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Convenient Representations of Structured Systems for Model Order Reduction

Aivar Sootla and Anders Rantzer

***Abstract*— In control theory there exist two convenient representations of a model, which are transfer functions and state-space matrices. However, structure in the transfer functions is not clearly seen in the state-space form, and on the other hand structure in the state-space form is not clearly seen in the transfer functions. The main reason for this is that Laplace transformation destroys most types of structures. The goal of this paper was introducing such representations that clearly reflect structure in both frequency and time domain. Such representations are obtained by introducing auxiliary signals which define the interactions within the structure. The auxiliary signals lower level of abstraction of input-output mappings, thus providing an insight into physical properties of a system.**

***Index Terms*— Interconnected systems, structured systems, coprime factorization, model order reduction.**

Model order reduction is an approximation tool typically used for simulation of complex systems, which takes considerable time and/or it has overwhelming memory requirements. Typically, approximation quality is measured by the $\mathbb{H}_\infty$ norm, which reflects the input-output mapping of a model. Most of the existing LTI model order reduction methods fall into two categories: singular value decomposition (SVD) based (balanced truncation [1] and Hankel model reduction [2]) and Krylov based methods ([3], [4], [5]). The SVD and Krylov methods can even preserve some specific types of structure, for example, a second order structure [6], [7], [8]. However, if structure induced by a block diagram is considered these methods generally cannot be applied. One of the first step to address this issue was made by Enns in [9]. The method extended the established balanced truncation algorithm ([1]) to a frequency-weighted problem. Essentially, a cascade interconnection of three systems was approximated. A number of approaches to address this problem have been proposed (e.g., [10], [11]). A method reported in [12] can be also seen as a generalized version of frequency-weighted model reduction problems. An important subclass of systems with a structure is a controller-plant interconnection. A number of methods address the controller reduction problem including those described in [13], [14], [15], [16] and recently proposed in [17], [18].

In this paper, the main focus is on an interconnection of *"subsystems"* in a typical block diagram. Interactions of subsystems define a certain topology of interconnections in a system. Clearly, a modeling procedure should account for such a structure. Therefore, the goal of model order reduction in this setting is to reduce certain subsystems,

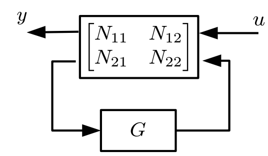

Fig. 1. An LFT loop with two subsystems G and N.

while preserving the overall structure of the system. This concept can be formalized in a concrete example. Consider two subsystems N and G interacting in a linear fractional transformation (or LFT) loop (see, [13]). The LFT system $\mathcal{F}_l\,(N,G)$ is depicted in Figure 1. Assume N should be preserved in the reduction procedure and G should be approximated by some $\widehat{G}$. It is also required that the resulting LFT is similar to the original one, which entails that the error $\|\mathcal{F}_l\,(N,G) - \mathcal{F}_l\,(N,\widehat{G})\|_{\mathbb{H}_\infty}$ has to be minimized. However, to author's best knowledge, there is no such method, that can guarantee finding a solution for an arbitrary order of $\widehat{G}$. Therefore, in order to simplify this problem, auxiliary input and output signals are introduced. By adding these signals, the transfer function $\mathcal{F}_l\,(N,G)$ will be replaced by an extended one $\mathcal{S}_e(N,G)$, which will be, in fact, reduced in the optimization problem. The extended system $\mathcal{S}_e$ depicts the input-output mapping in Figure 2. Finally, the structured model reduction problem is cast as a minimization one.

$$\min_{\text{low-order }\widehat{G}} \|\mathcal{S}_e(N,G) - \mathcal{S}_e(N,\widehat{G})\|_{\mathbb{H}_\infty} \tag{1}$$

This minimization problem is approached by rewriting the system $\mathcal{S}_e(N,\widehat{G})$ in a coprime factor form. These coprime factors will also represent the structure of interconnections. As shown in [19], this problem can be approached using semidefinite optimization techniques. The main goal of this paper is to generalize the coprime factor representation to multiple subsystems and thus generalize the optimization framework to these systems. The optimization framework [19] is not the only model reduction framework, which can take advantage of the proposed representation. The authors suppose the coprime factors can be directly approximated by Krylov methods and balanced truncation, again generalizing

$$\begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} = \mathcal{S}_e(N,G) \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix}$$

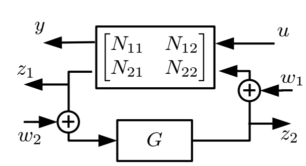

Fig. 2. An extended system $\mathcal{S}_e$ with two subsystems G and N.

