Close Approximations of Minimum Rectangular Coverings

Joachim Gudmundsson Christos Levcopoulos*

Department of Computer Science
Lund University, Box 118, S-221 00 Lund, Sweden

Abstract. We consider the problem of covering arbitrary polygons with rectangles. The rectangles must lie entirely within the polygon. (This requires that the interior angles of the polygon are all greater than or equal to 90 degrees.) We want to cover the polygon with as few rectangles as possible. This problem has an application in fabricating masks for integrated circuits.
In this paper we will describe the first polynomial algorithm, guaranteeing an $O(\log n)$ approximation factor, provided that the n vertices of the input polygon are given as polynomially bounded integer coordinates. By the same technique we also obtain the first algorithm producing a covering which is within a constant factor of the optimal in exponential time (compared to the doubly-exponential known before).

1 Introduction

The problem of covering polygons with various types of simpler polygons has a number of important practical applications [5, 6] and has received considerable attention from a theoretical perspective. One application for this problem is the fabrication of VLSI chips. According to [5], a common method for fabricating VLSI chips is the optical method of the automatic blockflasher, which exposes rectangles of, in practice, almost any size and any orientation. In order to minimize the cost of the fabrication it is desirable to cover the polygonal area of each layer of the circuit with as few rectangles as possible. For more information about the VLSI fabrication process see [5].

O'Rourke and Supowit [12] showed that the problems of covering polygons with a minimum number of convex polygons, star-shaped polygons or spiral polygons are all NP-hard, if the polygon contains holes. The rectilinear case, i.e. when the polygons, as well as the rectangles, have sides that are vertical or horizontal, has been treated in several papers [6, 4, 9]. Culberson and Reckhow [3] showed this case to be NP-hard even when the given polygon was hole-free. Berman and DasGupta [1] showed that if the rectilinear polygon has holes, the problem is also MAXSNP-hard. In the more general, non-rectilinear, case it is not known whether the optimal solution can be computed even in exponential

* E-mail: {christos,joakim}@dna.lth.se

time. Using the proof technique in [11] (see also [2] and [10]), Tarski's decidability results can be applied to prove that the decision version of the problem is computable, but no upper bounds better than $2^{2^{O(n)}}$ was known for its execution time. Hegedüs [5] implemented a program (the so-called GENCOV-algorithm) which covers general polygons with rectangles and presented some empirical results concerning its performance. The results show that the algorithm is rather good in practice, but no theoretical analysis is presented in [5]. Another approximation algorithm was presented in [7], and with refined analysis in [8], which produces $O(n \log n + \mu(P))$ rectangles to cover an arbitrary coverable polygon P in time $O(n \log n + \mu(P))$, where n is the number of vertices of P and $\mu(P)$ is the minimum number of rectangles required to cover P. (A polygon can be covered by rectangles iff it has no acute interior angles.)

A problem with the above algorithm is that, even when $\mu(P) = O(n^{0.5+\epsilon})$, where ϵ is any fixed constant greater than zero, the algorithm may produce a covering with $\Omega(n \log n)$ rectangles, as is very far from being optimal, as is shown in [8]. To get a reasonable approximation in the case when $\mu(P) \ll n$ we need a different approach. In order to find a better covering in the rectilinear case one has used the approach of finding a superset of rectangles which includes the rectangles in an optimal covering. If we could find such a set for the general case we could easily translate the polygon-covering problem into the set-covering problem. The main difference between the rectilinear case and the general case is that there are only $O(n^2)$ possible maximal rectangles in the rectilinear case versus infinite many possible maximal rectangles in the general case. Instead we will in this paper construct a set of rectangles, denoted C_R, and prove that C_R includes a covering which is within a constant factor of the optimal. In fact we show that C_R possesses an even stronger property, namely that every possible rectangle within a polygon P can be covered by a constant number of rectangles in C_R (Lemmas 4-6).

By reducing the original problem into the set-covering problem, by rectangles from C_R, we can find a covering which is within a constant factor of the optimal in exponential time (details in subsection 4.1). We also present the first polynomial algorithm, guaranteeing an $O(\log n)$ approximation factor, provided that the n vertices of the polygon are given as polynomially bounded integer coordinates.

2 Defining the pool of rectangles

Let P be an arbitrary polygon, possibly with polygonal holes. The shortest distance, within P, between two non-incident edges in P is denoted $s(P)$ and the largest distance between two points in P is denoted $d(P)$. When it is clear from the context which polygon we refer to, we shall simply write d and s instead of $s(P)$ and $d(P)$. Construct a $(6d/s \times 6d/s)$ grid such that every square in the grid has length and height $s/6$, and P lies entirely within the grid, Fig. 1.

Lemma 1. *Let P be a polygon of n vertices, without any acute interior angles. Then it holds that $n = O((\frac{d}{s})^2)$.*

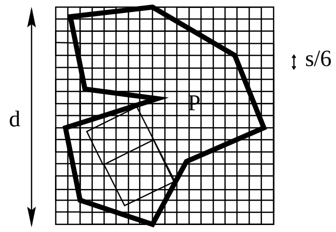

Fig. 1. The polygon P within the grid and two of the rectangles in C_R.

