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Protograph Design for Spatially-Coupled Codes to Attain an Arbitrary Diversity Order

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***Abstract*—This work focuses on the design of SC-LDPC codes for transmission over non-ergodic, block-fading channels. Our main contribution is an algorithm, allowing to start from a (J, K)-regular, uncoupled LDPC ensemble, from which one can recursively build up a protograph-based SC-LDPC ensemble having any target diversity order d. The diversity order is achieved assuming a low-complexity iterative decoding algorithm. The increase of d comes at the cost of increasing the memory constraint (i.e., the coupling parameter) of the SC-LDPC ensemble.**

I. Motivation: Codes of Flexible Diversity Order

The mobile-radio channel can be modelled as a slow, flat fading together with additive noise. In many cases (e.g., short-range, high-throughput data communications), the channel coherence interval (time where the channel fading is constant) is much longer than one symbol duration. Thus several symbols are affected by the same fading coefficient. An example of such a channel model is the *block-fading* channel introduced in [1]. In the block-fading channel, coded information is transmitted over a finite number of coherence intervals to provide *diversity*. The diversity order d of the code is an important parameter that gives the slope of the word error rate (WER) of the decoder.

In this paper, we consider transmission of a sequence of LN coded bits through a block-fading channel with a coherence interval of N bits. Hence, the large diversity $d \leq L$ can be achieved, but at a cost of using a long block code or a long interleaver (large latency). Another possibility would be to use N convolutional codes in parallel, with a memory constraint $m_{\mathrm{cc}} \ll L$. This would provide a relatively good latency but, unfortunately, would result in the performance degradation or require an increase of the decoder complexity.

In order to obtain a better trade-off between the decoding latency, decoding complexity and the diversity order d, we propose to use spatially coupled low-density parity-check (SC-LDPC) codes [2], [3]. It was observed in [3] that SC-LDPC codes, decoded using a latency constrained window decoder, have very good performance over the block-fading channel. This work is our investigation on a systematic design of SC-LDPC codes if some targeted (but arbitrary) diversity order d is required. In what follows, we a) study which maximum d is achievable for a given (J, K) SC-LDPC ensemble, and b) propose a protograph-based construction of SC-LDPC codes that can achieve an arbitrary d, at the cost of an increasing value of the coupling width and, thus, of the memory constraint m_{cc} for the underlying convolutional structure. In order to achieve the points above, an explicit connection between block stopping sets and diversity is established.

As a comparison reference for our codes, root-LDPC block codes are considered [4]. This is the most known construction of block-codes for non-ergodic channels, which motivated a number of further results in this area [5], [6]. However, all the existing constructions are designed for a single specified value of d (in most of cases $d = 2$), and will not work well if the number of fading coefficients per codeword changes. Furthermore, the boundedness of the root structure, on which they are built on, does not allow to treat an arbitrary d without blowing up the number of node classes in their multi-edge graph structure.

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II. System Model and Performance Metric

A. Channel Model

Assume transmission of a sequence of LN coded bits through a block-fading channel with a coherence interval of N bits. For such block-fading, the received symbols y_i are

$$y_i = \alpha_j x_i + n_i, \quad i = 0, \ldots, LN-1, \text{ and } j = \lfloor i/N \rfloor. \quad (1)$$

The input symbols x_i are chosen from the BPSK alphabet $\{\pm 1\}$, n_i are Gaussian random variables with zero mean and variance σ_n^2, and the fading coefficients α_j are Rayleigh distributed with $\mathbb{E}[\alpha_j^2] = 1$. Hence the signal-to-noise ratio γ of the received symbols is characterized only by σ_n^2.

B. SC-LDPC Coding Scheme

Let LN coded bits, sent through the block-fading channel as described above, be generated using a SC-LDPC code of overall code length LN (i.e., containing L coupled LDPC codewords of length N). Also, without loss of generality, let the bits, belonging to the same coupled LDPC codeword, be sent within the same coherence interval. The related SC-LDPC code ensemble is defined as follows.