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known techniques to multiple systems. The LFT case is discussed in detail in Sections I and II. In Section III, the reduction of multiple subsystems is discussed. There an LFT loop is replaced by an arbitrary block diagram, with multiple subsystems interacting with each other. Numerical examples can be found in [19].

I. Coprime Factorization in an LFT loop

Consider an LFT loop in Figure 2 and the minimization criterion (1). The biggest advantage of introducing the auxiliary signals w_i and z_i is the ability to create a convenient coprime factor representation. This representation is computed by a direct calculation using the coprime factorization of N and G. Indeed, the extended system $\mathcal{S}_e$ can be described by the following set of equations:

$$\begin{bmatrix} y \\ z_1 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} u \\ z_2 + w_1 \end{bmatrix}$$
$$z_2 = G(w_2 + z_1)$$

Factorize N and G using a left coprime factorization, i.e.:

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} Q_{N11} & Q_{N12} \\ Q_{N21} & Q_{N22} \end{bmatrix}^{-1} \begin{bmatrix} P_{N11} & P_{N12} \\ P_{N21} & P_{N22} \end{bmatrix} \tag{2}$$
$$G = Q_G^{-1} P_G$$

and substitute them into the equations above. Since z_1 and z_2 are treated as outputs, they are moved to the left hand side of the equations:

$$\begin{bmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \end{bmatrix} \begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P_{N11} & P_{N12} \\ P_{N21} & P_{N22} \end{bmatrix} \begin{bmatrix} u \\ w_1 \end{bmatrix}$$
$$-P_G z_1 + Q_G z_2 = P_G w_2$$

Both equations can be united into the following matrix equation

$$\mathcal{Q} \begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} = \mathcal{P} \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix}$$

where

$$\begin{aligned} \mathcal{P} &= \begin{pmatrix} P_{N11} & P_{N12} & 0 \\ P_{N21} & P_{N22} & 0 \\ 0 & 0 & P_G \end{pmatrix} \\ \mathcal{Q} &= \begin{pmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \\ 0 & -P_G & Q_G \end{pmatrix} \end{aligned} \tag{3}$$

Lemma 1: The transfer matrices $\mathcal{P}$ and $\mathcal{Q}$ are left coprime over $\mathbb{H}_\infty$.

Proof. To show coprimeness, the definition from [13] is used: transfer matrices $\mathcal{P}$ and $\mathcal{Q}$ are left coprime in $\mathbb{H}_\infty$ if there exist rational transfer matrices X and Y in $\mathbb{H}_\infty$ such that

$$\mathcal{P} \cdot X + \mathcal{Q} \cdot Y = I$$

Due to coprimeness of P_N and Q_N, there exist such X_{Nii} and Y_{Nii} that

$$\begin{pmatrix} P_{N11} & P_{N12} \\ P_{N21} & P_{N22} \end{pmatrix} \begin{pmatrix} X_{N11} & X_{N12} \\ X_{N21} & X_{N22} \end{pmatrix} + \begin{pmatrix} Q_{N11} & Q_{N12} \\ Q_{N21} & Q_{N22} \end{pmatrix} \begin{pmatrix} Y_{N11} & Y_{N12} \\ Y_{N21} & Y_{N22} \end{pmatrix} = I$$

similarly X_G and Y_G are defined trough P_G and Q_G as:

$$P_G X_G + Q_G Y_G = I$$

To prove coprimeness of $\mathcal{P}$ and $\mathcal{Q}$, the transfer matrices X and Y can be chosen as

$$X = \begin{pmatrix} X_{N11} & X_{N12} & 0 \\ X_{N21} & X_{N22} & Y_G \\ Y_{N21} & Y_{N22} & X_G \end{pmatrix} \quad Y = \begin{pmatrix} Y_{N11} & Y_{N12} & 0 \\ Y_{N21} & Y_{N22} & 0 \\ 0 & 0 & Y_G \end{pmatrix}$$

Finally, the relation $\mathcal{P}X + \mathcal{Q}Y = I$ is verified by direct computation. ■

Now examine the transfer matrices $\mathcal{P}$ and $\mathcal{Q}$ closely. Every block-row of each transfer matrix depends either on a coprime factor of N, either on a coprime factor of G. It can not be called a "sparsity structure", since some entries are repeated in $\mathcal{P}$ and $\mathcal{Q}$, however, this kind of structure can be exploited by techniques laid out in [20], [19]. Note also that $\mathcal{S}_e$ is stable if and only if $\mathcal{Q}$ has a stable inverse.

