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High-Rate QC LDPC Codes of Short and Moderate Length with Good Girth Profile

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***Abstract*—Irregular QC LDPC codes with parity-check matrices having different degree distributions are studied. A new algorithm for finding regular and irregular QC LDPC codes with a good girth profile as well as a good sliding-window girth is presented. As examples, simulation results for QC LDPC codes with good girth profile, rate R=4/5, and lengths about 1000, 2000, and 4000, constructed from base matrices with proper degree distributions are given. Their simulated BER and FER performances for belief propagation decoding are compared with the best previously known irregular QC LDPC codes of the same rate and length. It is shown that the constructed codes outperform the best previously known codes of same rate and lengths.**

I. Introduction

It is well-known that low-density parity-check (LDPC) codes with iterative low-complexity decoding achieve performances close to the Shannon limit. This fact attracts attention of many researches to this class of codes and makes LDPC codes a good choice for modern communication standards [1], [2], [3]. In this paper we will focus on quasi-cyclic (QC) LDPC codes. It is well-known that such codes can be represented in the form of tailbiting block codes, which supports searching for new low-complexity encoding codes. There exist two subclasses of QC LDPC block codes: *regular* and *irregular*. If each column of its parity-check matrix contains exactly J ones and each row contains exactly K ones the corresponding QC LDPC code is called (J, K)-regular; and irregular otherwise.

The length of the shortest cycle in the Tanner graph of a QC LDPC code, that is, its *girth*, is considered to be an important code parameter, as it determines the number of independent iterations in low-complexity belief-propagation decoding. A typical approach to constructing new both regular and irregular QC LDPC codes is based on searching for a suitable labeling of a proper *base* parity-check matrix (*base* Tanner graph [4]) which satisfies the target girth of its Tanner graph (see, for example, [5], [6], [7] and references therein). It is intuitively clear that the bit error rate performance of QC LDPC codes obtained in such a way depends also on the structure of the base matrix, more precisely on its column *degree distribution* [8], that is, the distribution of its column weights. Some publications state that a girth of 6 is sufficient to achieve bit error rate performance close to the theoretical limit (see, for example, [9] and references therein) and that QC LDPC codes with larger girth typically have smaller minimum distances resulting in higher error floors. Moreover, it is typically stated that for irregular codes the structure of the base matrix plays an even more important role than the girth of the Tanner graph. In other words, if the girth of the Tanner graph is 6 then by improving its degree distribution a bit error rate performance close to the theoretical limit can be achieved. For example, in the WIMAX standard [1], a set of such high-rate irregular QC LDPC codes of rates R=1/2, 2/3, 3/4, and 5/6 and rather short lengths were suggested. In [9], examples of similar high-rate irregular QC LDPC codes of short and moderate lengths with improved bit error rate performance are given. To the best of our knowledge nobody reported noticeably better results than [9], especially for rate $R = 4/5$ and lengths less than or equal to 4000.

In this paper we study the relation between the girth of the Tanner graph corresponding to irregular QC LDPC codes constructed from base matrices with good degree distributions and the achievable bit error performance. A new algorithm is presented to find labelings yielding a good *girth profile*, that is, a vector of girth values whose ith component is equal to the girth of the Tanner graph corresponding to the submatrix constructed by using i columns of the smallest weight of its parity-check matrix. Moreover, our algorithm guarantees that in a sliding-window containing w consecutive columns of the parity-check matrix ordered by their weights, the minimal girth value is larger than or equal to a predetermined value $g_w > g$, where g is the target girth. By applying a greedy algorithm [10] to obtain a base matrix for an irregular QC LDPC code with a good *degree distribution*, followed by our new algorithm for finding labelings with a good girth profile, we constructed a set of both regular and irregular QC LDPC codes of rate $R = 4/5$ and lengths about 1000, 2000, and 4000. Their bit error rate (BER) and frame error rate (FER) performances is compared with performances of QC LDPC codes from [9] with the same rate and lengths. Finally, some conclusions are drawn.