Proof. Let P be a polygon with n vertices. By definition of d we have that the area of P is less than $\frac{\pi \cdot d^2}{4}$. We know that the sum of all the interior angles of P is $180° \cdot (n-2)$. Now, construct for every vertex v_i, $1 \leq i \leq n$, a region, such that every point in P closer than $s/2$ to v_i belongs to the region $r(v_i)$, Fig. 1. According to the definition of s, we have that the shortest distance between two vertices of P is at least s, thus the regions are disjoint. We have that the total area covered by these regions is $(n-2) \cdot \frac{\pi \cdot s^2}{8}$, which is less than the area of P. Thus we have

$$(n-2)\frac{\pi \cdot s^2}{8} < \frac{\pi \cdot d^2}{4}, \quad \text{and hence} \quad n < 2 \cdot \left(\frac{d}{s}\right)^2 + 2.$$

Let C_P be a set of points, such that:

1. every vertex of P is a point in C_P,
2. every crossing between P's perimeter and the grid is a point in C_P, and
3. every crossing in the grid is a point in C_P.

The number of points in C_P is $O((d/s)^3)$, according to Lemma 1. We construct a set of rectangles, C_R, by first inserting into C_R each maximal rectangle which lies within P and which has a side with endpoints in C_P, Fig. 1. Thus the number of rectangles in C_R is now $O((d/s)^6)$.

We will need to extend the number of rectangles in C_R, with two sets of rectangles. First we add the following set.

For every pair of vertices v_i, v_j of P, where v_i is visible from v_j, let s_{ij} be the maximal segment, within P, that passes through or ends at v_i and v_j. For every segment s_{ij} denote every crossing between s_{ij} and a segment of the grid plus v_i and v_j by $p_1, \ldots, p_q$, (thus $q = O(d/s)$). For every possible pair of points (p_l, p_k), $1 \leq l, k \leq q$, on a segment s_{ij}, we insert into C_R all maximal rectangles within P (there are 0,1 or 2 such rectangles) which have a side with endpoints in p_l and p_k. The total number of rectangles inserted in this step is at most $n^2 \cdot (d/s)^2 = O((d/s)^6))$.

The second set of rectangles added to C_R is needed to cover the rectangles in Lemma 5, *Case d*. The rest of this section is devoted to the construction of these special rectangles. We will need the following definitions:

Definition 2. The shortest distance between a rectangle T, within P, and a point p, with a perpendicular projection on one of T's long sides (or any side if T is a square), is denoted the *v-distance* between T and p. If p doesn't have a perpendicular projection on T's long sides then the v-distance is infinite.

The v-distance between a rectangle T, within a polygon P, and P's perimeter is the shortest v-distance between T and a point on P's perimeter.

In order to facilitate an easier description, we assume w.l.o.g. that the two longer sides of a rectangle T, within P, lie horizontally, that is, we rotate T, P and the grid until T's long sides lie horizontally. Let A, B, C and D be the corners of T in a counter-clockwise order, where A and B are the endpoints of T's lowest long-side, Fig. 6. Thus the segments AB and CD are at least as long as AD and BC.

Definition 3. Let T be any rectangle within P with short sides of length at most $5s/6$ and long sides of length at least $4s$, whose opposite long sides touch two vertices v_1 respectively v_2, where v_1 touches AB and v_2 touches CD.

If there exist some other vertices of P, to the left or right of v_1 and v_2, closer than the v-distance $s/6$ from T, then let v_3 be the left- or rightmost vertex of P such that the v-distance between T and v_3 is less than $s/6$. We may assume, according to symmetry, that v_3 lies to the left of v_1 and v_2, and that v_3 lies closer to CD than to AB. Rotate P,T and the grid such that v_3 and v_2 have the same y-coordinate, Fig. 2.

Let Q be the vertical segment of length s with upper endpoint at v_3. Let H be a segment within P between v_2 and Q, such that the crossing between Q and H is as close as possible to v_3, and let, if possible, L be a segment within P between v_1 and Q such that L's and H's extensions cross to the right of v_1 and v_2 and the crossing between Q and L has the smallest possible y-coordinate. Let s' be the longest of the two vertical segments between v_2 and L (or its extension), respectively v_1 and H (or its extension), and let d' be the segment on Q between H and L. If there exists a trapezoid whose boundaries, the two vertical segments s' and d', and the segments of H and L between s' and d', is well-defined, as described above, then the trapezoid is denoted a *3v-trapezoid*. A 3v-trapezoid is hole-free since the shortest diagonal of a hole in P is at least s and $|d'| \leq s$. □