Remark 1: The state-space representation also manifests the structure in a convenient manner. Assume the space-space representations of N and G are given as follows.

$$N = \left[\begin{array}{c|cc} A_N & B_{N1} & B_{N2} \\ \hline C_{N1} & D_{N11} & D_{N21} \\ C_{N2} & D_{N21} & D_{N22} \end{array}\right] \quad G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] \tag{4}$$

where D_G is set to 0, which is a common assumption in control theory. To shorten the notation, additionally define

$$B_N^T = \begin{bmatrix} B_{N1}^T \\ B_{N2}^T \end{bmatrix} \quad D_N = \begin{bmatrix} D_{N11} & D_{N12} \\ D_{N21} & D_{N22} \end{bmatrix} \quad C_N = \begin{bmatrix} C_{N1} \\ C_{N2} \end{bmatrix}$$

A state space representation can be derived for $\mathcal{P}$ and $\mathcal{Q}$

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right]$$

where

$$\mathcal{A} = \text{blkdiag}(A_N + L_N C_N, A_G + L_G C_G)$$
$$\mathcal{B} = \begin{bmatrix} \text{blkdiag}(B_N + L_N D_N, B_G) & \begin{bmatrix} L_N & -B_{N2} \\ \begin{bmatrix} 0 & -B_G \end{bmatrix} & L_G \end{bmatrix} \end{bmatrix}$$
$$\mathcal{C} = \text{blkdiag}(C_N, C_G) \qquad \mathcal{D} = \begin{bmatrix} \text{blkdiag}(D_N, 0) & I \end{bmatrix}$$

where L_N and L_G are free parameters, which are used to stabilize the coprime factors. The dynamics of the subsystems N and G are completely decoupled. The structure manifests itself only in the input and output matrices. If model reduction in the time domain is considered, the transfer function $\begin{bmatrix} \mathcal{P} & \mathcal{Q} \end{bmatrix}$ can be approximated using a structured Gramian framework from [12]. However, stability of $\mathcal{S}_e(N, \widehat{G})$ is achieved if the reduced order $\mathcal{Q}$ has a stable inverse. This property is generally hard to accommodate.

II. Model Reduction in an LFT loop

Above, instead of solving

$$\underset{\text{low-order } \widehat{G}}{\operatorname{argmin}} \|\mathcal{F}_l(N,G) - \mathcal{F}_l(N,\widehat{G})\|_{\mathbb{H}_\infty} \tag{5}$$

it is proposed to address the following optimization problem

$$\underset{\text{low-order } \widehat{G}}{\operatorname{argmin}} \|\mathcal{S}_e(N,G) - \mathcal{S}_e(N,\widehat{G})\|_{\mathbb{H}_\infty} \tag{6}$$

The major feature in (6) is the ability of tracking signals w_i and z_i. It means that not only the behaviour of $\mathcal{F}_l$ is approximated as in (5), but also the interaction between N and G.

Now let us try to understand what kind of problem is being addressed. The transfer function $\mathcal{S}_e$ reads as

$$\mathcal{S}_e = \begin{pmatrix} \mathcal{F}_l(N,G) & N_{12}\Xi & N_{12}\Xi G \\ \Theta N_{21} & N_{22}\Xi & N_{22}\Xi G \\ G\Theta N_{21} & G\Theta N_{22} & G\Theta \end{pmatrix}$$

where $\Theta = (I - N_{22}G)^{-1}$ and $\Xi = (I - GN_{22})^{-1}$. Due to the structure of $\mathcal{S}_e$, it can be shown that the program (6) is equivalent to:

$$\min_{\text{low-order } \widehat{G}} \left\| H_o((I - GN_{22})^{-1}G - (I - \widehat{G}N_{22})^{-1}\widehat{G})H_i \right\|_{\mathbb{H}_\infty}$$

where $H_i = \begin{pmatrix} N_{12}^T & N_{22}^T & I \end{pmatrix}^T$ and $H_o = \begin{pmatrix} N_{21} & N_{22} & I \end{pmatrix}$. The LFT loop $\mathcal{F}_l(N,G)$ appears at the block entry $\{1,1\}$. Therefore (6) provides an estimate on (5).

