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A SCALE SEPARATION METHOD FOR LARGE FINITE PERIODIC STRUCTURES

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Abstract: We discuss the possibility of using codes developed for infinite periodic structures to analyze (large) finite periodic structures. The idea is based on the interpretation of the Floquet-Bloch transformation as a two-scale representation of the structure.

1 INTRODUCTION

Even though the expression "finite periodic structures" is a contradiction in terms, it is well understood that what is meant is a truncated periodic structure. Such geometries occur frequently in antenna technology (array antennas or frequency selective surfaces) and material modelling (composite materials with a periodic microstructure), since the structures have a local periodicity on one length scale, but are necessarily finite on a larger scale.

There are many more or less standard methods developed for solving the infinite case, that is for solving partial differential equations with periodic boundary conditions. Examples are the finite element method (FEM), method of moments (MoM), and finite differences in the time domain (FDTD). Physically, we expect that the infinite solution should be a good starting point for calculating the solution, and the finiteness should be a relatively small correction to this.

The downside with this hypothesis is that we know it is wrong. The reason it breaks down is that for certain frequencies, surface waves can propagate along the periodic structure. These correspond to freely propagating waves in the infinite periodic structure, whose amplitudes can only be determined from analysis of the finite structure [2]. This means the correction to the infinite case from the finite size may not be small, but instead on the same order of magnitude, possibly leading to large standing waves in the structure if they are not designed properly. In this contribution, we demonstrate a possible means of treating this situation.

2 FLOQUET-BLOCH TRANSFORMATION

Let an infinite periodic structure be described by lattice vectors $\boldsymbol{a}_n$ and reciprocal lattice vectors $\boldsymbol{b}_n$, such that $\boldsymbol{a}_n \cdot \boldsymbol{b}_{n'} = 2\pi\delta_{nn'}$, where $\delta_{nn'}$ is the Kronecker delta. The lattice vectors $\boldsymbol{a}_n$ make up the edges of the physical unit cell U, and the reciprocal lattice vectors $\boldsymbol{b}_n$ make up the edges of the reciprocal unit cell U'. Any field $\boldsymbol{E}(\boldsymbol{x})$ can then be represented by its Bloch amplitude $\widetilde{\boldsymbol{E}}(\boldsymbol{x},\boldsymbol{k})$ as

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{|U'|}\int_{U'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\widetilde{\boldsymbol{E}}(\boldsymbol{x},\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} \tag{1}$$

where $\widetilde{\boldsymbol{E}}(\boldsymbol{x},\boldsymbol{k})$ is U-periodic in $\boldsymbol{x}$ and $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\widetilde{\boldsymbol{E}}(\boldsymbol{x},\boldsymbol{k})$ is U'-periodic in $\boldsymbol{k}$. The Bloch amplitude can be defined either as a sum in the physical lattice or as a sum in the reciprocal lattice,

$$\widetilde{\boldsymbol{E}}(\boldsymbol{x},\boldsymbol{k}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \boldsymbol{E}(\boldsymbol{x}+\boldsymbol{x_n})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x_n}} = \frac{1}{|U|}\sum_{\boldsymbol{n}\in\mathbb{Z}^d} \widehat{\boldsymbol{E}}(\boldsymbol{k}+\boldsymbol{k_n})\mathrm{e}^{\mathrm{i}\boldsymbol{k_n}\cdot\boldsymbol{x}} \tag{2}$$

where $\widehat{\boldsymbol{E}}(\boldsymbol{k})$ is the standard Fourier transform of $\boldsymbol{E}(\boldsymbol{x})$, $\boldsymbol{x_n} = n_1\boldsymbol{a}_1 + \cdots + n_d\boldsymbol{a}_d$, and $\boldsymbol{k_n} = n_1\boldsymbol{b}_1 + \cdots + n_d\boldsymbol{b}_d$, with d being the dimension of the lattice (usually $d = 2$ for an array antenna, and $d = 3$ for a crystalline material). We can interpret the Floquet-Bloch transformation as a means of describing a structure in terms of a microscopic variable $\boldsymbol{x}$ for scales smaller than the unit cell, and a macroscopic variable $\boldsymbol{k}$ for scales larger than the unit cell.

