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Distributed Control Using Positive Quadratic Programming

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Building theoretical foundations for distributed control

We need methodology for
- Decentralized specifications
- Decentralized design
- Verification of global behavior

Example 1: A vehicle formation

\[
\begin{align*}
\dot{x}_1(t) &= 0, \quad x_1(t) = 0, \\
\dot{x}_2(t) &= 0, \quad x_2(t) = 0, \\
\dot{x}_3(t) &= 0, \quad x_3(t) = 0, \\
\dot{x}_4(t) &= 0, \quad x_4(t) = 0
\end{align*}
\]

The objective is to make \( E[Cx_{i+1} - Cx_i]^2 \) small for \( i = 1, \ldots, 4 \).

Example 2: A supply chain for fresh products

Fresh products degrade with time:

\[
\begin{align*}
\dot{x}_1(t) &= 0, \quad x_1(t) = 0, \\
\dot{x}_2(t) &= 0, \quad x_2(t) = 0, \\
\dot{x}_3(t) &= 0, \quad x_3(t) = 0, \\
\dot{x}_4(t) &= 0, \quad x_4(t) = 0
\end{align*}
\]

Example 3: A Wind Farm

\[
\begin{align*}
\dot{x}_1(t) &= 0, \quad x_1(t) = 0, \\
\dot{x}_2(t) &= 0, \quad x_2(t) = 0, \\
\dot{x}_3(t) &= 0, \quad x_3(t) = 0, \\
\dot{x}_4(t) &= 0, \quad x_4(t) = 0
\end{align*}
\]

Example 1: A vehicle formation

Example 2: A supply chain for fresh products

Example 3: A Wind Farm

Outline

- Why Distributed Control?
  - Distributed Control of Positive Systems
  - Example: Optimizing Electrical Power Flow
  - Solution using Positive Quadratic Programming
  - Finding Optimum by Distributed Control

Example 1: A vehicle formation

Example 2: A supply chain for fresh products

Example 3: A Wind Farm
Example 3: A Wind Farm

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Positive Systems and Nonnegative Matrices

Classics:
- Perron (1907) and Frobenius (1912)
- Leontief (1936)
- Hirsch (1985)

Books:
- Gantmacher (1959)
- Berman and Plemmons (1979)
- Luenberger (1979)

Recent control related work:

Stability can be Tested in a Distributed Way

Stability of Positive systems

Suppose the matrix $A$ has nonnegative off-diagonal elements. Then the following conditions are equivalent:

(i) The system $\frac{dx}{dt} = Ax$ is exponentially stable.

(ii) There exists a vector $x > 0$ such that $Ax < 0$. (The vector inequalities are elementwise.)

(iii) There is a diagonal matrix $P > 0$ such that $PA^T + AP < 0$

Example 4: Irrigation Channels

Positive systems have nonnegative impulse response

If the matrices $A$, $B$ and $C$ have nonnegative coefficients except possibly for the diagonal of $A$, then the system

$\frac{dx}{dt} = Ax + Bu$

$y = Cx$

has non-negative impulse response $Ce^{At}B$.

Examples:
- Ecological system with $x_4$ the population of species $k$.
- Chemical reaction with $x_4$ the concentration of reactant $k$.
- Economic system with $x_4$ the quantity of commodity $k$.
- Probabilistic model with $x_4$ the probability of state $k$.

Note: Off-diagonal elements are typically positive!
Suppose the matrices A, B and C have nonnegative coefficients except for the diagonal of A. Suppose A is Hurwitz. Then the following conditions are equivalent:

(i) \( \max_{\omega} |C(i\omega I - A)^{-1}B| < \gamma \)

(ii) \( |CA^{-1}B| < \gamma \)

(iii) There exists \( x > 0 \) such that \( Cx < \gamma, Ax + B = 0 \).

(iv) There is a diagonal matrix \( P > 0 \) such that

\[
PA^T + AP + PC^T CP + \gamma^{-2}BB^T < 0
\]

Note: The linear inequalities (iii) can be tested row by row.