A protograph P of an (uncoupled) LDPC ensemble is a bipartite graph consisting of n_c check sets' (CS) and n_v

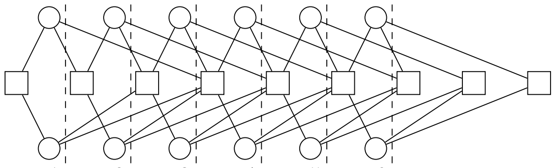


Fig. 1. Convolutional protograph P_c of Example 1. Circles (boxes) represent variables (check) set nodes. t's represent LDPC block instances.

variable sets' (VS) nodes. Each node in a protograph therefore represents a subset of variable or check nodes of the bipartite graph for any LDPC code from the ensemble. Edges in P establish a structure of connections which are allowed in a bipartite graph of the LDPC code (i.e., a variable node from subset V_i can be connected to a check node from subset C_j iff there is an edge between corresponding VS and CS nodes in P). P is usually given by its *base matrix* $\mathbf{B}$ of size $n_c \times n_v$. E.g., the protograph of a $(3,6)$-regular LDPC ensemble is $\mathbf{B} = [3,\ 3]$. Thus, $n_v = 2$, $n_c = 1$, and each check node from the unique set C_1 is connected three times to variable nodes from V_1 and three times – to V_2.

Basing on P, let us define a *convolutional protograph* P_c, having a memory constraint m_{cc}, $m_{\text{cc}} \geq 1$, and describing the structure of the SC-LDPC ensemble with L coupled LDPC blocks. Let $\mathbf{B}_0, \ldots, \mathbf{B}_{m_{\text{cc}}}$ be $n_c \times n_v$ matrices with elements from $\mathbb{N}$, s.t. $\sum_{i=0}^{m_{\text{cc}}} \mathbf{B}_i = \mathbf{B}$. Then, P_c is described by means of the following base matrix $\mathbf{B}_{[1,L]}$ of size $(L+m_{\text{cc}})n_c \times Ln_v$:

$$\mathbf{B}_{[1,L]} = \begin{bmatrix} \mathbf{B}_0 & & \\ \vdots & \ddots & \\ \mathbf{B}_{m_{\text{cc}}} & & \mathbf{B}_0 \\ & \ddots & \vdots \\ & & \mathbf{B}_{m_{\text{cc}}} \end{bmatrix}_{(L+m_{\text{cc}})n_c \times Ln_v} \quad (2)$$

Note that the submatrix $\mathbf{B}_i$, $0 \leq i \leq m_{\text{cc}}$ at time t, $1 \leq t \leq L$, defines the connection from LDPC block t to LDPC block $t+i$. Also, P_c can be given by a bipartite graph, whose connections are drawn accordingly to $\mathbf{B}_{[1,L]}$:

Example 1: Let $\mathbf{B} = [3,\ 3]$, and $\mathbf{B}_0 = \mathbf{B}_3 = [1,\ 1]$, $\mathbf{B}_1 = [1,\ 0]$, $\mathbf{B}_2 = [0,\ 1]$, i.e., $m_{\text{cc}} = 3$. Then the respective bipartite graph of P_c with $L = 6$ coupled blocks is presented in Fig. 1.

C. Definition of the Code Diversity

One of the main performance measures of a code family $\mathcal{F}$ over the block-fading channel is the diversity order d ([7]):

$$d = \sup_{C \in \mathcal{F}} \lim_{\gamma \to \infty} \frac{-\log P_e(\gamma, C)}{\log \gamma}. \quad (3)$$

Here C is a code belonging to $\mathcal{F}$, $P_e(\gamma, C)$ is the error probability of code C, under optimal (ML) decoding[1]. Then d is related to the *blockwise minimum Hamming distance* $d_{\min}(N)$ of the code family $\mathcal{F}$ as follows. If a codeword $\mathbf{c}$ from C, $C \in \mathcal{F}$, is affected by F fading gains in such a way that (a) the coherence interval is of size N bits, and (b) $\omega_i(\mathbf{c})$ is the Hamming weight of coded bits affected by the i-th fading values, then [4]

$$d_{\min}(N) = \min_{\mathbf{c} \in C \setminus \mathbf{0},\ C \in \mathcal{F}} |\{\omega_i(\mathbf{c}) \neq 0\}|. \quad (4)$$

Therefore, for a code of diversity/blockwise minimum distance $d_{\min}$, $d_{\min} - 1$ fades can be perfectly recovered, i.e., $d = d_{\min}$.

Note that the same definitions can be applied to convolutional-like codes (e.g., SC-LDPC codes), as they can be seen as block codes of code length LN. In this case, the diversity order d, $1 \leq d \leq L$ will be a function of the *memory* of the convolutional encoder m_{cc} rather than a function of L.