II. Regular and Irregular QC LDPC Codes

A rate $R = b/c$ parent convolutional LDPC code can be determined by its polynomial parity-check matrix $H(D)$ of

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$$H(D) = \begin{pmatrix} h_{11}(D) & h_{12}(D) & \dots & h_{1c}(D) \\ h_{21}(D) & h_{22}(D) & \dots & h_{2c}(D) \\ \vdots & \vdots & \ddots & \\ h_{(c-b)1}(D) & h_{(c-b)2}(D) & \dots & h_{(c-b)c}(D) \end{pmatrix} \quad (1)$$

where $h_{ij}(D)$ is either zero or a monomial entry, that is, $D^{w_{ij}}$ with w_{ij} being a nonnegative integer. Its *degree matrix* W follows as the $(c-b)\times c$ matrix with entries w_{ij} at the positions of the monomials $D^{w_{ij}}$ and -1 at the zero positions. If each column of $H(D)$ contains J nonzero elements and each row contains K nonzero elements the QC LDPC convolutional code is (J, K)-regular; and irregular otherwise.

By tailbiting the parent convolutional code to length $M > m$ we obtain the parity-check matrix

$$H_{\mathrm{TB}}^{\mathrm{T}} = \begin{pmatrix} H_0^{\mathrm{T}} & H_1^{\mathrm{T}} & \dots & H_{m-1}^{\mathrm{T}} & H_m^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & H_0^{\mathrm{T}} & H_1^{\mathrm{T}} & \dots & H_{m-1}^{\mathrm{T}} & H_m^{\mathrm{T}} \\ H_m^{\mathrm{T}} & \mathbf{0} & H_0^{\mathrm{T}} & H_1^{\mathrm{T}} & \dots & H_{m-1}^{\mathrm{T}} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ H_1^{\mathrm{T}} & \dots & H_{m-1}^{\mathrm{T}} & H_m^{\mathrm{T}} & \mathbf{0} & H_0^{\mathrm{T}} \end{pmatrix} \quad (2)$$

of an (Mc, Mb) QC LDPC block code, where

$$H(D) = H_0 + H_1 D + \dots + H_m D^m$$

and H_i, $i = 0, 1, \dots, m$, are binary $(c-b) \times c$ matrices.

The polynomial parity-check matrix $H(D)$ (1) can be interpreted as a base matrix B labeled by monomials, where B is the $(c-b) \times c$ binary matrix with ones at the positions of the nonzero entries of $H(D)$, satisfying

$$B = H(D)|_{D=1}$$

Interpreting both B and H_{TB} as *biadjacency matrices* [11] yields their corresponding Tanner graphs $\mathcal{G}_{\mathrm{B}}$ and $\mathcal{G}$, respectively. Moreover, the parity-check matrix $H(D)$ (1) of the parent convolutional code corresponds to an infinite Tanner graph obtained by unwrapping the Tanner graph $\mathcal{G}$ of the tailbiting block code and extending it to infinity in the time domain. Hence, the problem of finding new QC LDPC codes can be reduced to finding suitable base matrices B as well as proper labelings for the base Tanner graphs $\mathcal{G}_{\mathrm{B}}$ determined by B. The girth g is used as a target when searching for good QC LDPC codes. It is known that there exist a tailbiting length M and a set of edge labels, such that the girth g of the Tanner graph for the corresponding tailbiting block code of length $n = Mc$ satisfies the inequalities [7].

$$g \geq 2d_2 \geq g_{\mathrm{B}} + \lceil g_{\mathrm{B}}/2 \rceil \geq 3g_{\mathrm{B}} \quad (3)$$

where g_{B} is the girth of the base Tanner graph $\mathcal{G}_{\mathrm{B}}$ and d_2 is the minimum second generalized Hamming distance of the linear block code whose parity-check matrix is equal to the incidence matrix of $\mathcal{G}_{\mathrm{B}}$. Notice that the free girth g_{free}, that is, the length of the shortest cycle in the infinite Tanner graph determined by the polynomial matrix $H(D)$ upperbounds g, that is, $g \leq g_{\mathrm{free}}$.