If only the transfer matrix generated by signals w_1, w_2, z_1 and z_2 is considered (the lower two by two block of $\mathcal{S}_e$), it can be shown, that it is the loop depicted in Figure 3. It is a so called "gang-of-four" applied to G and N_{22}. It is also known that this loop admits a very convenient coprime factorization parametrization, which is exploited in $\mathbb{H}_\infty$ loop shaping (see, [21]). Adding these transfer functions into the objective takes also care of robust stability of the LFT loop.

Given these insights, the problem (6) can be modified as follows:

$$\underset{\text{low-order } \widehat{G}}{\operatorname{argmin}} \|W_o(\mathcal{S}_e(N,G) - \mathcal{S}_e(N,\widehat{G}))W_i\|_{\mathbb{H}_\infty}$$

where the weights W_o and W_i regulate the trade-off between performance (block-entry $\{1,1\}$) and robustness of the loop (block-entries $\{2,2\}$–$\{3,3\}$).

A. Model Reduction Based on Semidefinite Programming

Assume $\mathcal{S}_e(N,G)$ is an asymptotically stable discrete-time transfer function. The $\mathbb{H}_\infty$ optimization problem being addressed reads as

$$\begin{gathered}\min_{\mathcal{P},\,\mathcal{Q}} \|\mathcal{S}_e(N,G) - \mathcal{Q}^{-1}\mathcal{P}\|_{\mathbb{H}_\infty} \\ \text{subject to: } \mathcal{Q} \text{ has a stable inverse}\end{gathered} \tag{7}$$

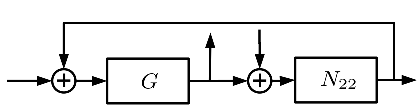

Fig. 3. Gang of Four Loop

where $\mathcal{Q}$ and $\mathcal{P}$ correspond to the left coprime factorization of $\mathcal{S}_e(N,\widehat{G}) = \mathcal{Q}^{-1}\mathcal{P}$ and $\widehat{G}$ is a low order approximation of G. The reduction procedure will be decoupled into two problems: finding a stability preserving low-order approximation of G by some $\widehat{G}_0$ without considering the quality of the loop $\mathcal{S}_e(N,\widehat{G}_0)$. There are quite a few reduction techniques of structured models which guarantee stability under certain conditions. However, none of those can guarantee finding a solution, if one exists. Given this initial point $\widehat{G}_0$, the approximation quality of (7) is minimized.

Let us return to the problem at hand: (7). The transfer matrices $\mathcal{Q}$ and $\mathcal{P}$ admit the representation as in (3)

$$\mathcal{P} = \begin{pmatrix} P_{N11} & P_{N12} & 0 \\ P_{N21} & P_{N22} & 0 \\ 0 & 0 & Y \end{pmatrix}$$

$$\mathcal{Q} = \begin{pmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \\ 0 & -Y & X \end{pmatrix}$$

where P_{Nii} and Q_{Nii} is a factorization of N as in (2) and X, Y constitute a left coprime factorization of $\widehat{G} = X^{-1}Y$ and are unknown and parameterized as

$$X = \sum_{i=0}^{r} X_i z^{-i} \quad Y = \sum_{i=0}^{r} Y_i z^{-i}$$

with the real matrices X_i, Y_i being the decision variables. The procedure is concluded in Algorithm 1. For every fixed $\mathcal{Q}_j$ the program is semidefinite. Convergence of the algorithm can be treated as in [19, Chapter 2]. The transfer matrix $\mathcal{Q}_0$ is computed based on the initial point $\widehat{G}_0$. It is reasonable to assume that $\mathcal{Q}_0$ should have the same structure as $\mathcal{Q}$, therefore:

$$\mathcal{Q} = \begin{pmatrix} Q_{N11} & Q_{N12} & -P_{N12} \\ Q_{N21} & Q_{N22} & -P_{N22} \\ 0 & -Y^0 & X^0 \end{pmatrix}$$

where X^0, Y^0 are FIR filters of order r, and $\widehat{G}_0 = (X^0)^{-1}Y^0$. Since, $\mathcal{S}_e(N,\widehat{G}_0)$ is stable, $\mathcal{Q}_0$ has a stable inverse, and so does $\mathcal{Q}$ (as shown in [19, Chapter 2]). Now,