As the main focus of our work is on LDPC codes, decoded iteratively, let us also define a diversity order d^{IT} *under iterative decoding*, which is given by $d^{\text{IT}} = D$, with D being the smallest number of deep fades (i.e., the number of $\alpha_j = 0$) that cannot be recovered under iterative decoding, in the limit of high SNRs (i.e., assuming $\sigma_n^2 = 0$). Clearly, for any code family $\mathcal{F}$, $d^{\text{IT}} \leq d$.

III. Bounds on the Diversity Order for SC-LDPC Codes

A. Estimation of d

Let us evaluate the blockwise minimum distance $d_{\min}$ of a SC-LDPC ensemble and find the maximum possible value of d. Under the channel model discussed in Section II, $d_{\min}$ of interest is in fact $d_{\min}(N)$. Given that one LDPC block of length N in the SC-LDPC code contains bits related to exactly n_v VS nodes, then, instead of searching $d_{\min}(N)$ based on the parity matrix of the SC-LDPC code, one can equivalently search for $d^{\mathbf{B}}_{\min}(n_v)$ based on $\mathbf{B}_{[1,L]}$. In other words, the calculation of the diversity order can be equivalently done over the protograph P_c of the SC-LDPC ensemble.

The procedure to calculate d comes from the theory of convolutional codes. Here we apply it to SC-LDPC codes:

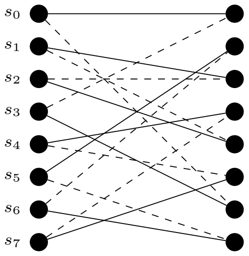
- Consider the trellis representation of $\mathbf{B}_{[1,L]}$, for which each trellis section is labeled with n_v bits. This trellis representation is obtained based on $\mathbf{B}_{[1,L]}$ using the approach from [8].
- According to the structure of one trellis section, define an adjacency matrix $\mathbf{A}$ to be a $2^{n_c(m_{\text{cc}}+1)} \times 2^{n_c(m_{\text{cc}}+1)}$ matrix constructed as follows. If there exists a transition between the state i and the state j in the trellis, then the ij-th element of $\mathbf{A}$ is $x^{\omega(\mathbf{e}_{ij})}$, where $\mathbf{e}_{ij}$ is the transition label, and $\omega(\mathbf{e})$ is the vector Hamming weight of $\mathbf{e}_{ij}$. So, as $\omega(\mathbf{e})$ can only take values 0 or 1, $x^{\omega(\mathbf{e}_{ij})}$ can only be either 1 or x. Finally, if there is no transition between the state i and the state j, $\mathbf{A}_{ij} = 0$.
- Given both-side termination for the SC-LDPC ensemble, its Hamming weight distribution is $W(x) = [\mathbf{A}^L]_{0,0}$.
- The minimum Hamming distance is the smallest non-zero power in $W(x)$. So,

$$d = \min_{i>0} \{i \in \mathbb{N} : \text{coeff}(W(x), x^i) \neq 0\}.$$

[1]In order to simplify the analysis, this paper assumes that P_e is computed over the *whole codeword*, and not over the information part of it, as it was previously assumed in [4]–[6]. So, the values of d, obtained in the present paper, are in fact lower bounds (one might still increase the diversity for information bits by placing them carefully within the codeword).

Note that, as we are interested not to find the whole $W(x)$, but only d, we might simplify the calculation of $\mathbf{A}^L$ by keeping track only of a few polynomial terms with the smallest power and by neglecting all others.

Example 2: Assume the $(3,6)$-regular SC-LDPC ensemble, defined in Example 1. As a first step, a section of a syndrome trellis corresponding to $\mathbf{B}_{[1,L]}$ is drawn in Fig. 2 (left).



$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 1 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 1 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x & 0 & x & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & x \\ 0 & x & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & x & 0 & x & 0 & 0 \end{pmatrix}$$

Fig. 2. Trellis section (left) and corresponding matrix $\mathbf{A}$ (right). Solid (resp. dashed) transitions for the trellis section represent transitions for which the related information bit $= 0$ (resp. $= 1$).

Further, the corresponding matrix $\mathbf{A}$ is given in Fig. 2 (right). The minimum Hamming distance can then be calculated by computing the L'th power of $\mathbf{A}$, $W(x) = 1 + 24x^4 + O(x^5)$. Hence in this case, $d = d_{\min} = 4$.