For irregular QC LDPC codes, let $J_{\min}$ and $J_{\max}$ denote the minimum and maximum numbers of the nonzero entries in any column of $H(D)$, respectively. Then the degree distribution of the parity-check matrix $H(D)$ as well as of the base matrix B is given by the vector

$$\boldsymbol{d}^{\mathrm{p}} = \left(n_{J_{\min}} \, n_{J_{\min}+1} \, \dots \, n_{J_{\max}} \right)$$

where n_i denotes the number of columns having i nonzero elements. Since the degree distribution can contain zeros, it can be efficiently represented as a set

$$\lambda_s = \left\{ d^{(1)}(n_1), d^{(2)}(n_2), \dots, d^{(l)}(n_l) \right\}$$

where n_i is the number of columns with $d^{(i)}$ nonzero elements, $d^{(1)} = J_{\min}$, $d^{(l)} = J_{\max}$, and $\sum_{i=1}^{l} n_i = c$. Notice that the degree distribution is invariant with respect to column permutations.

Denote as H_{TB}^i, $i = 1, 2, \dots, Mc$, a submatrix of H_{TB} formed by its i columns with the smallest column weight and let g_i be the girth of the corresponding Tanner graph with biadjacency matrix H_{TB}^i. Then we call the Mc-tuple

$$\boldsymbol{g}^{\mathrm{p}} = (g_1 \, g_2 \, \dots \, g_{Mc} = g)$$

where $g_i > g_{i+1}$, $i = 1, 2, ..., Mc$, the *girth profile* of the QC LDPC block code determined by H_{TB}. It can be efficiently represented as the ordered set

$$\mathcal{S}_g = \left\{ g^{(1)}(n_1), g^{(2)}(n_2), \dots, g^{(l)}(n_l) \right\} \quad (4)$$

where $n_j < n_{j+1}$, $n_l = Mc$, and $g_i = g^{(j)}$ for $n_{j-1} < i \leq n_j$ with $n_0 = 0$. A girth profile $\boldsymbol{g}^p$ is said to be superior to $(>)$ another profile $\boldsymbol{g}^{p^*}$ if there exists a positive integer p such that

$$g_j = \begin{cases} = g_j^* & j = 1, 2, \dots, p-1 \\ > g_j^* & p \leq j \leq Mc \end{cases}$$

Similarly, for the infinite Tanner graph with biadjacency matrix $H(D)$ the *free girth profile* is defined [10] as

$$\boldsymbol{g}_{\mathrm{free}}^p = (g_{\mathrm{free},1} \, g_{\mathrm{free},2} \, \dots \, g_{\mathrm{free,c}})$$

Note that the inequality $\boldsymbol{g}_{\mathrm{free}}^p \geq \boldsymbol{g}^p$ always holds.

Next we will present an algorithm for searching for QC LDPC codes having a sliding-window girth g_w. For simplicity, we restrict ourselves to girth profiles for which n_i in (4) are multiples of M and represent the corresponding girth profiles as vectors of length c

$$\widetilde{\boldsymbol{g}}^p = (\widetilde{g}_1 \, \widetilde{g}_2 \, \dots \, \widetilde{g}_c = g)$$

that is, we use the free girth profile $\boldsymbol{g}_{\mathrm{free}}$ but take also the tailbiting length M into account. Hence, the corresponding set $\mathcal{S}_g$ contains at most c elements.

III. A Sliding-Window Algorithm for Searching QC LDPC Block Codes with Good Girth Profile

Assume that by applying the greedy algorithm given in [10] we have constructed a base matrix B of size $(c-b)\times c$ with a given degree distribution

$$\lambda_s = \left\{ d^{(1)}(n_1), d^{(2)}(n_2), \ldots, d^{(l)}(n_l) \right\}$$

Moreover, let M be the tailbiting length and g be the target girth value. Our goal is to obtain a proper labeling for the base matrix B such that a parity-check matrix $H(D)$ of the parent convolutional code has a good free girth profile $\boldsymbol{g}^p_{\text{free}}$ with $g_{\text{free},c} \geq g$. Moreover, by tailbiting $H(D)$ to length M we obtain the parity-check matrix H_{TB} of the (Mc, Mb) QC LDPC block code whose girth profile $\boldsymbol{g}^p$ satisfies $g_{Mc} = \widetilde{g}_c \geq g$.

