Optimal Control over Networks with Long Random Delays

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Abstract

This paper studies the effects of stochastic time delays on automatic control systems which uses communication networks. We assume a linear process to be controlled and known delay probability distributions.

The contribution of this paper is to extend the theory in [6] to delays that may be longer than one sample period. Using a quadratic cost we find the optimal full-state-information controller. We also show that the standard Kalman filter is an optimal observer, and that the separation principle holds.

1. Introduction

Measurement and control signals that are sent over a communication network or a field bus are often subject to undeterministic time delays. The design of a high performance control system demands that the effect of these delays are analyzed, see for example [2, 3, 4, 5, 7, 8, 9, 10, 11]. An increased understanding of the effects of stochastic time delays is also needed to compare different suggestions for future network protocols for control, for example in next generation wireless networks such as Bluetooth [1].

In this paper we study the effects of time delays that are longer than one sampling period on a single loop linear system. This continues the work in [6]. We find the optimal time-stamped controller under a linear quadratic criterion, and show that the separation property holds between control and estimation even under the more complicated information structure present here.

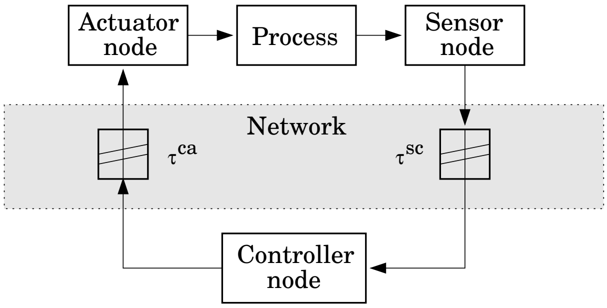

Figure 1 The control system setup. The sensor node is assumed to sample regularly at a rate of h. The controller node and actuator nodes are event driven. The time delays τ^{sc} and τ^{ca} are stochastic with known probability distributions. It is assumed that older values of time delays are known when the control signal is calculated in the controller node.

2. Problem Formulation

The controlled plant studied is assumed to be linear with m states and p inputs,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + v(t) \\ y(t) &= Cx(t) + w(t), \end{aligned} \tag{1}$$

where $v(t)$ and $w(t)$ are independent gaussian random processes. The cost-to-go function at time t_n, denoted V_n, is defined by

$$V_n = \mathbf{E}\left\{ \int_{t_n}^{t_f} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + x(t_f)^T Q_f x(t_f) \right\} \tag{2}$$

The control system studied (see Figure 1) has the following properties:

- The actuator is event-driven, and outputs the latest control signal until a new is received.
- The sensor is periodic, and samples the process output at intervals of h seconds. The sample times are $t_i = ih$.
- The controller calculates the control signal at the time the sample packet arrives from the sensor.
- The time for sample n to go from sensor to controller is denoted τ_n^{sc}, and the time for control signal n from controller to actuator is denoted τ_n^{ca}. The time delays are stochastically independent of the state of the process (this enables a quadratic-form description of the cost V_n).
- The probability distributions of the delays are known by the designer of the controller. The delays are mutually independent.
- The controller gets sample and actuation time information from the sensor. Thus τ_i^{sc} and $\{\tau_i^{ca} \mid u_i$ has been actuated at $t_n\}$, are known by the controller when control signal u_n is calculated (see the example in Figure 2).

