begin{bmatrix} W_1 & W_2\end{bmatrix}\\ &= \begin{bmatrix} -(A^TP+PA) & PB - C^T\kappa^T\\ B^TP-\kappa C & 2I_m\end{bmatrix} && (13)\end{aligned}$$

It is sufficient to make the matrix $W$ positive definite so that stability can be guaranteed by making the derivative $dV/dt$ negative definite—*i.e.*,

$$\frac{dV}{dt} = -\begin{bmatrix} x \\ \psi(\sigma,t) \end{bmatrix}^T W \begin{bmatrix} x \\ \psi(\sigma,t) \end{bmatrix} < 0, \quad \|x\| \neq 0 \tag{14}$$

The circle criterion assures an asymptotically stable solution for the time-varying case under the assumption that $\psi(\cdot,t)$ belongs to the cone $[0,\infty)$ and that $\inf_\omega \mathrm{Re}\ G(j\omega) > 0$. As guaranteed by the Yakubovich-Kalman-Popov (YKP) lemma, the existence of $W > 0$ leading to the stability condition $\dot{V} \leq 0$ holds under the fairly restrictive strictly positive real (SPR) [18], [19], [20]. Then, the system will be asymptotically stable and $\mathcal{L}_2$−stable as

$$\begin{aligned} 0 &\leq \int_0^T \epsilon x^T P x \leq \int_0^T -\dot{V}(x,t)dt \qquad (15) \\ &= V(x(0),0) - V(x(T),T) \qquad (16) \end{aligned}$$

When SPR (relative degree) and measurement conditions of some output $y = Cx$ prevent realization of $u = -Sx$, approximate control can be made with $u = -S\hat{x}$ for some state estimate $\hat{x}$. As $\hat{x} \neq x$, it is necessary to investigate whether some degradation in performance and stability may occur. To that purpose, introduce a full-order observer for the state vector $x$ so that

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x}) \tag{17}$$

where $K \in \mathbb{R}^{n\times m}$ is an observer-gain matrix that multiplies the estimation error. By substitution of actual, unmeasured states $x$ by estimated states $\hat{x}$ in the feedback, the system dynamics will be

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ KC & A-KC \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u \qquad (18) \\ y &= Cx, \quad \hat{\sigma} = S\hat{x}, \qquad (19) \\ u &= -\psi(\hat{\sigma},t) = -\psi(S\hat{x},t) \qquad (20) \end{aligned}$$

As the augmented system of control object and observer of Eqs. (18-20) will not be controllable—*i.e.*, the estimation error $\tilde{x} = \hat{x} - x$ will not be controllable from $u$. Thus, attempts of application of the Molander-Willems result to the observer-supported system (18–20) will fail due to violation of the controllability condition.

We will show that there exist Lyapunov functions that assure asymptotic stability for the closed-loop system of Eqs. (18–20).

*Lyapunov Design for Nonlinear Observer Feedback*

To the purpose of stability analysis, equip the state-space system with a new output $z$ formed by means of a full-order observer.

*Proposition 1 (Dynamic Feedback Circle Theorem):* For a nonlinear function $\psi(\cdot,\cdot)$ fulfilling the sector condition $\psi^T(\sigma,t)(\psi(\sigma,t) - \kappa\sigma) \leq 0$, $\kappa > 0$ and a linear time-invariant system $\dot{x} = Ax + Bu, y = Cx$ such that $(A,B)$ is controllable and $(A,C)$ is observable, there exist a full-order observer with observer gain $K$ and an observer state feedback $\hat{\sigma} = S\hat{x}$ with gain $S$ such that the closed-loop system