For every 3v-trapezoid t in P, construct at most $\lceil |d'|/|s'| \rceil$ rectangles with short-sides of length $|s'|$ where every rectangle has one endpoint on s''s lower endpoint and one on d'. We place the rectangles in such a way that a rectangle's short side, closest to d', touches both d' and the upper long-side of the rectangle below, Fig. 3b. The rectangles will at least cover the part of t at distance more than $s/2$ from s' and d', Fig. 3b. Add these rectangles to C_R. These rectangles can always be constructed since a 3v-trapezoid is hole-free. Rotate the 3v-trapezoid in such a way that s' and d' are vertical. Let l_1 be the horizontal distance between d' and the crossing of H's and L's extensions, and let l_2 be the horizontal distance between s' and the crossing of H's and L's extensions, Fig. 3a. Note that $l_2 > s/2$ since the shortest distance between v_1 and v_2 is s and the vertical distance is at most $5s/6$. So for every 3v-trapezoid in P we

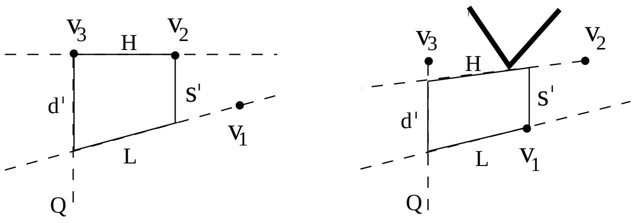

Fig. 2. Two $3v$-trapezoids.

will construct at most $|d'|/|s'|$ rectangles, and by simple trigonometry we have: $|d'|/|s'|=(l_1/l_2)<(d/l_2)<2{\cdot}d/s$. Since P consists of n vertices there can be at most $O(n^3)$ $3v$-trapezoids in P, and since $n=O((d/s)^2)$, according to Lemma 1, we will add at most $n^3{\cdot}2\cdot(d/s)=O((d/s)^7)$ rectangles to C_R. So the total number of rectangles in C_R will be $O((d/s)^7)$. It is easily seen that C_R can be constructed in time linear with respect to the number of rectangles in C_R.

3 Covering Rectangles

In Lemmas 4-6 we will show that every rectangle T within an arbitrary polygon P can be covered by a constant number of rectangles in C_R. For convenience we divide the rectangles into three cases according to their shape.

Lemma 4. *Every rectangle T, within P, whose longest side is shorter than $s/2$ can be covered by a constant number of rectangles in C_R.*

Proof. We have two cases:

Case a: The shortest distance between T and P's perimeter is at least $s/\sqrt{18}$.

The distance between T and P's perimeter is at least as long as the diagonal of a square in the grid, so T will never lie within the same square as P's perimeter.

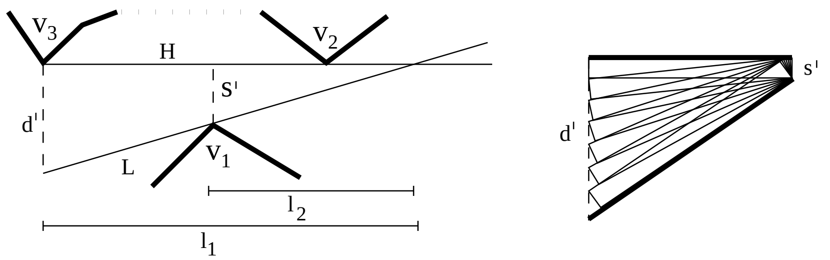

Fig. 3. (a) Finding a $3v$-trapezoid. (b) A $3v$-trapezoid covered by rectangles.

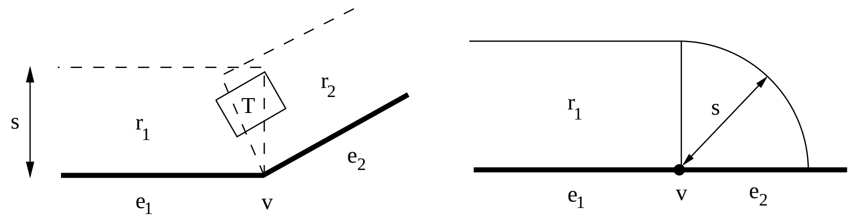

Fig. 4. (a) T is covered by r_1 and r_2. (b) The quadrant bounds the area which T may overlap.

Since the diagonal of T is at most $1/\sqrt{2}s$, T can at most overlap six rows or columns of squares in the grid. And since every side of a square is the base of a rectangle in C_R, the rectangle T can be covered by at most six rectangles in C_R.

Case b: The rectangle T lies closer than $s/\sqrt{18}$ from P's perimeter.