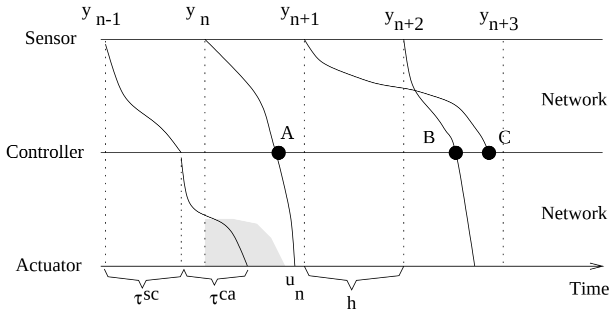

Figure 2 Illustration of packet arrivals when long delays are present. At point **A**, the controller receives the process sample y_n. It also receives information that the last control signal u_{n-1} had not been actuated at time t_n. Therefore, the sending time of u_{n-1} is also sent. The latter is used to calculate the conditional probability that u_{n-1} will be actuated at a certain time instant (possibilities are represented by the shaded area in the figure). At point **B**, y_{n+2} arrives before y_{n+1}. The state estimate is calculated using the available information (i.e. without y_{n+1}), and a new control signal is sent. When y_{n+1} arrives at point **C**, the state estimate is updated but no new control sample is sent.

All assumptions except the last one are natural and describe how many control systems work. The last requirement about time-stamped signals can be satisfied by attaching timing information to all signals. In many applications this can be achieved with neglectable overhead cost. Synchronized clocks must be implemented at the different nodes, but this is becoming more and more common. For a discussion of implementation aspects see [6]. The information on actuation and sample times is crucial for the separation property to hold.

Earlier work in [6] has described the optimal linear quadratic time-stamp controller when the total delay $\tau_n^{sc} + \tau_n^{ca}$ was smaller than the sampling period, h. We will now describe how to extend this work to longer time delays.

3. Optimal Controller

For this problem the maximum delay is restricted to

$$\tau_n^{sc} + \tau_n^{ca} < Nh, \qquad N \in \mathcal{Z}_+. \tag{3}$$

The linear process with the zero-order-hold actuator on the input, periodically sampled, defines a discrete-time time-varying linear system parameterized by the time delays.

$$\begin{aligned} x_{n+1} &= \Phi x_n + \Gamma(\tau_n) \begin{bmatrix} u_{n-N} \\ \vdots \\ u_n \end{bmatrix} = \\ &= [\Phi \;\; \Gamma_1(\tau_n)] z_n + \Gamma_2(\tau_n) u_n \end{aligned} \tag{4}$$

where

$$z_n = \begin{bmatrix} x_n^T & u_{n-N}^T & \dots & u_{n-1}^T \end{bmatrix}^T$$

and

$$\tau_n = \begin{bmatrix} \tau_{n-N}^{sc} & \tau_{n-N}^{ca} & \dots & \tau_n^{sc} & \tau_n^{ca} \end{bmatrix}.$$

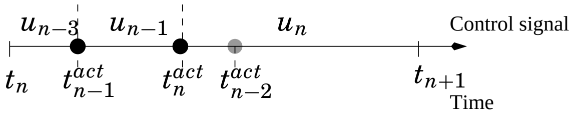

Figure 3 An example of arrivals in $[t_n, t_{n+1}]$. In this case, $K(\tau_n) = \{n-1, n\}$. As u_{n-2} arrives later than u_n (and u_{n-1}) it is ignored. $K_{start} = n-3$ in this case, as u_{n-3} is active in the beginning.

Thus z_n is the process state at sample n augmented with the control signals which may not yet have arrived to the actuator. The vector τ_n contains the delay data for each control signal. If one control signal has already been actuated, the corresponding τ^{sc} and τ^{ca} are set to 0.

For the calculations in this paper, it useful to have a transition matrix $X(t, \tau_n)$ implicitly defined as

$$\begin{bmatrix} x(t) \\ u_{n-N} \\ \vdots \\ u_n \end{bmatrix} = X(t, \tau_n) \begin{bmatrix} z_n \\ u_n \end{bmatrix} \tag{5}$$

To be able to define $X(t, \tau_n)$ explicitly, we have to specify some policies (due to the long delays):

- Control signal packets and sample packets can switch places in time. If the control signal u_i arrives after u_j, and $i < j$, u_i should be ignored by the actuator. In the same way, the controller should not send a new control signal if process sample y_i arrives after y_j, and $i < j$, although the state estimate is updated (see point **C** in Figure 2).
- When the controller receives sample y_n, it only has information on the process up to time $t_n = nh$. Old control signals may not have arrived at the actuator when the process is sampled. Thus, the controller will have to estimate the state of the process at the control decision using information on when the not-yet-arrived control signals was sent. This timing information creates a parameterized cost function $V_n(z_n, \tau_{n-1}^{sc}, \ldots, \tau_{n-N+1}^{sc})$.