3 ITERATIVE SOLUTION AND CHARACTERIZATION OF SURFACE WAVES

The finiteness of the structure can be taken into account by defining a projection operator,

$$P\widetilde{\boldsymbol{E}} = \widetilde{\zeta} \overset{U'}{*} \widetilde{\boldsymbol{E}} = \frac{1}{|U'|}\int_{U'} \widetilde{\zeta}(\boldsymbol{x}, \boldsymbol{k} - \boldsymbol{k}')\widetilde{\boldsymbol{E}}(\boldsymbol{x}, \boldsymbol{k}')\,\mathrm{d}\boldsymbol{k}' \tag{3}$$

where $\widetilde{\zeta}(\boldsymbol{x}, \boldsymbol{k})$ is the Bloch amplitude of the characteristic function (or window function) $\zeta(\boldsymbol{x})$ of the finite periodic structure, that is, $\zeta(\boldsymbol{x}) = 1$ when $\boldsymbol{x}$ is inside the structure, and $\zeta(\boldsymbol{x}) = 0$ when $\boldsymbol{x}$ is outside. This means the equation for the currents becomes

$$ZI = V \quad \Rightarrow \quad \boxed{P\widetilde{Z}P\widetilde{I} = P\widetilde{V}} \tag{4}$$

where $\widetilde{Z}(\boldsymbol{k})$ is the impedance matrix for the infinite periodic structure at phase shift $\boldsymbol{k}$. We expect that for a large structure, the operator $P\widetilde{Z}P$ is "close" to $\widetilde{Z}$, that is, the operator for the infinite case. By rewriting this equation as $\widetilde{I} - (1 - \widetilde{Z}^{-1}P\widetilde{Z}P)\widetilde{I} = \widetilde{Z}^{-1}P\widetilde{V}$, we obtain

$$\widetilde{I} = [1 - (1 - \widetilde{Z}^{-1}P\widetilde{Z}P)]^{-1}P\widetilde{V} = \sum_{n=0}^{\infty}(1 - \widetilde{Z}^{-1}P\widetilde{Z}P)^n P\widetilde{V} \tag{5}$$

where we used the Neumann series for the inverse, expecting that $1 - \widetilde{Z}^{-1}P\widetilde{Z}P$ is a small operator. This leads to the iterative scheme

$$\widetilde{I} = \sum_{n=0}^{\infty}\widetilde{I}_n, \quad \widetilde{Z}\widetilde{I}_0 = P\widetilde{V}, \quad \widetilde{Z}\widetilde{I}_{n+1} = (\widetilde{Z} - P\widetilde{Z}P)\widetilde{I}_n \tag{6}$$

The first term in this series corresponds to the windowing technique [1, 4]. We have written it in a form without inverses on the impedance matrix to emphasize that we are usually not computing the inverse of the impedance matrix, but rather solving the linear system of equations it defines. For large systems with many degrees of freedom, this is typically done in an iterative manner for each fixed $\widetilde{Z}(\boldsymbol{k})$. Thus, as soon as the impedance matrices are computed and means of solving associated equations are defined, the extra computational effort to take the finite periodic structure into account is not great. Indeed, what we have outlined is a means of factorizing the problem into independent parts (one for each $\boldsymbol{k}$), which is highly suited for parallel computations.

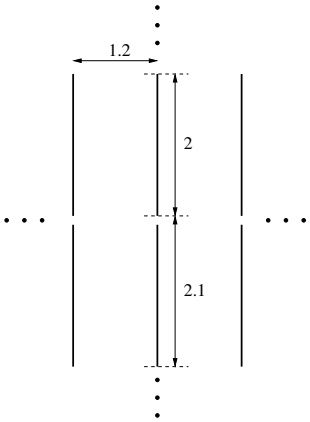
Obviously, the solution of the equation $\widetilde{Z}\widetilde{I} = \widetilde{V}$ cannot be unique if there is a nullspace in $\widetilde{Z}$, that is, there exist solutions to the equation