Algorithm 1 Structured Model Reduction

Compute P_N, Q_N as described above. Obtain $\mathcal{Q}_0$, e.g., using a stability preserving heuristics from [13]. Set $\mathcal{Q}_j = \mathcal{Q}_0$ and $j = 1$

Introduce X, Y, $\mathcal{P}$ and $\mathcal{Q}$ as above.

repeat

Solve a semidefinite problem

$$\min_{X_i,\,Y_i,\,\gamma} \gamma^2 \quad \text{subject to for all } \omega \in [0,\pi]$$

$$\begin{bmatrix} \mathcal{Q}_j\mathcal{Q}^\sim + \mathcal{Q}\mathcal{Q}_j^\sim - \mathcal{Q}_j\mathcal{Q}_j^\sim & \mathcal{Q}\mathcal{S}_e(N,G) - \mathcal{P} \\ (\mathcal{Q}\mathcal{S}_e(N,G) - \mathcal{P})^\sim & \gamma^2 I \end{bmatrix} \geq 0$$

Set $\mathcal{Q}_{j+1} = \mathcal{Q}$ and $j = j + 1$

until $\|\mathcal{Q}_{j+1} - \mathcal{Q}_j\|_\infty \leq \varepsilon$

Compute the reduced model as $\widehat{G} = X^{-1}Y$

given an initial point $\mathcal{Q}_0$, it is possible to find another feasible point $\mathcal{Q}$ with an improved approximation quality. A semidefinite program with a finite number of constraints can be obtained using the KYP lemma ([22]) or using a frequency griding approach as in [19].

III. Generalization to Multiple Subsystems

Let us start with an example, to see what kind of problems can occur, if a generalization is not performed carefully.

Consider a block diagram with three subsystems G_1, G_2 and G_3 in Figure 4. In order to obtain a coprime factorization as before, excite every subsystem G_i by an additional signal w_i and measure its output by an additional signal z_i. Note that three signals u, w_1 and w_3 are exciting only two subsystems G_1 and G_3. Additionally, the output of the system y is a sum of z_1 and z_3. This way some degree of redundancy appears in the extended supersystem. These simple observations raise a question: how the is model reduction problem affected? Assume the extended transfer function is introduced as follows:

$$\begin{bmatrix} y^T & z_1^T & z_2^T & z_3^T \end{bmatrix}^T = \mathcal{S}_e \begin{bmatrix} u^T & w_1^T & w_2^T & w_3^T \end{bmatrix}^T$$

Note that the sum of last two rows of $\mathcal{S}_e$ will be equal to the first one, since $y = z_1 + z_3$. The sum of the second and fourth columns of $\mathcal{S}_e$ will be equal to the first one, since u excites the system in the same manner as the sum of w_1 and w_3. Thus, the transfer matrix $\mathcal{S}_e$ will have at least one zero singular value for all the frequencies ω in $[0, \pi]$. Solving a model reduction problem in this setting is problematic, since a rank-deficient matrix is approximated. On the other hand, all zero singular values can be eliminated while designing the extended system. Therefore, some signals should be eliminated to provide a full rank $\mathcal{S}_e$ for all the frequencies. Here, signals z_1 and w_3 were chosen to be eliminated, providing the equations:

$$\begin{aligned} y &= z_3 + G_1(u + y + w_1 + z_2) \\ z_2 &= G_2(w_2 + y - z_3) \\ z_3 &= G_3(u + y) \end{aligned}$$

Factorize the subsystems G_i using left coprime factorizations $Q_i^{-1}P_i$. Substitute them into the equations above, while multiplying both sides with Q_i. Also, separate the outputs and inputs on the different sides of the equations:

$$\begin{aligned} (Q_1 - P_1)y + P_1 z_2 - Q_1 z_3 &= P_1(u + w_1) \\ -P_2 y + Q_2 z_2 + P_2 z_3 &= P_2 w_2 \\ -P_3 y + Q_3 z_3 &= P_3 u \end{aligned}$$