A formal definition of $X(t, \tau_n)$ requires some help variables. First, we need the actuation times of control signals explicitly

$$t_i^{act} = \tau_i^{sc} + \tau_i^{ca} + t_i, \tag{6}$$

where t_i^{act} is the actuation time of the control signal sent at sample i. Then define an ordered index set $K(\tau_n)$ of the control signals which will be actuated in $[t_n, t_{n+1}]$ in sorted order, i.e. such that

$$\begin{aligned} & t_{K_i}^{act} \le t_{K_j}^{act} < t_{n+1}, \ \forall j > i \\ & K_i > K_j, \ \forall i > j \text{ (ignore old samples).} \end{aligned}$$

Also, let K_{start} be the index of the control signal which is active at the start of the sample period. Figure 3 illustrates the idea.

We can now write the transition matrix $X(t, \tau_n)$ which brings the state from t_n to t as

$$X(t, \tau_n) = \begin{cases} \Phi(t - t_n, K_{start} + N - n) & t_n \leq t \leq t^{act}_{K_1} \\ \Phi(t - t^{act}_{K_1}, K_1 + N - n) \times X(t^{act}_{K_1}, \tau_n) & t^{act}_{K_1} < t \leq t^{act}_{K_2} \\ \Phi(t - t^{act}_{K_2}, K_2 + N - n) \times X(t^{act}_{K_2}, \tau_n) & t^{act}_{K_2} < t \leq t^{act}_{K_3} \\ \quad\vdots \end{cases} \tag{7}$$

where

$$\Phi(\Delta, i) = \begin{bmatrix} e^{A\Delta} & \mathbf{0}_{m\times ip} & \int_0^\Delta e^{A(\Delta - s)} B ds & \mathbf{0}_{m\times(N-i)p} \\ \mathbf{0} & & I_{p(N+1)\times p(N+1)} & \end{bmatrix}$$

We also define $Q(t, \tau_n)$ as the positive definite cost matrix from t_n to t

$$Q(t, \tau_n) = \begin{cases} \Omega(t - t_n, K_{start} + N - n) & t_n \leq t \leq t^{act}_{K_1} \\ \Omega(t - t^{act}_{K_1}, K_1 + N - n) \cdot X(t^{act}_{K_1}, \tau_n) + & \\ \quad Q(t^{act}_{K_1}, \tau_n) & t^{act}_{K_1} < t \leq t^{act}_{K_2} \\ \Omega(t - t^{act}_{K_2}, K_2 + N - n) \cdot X(t^{act}_{K_2}, \tau_n) + & \\ \quad Q(t^{act}_{K_2}, \tau_n) & t^{act}_{K_2} < t \leq t^{act}_{K_3} \\ \quad\vdots \end{cases} \tag{8}$$

where

$$\Omega(\Delta, i) = \int_0^\Delta [\ldots]^T Q \begin{bmatrix} e^{As} & \mathbf{0}_{m\times ip} & \int_0^s e^{A(s-\sigma)} B d\sigma & \mathbf{0}_{m\times(N-i)p} \\ \mathbf{0} & \mathbf{0}_{p\times ip} & I_{p\times p} & \mathbf{0}_{p\times(N-i)p} \end{bmatrix} ds$$