Let L_w denote the current number of columns of the sliding-window within the base matrix B which should be labeled with the target girth $g_w > g$, then choose the maximum number of trials to be $N_{\max}$, as well as the minimal and maximal number of the surviving candidates during each round to be $S_{\max}$ and $S_{\min}$, respectively.

Moreover, let B_i denote a submatrix of B formed by its first i columns, where $i \leq c$, and let $B_i^{(j)}$ denote the submatrix of B_i containing the j last columns of B_i, $j = \min\{i, L_w\}$.

1. Choose the initial number of columns i_0 and set $i = i_0$. Perform a random labeling of the nonzero positions of B_i with target girth values g and g_w for B_i and $B_i^{(j)}$, respectively. Generate s source candidates W_i, $S_{\min} \leq s \leq S_{\max}$. Set the number of candidates $n = 0$ and the number of trials $N = 0$.
2. While $i \leq c$
   - 2.1 Choose randomly one of the s source candidates W_i. Assign random labels to the column b_{i+1} of the base matrix B and append it to the matrix W_i, that is, form $W_{i+1} = [W_i; \boldsymbol{w}_{i+1}]$ and set $N \leftarrow N+1$.
   - 2.2 If the girth constraints g and g_w (for B_i and $B_i^{(j)}$) are satisfied, then add W_{i+1} to the set of target candidates, set $n \leftarrow n+1$, and, as long as $N < N_{\max}$ and $n < S_{\max}$, goto step 2.1.
   - 2.3 If $n \geq S_{\min}$, replace the set of source candidates W_i by the set of target candidates W_{i+1}, set $i \leftarrow i+1$, set $s = n$, $n = 0$, and goto step 2. Otherwise, stop the algorithm and revise L_w and g_w.

Notice that labeling a base matrix using the algorithm above implies constructing a system of girth inequalities for $[W; w]$ with target girth g using algorithm A or B presented in [7] while taking the tailbiting length M additionally into account.

In Fig. 1, the simulated bit error rate (BER) and frame error rate (FER) performances for belief propagation decoding of QC LDPC blocks codes with rate $R = 4/5$ and lengths about 1000, 2000, and 4000 with random labeling are compared with the QC LDPC codes with the same parameters but improved labelings obtained by using the above algorithm (Table I). As clearly seen from Fig. 1, improving the girth profile according to the proposed algorithm yields significantly better BER and FER performances, especially for higher signal-to-noise ratios (SNRs).

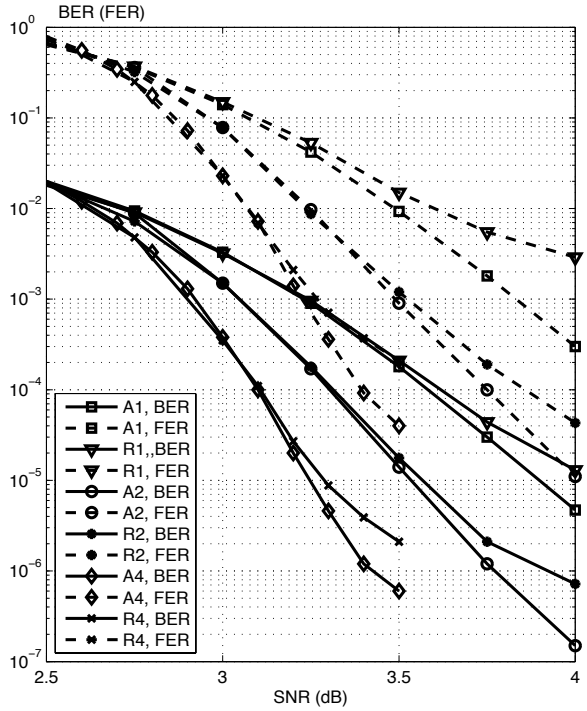

Fig. 1. Comparison of bit and frame error rates for QC LDPC codes with random and good labelings.