Let e_1 be an edge closest to T. Let r_1 be the rectangle in C_R with base e_1. If r_1 covers T then we are finished, otherwise let e_2 be the edge incident to e_1 that lies on the same side of e_1 as the part of T that isn't covered by r_1, Fig. 4a. The vertex connecting e_1 and e_2 is denoted v. Now, two cases can occur:

(i) The angle (e_1, e_2) is less than or equal to 180 degrees. Since T lies closer than $s/\sqrt{18}$ from e_1, the largest distance between a point in T and e_1 is $\frac{s}{\sqrt{18}}+\frac{s}{\sqrt{2}}=\frac{\sqrt{8}}{3}s<s$. The height of r_1 is at least s. That is, the part of T with a perpendicular projection on e_1 is covered by r_1. It remains to show that there exists a rectangle r_2 in C_R that covers the rest of T. Since every point in T lies within distance less than s from e_1, the upper right quadrant of a circle with its center in v and radius s bounds the area outside r_1 where T partly can overlap, Fig. 4b. The rectangle r_2 covers the entire area described by the quadrant, so the remaining part of T that isn't covered by r_1 is covered by r_2.

(ii) If the interior angle (e_1, e_2) is greater than 180 degrees then partition T into nine rectangles, $T_1, \ldots, T_9$ of equal size, such that the longest side of each sub-rectangle T_i, $1 \leq i \leq 9$, is at most of length $s/6$. Now we can cover every sub-rectangle T_i separately. Every sub-rectangle that lies at least $s/\sqrt{18}$ from P's perimeter is covered according to *Case a*. Let T' be any of the sub-rectangles, T_i, that lies closer than $s/\sqrt{18}$ from P's perimeter. Let v_1 (v_2) and v be the vertices of e_1 (e_2). Recall that on every edge and its extensions there exist $O((d/s)^2)$ rectangles in C_R. Let r_1 (r_2) be the maximal rectangle in C_R with base on e_1 (e_2) and its extension, such that one corner lies in v_1 (v_2) and one corner on the extension of e_1 (e_2) between $s/2$ and $(s/2+s/\sqrt{18})$ from v, Fig. 5. Since the shortest distance from v to a non-incident edge is s, it follows that the height of r_1 and r_2 is at least $s/2$. Also, since the length of T's diagonal is at most $s/\sqrt{18}$, we have that a point in T' lies within distance not greater than $(s/\sqrt{18}+s/\sqrt{18})<s/2$ from e_1. Thus r_1 and r_2 covers T', and T can be covered by a constant number of rectangles in C_R.

When the rectangles are thin we need to cover the rectangles in different ways depending on the v-distance between the rectangle and P's perimeter.

Lemma 5. *Every rectangle T, within P, whose shortest side is at most of length $s/2$, can be covered by a constant number of rectangles in C_R.*

Proof. If $|AB|\leq 4s$ then partition T into eight equal rectangles, whose long sides are of length at most $s/2$, which are then covered according to Lemma 4. Otherwise insert four points A', B', C' and D' on T's perimeter, where A' and B' are points on AB at a distance s from A, respectively B, and C' and D' are points on CD at a distance s from C, respectively D. Partition T into three rectangles by inserting two segments $A'D'$ and $B'C'$ in T. We denote these partitions, from left to right, T_l, T_m and T_r, Fig. 6. Partition T_l, respectively T_r, into two equal rectangles with long sides of length $s/2$ and cover these according to Lemma 4. We distinguish four cases:

Case a: The v-distance between T and P's perimeter is at least $s/6$.

We claim that there exists a rectangle r in C_R that covers T_m. To prove this claim, let p_1 be the point in C_P with largest y-coordinate below or on AA', such that the horizontal distance from p_1 to the extension of DA is between $s/6$ and $5s/6$, and symmetrically, let p_2 be the point in C_P with largest y-coordinate below or on BB', such that the horizontal distance from p_2 to the extension of CB is between $s/6$ and $5s/6$, Fig. 6.

Let r be the rectangle in C_R with base p_1p_2. Since the vertical differenc between p_1 and p_2 is less than $s/6$ and since $|p_1p_2|>s$, it follows that the slope of r's left and right side is greater than six and, hence r covers T_m.

Case b: The v-distance between one of T's long sides and P's perimeter is less than $s/6$, and the v-distance between the opposite long side and P's perimeter is at least $s/6$.

We can w.l.o.g. assume that AB is the long side of T with shortest v-distance to P's perimeter. Let p_1 be the point in C_P with largest y-coordinate below or on AA', such that the horizontal distance from p_1 to the extension of DA is between $s/6$ and $5s/6$. Symmetrically, let p_2 be the point in C_P with largest

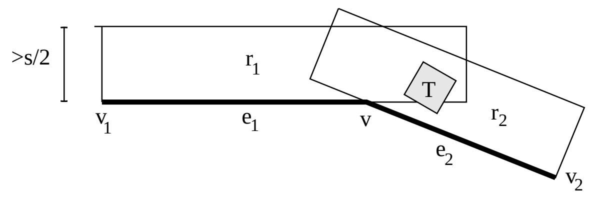

Fig. 5. r_1 and r_2 cover every small rectangle that lies closer than $s/\sqrt{18}$ from P's perimeter.

y-coordinate below or on BB', such that the horizontal distance from p_2 to the extension of CB is between $s/6$ and $5s/6$, Fig. 6.