Optimal Full-State-Information Controller

Assume we have full state information, i.e. $y_n = x_n$. Also assume the cost from sample t_{n+1} to t_f can be written as

$$V_{n+1}(z_{n+1}, \tau^{sc}_n, \ldots, \tau^{sc}_{n-N+2}) = z^T_{n+1} S_{n+1}(\tau^{sc}_n, \ldots, \tau^{sc}_{n-N+2}) z_{n+1} \tag{9}$$

where $S_{n+1}(\tau^{sc}_n, \ldots, \tau^{sc}_{n-N+2})$ is a symmetric, positive-definite cost-matrix. We then move one step backwards in time

$$
\begin{aligned}
&V_n(z_n, \tau^{sc}_{n-1}, \ldots, \tau^{sc}_{n-N+1}) = \\
&\quad \mathop{\mathbf{E}}_{\tau^{sc}_n} \left(\min_{u_n} \mathop{\mathbf{E}}_{\tau^{ca}_n,\ldots,\tau^{ca}_{n-N+1}} \left\{ \begin{bmatrix} z_n \\ u_n \end{bmatrix}^T Q(t_{n+1}, \tau_n) \begin{bmatrix} z_n \\ u_n \end{bmatrix} + \right.\right. \\
&\quad \begin{bmatrix} z_n \\ u_n \end{bmatrix}^T X(t_{n+1}, \tau_n)^T D^T S_{n+1}(\tau^{sc}_n, \ldots, \tau^{sc}_{n-N+2}) \cdot \\
&\quad \left.\left. DX(t_{n+1}, \tau_n) \begin{bmatrix} z_n \\ u_n \end{bmatrix} \,\middle|\, \tau^{sc}_n, \ldots, \tau^{sc}_{n-N+1} \right\} \right) = \\
&\quad \mathop{\mathbf{E}}_{\tau^{sc}_n} \left(\min_{u_n} \mathop{\mathbf{E}}_{\tau^{ca}_n,\ldots,\tau^{ca}_{n-N+1}} \left\{ \underbrace{\begin{bmatrix} z_n \\ u_n \end{bmatrix}^T Q(t_{n+1}, \tau_n) \begin{bmatrix} z_n \\ u_n \end{bmatrix}}_{\text{Cost during this period}} + \right.\right. \\
&\quad \left.\left. \underbrace{\begin{bmatrix} z_n \\ u_n \end{bmatrix}^T \tilde{S}(\tau_n) \begin{bmatrix} z_n \\ u_n \end{bmatrix}}_{\text{Remaining cost}} \,\middle|\, \tau^{sc}_n, \ldots, \tau^{sc}_{n-N+1} \right\} \right) = \\
&\quad \mathop{\mathbf{E}}_{\tau^{sc}_n} \left(\min_{u} \begin{bmatrix} z_n \\ u_n \end{bmatrix}^T F_n(\tau^{sc}_n, \ldots, \tau^{sc}_{n-N+1}) \begin{bmatrix} z_n \\ u_n \end{bmatrix} \right) = \\
&\quad \mathop{\mathbf{E}}_{\tau^{sc}_n} z_n^T \left(F_{zz_n} - (F^{-1}_{uu_n} F^T_{zu_n})^T F^T_{zu_n} \right) z_n = \\
&\quad z_n^T S_n(\tau^{sc}_{n-1}, \ldots, \tau^{sc}_{n-N+1}) z_n.
\end{aligned} \tag{10}
$$

where

$$
D = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{11}
$$

is a matrix which translates the control signals u_i one step back in time. At the third equality, the expectation over $\tau_n, \ldots, \tau_{n-N+1}$ is moved inside the state since τ_i is conditionally independent of x_j and u_j for $i, j \in [n-N+1, n]$ given information on $\tau^{sc}_n, \tau^{sc}_{n-1}, \ldots, \tau^{sc}_{n-N+1}$. As τ^{sc}_n is known by the controller at the control decision, the expectation over it is done outside the u_n optimization. The quadratic form is minimized by

$$
\begin{aligned}
u_n &= -L_n(\tau^{sc}_n, \ldots, \tau^{sc}_{n-N+1}) z_n = \\
&-F^{uu}_n(\tau^{sc}_n, \ldots, \tau^{sc}_{n-N+1})^{-1} F^{zu}_n(\tau^{sc}_n, \ldots, \tau^{sc}_{n-N+1})^T z_n.
\end{aligned} \tag{12}
$$

Thus, the optimal control is a linear feedback, where the feedback gain has N parameters.