Notice that our search criteria were entirely based on girth constraints for the QC LDPC code parity-check matrix and its submatrices. However, the proposed algorithm implicitly achieves rather good approximate cycle extrinsic (ACE) message degrees since it yields larger girth values for cycles formed by low-weight columns and thereby low-weight columns have larger connectivity [12].

IV. Simulation Results and Comparison

In the following we will present the BER and FER performances of regular and irregular QC LDPC codes with rate $R = 4/5$ and lengths about 1000, 2000, and 4000 constructed by applying the algorithm given above to base matrices with a good degree distribution and compare those results with QC LDPC codes from [9] with the same rate and lengths. Using the greedy algorithm in [10] we constructed a set of base matrices of size $r \times 5r$, $r = 3, 4, \ldots, 7$, with good degree distributions including weights from 2 up to r. By using the proposed algorithm, we obtained suitable labelings for these base matrices, yielding an improved girth profile $\widetilde{\boldsymbol{g}}^p$, from which we selected the ones with the best BER and FER performances. In Fig. 2, BER and FER performances for four QC LDPC codes of rate $R = 4/5$ and lengths about 4000 (Table I) are compared. The corresponding code parameters, such as the girth g of the corresponding Tanner graph, girth profile $\widetilde{\boldsymbol{g}}^p$, sliding-window girth g_w, sliding-window size L_w, and degree distribution λ_s (for irregular codes), are presented in Table I. Clearly, irregular codes have better

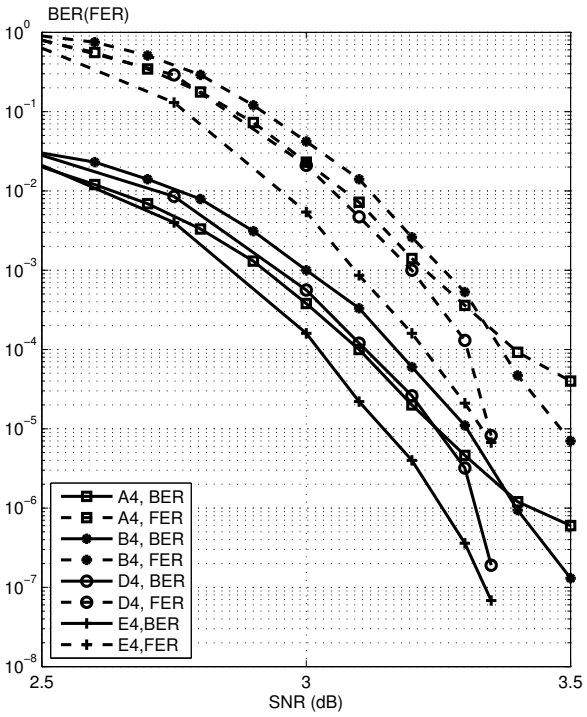

Fig. 2. Comparison of bit and frame error rates for QC LDPC codes of length 4000 with different base matrices and good labelings.

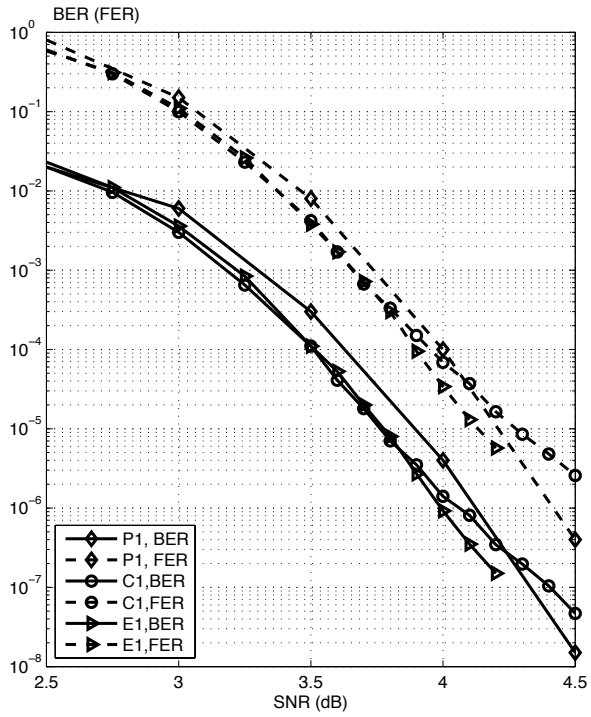

Fig. 3. Bit and frame error rates for the QC LDPC block codes of $R = 4/5$ and length 1000.