The relationship between the signals can be computed as:

$$\begin{bmatrix} Q_1 - P_1 & P_1 & -Q_1 \\ -P_2 & Q_2 & P_2 \\ -P_3 & 0 & Q_3 \end{bmatrix} \begin{bmatrix} y \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} P_1 & 0 & P_1 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}$$

This representation is left coprime, to show this let:

$$\mathcal{P} = \begin{bmatrix} P_1 & 0 & P_1 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \quad \mathcal{Q} = \begin{bmatrix} Q_1 - P_1 & -P_2 & -Q_1 \\ -P_2 & Q_2 & P_2 \\ -P_3 & 0 & Q_3 \end{bmatrix}$$

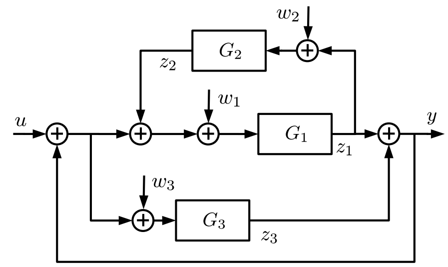

Fig. 4. A block diagram with three subsystems

Algorithm 2 Coprime factorization of an arbitrary block diagram

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- Given a block diagram with interconnected subsystems $G_1, G_2, \ldots, G_n$, introduce signals w_i, z_i. Every w_i excites the subsystem G_i, and every z_i measures its output
- If u excites the system $\mathcal{S}_e$ the same way as a linear combination of w_j, then one of the signals w_j is set to zero
- If a signal z_j is a linear combination of other output signals it is eliminated by $z_j = y - \sum_{j \neq i}^{n} \gamma_j z_j$, where γ_j are zeros or ones
- Write down the equations describing dependence of z_i on the signals w_j, u, y and z_j, while replacing G_i by its left coprime factorization $Q_i^{-1}P_i$
- Compute the left coprime factors $\mathcal{P}$ and $\mathcal{Q}$

$$\mathcal{P}_G = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \quad \mathcal{Q}_G = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix}$$

Given the notations, it can be computed that

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_G & \mathcal{Q}_G \end{bmatrix} T$$

where the transformation T is invertible. Note that $\mathcal{P}_G$ and $\mathcal{Q}_G$ are left coprime, since they are block diagonal with left coprime factors of the block diagonal and there exist X_G and Y_G such that:

$$\begin{bmatrix} \mathcal{P}_G & \mathcal{Q}_G \end{bmatrix} \begin{bmatrix} X_G \\ Y_G \end{bmatrix} = I$$

Thus, there exist stable X and Y such that:

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = I \quad \text{where} \quad \begin{bmatrix} X \\ Y \end{bmatrix} = T^{-1} \begin{bmatrix} X_G \\ Y_G \end{bmatrix}$$

This proves that $\mathcal{P}$ and $\mathcal{Q}$ are left coprime if and only if $\mathcal{P}_G$ and $\mathcal{Q}_G$ are left coprime. Inspired by this simple example, Algorithm 2 is formulated. The algorithm is constructive, however, it has to be shown that it always produces a left coprime factorization.

Lemma 2: The transfer matrices $\mathcal{P}$ and $\mathcal{Q}$ obtained in Algorithm 2 are left coprime.

Proof. The proof is a generalization of the technique described for the block diagram investigated above. Introduce

$\mathcal{P}_G$ and $\mathcal{Q}_G$:

$$\mathcal{P}_G = \begin{bmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_n \end{bmatrix} \qquad \mathcal{Q}_G = \begin{bmatrix} Q_1 & & 0 \\ & \ddots & \\ 0 & & Q_n \end{bmatrix}$$

where Q_i and P_i are left coprime and $G_i = Q_i^{-1}P_i$. The equations describing the relationships between the signals read as:

$$Q_i z_i = P_i(\alpha_0^i u + \alpha_i w_i + \beta_0^i y + \sum_{j=1}^{n} \beta_j^i z_j) \tag{8}$$

$$Q_i(y - \sum_{j\neq i}^{n} \gamma_j z_j) = P_i(\alpha_0^i u + \alpha_i w_i + \beta_0^i y + \sum_{j=1}^{n} \beta_j^i z_j) \tag{9}$$

where α_0^i, α_i, β_j^i and γ_j are equal to zero or one, depending on a particular block diagram. The equation (9) appears when a signal z_i is a linear combination of other outputs, i.e., $z_i = y - \sum_{j\neq i}^{n} \gamma_j z_j$, and it is eliminated. Hence, only one equation has the form (9). Note that $\mathcal{P}$ will not depend on $\mathcal{Q}_G$ since none of the signals u, w_i are multiplied with Q_i. The static transformation between the transfer matrices $\mathcal{P}_G$, $\mathcal{Q}_G$ and $\mathcal{P}$, $\mathcal{Q}$ are given as follows.