Optimal State Estimate

If the full state is not available at the sampling instant, the state vector has to be estimated. Assume the system can be described by (4) plus noise

$$\begin{aligned} x_{n+1} &= \Phi x_n + \Gamma(\tau_n) \begin{bmatrix} u_{n-N} \\ \vdots \\ u_n \end{bmatrix} + v_n \\ y_n &= C x_n + w_n, \end{aligned} \tag{13}$$

where $\mathbf{E}(v_i v_j^T) = R_1 \delta(i-j)$, $\mathbf{E}(e_i e_j^T) = R_2 \delta(i-j)$, and $\mathbf{E}(v_i e_j^T) = 0$. Then, since all actions in the actuator are immediately reported to the sensor, there is full information on all control actions which have been actuated so far. Thus the standard Kalman filter is optimal. The gains K and $\bar{K}$ will not depend on the delays, since only Φ is involved in the calculations.

For the estimation of $\hat{x}_n$, almost all y_k, $k \leq n$ are known. Since sensor-to-controller packets can switch places in time, it is possible that $\hat{x}_n$ has to be estimated without e.g. y_{n-1}. This can be handled by letting $R_2^{-1} \to 0$ when estimating $\hat{x}_{n-1}$ (i.e. no new information on the state). The estimate $\hat{x}_n$ can then be calculated as usual. When (if) y_{n-1} eventually arrives, the estimation is redone for future control samples.

$$\hat{x}_{n|n} = \hat{x}_{n|n-1} + \bar{K}_n (y_n - C\hat{x}_{n|n-1}) \tag{14}$$

$$\begin{aligned} \hat{x}_{n+1|n} &= \Phi x_n + \Gamma(\tau_n) \begin{bmatrix} u_{n-N} \\ \vdots \\ u_n \end{bmatrix} + \\ &\quad K_n (y_n - C\hat{x}_{n|n-1}) \end{aligned} \tag{15}$$

$$\bar{K}_n = P_{n|n-1} C^T \left(C P_{n|n-1} C^T + R_2 \right)^{-1} \tag{16}$$

$$K_n = \Phi P_{n|n-1} C^T \left(C P_{n|n-1} C^T + R_2 \right)^{-1} \tag{17}$$

$$\begin{aligned} P_{n|n} &= P_{n|n-1} - \\ &\quad P_{n|n-1} C^T \left(C P_{n|n-1} C^T + R_2 \right)^{-1} C P_{n|n-1} \end{aligned} \tag{18}$$

$$\begin{aligned} P_{n+1|n} &= \Phi P_{n|n-1} \Phi^T + R_1 - \\ &\quad K_n \left(C P_{n|n-1} C^T + R_2 \right) K_n^T \end{aligned} \tag{19}$$

$$\hat{x}_{0|-1} = \mathbf{E}(x_0) \tag{20}$$

$$P_{0|-1} = \mathbf{E}(x_0 x_0^T) \tag{21}$$

Optimal State Feedback

The optimal state feedback is the standard combination of optimal full-state-information controller and optimal observer. This is stated in the following theorem:

THEOREM 1—SEPARATION PROPERTY
Given the system in (13), and the measurements
$Y_n = \{y_n\} \cup \{y_i \mid y_i \text{ received at controller before } y_n\}$. The optimal controller with

respect to the cost function in (2) is

$$u_n = -L_n(\tau_n^{sc}, \ldots, \tau_{n-N+1}^{sc})\hat{z}_{n|n} = -L_n(\ldots)\begin{bmatrix} \hat{x}_{n|n} \\ u_{n-N} \\ \vdots \\ u_{n-1} \end{bmatrix}, \tag{22}$$

i.e. the optimal controller gain in (12) applied on the optimal state estimate in (14). □