### Synthesizing Positive Systems

\[
A + BL = \begin{bmatrix} a_{11} + \ell_1 & a_{12} & 0 & 0 \\ a_{21} - \ell_1 & a_{22} + \ell_2 & a_{23} & 0 \\ 0 & a_{32} - \ell_2 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}
\]

is stable and nonnegative if and only if \( p_k \geq 0 \) and

\[
(A + BL)P = \begin{bmatrix} (a_{11} + \ell_1)p_1 & a_{12}p_2 & 0 & 0 \\ (a_{21} - \ell_1)(a_{22} + \ell_2)p_4 & a_{22}p_2 + a_{23}p_3 & 0 & 0 \\ 0 & (a_{32} - \ell_2)p_4 & a_{33}p_3 & a_{34}p_4 \\ 0 & 0 & a_{43}p_3 & a_{44}p_4 \end{bmatrix}
\]

make \((A + BL)P + P(A + BL)^T\) negative definite with nonnegative off-diagonal elements. Solve using convex optimization in the pair \((P, PL)\)!

[Tanaka and Langbort, ACC 2010]

### Outline

- Why Distributed Control?
- Distributed Control of Positive Systems
  - Example: Optimizing Electrical Power Flow
    - Solution using Positive Quadratic Programming
    - Finding Optimum by Distributed Control

### Optimal Allocation for Example A

Both transmission lines serving the load need to be used at full capacity to meet the demand \( p_3 = 2 \).

Optimal profit: \( 10p_3 - p_1 = 18 \)

In real power networks, electrons flow according to Kirchhoff's laws. The allocation above is not feasible when all three lines are identical. Why?

### Distributed Control Synthesis

Suppose the matrix

\[
\begin{bmatrix} a_{11} + \ell_1 & a_{12} \\ a_{21} - \ell_1 & a_{22} + \ell_2 \\ 0 & a_{32} - \ell_2 \\ 0 & 0 \end{bmatrix}
\]

is nonnegative for all \( \ell_1, \ell_2 \in [0, 1] \). For stabilizing gains \( \ell_1, \ell_2 \), find \( 0 \leq u_3 \leq u_6 \) such that

\[
\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}
\]

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}
\]

and set \( \ell_1 = u_1/x_1 \) and \( \ell_2 = u_2/x_2 \). Every row gives a local test.

Note: Positivity assumed a priori. What if \( \ell_1, \ell_2 \in \mathbb{R} \)?

### Positivity versus Passivity

- Passivity can be described naturally in frequency domain.
- Positivity can be described naturally in time-domain.
- Negative feedback loops preserve passivity.
- Positive feedback loops preserve passivity.
- Parallel connections preserve both passivity and positivity.
- Series connections preserves positivity, but not passivity.

### Example A: Electrical Power Transmission

Two generators with generation cost 1 and 9 respectively. One load willing to buy \( p_3 = 2 \) at the price 10:

Maximize profit:

\[
10p_3 - 9p_2 - p_1
\]

subject to capacity constraints:

\[
|p_1| \leq 1, p_1 \geq 0, p_2 \geq 0, p_3 \geq 2
\]

and conservation laws:

\[
p_1 = p_2 + p_3 \\
p_2 = p_1 + p_3 \\
p_3 = p_1 + p_2
\]

### Example B: Optimal Potential Flow

Power flow is driven by potential differences:

Maximize profit:

\[
10p_3 - 9p_2 - p_1
\]

subject to capacity constraints:

\[
|u_1 - u_3| \leq 1, p_1 \geq 0, p_2 \geq 2
\]

and conservation laws:

\[
p_1 = (u_4 - u_2) + (u_1 - u_3) \\
p_2 = (u_4 - u_1) + (u_2 - u_3) \\
p_3 = (u_1 - u_3) + (u_2 - u_3)
\]
Optimal Allocation for Example B

Both transmission lines serving the load need to be used at full capacity to meet the demand \( p_3 = 2 \). Hence \( v_1 = v_2 \) and there is no flow between node 1 and node 2!

\[
\begin{align*}
p_1 &= 1 \\
p_2 &= 1 \\
p_3 &= 2
\end{align*}
\]

The optimal profit is much smaller: \( 10p_3 - p_1 - 9p_2 = 10 \)

When transmission lines operate near capacity limits, losses are big. Can we take losses into account in the optimization?

Example C: Optimal Power Flow with Losses

Maximize profit: \( 10p_3 - 9p_2 - p_1 \)

subject to capacity constraints: \( 0 \leq v_i \leq 2 \)

\[
\begin{align*}
p_1 &= v_1(v_1 - v_2) + v_1(v_1 - v_3) \\
p_2 &= v_2(v_2 - v_1) + v_2(v_2 - v_3) \\
p_3 &= v_3(v_3 - v_1) + v_3(v_3 - v_2)
\end{align*}
\]

Profit Versus Power Demand

Power Losses in a DC Transmission Line

For a DC transmission line with admittance \( y \), input voltage \( v_1 \) and output voltage \( v_2 \), we have:

Line current: \( i = y(v_1 - v_2) \)

Injected power: \( p_1 = yv_1(v_1 - v_2) \)

Delivered power: \( p_2 = yv_2(v_1 - v_2) \)

Power loss: \( p_1 - p_2 = y(v_1 - v_2)^2 \)

If the voltages are bounded from above by \( \pi \), there is an upper bound on how much power the transmission line can deliver:

\[
p_2 = yv_2(v_1 - v_2) \leq yv_2(\pi - v_2) \leq y\pi^2/4
\]

At the capacity limit, the power loss equals the delivered power.