If p_1 and p_2 can be connected by a straight line segment within P, then let r be the rectangle in C_R with base p_1p_2. Since the vertical differenc between p_1 and p_2 is less than $s/6$ and since $|p_1p_2|>s$, the slope of r's left and right side is greater than six, hence r covers T_m, Fig. 7(left).

Otherwise, if p_1 and p_2 can't be connected by a straight line segment within P, then there exists a vertex v, the closest to AB and lying between AB and p_1p_2. If $|p_1v|<s$ or $|p_2v|<s$ then we partition the area in T with x-coordinates between p_1 and v (or p_2 and v), into two rectangles which are then covered according to Lemma 4. We may assume w.l.o.g. that $|p_1v|$ is longer than s. We will now show that the uncovered area of T_m can be covered by a rectangle in C_R. If $|p_2v|$ is also longer than s we do the corresponding procedure to the right of v. The region of T at most $s/4$ to the left and right of v is covered according to Lemma 4, Fig. 7(right). Recall that there exist rectangles in C_R that lie on the visibility lines and their extensions for every two vertices of T. Thus, there exists a rectangle r in C_R with lower right corner in v and with lower left corner, denoted p', between 0 and $s/6$ below AA' and with horizontal distance between $s/6$ and $5s/6$ from AD. Note that p' and p_1 don't necessarily coincide. Since the vertical differenc between p' and v is less than $s/6$, and since $|p'v|>s$, the slope of r's left and right side is greater than six, thus r covers the uncovered area of T_m to the left of v, Fig. 7(right).

Before we show the case when the v-distance between both T's long sides and P's perimeter is less than $s/6$, we need to prove the following special case.

Case c: Two diagonally opposite corners of T touch a vertex in P.

We can assume w.l.o.g. that the vertex v_1 coincides with A, and the vertex v_2 coincides with C, Fig. 8. Let p_1 be the point in C_P with largest y-coordinate below or on BB', such that the horizontal distance from p_1 to the extension of BC is between $s/4$ and s. Let r_1 be the rectangle in C_R with base v_1p_1 and let α be the angle (v_1p_1, AB). If r_1 covers T_m then we are finished, otherwise there exists a vertex v_3 above CD that prevented r_1 from covering T_m. Recall that there exist rectangles in C_R that lie on the visibility lines and their extensions for every two vertices of T. Thus there exists a rectangle r_2 in C_R with upper

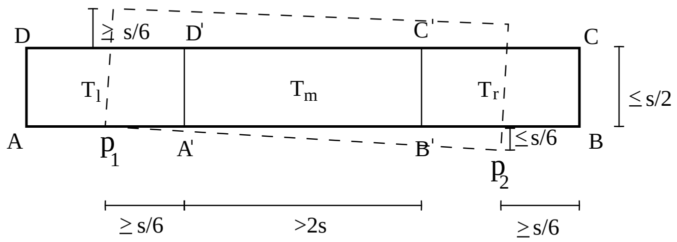

Fig. 6. T partitioned into sub-rectangles and, the rectangle with base p_1, p_2 covers T_m.

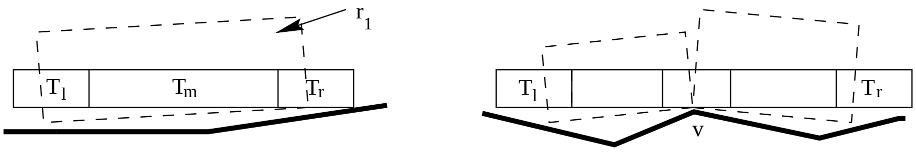

Fig. 7. Covering T according to Lemma 5, Case b.

right corner at v_2 and upper left corner, denoted p_2, between 0 and $s/6$ above DD' and at horizontal distance between 0 and $s/4$ from AD, Fig. 8(right). Let β be the angle (v_2p_2, CD). We know that $\beta<\alpha$, since β is at most equal to the angle (v_2v_3, CD), (i.e. v_3, v_2, D). Thus r_2 can't be stopped by any part of the perimeter above r_1's base, and it follows that r_2 covers T_m.

Case d: The v-distance between both T's long sides and P's perimeter is less than $s/6$.

Let v_2 be the vertex in P, above or on $C'D'$, with smallest v-distance to $C'D'$, and let v_1 be the vertex in P, below or on $A'B'$, with smallest v-distance to $A'B'$. We may assume w.l.o.g. that v_1 lies to the right of v_2 . The region of T between v_1 and v_2 is covered according to *Case c*, since the part of T between v_1 and v_2 can be expanded in such a way that v_1 and v_2 touch T's opposite long sides. Let T_l' be the uncovered region of T to the left of v_2 and to the right of T_l and let T_r' be the uncovered region of T to the right of v_1 and to the left of T_r. Assume that T_l''s both long sides lie closer to P's perimeter than $s/6$. We will now prove that it is possible to cover T_l' with a constant number of rectangles in C_R. If T_r''s both long sides lie closer to P's perimeter than $s/6$ then we do the corresponding procedure on T_r', otherwise we cover T_r' according to *Case a* or *b*.