The proof is following the lines of the similar proof in [6]. First, we present a lemma (5.2 in [6]) which simplifies the proof:

LEMMA 1—ESTIMATE RECURSION

$$\begin{aligned}
&\mathop{\mathbf{E}}_{\tau_n^{ca},\ldots,\tau_{n-N+1}^{ca},v_n,w_{n+1}} \left\{ \hat{z}_{n+1|n+1}^T S_{n+1}(\ldots)\hat{z}_{n+1|n+1} \,\middle|\, Y_n \right\} = \\
&[\ldots]^T \underbrace{\mathop{\mathbf{E}}_{\tau_n^{sc},\ldots,\tau_{n-N+1}^{sc}} \left\{ [\ldots]^T S_{n+1}(\ldots) DX(\ldots) \right\}}_{\tilde{S}_{n+1}()} \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix} + \\
&\mathop{\mathbf{E}}_{v_n,w_{n+1}} \left\{ [\ldots]^T S_{n+1}(\ldots) \begin{bmatrix} H_{n+1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{x}_n \\ v_n \\ w_{n+1} \end{bmatrix} \,\middle|\, Y_n \right\} = \\
&\mathop{\mathbf{E}}_{\tau_n^{ca},\ldots,\tau_{n-N+1}^{sc}} \left\{ \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix}^T \tilde{S}_{n+1}(\ldots) \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix} \right\} + \\
&\underbrace{\mathrm{tr}(P_{n|n}\Phi^T C^T \bar{K}_{n+1}^T S_{xx_{n+1}}()\bar{K}_{n+1} C\Phi)}_{\text{Cost from } \tilde{x}_n} + \\
&\underbrace{\mathrm{tr}(R_1 C^T \bar{K}_{n+1}^T S_{xx_{n+1}}(\ldots)\bar{K}_{n+1} C)}_{\text{Cost from } v_n} + \\
&\underbrace{\mathrm{tr}(R_2 \bar{K}_{n+1}^T S_{xx_{n+1}}()\bar{K}_{n+1})}_{\text{Cost from } w_{n+1}},
\end{aligned} \tag{23}$$

where

$$H_{n+1} = \begin{bmatrix} \bar{K}_{n+1}C\Phi & \bar{K}_{n+1}C & \bar{K}_{n+1} \end{bmatrix} \tag{24}$$

□

Proof of Lemma 1

The proof is built on the fact that $\hat{x}_{n+1|n+1}$ can be calculated as

$$
\begin{aligned}
&\hat{x}_{n+1|n+1} = (I-\bar{K}_{n+1}C)\hat{x}_{n+1|n} + \bar{K}_{n+1}y_{n+1} = \\
&(I-\bar{K}_{n+1}C)\left(\Phi\hat{x}_{n|n} + \Gamma(\tau_n)\begin{bmatrix} u_{n-N} \\ \vdots \\ u_n \end{bmatrix}\right) + \\
&\bar{K}_{n+1}\left(C\left(\Phi x_n + \Gamma(\tau_n)\begin{bmatrix} \vdots \end{bmatrix} + v_n\right) + w_{n+1}\right) = \\
&\Phi\hat{x}_{n|n} + \Gamma(\tau_n)\begin{bmatrix} u_{n-N} \\ \vdots \\ u_n \end{bmatrix} + H_{n+1}\begin{bmatrix} \tilde{x}_n \\ v_n \\ w_{n+1} \end{bmatrix}
\end{aligned}
\tag{25}
$$

The optimal state estimate gives that $\hat{x}_{n|n}$ and $\tilde{x}_n$ are orthogonal, and causality gives independence of w_{n+1}, v_n and $\hat{x}_n$. Thus the expectation in (23) can be split. □