B. Estimation of d^{IT}

As SC-LDPC codes are decoded using a low-complexity iterative decoding, we are mainly interested in d^{IT} (while d serves as an upper bound on d^{IT}). d^{IT} is related to the *blockwise minimum stopping distance* $s_{\min}(N)$ of the SC-LDPC ensemble. Similarly as for $d_{\min}(N)$, the diversity is completely determined by the protograph, i.e., $s_{\min}(N) = s^{\mathbf{B}}_{\min}(n_v)$, where $s^{\mathbf{B}}_{\min}(n_v)$ is related to P_c and is defined below.

Definition 1 (Blockwise stopping set): A blockwise stopping set in a protograph P_c is a subset $\mathcal{S}$ of VS nodes in P_c such that: 1) if a VS node at time t belongs to $\mathcal{S}$, then any other VS node at t also belongs to $\mathcal{S}$; 2) VS nodes from $\mathcal{S}$ are connected to a set $\mathcal{C}$ of CS nodes, and each node from $\mathcal{C}$ is connected at least twice. $s^{\mathbf{B}}(n_v) = |\mathcal{S}|/n_v$ denotes the size of the blockwise stopping set $\mathcal{S}$ (in blocks of n_v VS nodes). Note that, by condition 1) above, if bits of the t-th LDPC block in the SC-LDPC code are related to a class V_i, $i = 1, \ldots, n_v$, and they are in deep fade, then all the other bits in this codeword are also in deep fade, because they belong to the same coherence interval. Moreover, if the related VS nodes of $\mathcal{S}$ satisfy the condition 2) above, then they cannot be corrected by iterative decoding.

Definition 2 (Blockwise minimum stopping distance): The blockwise minimum stopping distance $s^{\mathbf{B}}_{\min}$ is the smallest size of the stopping set in P_c.
Note that, with a blockwise minimum stopping distance $s^{\mathbf{B}}_{\min}$,

$$d^{\text{IT}} = s^{\mathbf{B}}_{\min}.$$

Example 3: Consider P_c from Fig. 1. Then $s^{\mathbf{B}}_{\min} = 2$. To check it, Fig. 3 shows the induced graph when the blocks at

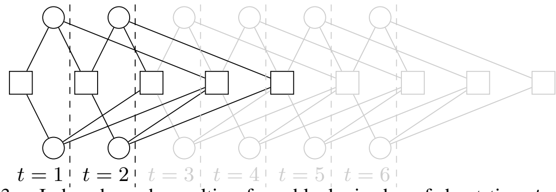


Fig. 3. Induced graph resulting from blocks in deep fade at time $t = 1, 2$ of P_c in Fig. 1. The highlighted nodes and edges form a stopping set.

time $t = 1$ and 2 are in deep fade: all the CS nodes in the graph are connected at least twice to the set of VS nodes.

In order to find $s^{\mathbf{B}}_{\min}$ for a given protograph-based SC-LDPC ensemble, one might use the exhaustive search of stopping sets, by using peeling decoder, applied over P_c. Some bounds on $s^{\mathbf{B}}_{\min}$ are also available in [9].

Notice that, for any n, $s_{\min}(n) \leq d_{\min}(n)$ and also $s^{\mathbf{B}}_{\min}(n) \leq d^{\mathbf{B}}_{\min}(n)$. In this work we aim to design the protographs such that $s^{\mathbf{B}}_{\min}(n_v)$ can be increased to $d^{\mathbf{B}}_{\min}(n_v)$, hence approaching the ML decoding performance by using the sub-optimal BP decoding.

IV. Protograph Design for a Targeted d^{IT}

At our knowledge, there does not exist a method to design a code with a fixed (but arbitrary) diversity order. Even for convolutional codes, supposed to be well-studied, the search of a code with given $d_{\min}$ is performed by computer search (or using available results of the computer search). But for block codes, the situation is even worse, and only few, very particular code designs, are proposed in the literature.

In this section, we propose a recursive algorithm for generating a class of codes with increasing diversity order d^{IT}. The algorithm generates the protograph for a SC-LDPC code. The advantage of working with d^{IT} instead of d is in fact that one can use a low-complexity iterative decoder, instead of the optimum ML decoder. From another side, as $d^{\text{IT}} \leq d$, it might be a challenging task to obtain a high enough value for d^{IT}.