Let v_3 be the leftmost vertex in P, such that the v-distance from v_3 to T_l' is less than $s/6$. Let Q be the vertical segment of length s with one endpoint in v_3, Fig. 9. The part of T_l' to the left of Q can be covered according to *Case a* or *b*, since the shortest v-distance from P's perimeter to one of T_l''s long sides

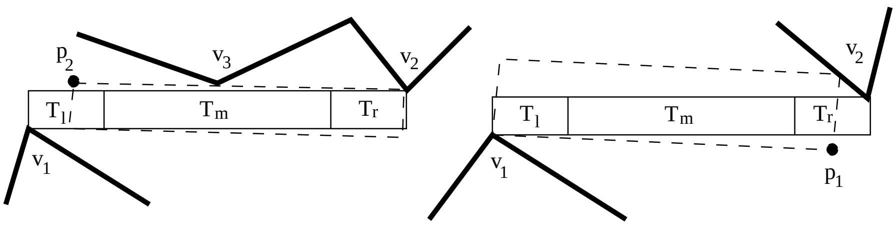

Fig. 8. T_m is covered by a rectangle in C_R.

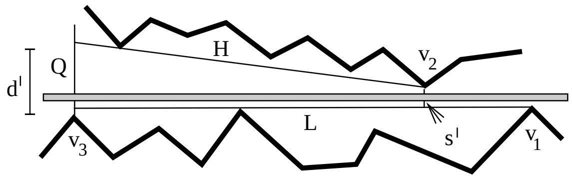

Fig. 9. A $3v$-trapezoid within P.

to the left of Q is greater than $s/6$. Let L be the lowest segment between v_1 and Q and let H be the highest segment between v_2 and Q, such that H's and L's extensions cross to the right of v_1. The segment on Q between H and L is denoted d'.

The shortest vertical segment from v_2 to L is denoted s'. The length of s' is the largest possible thickness of T. The two areas between s' and d' of T closer than $s/2$ from s' or d' are covered according to Lemma 4. Let t be the trapezoid bounded by the two parallel segments s' and d', and the part of the segments H and L between s' and d'. This trapezoid is a $3v$-trapezoid, according to Definition 3, and according to the definition of C_R, the uncovered area of t is entirely covered by rectangles in C_R. Therefore, since the thickness of these rectangles is s', the part of T in t can be covered by two of these rectangles in C_R. Thus the original rectangle T can be covered by a constant number of rectangles in C_R.

Lemma 6. *Every rectangle T, within a polygon P, whose shortest side is longer than $s/2$, can be covered by a constant number of rectangles in C_P.*

Proof. If $|AB| \leq 3s$ we partition T into six equal rectangles which are then covered according to Lemma 5. Otherwise partition T in such a way that we can cover the border of T, that is the region in T that lies at most s from T's perimeter, according to Lemma 5. Let T_m be the uncovered region of T. Let p_1 be the point in C_P with largest y-coordinate below or on the extension of T_m's base, such that p_1 lies between $s/4$ and $3s/4$ from AD, and let p_2 be the point in C_P with largest y-coordinate below or on the extension of T_m's base, such that p_2 lies between $s/4$ and $3s/4$ from BC. Let r be the rectangle in C_R with base p_1p_2. Since the vertical distance between p_1 and p_2 is less than $s/6$, according to the definition of C_P, and since $|p_1p_2|$ is greater than T_m's height, the rectangle r will cover T_m.

4 Two algorithms and their complexity

In Lemmas 4-6, we have shown that C_R includes a covering which is within a constant factor of the optimal. Thus it is possible to translate the original geometrical covering problem into the **set-covering problem**, as follows.

An instance (X, F) of the **set-covering problem** consists of a finite set X and a family F of subsets of X, such that every element of X belongs to at least one subset in F. We want to cover the polygon P with a minimum number of rectangles in C_R. Define X to be the set of all cells in the partition induced by the perimeters of all the rectangles in C_R, (thus $|X|{=}O(|C_R|^2)$), and we define F, such that every element $f{\in}F$ corresponds to a rectangle r in C_R, where f is the subset of X that is included in r. In the following two sections we will show how one can use this translation to find two algorithms for the polygon-covering problem by using two known set-covering algorithms. Before we show the algorithms we note the following observation:

Observation 7. *If the input polygon is given as integer coordinates in the universe* $[0..u]$ *then* $s{\geq}\frac{1}{\sqrt{2}\cdot u}$.