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ KC & A-KC \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u \qquad (21) \\ y &= Cx, \quad \hat{\sigma} = S\hat{x}, \qquad (22) \\ u &= -\psi(\hat{\sigma},t) = -\psi(S\hat{x},t) \qquad (23) \end{aligned}$$

is asymptotically stable. For this system, there exist matrices $P = P^T > 0$, $Q = Q^T > 0$ and a Lyapunov function

$$\begin{aligned} V(\xi) &= \xi^T P_0 \xi, \quad \xi = \begin{bmatrix} x \\ \hat{x} - x \end{bmatrix} \qquad (24) \\ \frac{dV}{dt} &= -\begin{bmatrix} \xi \\ S\hat{x} - \psi(\hat{\sigma},t) \end{bmatrix}^T Q_0 \begin{bmatrix} \xi \\ S\hat{x} - \psi(\hat{\sigma},t) \end{bmatrix} \\ &< 0, \quad \|x\| \neq 0 \qquad (25) \end{aligned}$$

*Proof:* —See [25]

Recently, it was shown that for $Q_0 > 0$ there exist solution $P_0 > 0$ and a constructive procedure was provided [25]. Actually, a solution satisfying the Yakubovich-Kalman-Popov may be obtained [31]. If $P$ is a solution to the Molander-Willems equation and $P_K$ is a weighting matrix for the Lyapunov function of the observer error dynamics $\dot{\tilde{x}} = (A - KC)\tilde{x}$ , then $P_0$ may be composed as

$$\begin{aligned} P_0 &= \begin{bmatrix} P & P \\ P & \mu P_K \end{bmatrix}, \qquad (26) \\ Q_0 &= \begin{bmatrix} Q + S^TRS & Q + PKC + S^TRS \\ Q + C^TK^TP + S^TRS & \mu Q_K \end{bmatrix} \end{aligned}$$

for $\mu > 0$ and sufficiently large in magnitude where

$$-Q_K = P_K(A-KC) + (A-KC)^T P_K \tag{27}$$

Moreover, $P_0$ satisfies Eq. (5) with the Yakubovich-Kalman-Popov equations

$$P_0A_0 + A_0^T P_0 = -Q_0, \quad P_0B_0 = C_0^T \tag{28}$$

for the system matrices

$$\begin{aligned} A_0 &= \begin{bmatrix} A-BS & -BS \\ 0 & A-KC \end{bmatrix}, \quad B_0 = \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad (29) \\ C_0 &= \begin{bmatrix} C & C \end{bmatrix} \qquad (30) \end{aligned}$$

Note that there exist solutions $P_0 > 0$ also for $(A_0, B_0)$ not controllable. Thus, assume

$$\begin{aligned} V(\xi) &= \xi^T P_0 \xi \qquad (31) \\ \frac{dV}{dt} &= \frac{\partial V}{\partial \xi}\dot{\xi} = 2\xi^T P_0(A_0\xi + B_0 u) \qquad (32) \\ u &= -R^{-1}\mathrm{sgn}(\frac{\partial V}{\partial \xi}) = -R^{-1}\mathrm{sgn}(\hat{\sigma}) \qquad (33) \\ \hat{\sigma} &= S\hat{x} = B^T P\hat{x} \qquad (34) \end{aligned}$$

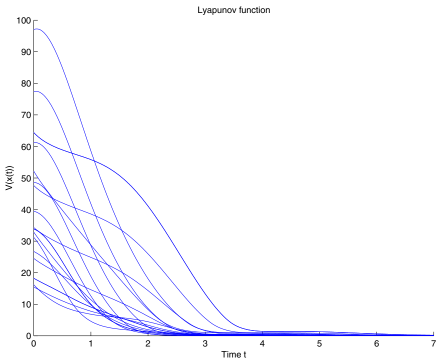


Fig. 3. Lyapunov function trajectories from switching output feedback control of double integrator dynamics.

for $P$ solving the Riccati equation

$$PA + A^TP + Q - PBR^{-1}B^TP = 0 \tag{35}$$

The closed-loop system will satisfy

$$\begin{aligned} \frac{dV}{dt} &= 2\xi^T P_0(A_0\xi + B_0u) && (36)\\ &= \xi^T(P_0A_0 + A_0P_0)\xi^T && (37)\\ &- 2\xi^T P_0B_0R^{-1}\text{sgn}(B_0^T P_0\xi) && (38)\end{aligned}$$

which permits asymptotically stable switching output feedback control.