Optimal Allocation for Example C

Both transmission lines serving the load need to be used at full capacity to meet the demand \( p_3 = 2 \). Hence \( v_1 = v_2 = \pi \) and there is no current between node 1 and node 2!

\[
\begin{align*}
p_1 &= 2 \\
p_2 &= 2 \\
p_3 &= 2
\end{align*}
\]

There is no room for profit: \( 10p_3 - p_1 - 9p_2 = 0 \)

Notice that half of the generated power is lost in transmission!

Analogies to Electric Power Flow

Water distribution systems: Electrical voltage corresponds to water pressure. Differences in pressure creates flow.

Gas diffusion: Electrical voltage corresponds to partial pressure. Gradients in partial pressure creates diffusion.

Exchange economy: Voltages correspond to inverse prices. Price differences drive commodity flows. Delivered electric power corresponds to delivered commodity volume.

Two kinds of flow of simultaneous interest.

In power transmission networks, electric current is conserved, but electric power is dissipated due to transmission losses.

In economic systems the commodity value is conserved, but the commodity volume is dissipated due to transportation losses.

A General Power Transmission Network

\[
\begin{pmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\mathbf{v}_3 \\
\mathbf{v}_4
\end{pmatrix} =
\begin{pmatrix}
Y_{12} + Y_{14} & -Y_{12} & Y_{12} & 0 \\
-Y_{23} & Y_{21} + Y_{23} + Y_{24} & -Y_{23} & Y_{24} \\
0 & -Y_{32} & Y_{32} & 0 \\
-Y_{42} & -Y_{42} & 0 & Y_{41} + Y_{42}
\end{pmatrix}
\begin{pmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4
\end{pmatrix}
\]

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An Optimal Flow Problem for AC Power

\[
\begin{align*}
I_1 & \to V_1 \\
& \to V_2 \\
& \to V_3 \\
& \to V_4 \\
& \to I_4 \\
& \to I_3 \\
& \to I_2 \\
I_5 & \to V_5 \\
V_6 & \in \mathbb{C} \\
I_6 & \in \mathbb{C}
\end{align*}
\]

Minimize \[ \text{Re} \sum_k I_k^* V_k \]
subject to \[ I = Y V \] and \[ P_k \leq \text{Re} (I_k^* V_k) \leq P_k \]
\[ Q_k \leq \text{Im} (I_k^* V_k) \leq Q_k \]
\[ |V_k| \leq b_k \]
for \( k = 1, \ldots, 4 \)

(Convex relaxation by Lavaei/Low inspired this talk.)

Optimizing DC Power Flow

\[
\begin{align*}
i_1 & \to (1) \\
& \to (3) \\
i_3 & \to (2) \\
i_2 & \to (4) \\
i_4 & \to \in \mathbb{R} \quad v_k \in \mathbb{R}
\end{align*}
\]

Minimize \[ \sum_k i_k v_k \]
subject to \[ i = Y v \] and \[ i_k v_k \leq v_k \]
\[ (v_k - v_j)^2 \leq c_{kj} \]
\[ v_k \leq v_j \leq v_k \]
for all \( k, j \)

Notice: \( v_k \) negative at loads, positive at generators.

Positive Quadratic Programming

Given \( A_1, \ldots, A_K \in \mathbb{R}^{n \times n} \) with nonnegative off-diagonal entries and \( b_1, \ldots, b_K \in \mathbb{R} \), the following equality holds:

\[
\begin{align*}
\max_x x^T A_0 x &= \max_x \text{trace}(A_0 X) \\
\text{subject to} \quad x &\in \mathbb{R}_+^n \\
x^T A_k x &\geq b_k \\
&\quad k = 1, \ldots, K
\end{align*}
\]

Proof

If \( X = \begin{bmatrix} |x_1|^2 & \cdots \\ \cdots & \cdots \end{bmatrix} \) maximizes the right hand side, \( x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \) maximizes the left.

Note: The problem is convex in \( |v_i|^2, \ldots, |v_i|^2 \).