BER and FER performances than their regular competitors for relatively low SNRs. However, for higher SNR values the $(J = 4, K = 20)$-regular QC LDPC code of rate $R = 16/20$ yields surprisingly good BER and FER performances, while the $(J = 3, K = 15)$-regular rate $R = 12/15$ QC LDPC code has significantly worse performances even for higher SNRs. Among all irregular QC LDPC codes of length about 4000, the best performances are obtained by using the QC LDPC code of rate $R = 28/35$.

The high efficiency of the irregular rate $R = 28/35$ QC LDPC code can be explained as follows: The girth of the Tanner graph corresponding to the submatrix which contains all columns with two nonzero elements and some of the columns with three nonzero elements is equal to 12, while using all other columns with three nonzero elements we obtain a girth larger than or equal to 10 (see the girth profile in Table I). Moreover, for a sliding-window size of $L_w \geq 3$ among all columns with four nonzero elements, the sliding-window girth follows as $g_w = 8$. Finally, all remaining columns have a stronger protection due to their relatively large column weight. Thus, most of the cycles corresponding to $g = 6$ include code symbols which participate in at least 5 different parity-checks, yielding an overall improved BER and FER performance.

In Figs. 3–5, the simulated BER and FER performances of the constructed QC LDPC codes (Table I) of lengths about 1000, 2000, and 4000, respectively, are compared with the best known competing codes [9]. Typically, the selected irregular codes whose base matrix has a good degree distribution yield better BER and FER behavior for lower SNRs and lose compared to the regular codes with column weight larger than 3 for higher SNRs. In a wide range of SNRs, QC LDPC codes of lengths 1000 and 4000 outperform the best previously known codes of the same rate and lengths. For lengths about 2000 we were unable to find any competing QC LDPC code of rate $R = 4/5$.

TABLE I
PARAMETERS OF RATE $R = 4/5$ QC LDPC BLOCK CODES

Code	Base matrix H	n, k	g, g_w, L_w	$\boldsymbol{g}^p$
A1	3×15	(1005,804)	6,8,9	10(4),8(9),6(15)
A2	all ones	(2010,1608)	8,10,4	10(5),8(15)
A4		(4005,3204)	8,10,6	12(5),10(8) ,8(15)
B1	4×20	(1000,800)	6,8,6	8(7),6(15)
B2	all ones	(2020,1616)	6,8,9	8(10),6(15)
B4		(4000,3200)	6,8,13	8(13),6(15)
C1	5×25,	(1000,800)	6,8,4	8(16),6(25)
C2	2(4),3(10),	(2025,1620)	6,8,7	8(19)6(25)
C4	4(5),5(6)	(4000,3200)	6,8,10	12(7),10(14),8(21),6(25)
D1	6×30,	(1020,816)	6,8,4	12(6),10(8),8(9),6(30)
D2	2(5),4(15),	(2010,1608)	6,8,8	12(7),10(8),8(11),6(30)
D4	5(6),6(4)	(4020,3216)	6,8,7	12(9), 10(10),8(20),6(30)
E1	7×35,	(1015,812)	6,8,2	10(15),8(22),6(35)
E2	2(6),3(11),	(2030,1624)	6,8,3	12(9),10(17),8(24)
E4	4(7),6(7),7(4)	(4025,3220)	6,8,3	12(11),10(18),8(21),6(35)
R1	3×15	(1005,804)	$g = 6$	
R2	all ones	(2010,1608)	$g = 6$	
R4	random	(4005,3204)	$g = 8$	
P1	[9], ex. 1	(961,765)	$g = 6$	
P4	[9], ex. 8	(3969,3213)	$g = 6$	