Proof. Let v_1 and v_2 be two vertices of P connected by an edge e, and let v' be an arbitrary vertex of P. We want to compute the shortest possible distance between v' and e. Assume that there exists a point p on e with the same y-coordinate as v', then the distance between p and v' is:

$$|v'.x - p.x| = |v'.x{-}(v_1.x{+}\frac{|v_1.x{-}v_2.x|}{|v_1.y{-}v_2.y|}(p.y{-}v_1.y))|.$$

Let $\alpha{=}|v_1.x{-}v_2.x|$, $\beta{=}|v_1.y{-}v_2.y|$ and let $\gamma{=}|p.y{-}v_1.y|$. Thus we have:

$$|v'.x - p.x| = |v'.x - (v_1.x{+}|\frac{\alpha}{\beta} \cdot \gamma|)|.$$

Since α, β and γ are in the universe $[0..u]$ and if $p \neq v'$, the horizontal distance between p and v' is:

$$|p.x - v'.x|{\geq}0 + \frac{1}{u} \cdot 1 = \frac{1}{u}$$

According to symmetry we will get the same result if p's and v''s x-coordinates are equal. Since e can cross the horizontal line $v'.x$ and the vertical line $v'.y$ at a distance $1/u$ from p, we have that the shortest distance, s, between v' and e is $\frac{1}{\sqrt{2}u}$, Fig. 10.

4.1 An exponential algorithm

This is a simple straight-forward algorithm which checks if there is any subset of k rectangles in C_R that cover P. If such a subset is found then we are finished, otherwise increment k and continue as described above. The initial value of k is 1 and from Lemmas 4-6 we know that C_R includes a covering which is within a constant factor of the optimal, thus $k{=}1,\ldots,O(\mu(P))$. Since there are no more than $((d/s)^7)^{O(\mu(P))}$ subsets of C_R with $O(\mu(P))$ rectangles, we can find a covering with $O(\mu(P))$ rectangles in time:

$$\sum_{k=1}^{O(\mu(P))} ((d/s)^7)^k{=}(d/s)^{O(\mu(P))}{=}2^{O(\mu(P)\cdot \log d/s)}.$$

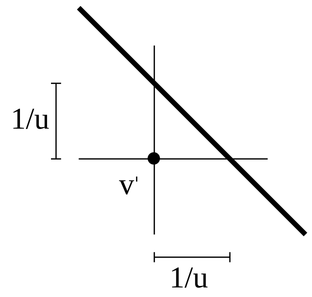

Fig. 10. The shortest distance between a vertex v' and an edge of P is $\frac{1}{\sqrt{2}u}$.

According to Theorem 1 in [7] it holds that $\mu(P){=}O(n\cdot \log d/s)$, thus we get the time complexity $2^{O(n\cdot(\log d/s)^2)}$. Hence by Observation 7 we obtain the following theorem:

Theorem 8. *When the vertices of the input polygon are given as integer coordinates in the range* $[0..u]$*, the algorithm described above will produce a covering within a constant factor of the optimal in* $2^{O(n\cdot(\log u)^2)}$ *time.*

4.2 A polynomial algorithm

A natural approach to find an approximation algorithm would be to use a known set-covering algorithm, for example a greedy algorithm. A Greedy-Set-Cover algorithm [13] can easily be implemented to run in time $O(|X||F|min(|X|,|F|))$, with a ratio bound of $(\ln|X|+1)$. Recall that $|X|{=}|C_R|^2$ and $F{=}C_R$.

If the input polygon P is given in integer coordinates, where the coordinates are in the universe $[0..u]$, the above greedy algorithm produces a covering which is within a logarithmic factor of the optimal, $(\ln|C_R|^2+1)$, in pseudo-polynomial time $O(((d/s)^7)^4){=}O((d/s)^{28})$. Since $s{\geq}\frac{1}{\sqrt{2}\cdot u}$ and $d{\leq}\sqrt{2}\cdot u$ we have that the time-complexity for the greedy algorithm is $O(u^{56})$. So we obtain the following:

Theorem 9. *When the n vertices of the input polygon are given as polynomially bounded integer coordinates, the algorithm described above will produce a covering with a $O(\log n)$ approximation factor in polynomial time.*

5 Lower Bounds

From Lemmas 4-6 we derived the following proposition:

Proposition 10. *There exists some constant, say c, such that for any polygon P there exists a finite set, C, of rectangles, lying within P, where every other rectangle in P can be covered by c rectangles from C.*

Two questions arise naturally in connection with these proofs:

a) What is the minimum constant c, for which the above proposition holds? That is, if we don't have any restrictions on C except that it should be finite?

b) How large does C have to be in the worst case, in terms of n, d and s, in order for the proposition to hold?

Therefore, in this section we give lower bounds for (*a*) and (*b*).

Theorem 11. *For all integers $n \geq 8$, there exists a polygon P_n with n vertices, such that for any finite set of rectangles, C lying inside P_n, there exists a rectangle T within P that can't be covered by less than six rectangles in C.*

Proof. To prove the theorem we consider a polygon P_8 with eight sides, two horizontal parallel long sides of length $2 \cdot l$ and six sides of length l, such that the angle between every pair of incident edges is 135 degrees, Fig. 11a. We place a maximal rectangle T within P, such that (1) its corners touch each one of the non-horizontal and non-vertical edges of P, and (2) its corners don't coincide with any of the corners of any rectangle in C.