For simplicity, we describe the construction through an example of a $(3,6)$ SC-LDPC code with $\mathbf{B} = [3\ \ 3]$.

A. Initialization

In the initialization step, we first define $\mathbf{B}_i$, with i varying from 0 up to some $l > 0$, that fulfill the constraint on m_{cc}:

Definition 3 (Block locality): Assume that a block $\mathbf{v}_t$ at time t with n_v VS nodes is in deep fade. Locality is defined as the minimum number of component matrices l, required to recover n_v VS nodes.

In order to guarantee some diversity $d^{\text{IT}} \geq 2$, one should choose a protograph P_c with $m_{cc} \geq l$. This implies a minimum number of component matrices $\mathbf{B}_i$s which should be chosen.

Proposition 1: Given a SC-LDPC code with a base matrix $\mathbf{B}_{[1,L]}$, constructed using a matrix $\mathbf{B}$ of size $n_c \times n_v$, and considering the channel model as in (1), at least $l = \left\lceil \frac{n_v}{n_c} \right\rceil$ components $\mathbf{B}_i$s are required in order to achieve $d^{\text{IT}} = 2$.

Proof: Let $\mathbf{I}_0 = \mathcal{G}(\mathcal{C}, \mathcal{V})$ represent an induced graph consisting of a set of check and variable nodes $\mathcal{C}$ and $\mathcal{V}$, respectively. Then, choose $\mathbf{B}_i$s such that there exists at least one degree 1 check node $c_r \in \mathcal{C}$, such that $\mathbf{I}_0 = \mathcal{G}(\mathcal{C} \backslash c_r, \mathcal{V} \backslash \{v_r \in \mathcal{N}(c_r)\})$

recursively contains at least one degree 1 check node. Here $\mathcal{N}(c_r)$ denotes the set of variable nodes connected to check node c_r. Now, note that each check node in a component $\mathbf{B}_i$ can be a root to only one variable node, i.e., at least n_v check nodes are required to be of degree 1. Now, since each $\mathbf{B}_i$ consists of n_c check nodes, therefore $l \geq \left\lceil \frac{n_v}{n_c} \right\rceil$.

Continuing the example of the $(3, 6)$ SC-LDPC ensemble, there are two distinct possibilities to select the component matrices fulfilling the above constraint on m_{cc}

Option 1: $\mathbf{B}_0 = [1,\ 1], \mathbf{B}_1 = [0,\ 1], \mathbf{B}_{m_{cc}} = [2,\ 1]$,
Option 2: $\mathbf{B}_0 = [1,\ 0], \mathbf{B}_1 = [0,\ 1], \mathbf{B}_{m_{cc}} = [2,\ 2]$.

For the purpose of this example, let us choose the first option.

B. Splitting

In the following, the goal is to increase the diversity. This is accomplished in two steps which will be recursively applied. The first step is to split the $\mathbf{B}_{m_{cc}}$ into two parts, such that we use the minimum number of edges. Let us denote the first part as $\mathbf{B}_s$ and the second part is $\mathbf{B}_{m_{cc}}$. The $\mathbf{B}_s$ must be chosen such that together with $\mathbf{B}_0$, it also satisfies the block locality. One choice for the $\mathbf{B}_s$ in this case is $\mathbf{B}_s = [1,\ 0]$. The resulting edge spreading is then given as:

$$\mathbf{B}_0 = [1,\ 1], \mathbf{B}_1 = [0,\ 1], \mathbf{B}_s = [1,\ 0], \mathbf{B}_{m_{cc}} = [1,\ 1].$$

C. Add All-Zero Matrix

The splitting step here is not intended to change the diversity order, or at least must not decrease d^{IT}. Therefore, a second step is required to increase d^{IT}. This is accomplished by inserting an all zero matrix[2] (e.g., here $[0,\ 0]$). This makes sure that all combinations of 2 block erasures are covered either by $\{\mathbf{B}_0,\ \mathbf{B}_1\}$ or $\{\mathbf{B}_0,\ \mathbf{B}_s\}$. Note that $\{\mathbf{B}_0,\ \mathbf{B}_1\}$ is responsible to protect against all single block erasures (block locality), and $\{\mathbf{B}_0,\ \mathbf{B}_s\}$ must reduce the problem from two erasures to a single erasure by correcting one deep fade or vice versa. In our example the diversity at this step can be increased to $d^{\text{IT}} = 3$ by inserting a single all zero matrix and is given below,

$$\mathbf{B}_0 = [1\ 1], \mathbf{B}_1 = [0\ 1], \mathbf{B}_2 = [0\ 0], \mathbf{B}_3 = [1\ 0], \mathbf{B}_{m_{cc}} = [1\ 1].$$