### Example 1

Consider observer-based feedback control of a system with the double integrator dynamics

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix} x + \begin{bmatrix} 1\\ 0\end{bmatrix} u && (39)\\ \dot{\hat{x}} &= \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix} \hat{x} + \begin{bmatrix} 1\\ 0\end{bmatrix} u + K(y - C\hat{x}),\ K = \begin{bmatrix} 1\\ 1\end{bmatrix} && (40)\\ y &= Cx = \begin{bmatrix} 0 & 2\end{bmatrix} x && (41)\\ u &= -\text{sgn}(S\hat{x}),\quad S = \begin{bmatrix} 1.7321 & 1.000\end{bmatrix} && (42)\end{aligned}$$

where $S = B^TP$ has been calculated based on the weighting matrices

$$\begin{aligned} Q &= Q_K = I_2,\quad R = 1,\quad \mu = 100 && (43)\\ P &= \begin{bmatrix} 1.732 & 1.000\\ 1.000 & 1.732\end{bmatrix},\ P_K = \begin{bmatrix} 0.875 & -0.500\\ -0.500 & 0.750\end{bmatrix} && \\ P_0 &= \begin{bmatrix} 1.732 & 1.000 & 1.732 & 1.000\\ 1.000 & 1.732 & 1.000 & 1.732\\ 1.732 & 1.000 & 87.5 & -50.0\\ 1.000 & 1.732 & -50.0 & 75.0\end{bmatrix} && (44)\\ Q_0 &= \begin{bmatrix} 4.000 & 1.732 & 4.000 & 7.196\\ 1.732 & 2.000 & 1.732 & 7.464\\ 4.000 & 1.732 & 106.00 & 3.464\\ 7.196 & 7.464 & 3.464 & 102.0\end{bmatrix} && (45)\\ V(\xi) &= \xi^T P\xi && (46)\\ P_0B_0 &= \begin{bmatrix} S^T\\ S^T\end{bmatrix},\ P_0A_0 + A_0^TP_0 = -Q_0, && (47)\end{aligned}$$

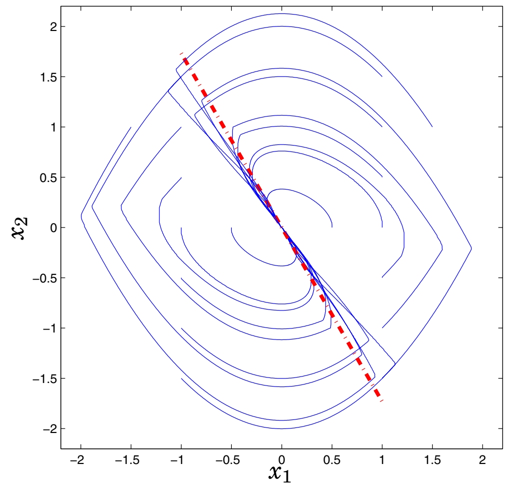


Fig. 4. Phase portrait of states $x_1$ and $x_2$ in Example 1 under closed-loop control using state estimation. The switching surface $\sigma = Sx$ is indicated (*dash-dot red*). As the state-space dimension is greater than two, the trajectories may exhibit intersections.

Figures 3-5 demonstrate the asymptotic stability achieved for observer-supported high-gain feedback.