Dual Positive Quadratic Programming

Given \( A_0, \ldots, A_K \in \mathbb{R}^{n \times n} \) with nonnegative off-diagonal entries and \( b_1, \ldots, b_K \in \mathbb{R} \), the following equality holds:

\[
\begin{align*}
\min_x x^T A_0 x &= \min_x -\sum_k \lambda_k b_k \\
\text{subject to} \quad x &\in \mathbb{R}_+^n \\
x^T A_k x &\geq b_k \\
&\quad k = 1, \ldots, K
\end{align*}
\]

Interpretation:
In the power flow example, \( \lambda_k \) is the price of power at node \( k \).

Future DC Power Transmission Network in Europe?

From Cigre Conference 2010, "Continental Overlay HVDC-Grid" by ABB

Positive Quadratic Programming

Given \( A_0, \ldots, A_K \in \mathbb{R}^{n \times n} \) with nonnegative off-diagonal entries and \( b_1, \ldots, b_K \in \mathbb{R} \), the following equality holds:

\[
\begin{align*}
\max_x x^T A_0 x &= \max_x \text{trace}(A_0 X) \\
\text{subject to} \quad x &\in \mathbb{R}_+^n \\
x^T A_k x &\geq b_k \\
&\quad k = 1, \ldots, K
\end{align*}
\]

Proof

If \( X = \begin{bmatrix} |x_1|^2 & \cdots \\ \cdots & \cdots \end{bmatrix} \) maximizes the right hand side, then \( x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \) maximizes the left.


Optimizing DC Power Flow

\[
\begin{align*}
i_1 & \to (1) \\
& \to (3) \\
i_3 & \to (2) \\
i_2 & \to (4) \\
i_4 & \to \in \mathbb{R} \quad v_k \in \mathbb{R}
\end{align*}
\]

Minimize \[ \sum_k i_k v_k \]
subject to \[ i = Y v \] and \[ i_k v_k \leq v_k \]
\[ (v_k - v_j)^2 \leq c_{kj} \]
\[ v_k \leq v_j \leq v_k \]
for all \( k, j \)

Notice: All mixed terms have the right sign!

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The convex problem
\[
\min_{x} [V_1(x_1, x_2, x_3) + V_2(x_1, x_2, x_3, x_4) + V_3(x_2, x_3) + V_4(x_1, x_2, x_4)]
\]
can be solved by the following distributed iteration:
\[
\begin{align*}
x_1^+ &= \arg \min_{x_1} [V_1(x_1, x_2, x_3) + V_2(x_1, x_2, x_3, x_4)] \\
x_2^+ &= \arg \min_{x_2} [V_1(x_1, x_2, x_3) + V_2(x_1, x_2, x_3, x_4) + V_3(x_2, x_3) + V_4(x_1, x_2, x_4)] \\
x_3^+ &= \arg \min_{x_3} [V_1(x_1, x_2, x_3) + V_2(x_1, x_2, x_3, x_4) + V_3(x_2, x_3) + V_4(x_1, x_2, x_4)] \\
x_4^+ &= \arg \min_{x_4} [V_1(x_1, x_2, x_3) + V_2(x_1, x_2, x_3, x_4) + V_3(x_2, x_3) + V_4(x_1, x_2, x_4)]
\end{align*}
\]

The dynamics
\[
\nu^+_k = \arg \min_{\nu_k} \sum_j \left[ \lambda_{k,j} y_{kj} (v_k - v_j) - \lambda_{j,k} y_{kj} (v_k - v_j) \right]
\]
has the form
\[
\nu^+ = \min \{ \nu, A \nu \}
\]
where \( A \) has nonnegative coefficients.

---

Primal Decomposition

\[
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
\end{array}
\]

Finding Optimum by Distributed Control

\[
\begin{array}{ccc}
i_1 & i_2 & i_3 \\
i_2 & i_2 & i_2 \\
i_3 & i_3 & i_3 \\
\end{array}
\]

Can it pay off to disconnect a line?

\[
\begin{array}{ccc}
i_1 & i_2 & i_3 \\
i_2 & i_2 & i_2 \\
i_3 & i_3 & i_3 \\
\end{array}
\]

Summary

- Why Distributed Control?
- Optimizing Electrical Power Flow
- Positive Quadratic Programming
- Distributed Control of Positive Systems
- Finding Optimum by Distributed Control

To read:
Slides on www.control.lth.se/Staff/anders rantzer.html
Extended abstract in Proceedings of CCC 2011
Upcoming paper in CDC 2011

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