The recursive process continues until the targeted d^{IT} is obtained. The steps to obtain a family of codes with increasing diversity d^{IT} for $(3, 6)$-regular code with base matrix $\mathbf{B} = [3,\ 3]$ is listed in the Table I. The matrix $\mathbf{B}_s$ is shown in red at each splitting step. Note that, the addition step can add more than one all-zero matrix.

D. Discussion

As mentioned above, d^{IT} is bounded by the blockwise minimum Hamming distance d_{min} of the code, which can be calculated by means of $W(x)$. The weight enumerator corresponding to the final protograph with $d^{\text{IT}} = 5$ is $W(x) = 1 + 9x^5 + O(x^8)$. Thus, $d_{\text{min}} = 5$ and, hence, up to $d - 1 = 4$ deep fades can be perfectly recovered under ML decoding. Hence, the best possible iterative diversity, i.e., $d^{\text{IT}} = d$, is achieved at the final step of the construction.

[2]In general, the all zero matrix can be inserted at any position except the first and the last one, as these two options do not change the protograph structure and, thus, do not increase d^{IT}.

TABLE I
DECOMPOSITION OF A $(3, 6)$-REGULAR CODE.

Step		m_{cc}	d^{IT}	**Component Matrices**	d
0	Init.	2	2	[1, 1][0, 1][2, 1]	
1	Split	3	-	[1, 1][0, 1][1, 0][1, 1]	-
	Add	4	3	[1, 1][0, 1][0, 0][1, 0][1, 1]	4
2	Split	5	-	[1, 1][0, 1][0, 0][1, 0] [0, 1][1, 0]	-
	Add	6	4	[1, 1][0, 1][0, 0][1, 0] [0, 0][0, 1][1, 0]	5
3	Split	-	-	not possible to split	-
	Add	7	5	[1, 1][0, 1][0, 0][1, 0] [0, 0][0, 0][0, 1][1, 0]	5

It is also noteworthy that the code construction does not achieve the upper bound on d^{IT} at each step of the recursive algorithm. This can be easily explained by the fact that at all the steps before the final one, component matrices $\mathbf{B}_i, i = 0, \ldots, m_{cc} - 1$ are designed such that these provide the corresponding iterative diversity d^{IT}. Hence, the maximum in terms of diversity is achieved only when all the component matrices $(\mathbf{B}_i, i = 0, \ldots, m_{cc})$ are involved in providing the diversity order. Since this algorithm recursively approaches the upper bound on the diversity, the component matrices in the intermediate steps are not necessarily achieving the upper bound on iterative diversity (see column 4 and 6 of Table I).

Consider as an example the edge spreading as a result of step 1 in Table I. For this, $d^{\text{IT}} = 3$, while the ML decoding diversity is $d = 4$. If we ignore the component matrix $\mathbf{B}_{m_{cc}}$, the resultant code becomes a $(2, 4)$-regular code with iterative diversity of 3. The weight enumerator function of the resultant $(2, 4)$ protograph is $W(x) = 1 + 27x^3 + O(x^4)$, i.e., ML diversity of 3. This shows that the component matrices excluding $\mathbf{B}_{m_{cc}}$ at each step achieves the upper bound on the iterative diversity i.e., $d^{\text{IT}} = d$.

V. RESULTS AND COMPARISONS

This section presents the results of the designed codes using the proposed algorithm.

A. Some Designed Codes over the Block Erasure Channel (BLEC)

The proposed recursive algorithm above can be used to design the protographs for a wide variety of rates and values of d. It is applied with multiple initial steps to find the codes with maximum diversity for rate $R = 1/2$, $2/3$ and $3/4$ with variable node degree 3 and 4. The results are given in Table II. Let us take an example of a variable node degree 3. It can be seen that the designed protographs achieve the maximum diversity of $d^{\text{IT}} = 6$[3] for all considered rates. However, as the

[3]Note that the maximum diversity of d^{IT} in Table II for $(3, 6)$ code is obtained by starting with Option 2 as described in Section IV-A. Since the Option 2 consumes less edges to fulfill the constraint on m_{cc}, a higher diversity order is possible in this case compared to the Option 1 in Table I.