It is easily seen that there always exists such a rectangle T, independently of which method we employ to choose the rectangles in C. Now we can see that the minimum number of rectangles in C that is needed to cover T is at least six.

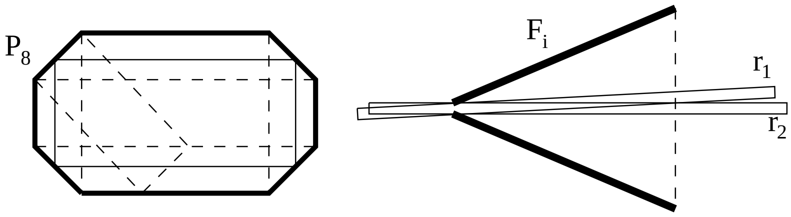

Fig. 11. (a) T is described by the solid thin lines within the polygon. (b) Two rectangles overlapping at F_i's base and top.

Theorem 12. *For every $n \geq 5$, and $d > s > 0$ there exists a polygon P, such that for any set of rectangles, C, for which it holds that every possible rectangle within P can be covered by at most c rectangles of C, it holds that $|C| = (\frac{d}{s})^{\Omega(1/c)}$.*

Proof. Consider a funnel F whose arms forms an angle of 60 degrees. Partition F into $10c$ sub-funnels, $F_1, \ldots, F_{10c}$, such that the ratio between the length of every sub-funnel's base (d') and top (s') is $(d'/s') = (d/s)^{1/10c}$. Let F_i be an arbitrary sub-funnel of F. If two rectangles r_1 and r_2 in C, that pass through F_i, should overlap each other in F_i's top and base, the angle between r_1's and r_2's long sides would have to be less than $\frac{200 s'}{d'}$ degrees (i.e. $\approx \frac{60^\circ s'}{d'}$), Fig. 11b.

Assume that $|C|<d'/10s'$. Then we can find a rectangle T that passes through F, and for all i, $1 \leq i \leq 10c$, the long side of T doesn't have an angle of less than $200s'/d'$ degrees to any of the rectangles' long sides in C that pass through F_i. That is, this rectangle T isn't covered in F_i by the rectangles that pass through F_i. So there exists at least one local rectangle, i.e. a rectangle that doesn't intersect the top of F_i, in C that covers the part of T in F_i. So, for every sub-funnel F_i, $1 \leq i \leq 10c$, a distinct local rectangle has to be used to cover the uncovered part of T in F_i. That is, we can't cover T by c rectangles in C since there are $10c$ sub-funnels.

References

1. P. Berman and B. DasGupta. Approximating the rectilinear polygon cover problems. In *Proc. 4th Canadian Conference Computational Geometry*, pages 229–235, 1992.
2. B.M. Chazelle. *Computational Geometry and Convexity.* PhD thesis, Dept. Comp. Sci., Yale University, New Haven, CT, 1979. Carnegie-Mellon Univ. Report CS-80-150.
3. J.C. Culberson and R.A. Reckhow. Covering polygon is hard. *Journal of Algorithms*, 17:2–44, 1994.
4. D. Franzblau and D. Kleitman. An algorithm for constructing regions with rectangles. In *Proc. 16th Annual ACM Symp. on Theory of Comp.*, pages 167–174, 1984.
5. A. Hegedus. Algorithms for covering polygons by rectangles. *Computer Aided Design*, 14(5), 1982.
6. J.M. Keil. Minimally covering a horizontally convex orthogonal polygon. In *Proc. 2nd Annual ACM Symposium on Computational Geometry*, pages 43–51, 1986.
7. C. Levcopoulos. A fast heuristic for covering polygons by rectangles. In *Proceedings Fundamentals of Computer Theory*, volume 199 of *LNCS*, 1985.
8. C. Levcopoulos. Improved bounds for covering general polygons with rectangles. In *FST & TCS'87*, volume 287 of *LNCS*, 1987.
9. T. Ibaraki M. Yamashita and N. Honda. The minimum number cover problem of a rectilinear region by rectangles. In *EATCS'84*, 1984.
10. L. Monk. *Elementary-recursive Decision Procedures.* PhD thesis, University of California, Berkley, 1975.
11. J. O'Rourke. The decidability of covering by convex polygons. Report JHU-EE 82-1, Dept. Elect. Engrg. Compu. Sci., Johns Hopkins University, Baltimore, MD, 1982.
12. J. O'Rourke and K.J. Supowit. Some np-hard polygon decomposition problems. *IEEE Transactions on Information Theory*, IT-29:181–190, 1983.
13. C. E. Leiserson T. H. Carmen and R. L. Rivest. *Introduction to Algorithms.* MIT Press, 1990.

This article was processed using the LaTeX macro package with LLNCS style