Proof of Theorem 1

For the optimal estimate defined in (14), it holds that

$$
\begin{aligned}
&\mathbf{E}\left\{\begin{bmatrix} z_n \\ u_n \end{bmatrix}^T Q \begin{bmatrix} z_n \\ u_n \end{bmatrix} \,\Big|\, Y_n\right\} = \\
&\mathbf{E}\left\{\begin{bmatrix} \hat{z}_{n|n} + \tilde{z} \\ u_n \end{bmatrix}^T Q \begin{bmatrix} \hat{z}_{n|n} + \tilde{z} \\ u_n \end{bmatrix} \,\Big|\, Y_n\right\} = \\
&\mathbf{E}\left\{\begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix}^T Q \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix} \,\Big|\, Y_n\right\} + \mathrm{tr}(P_{n|n}Q_{xx}()) = \\
&\mathbf{E}\left\{\begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix}^T Q \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix} \,\Big|\, Y_n\right\} + c_n,
\end{aligned}
\tag{26}
$$

i.e. the expected value of the cost is the cost function evaluated on the optimal estimates plus a constant (noise) term.

Now let $V_n(\hat{z}_{n|n}, \tau_n, P_{n|n})$ denote the optimal cost from x_n at time t_n to t_f, and the state estimation error has variance $P_{n|n}$. Also assume $V_{n+1}(\hat{z}_{n+1|n+1}, \tau_n, P_{n+1|n+1}) = \hat{z}^T_{n+1|n+1}S_{n+1}(\tau_n)\hat{z}_{n+1|n+1}$ for some n. Then (26) and Lemma 1 allows us to do a cost recursion step when only optimal estimates $\hat{x}$ of x are known

$$
\begin{aligned}
&V_n(\hat{z}_{n|n}, \tau_n, P_{n|n}) = \\
&\mathop{\mathbf{E}}_{\tau_n^{sc}} \min_{u_n} \mathbf{E} \left\{ \begin{bmatrix} z_n \\ u_n \end{bmatrix}^T Q(t_n, \tau_n) \begin{bmatrix} z_n \\ u_n \end{bmatrix} + \right. \\
&\left. V_{n+1}(\hat{z}_{n+1|n+1}, \tau_n, P_{n+1|n+1}) \;\middle|\; Y_n \right\} = \\
&\mathop{\mathbf{E}}_{\tau_n^{sc}} \min_{u_n} \mathbf{E} \left\{ \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix}^T Q(t_n, \tau_n) \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix} + c_n + \right. \\
&\left. \hat{z}^T_{n+1|n+1} S_{n+1}(\tau_n) \hat{z}_{n+1|n+1} \;\middle|\; Y_n \right\} = \\
&\mathop{\mathbf{E}}_{\tau_n^{sc}} \min_{u_n} \mathbf{E} \left\{ \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix}^T Q(t_n, \tau_n) \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix} + \right. \\
&\left. \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix}^T \tilde{S}_{n+1}(\tau_n) \begin{bmatrix} \hat{z}_{n|n} \\ u_n \end{bmatrix} + s_n \;\middle|\; Y_n \right\}
\end{aligned}
\tag{27}
$$

Minimizing this quadratic form with respect to u_n leads to the same feedback gain as in the full-information case (12). Thus the separation property holds. $\square$

4. Examples

The computation of the optimal controller has been implemented in Matlab. To show the properties of the optimal controller, two systems have been simulated. The first is taken from [6], Example 5.5 p. 72. It is a stable system.

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 35 \\ -61 \end{bmatrix} \xi \\
y &= \begin{bmatrix} 2 & 1 \end{bmatrix} x + \eta \\
Q &= 80 \begin{bmatrix} 35 & \sqrt{35} & 0 \\ \sqrt{35} & 1 & 0 \\ 0 & 0 & 1/80 \end{bmatrix} \\
h &= 0.05
\end{aligned}
$$

To simplify, the variances of the sampled noise, ξ_n and η_n, were set to unity. The second example is an unstable system (in fact, a linearized inverted pendulum).