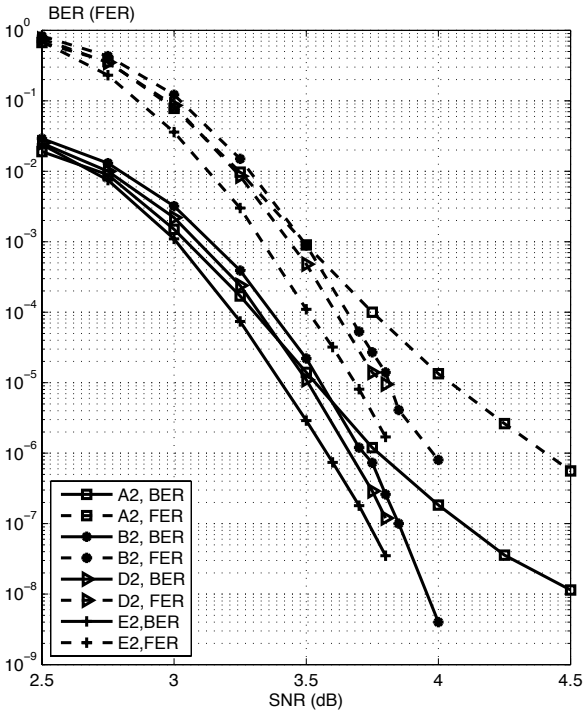

Fig. 4. Bit and frame error rates for the QC LDPC block codes of $R = 4/5$ and length 2000.

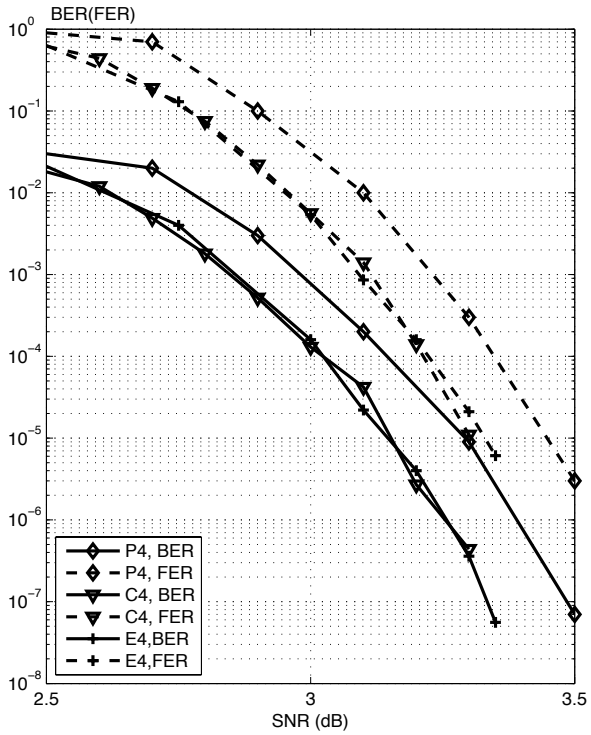

Fig. 5. Bit and frame error rates for the QC LDPC block codes of $R = 4/5$ and length 4000.

V. Conclusion

In this paper we suggested a computer-search based approach to improve the girth profile of QC LDPC codes. This approach consists of selecting QC LDPC codes with large girth values for subsets of columns of the code parity-check matrix. QC LDPC codes with different degree distributions of their base matrices were analyzed. A set of both regular and irregular QC LDPC codes of rate $R = 4/5$ of rather short lengths with good degree distributions and improved girth profile were constructed. Comparisons of the simulated BER and FER performances of the selected codes show that irregular codes are typically better for lower SNRs and lose compared to regular codes with column weight larger than 3 for higher SNRs. However, increasing the size of the base matrix gives more possibilities for improving its structure and could yield better irregular codes. This study brought us to the conjecture that, besides the girth profile and the degree distribution, the maximum column weight is an important parameter for QC LDPC codes. In particular, we note that even regular QC LDPC codes with larger column weight yield surprisingly good BER and FER performances.

Acknowledgements

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