### Dissipativity and Passivity

Following [32] and [33], a dynamical system is said to be dissipative if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}^+$, called a storage function such that for all $t_0$, $t_1$, $x \in \mathbb{R}^n$ and $u \in \mathbb{U}$, $y \in \mathbb{Y}$, $t_1 \geq t_0$ satisfying the inequality

$$V(x(t_0)) + \int_{t_0}^{t_1} w(u,y)dt \geq V(x(t_1)) \tag{48}$$

where $w(u,y)$ is a real-valued function called the supply rate—*i.e.*, $w : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$. Strict dissipativity holds if the inequality (48) is a strict inequality. For $V(x) = x^TPx$, $P = P^T > 0$ and

$$w(u,z) = \begin{bmatrix} z\\ u\end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12}\\ Q_{12}^T & Q_{22}\end{bmatrix} \begin{bmatrix} z\\ u\end{bmatrix} \tag{49}$$

with derivative

$$\frac{dV(x)}{dt} = \begin{bmatrix} x\\ u\end{bmatrix}^T \begin{bmatrix} PA + A^TP & PB\\ B^TP & 0\end{bmatrix} \begin{bmatrix} x\\ u\end{bmatrix} \tag{50}$$

the system is dissipative with respect to the supply rate $w(u,z)$ if

$$\frac{dV}{dt} \leq -w(u,z) = -\begin{bmatrix} z\\ u\end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12}\\ Q_{12}^T & Q_{22}\end{bmatrix} \begin{bmatrix} z\\ u\end{bmatrix}$$

$$\int_{t_0}^{t_1} w(u,z)dt \geq V(x(t_1)) - V(x(t_0)) \tag{51}$$

Moreover, the system is said to be passive if there is storage function $V$ and coefficients $\epsilon \geq 0, \delta \geq 0, \rho \geq 0$ and supply rate $w = u^Tz$ satifying

$$u^Tz \geq \frac{\partial V}{\partial x}\frac{dx}{dt} + \epsilon u^Tu + \delta z^Tz + \rho x^Tx, \tag{52}$$

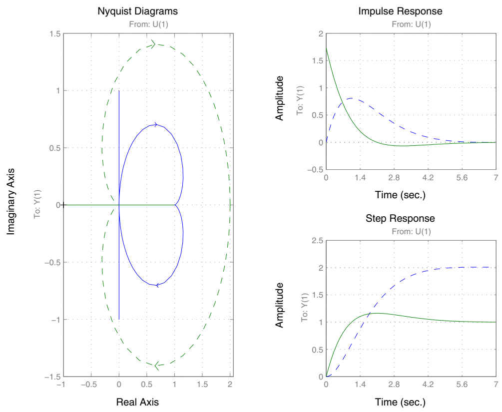


Fig. 5. Nyquist diagram of Example 1 depicting $G_0(s) = C(sI - (A - BS))^{-1}B)$ (*dashed line*) and loop transfer function $H_0(s) = S(sI - (A - BS))^{-1}B)$ (*solid line*) of observer-based feedback with SPR property.

The system is input strictly passive if $\epsilon > 0$, output strictly passive if $\delta > 0$ and state strictly passive if $\rho > 0$. For $V(x) = x^T P x$ and the system

$$\frac{d}{dt}\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} = \begin{bmatrix} A & 0 \\ KC & A - KC \end{bmatrix}\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u \quad (53)$$

$$y = Cx, \quad \widehat{\sigma} = S\widehat{x}, \quad u = -\psi(\widehat{\sigma}, t) \quad (54)$$

we have for the input-output map from $u$ to $\sigma$ that

$$\begin{aligned} 2u^T\widehat{\sigma} - \frac{\partial V}{\partial x}\frac{dx}{dt} &= -\begin{bmatrix} x \\ u \end{bmatrix}^T \left[\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P}\right]\begin{bmatrix} x \\ u \end{bmatrix} \quad (55) \\ &= \begin{bmatrix} x \\ u \end{bmatrix}^T \mathcal{Q}\begin{bmatrix} x \\ u \end{bmatrix} \geq 0 \text{ for } \mathcal{Q} > 0 \end{aligned}$$

and

$$\int_{t_0}^{t_1} 2u^T z dt = \int_{t_0}^{t_1} \dot{V}(x(t))dt + \int_{t_0}^{t_1} \begin{bmatrix} x \\ u \end{bmatrix}^T \mathcal{Q}\begin{bmatrix} x \\ u \end{bmatrix} dt$$

Thus, the dissipative properties of the observer-based systems appear to be formally similar to those of state-feedback control.