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0.2 & -0.1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 10 \end{bmatrix} \xi \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + \eta \\
Q &= I \\
h &= 0.5
\end{aligned}
$$

The total time delay $\tau = \tau^{sc} + \tau^{ca}$ is uniformly distributed between 0 and τ_{max}. This may seem unintuitive, but it forces all aspects of packet-switching to appear even when $N = 2$. Monte-Carlo simulations of 20000 samples were run with varying τ_{max}. The resulting cost is plotted in Figure 4a, where τ_{max} has been varied between 0 and $2h$. Using the same network parameters, an unstable system was also simulated (a linearized inverted pendulum). The results can be seen in Figure 4b. Both simulations have been run with three different controllers:

- The optimal controller presented in this paper.
- A fixed LQ controller was calculated assuming a deterministic system based on the average delay. The resulting feedback gain was applied on the optimal state estimate (i.e. for the real, stochastic system) described in Section 3.
- Again, a fixed LQ controller was calculated assuming a deterministic system based on the average delay. Also, a fixed Kalman filter based on the deterministic system was calculated (a suboptimal estimator). The controller was based on the LQ feedback gain applied on the suboptimal state estimate.

Notice the difference between the results as seen in Figure 4. In Example a) the main advantage of the optimal controller seems to be the correct state estimation, whereas in Example b) the main performance improvement comes from better linear feedback gains. The code for the two examples is available at `http://www.control.lth.se/~lincoln/mtns2000/`

5. Conclusions

The optimal controller has been found for a stochastic optimal control problem where communication delays are varying. A separation property holds so that the optimal controller consists of a linear gain scheduled feedback combined with the optimal state estimate. The complexity of the presented controller grows, however, rapidly when the maximal controller time delay $\tau^{sc} + \tau^{ca}$ is larger than h. It is assumed that previous values of the time delays are known at the time of computation of the control signal, something which can be obtained by letting the actuator send timestamps to the controller (possibly in the process sample packet). The theory was illustrated with two simulated examples, showing the advantages of the optimal controller.

6. References

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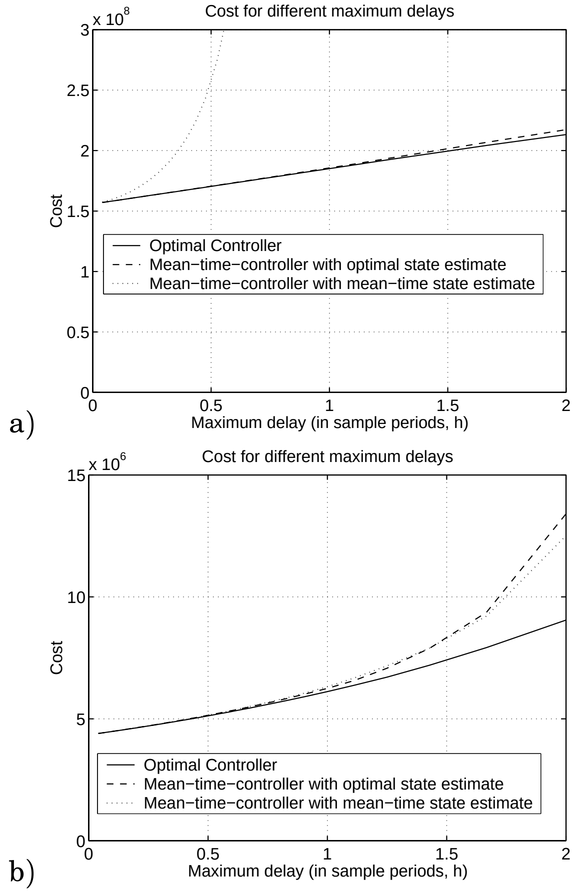

Figure 4 Monte-Carlo simulations of three controllers on the two example systems. Figure a) shows the cost of the stable system, and figure b) the unstable system. In a) the main advantage of the optimal controller seems to be the improved state estimation, where as in b) the main performance improvement comes from better linear feedback gains.

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