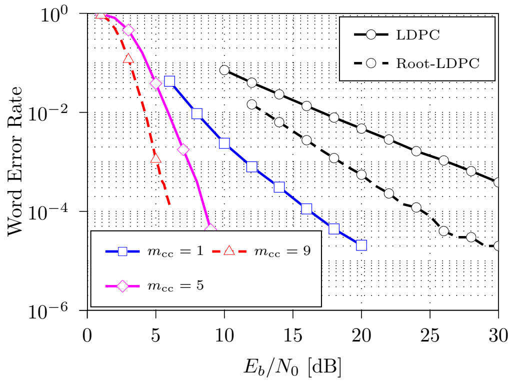


Fig. 4. Density evolution outage over block-fading channel for the $(3,6)$-regular code with increasing m_{cc}. Diversity orders in Table III have been estimated numerically from the slopes of WER curves.

rate increases from $1/2$ to $3/4$, the required m_{cc} to achieve diversity order of $d^{IT} = 6$ also increases from 9 to 23. One of the main reasons for this is the increase in the number of variable node sets (n_v) moving from rate $1/2$ to $3/4$.

TABLE II
CODES AND THE MAXIMUM ITERATIVE DIVERSITY UNDER BLEC.

Code	m_{cc}	d^{IT}	Code	m_{cc}	d^{IT}
$(3,6)$	9	6	$(4,8)$	15	8
$(3,9)$	14	6	$(4,12)$	27	8
$(3,12)$	23	6	$(4,16)$	27	6

Comparing the rate $1/2$ code with variable node degree 3 and 4, it can be seen that d^{IT} depends on the node degree. A maximum of $d^{IT} = 6$ is achieved with a $(3,6)$ code whereas, for a $(4,8)$ code $d^{IT} = 8$ is achievable. Hence we can conclude that the proposed construction provides a way to achieve arbitrary values of diversity given that the node degrees and the memory are allowed to increase. This observation is very similar to the increase of the average minimum distance [10] of the SC-LDPC ensemble, when m_{cc} grows.

B. WER over the Block-Fading Channel

The performance analysis over the block-fading channel is the exactly same as over the BLEC. In order to characterize the codes numerically, a density evolution outage probability is calculated as detailed in [4] [3]. It provides an upper bound on the WER. Fig. 4 shows the density evolution outage (DEO) of the designed protograph for a $(3,6)$-regular code. The results are compared with the DEO for a $(3,6)$ root-LDPC code and a $(3,6)$ uncoupled LDPC block code[4] of the same codelength. Thus, the uncoupled LDPC code, the root-LDPC code and the SC-LDPC code with $m_{cc} = 1$ are the codes, having the same decoding latency, and can be compared to each other (as for the decoding complexity, see Section VI for comments). Table III shows estimated d^{IT} vs. latency for the codes presented in Fig. 4. Note that the estimated diversity in Table III is larger than the computed one from Table II. This comes from the fact that the fading coefficients take soft values between 0 and ∞.

[4]For latency constrained comparison using window decoding, see [3], [11].

The increase of d^{IT} for SC-LDPC codes come at increase of m_{cc} and, thus, of the decoding latency.

TABLE III
ESTIMATED d^{IT} VS. LATENCY FOR VARIOUS $(3,6)$ LDPC CODES

Code	Estimated d^{IT}	Latency(in bits)
Uncoupled LDPC	1.3	$2N$
Root-LDPC	2	$2N$
SC-LDPC, $m_{cc} = 1$	3	$2N$
SC-LDPC, $m_{cc} = 5$	6.7	$6N$
SC-LDPC, $m_{cc} = 9$	10	$10N$

VI. CONCLUSION

SC-LDPC codes are suitable for transmission over a non-ergodic channel [3]. In this work we propose a systematic way of constructing protographs for SC-LDPC codes with any target diversity order d. At the best of our knowledge, this is the first systematic code design, addressing an arbitrary diversity order. Using our code design, any value of d can be obtained, for a moderate value of the memory constraint m_{cc}, so the proposed SC-LDPC codes have a good latency-diversity tradeoff. As for their decoding complexity, we believe that it can be kept (almost) constant, irrespectively of m_{cc}, if one uses the decoding schedules proposed in [12].

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