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_G & \mathcal{Q}_G \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

Since the transformation is block triangular, it remains to show that T_{11} and T_{22} are invertible. For most of the signals, the relation (8) is valid. The transfer matrix $\mathcal{Q}_G T_{22}$ will be affected only by the summands containing multiplication of Q_i and the outputs y or z_i. Therefore, most of the block rows of $\mathcal{Q}_G T_{22}$ will have only one non-zero entry Q_i and it will appear on the block diagonal.

For a single equation, the relation (9) is valid and therefore one block row of $\mathcal{Q}_G T_{22}$ consists of multiple Q_i with different signs. Without loss of generality, we can assume that z_i with $i = 1$ is eliminated, therefore T_{22} is a block-triangular matrix with the identity matrices on the block diagonal. Only the first block row has non-zero off diagonal entries. Given these facts T_{22} is invertible.

It can be similarly shown that T_{11} is an invertible block-triangular matrix, where only one block-column has non-zero entries except for the block-diagonal elements. Since T_{11} and T_{22} are invertible so is the whole transformation. Due to coprimeness of $\mathcal{P}_G$ and $\mathcal{Q}_G$, there exist X_G and Y_G such that:

$$\begin{bmatrix} \mathcal{P}_G & \mathcal{Q}_G \end{bmatrix} \begin{bmatrix} X_G \\ Y_G \end{bmatrix} = I$$

Finally, there exist stable X and Y such that

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = I \quad \text{where} \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}^{-1} \begin{bmatrix} X_G \\ Y_G \end{bmatrix}$$

Therefore $\mathcal{P}$ and $\mathcal{Q}$ are left coprime if and only if $\mathcal{P}_G$ and $\mathcal{Q}_G$ are left coprime, which is satisfied by construction. ■

The statement of this lemma is not unexpected. If there is just one subsystem with one input and one output, then the coprime factors $\mathcal{P}$, $\mathcal{Q}$ should have one input and one output. It stands to reason that the extended system $\mathcal{S}_e$ should have n inputs and n outputs, if there is n subsystems in $\mathcal{S}_e$ with one input and one output each. Stability of the system is equivalent to $\mathcal{Q}$ and $\mathcal{Q}^{-1}$ being stable, since the transfer matrices $\mathcal{P}$ and $\mathcal{Q}$ constitute a coprime factorization of $\mathcal{S}_e$.

Addressing the model reduction problem can be done similarly as was done before. Assume without loss of generality, that subsystems $G_{k+1}, \ldots, G_n$ have to be preserved during the approximation procedure and $G_1, \ldots, G_k$ are being reduced. All the subsystems $G_{k+1}, \ldots, G_n$ are treated as one subsystem N and the problem is cast a minimization one as follows.

$$\min_{\widehat{G}_1,\ldots,\widehat{G}_k} \|\mathcal{S}_e(N, G_1, \ldots, G_k) - \mathcal{S}_e(N, \widehat{G}_1, \ldots, \widehat{G}_k)\|_{\mathbb{H}_\infty}$$

where $\widehat{G}_1$, $\ldots$, $\widehat{G}_k$ are low-order approximations of G_1, $\ldots$, G_k. After that, Algorithm 1 can be modified in a straightforward manner in order to address this problem.

A. Structures in the State Space

In the state-space case it is assumed that the vector x is partitioned, providing us with a structure. Our goal is to reduce this representation to the coprime factorization introduced for block-diagrams. Due to space limitation we are going to explain only the main idea on a simple example. Let $\mathcal{S}$ a system with a state-space representation $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$.