## Discussion

Doyle and Stein [34] have pointed out the brittle robustness of a state-feedback control design modified by replacement of state feedback by observer feedback. Here, the stability and robustness results have been extended to class of variable structure control based on state estimation. The algorithmic approach is a sequential design of weighting matrices for Lyapunov functions for the SPR/FPR feedback control and for the observer design. The stability analysis and Lyapunov designs apply with or without the Lur'e term added as required in the Popov criterion and the circle criterion, respectively. Moreover, the nominal pole assignment for control and for observer dynamics can be made independently—a property similar to that of the separation principle.

The approach to modification of the relative-degree and SPR properties is related to the 'parallel feedforward' as proposed in the context of adaptive control [35]. Another related idea is passification by means of shunting introduced by Fradkov [36]. All these approaches represent derivation of a loop-transfer function with SPR properties for a control object without SPR properties by means of dynamic extensions or observers. Arcak and Kokotović made observer design for systems with monotone sector nonlinearities in the unmeasured states [37]. Interconnection of a multivariable sector nonlinearity and a linear system was made so that observer matrices could be calculated to satisfy the circle criterion. Subsequent control design was made by backstepping design.

Apart from its relevance to observer-based feedback control, we expect that the method will have application to hybrid systems with switching feedback control and to high-gain feedback systems controlled by logic-based switching devices. The circle criterion design provides implicit choices of switching surfaces

$$\widehat{\sigma}(\widehat{x}) = S\widehat{x} \quad (56)$$

In many cases, by 'inverse optimality' it is also possible to choose other switching surfaces corresponding to the solution of some Riccati equation provided that the SPR condition be satisfied in the transfer function from $u$ to $S\widehat{x}$ (though without SPR requirement on the transfer function from $u$ to $y$). An example is given in Fig. 6 where observer-based VSS control trajectories are shown for a switching surface $\widehat{\sigma}(\widehat{x}) = S\widehat{x}$.

## Conclusions

In this paper, some results on asymptotic stability of variable structure systems using dynamic output feedback and state estimation were established. The results extend previous results on dynamic output feedback for variable structure systems [6], [4], [7], [8], [15], [9], [16], [13]. Moreover, separation-like properties hold. The stability and robustness results of [23], [25] and [28] have been extended to a case with observer-based feedback control with resulting non-minimal loop transfer functions. A design procedure to find full-state observers and Lyapunov functions is provided. A new feature for switching output feedback is that one Lyapunov function serves for stability analysis for all switching modes.

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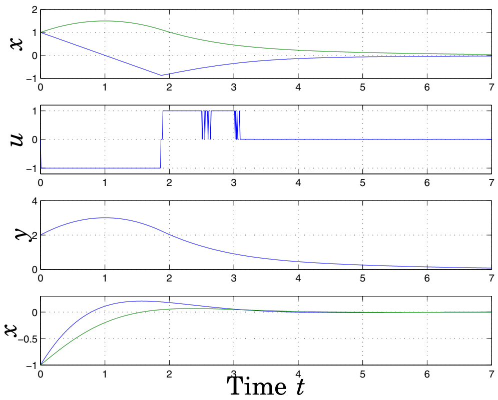


Fig. 6. VSS control trajectories of a double integrator using an observer-based switching surface $\sigma(\hat{x}) = S\hat{x}$. State trajectories $x$, input $u$, output $y$ and state-estimation errors $\tilde{x} = \hat{x} - x$ vs. time.


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