$$\widetilde{Z}\widetilde{I} = 0 \tag{7}$$

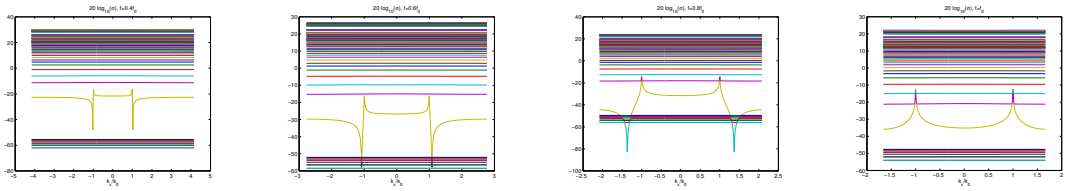
These solutions are precisely the surface waves. The equation can be satisfied only for certain phase shift vectors $\boldsymbol{k}$. Typically, for an isotropic structure surface waves exist only for $\boldsymbol{k} \in U'$ satisfying a condition of the kind $|\boldsymbol{k}|/\omega = \text{const}$, which can be considered as a dispersion relation for the surface wave.

The surface waves can also be characterized by a singular value decomposition of $\widetilde{Z}$, where the existence of surface waves corresponds to the smallest singular value being zero, $\sigma_1(\boldsymbol{k}) = 0$. An integral equation can then be formulated for the subspace corresponding to the surface waves [3].

4 NUMERICAL RESULTS

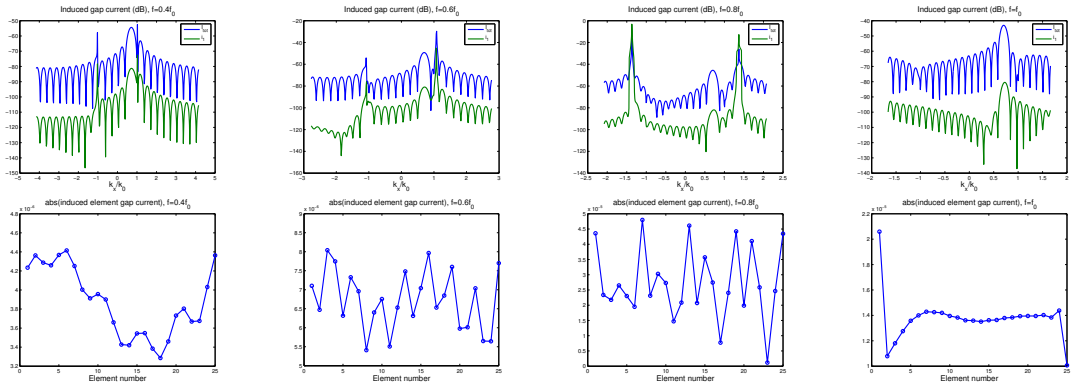


To demonstrate the effects of the presence of surface waves, we study an array of dipoles as indicated in the figur to the right [2]. The array is infinite in the y-direction and has 25 elements in the x-direction. We have computed the singular values for the (infinite periodicity case) impedance matrices $\widetilde{Z}(\boldsymbol{k}) = \widetilde{Z}(k_x\hat{\boldsymbol{x}})$ as functions of $k_x \in (-\pi/a, \pi/a)$, where a is the element spacing in the x direction. This is done for a number of frequencies as shown below, where f_0 is the frequency corresponding to a half-wavelength dipole. The singular values are shown in dB scale.



$f = 0.4f_0$ $\quad$ $f = 0.6f_0$ $\quad$ $f = 0.8f_0$ $\quad$ $f = f_0$

As can be seen, there is a clear macroscopic degree of freedom corresponding to the nonconstant curve in these figures. For the frequency $f = 0.8f_0$, the singular values are small enough to represent a numerical surface wave, which is evident from the oscillating nature of the currents induced in the different elements depicted below. These figures depict the spectral (top) and spatial (bottom) distribution of the currents in the array when illuminated by a plane wave of $45°$ incidence.



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