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & A_{42} & 0 & A_{44} \end{pmatrix} \\ \mathcal{B} &= \text{blkdiag}\{B_1, B_2, B_3, B_4\} \\ \mathcal{C} &= \text{blkdiag}\{C_1, C_2, C_3, C_4\} \\ \mathcal{D} &= \text{blkdiag}\{D_1, D_2, D_3, D_4\} \end{aligned}$$

The matrices A_{ij} represent the interconnections between the subsystems with dynamic matrices A_{ii}. The obvious step is to factorize A_{ij} into matrices F_{ij} and L_{ij}, which will define the outputs and the inputs in corresponding subsystems. However, if the factorization is not performed carefully it can lead to S_e, which is rank-deficient. This case is easy to obtain. Assume the singular value decomposition of A_{ij} gives:

$$\mathcal{A} = \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ U_{31}S_1V_{31}^T & U_{31}S_3V_{32}^T & A_{33} & 0 \\ U_{41}S_2V_{31}^T & U_{41}S_4V_{32}^T & 0 & A_{44} \end{pmatrix}$$

where U_{ij} and V_{ij} are columns and S_i are distinct, not equal to one, positive scalars. Now we can try to introduce the subsystems. The matrices S_1V_{31} and S_3V_{31} can be treated as output matrices from the subsystem 1, which entails its state-space representation:

$$G_1 = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline S_1V_{31}^T & 0 \\ S_3V_{31}^T & 0 \\ C_1 & D_1 \end{array}\right]$$

Whatever we do next the extended system S_e will be rank deficient, since the output matrix of G_1 is row-rank deficient. In order to avoid such issues, it is required to keep track of the columns V_{ij}, U_{ij} and introduce new output matrices only when it is required. For example, if we introduce matrices $L_1 = S_1^{1/2} V_{31}^T$, $F_3 = U_{31} S_1^{1/2}$, $L_2 = S_1^{-1/2} S_3 V_{32}^T$ and $F_4 = U_{41} S_2 S_1^{-1/2}$, the system $\mathcal{S}_e$ will not be rank-deficient for all frequencies. The subsystems are defined as:

$$G_1 = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline L_1 & 0 \\ C_1 & D_1 \end{array}\right] \quad G_2 = \left[\begin{array}{c|c} A_{22} & B_2 \\ \hline L_2 & 0 \\ C_2 & D_2 \end{array}\right]$$

$$G_3 = \left[\begin{array}{c|cc} A_{33} & F_3 & B_3 \\ \hline C_3 & 0 & D_3 \end{array}\right] \quad G_4 = \left[\begin{array}{c|cc} A_{44} & F_4 & B_4 \\ \hline C_4 & 0 & D_4 \end{array}\right]$$

Note, at the block-entry $\{4, 2\}$ we have a term $F_4 D L_2$, where $D = S_4 S_1 / (S_2 S_3)$. The static term D cannot be included into any of the subsystems or eliminated. We can define

$$G_4 = \left[\begin{array}{c|cc} A_{44} & F_4 & B_4 \\ \hline C_4 & 0 & D_4 \\ 0 & D & 0 \end{array}\right]$$

however, it seems as a strange definition. Therefore the term D appears in the block diagram (the right-most block-diagram in Figure 5) and thus in the structured coprime factorization. The subsystems can be introduced in a number of other ways. For example, if matrices F_i and L_i are introduced directly through U_{ij} and V_{ij}, and accordingly the subsystems $\widetilde{G}_i$. In this case all S_i are static signal transformations between the subsystems. Both block-diagrams in Figure 5 depict the same state-space representation. Nevertheless in the structured coprime factor representation these block-diagrams will be equivalent.

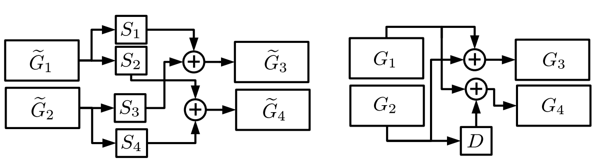

Fig. 5. Two equivalent block diagrams with static signal transformations

This algorithm can be readily extended to a more general case for A_{ij} and $i \neq j$ with rank larger than one and full $\mathcal{A}$ matrix. In this case, every A_{ij} is factorized into $U_{ij} S_{ij} V_{ij}^T$ using singular value decomposition. After that, new output L (or input F) matrices are introduced only for linearly independent columns of U_{ij} (or V_{ij}) for all $i \neq j$. After the subsystems are introduced the coprime factors can be computed as before and model reduction algorithms can be applied.

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