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# Spatially-Coupled Nearly-Regular LDPC Code Ensembles for Rate-flexible Code Design

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**Abstract**—Spatially coupled regular LDPC code ensembles have outstanding performance with belief propagation decoding and can perform close to the Shannon limit. In this paper we investigate the suitability of coupled regular LDPC code ensembles with respect to rate-flexibility. Regular ensembles with good performance and low complexity exist for a variety of specific code rates. On the other hand it can be observed that outside this set of favorable rational rates the complexity and performance penalty become unreasonably high. We therefore propose ensembles with slight irregularity that allow us to smoothly cover the complete range of rational rates. Our simple construction allows a performance with negligible gap to the Shannon limit while maintaining complexity as low as for the best regular code ensembles. At the same time the construction guarantees that asymptotically the minimum distance grows linearly with the length of the coupled blocks.

## I. INTRODUCTION

Low-density parity-check (LDPC) codes are widely used as they exhibit outstanding performance with the belief propagation decoding algorithm. While the BP decoder is suboptimal compared to the optimal *maximum a-posteriori* (MAP) decoder in terms of bit error probability, its complexity is greatly reduced compared to the optimal one. With the introduction of spatially coupled LDPC codes, also known as LDPC convolutional codes [1], this sub-optimality can be overcome due to a phenomenon called threshold saturation. The remarkable threshold improvement of spatially coupled ensembles was first investigated in [2][3]. In [4] it was shown for the BEC by Kudekar *et al.* that the BP threshold of a coupled regular LDPC ensemble actually converges to the MAP threshold of the corresponding uncoupled ensemble. More recently, potential functions have been identified as a powerful tool for characterizing the connection between MAP thresholds and BP thresholds [5][6]. Since the discovery of this threshold saturation phenomenon the concept of spatial coupling has been applied to several other fields such as compressed sensing [7], multiuser communication [8], relay channels [9] and wiretap channels [10].

In order to provably obtain capacity achieving performance for a family of rate-compatible codes, a code construction based on the extension of the parity-check matrices of spatially-coupled regular LDPC codes was introduced in [11]. Although this method was able to cover the whole rate region  $R \in [0, 1]$ , at some rates the variable and check degrees become very high which is disadvantageous for implementation. In [12], a rate compatible ensemble based on irregular protograph-based LDPC codes was considered, that significantly lowered the complexity of implementation while

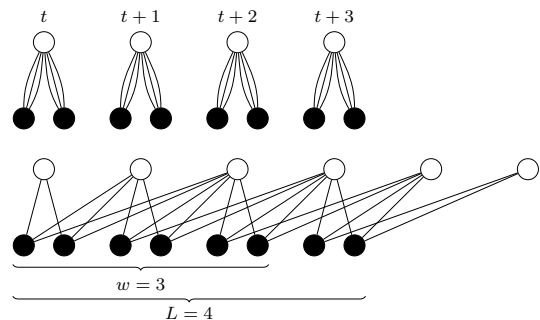


Fig. 1. A chain of  $L$  regular  $(3, 6)$  LDPC block codes at different time instants  $t$  (top) is uniformly interconnected over  $w$  time instants to form a LDPC convolutional code (bottom). Note that the two check nodes at the lower right are responsible for the rate loss but needed to cover this uniform edge spreading

keeping the good performance of spatial coupling. Spatially coupled rateless codes were investigated in [13]. In this work, motivated by the ideas of [11][12], we propose a simple method to overcome the complexity issues of rate-flexible regular codes. A code designer might ask how to construct a code for a given rate or a family of rates with good performance as well as low decoding complexity. This paper describes a simple approach and guideline to obtain such codes. The key is to allow for a slight irregularity in the code graph to add a degree of freedom that can be used for supporting arbitrary rational rates in  $[0, 1]$  as accurate as needed while keeping the check and variable degrees as low as possible. The good performance can then be achieved by spatial coupling of the constructed nearly-regular LDPC code ensembles. The paper is organized as follows. In Section II, we shortly review the concept of spatial coupling and introduce the terminology and some tools for later evaluation. Section III analyses the complexity and performance of a simple rate-flexible LDPC code construction with regular LDPC codes and unveils the complexity issues for unfavorable rates. In Section IV, we then introduce a new approach to the code construction of rate-flexible LDPC codes via mixing favorable regular LDPC codes into a slightly irregular ensemble. This is followed by a detailed discussion of the results in terms of complexity and performance in Section V. The paper is finally concluded.

## II. COUPLED LDPC CODE ENSEMBLES

Consider the transmission of a sequence of codewords  $\mathbf{v}_t$  with  $t = 1 \dots L$  using an LDPC block code with rate  $R = 1 - \frac{J}{K}$ . The fundamental difference between an LDPC

block code and its convolutional version is that, in the latter case, individual codewords of different time instants are coupled together. This coupling is done over  $w$  time instants yielding a chain of length  $L$  where codewords  $\mathbf{v}_t$  to  $\mathbf{v}_{t+w-1}$  are connected. The procedure is depicted in Fig. 1.

We consider in the following the coupled regular LDPC code ensembles as defined in [4]. At every position  $t \in [0, L-1]$ ,  $N \in \mathbb{N}$  variable nodes and  $M = N \frac{J}{K}$  check nodes are placed, where  $J$  and  $K$  denote the variable and check degree, respectively. We assume that all of the  $J$  edges emanating from a variable node at time instant  $t$  will be connected uniformly and independently at random to check nodes at positions  $t \in [t, t+w-1]$ . In this sense, the edge connections between time instants are randomized. In the same manner, the  $K$  edges from check nodes at position  $t$  are connected to variable nodes in the range  $[t-w+1, t]$ . Thus, a  $(J, K, L, w)$  coupled regular LDPC code ensemble is defined. Due to the termination of the coupled code chain, the convolutional ensemble exhibits a rate loss in the finite  $L$  regime. If  $L \rightarrow \infty$ , the rate converges to the design rate of the underlying regular ensemble.

We further assume that transmission takes place over the *binary erasure channel* (BEC). The asymptotic behavior for infinite block length is then described by the following *density evolution* (DE) equations.

**Definition II.1** (DE for  $(J, K, L, w)$  LDPC code ensembles). Assuming transmission over the BEC with erasure probability  $\epsilon$ , the erasure probability of variable nodes at position  $t$  after  $l$  iterations is given by

$$x_t^{(l)}(\epsilon) = \epsilon \left( 1 - \frac{1}{w} \sum_{j=0}^{w-1} \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{t+j-k}^{(l-1)} \right)^{K-1} \right)^{J-1} \quad (1)$$

for  $t \in [0, L-1]$ . We set  $x_t = 0$  for  $t \notin [0, L-1]$ .

Using DE, a unique decoding threshold for *belief propagation* (BP) decoding can be obtained and is defined as follows.

**Definition II.2** (BP Threshold). The BP threshold of a  $(J, K, L, w)$  LDPC ensemble is defined as

$$\epsilon^{BP} = \sup\{\epsilon \in [0, 1] | x_t^{(\infty)}(\epsilon) = 0; \forall t \in [0, L-1]\} \quad (2)$$

The optimal decoding strategy is MAP decoding, for which a similar threshold can be defined. The MAP threshold  $\epsilon^{MAP}$  is defined as the maximum erasure probability  $\epsilon$  under which the decoding error probability is equal to zero. By applying the aforementioned coupling procedure, the BP threshold of the coupled code ensemble is improved in comparison to the threshold of the uncoupled block ensemble [2][3]. This improvement can be explained by the specific structure at the boundaries of the convolutional code chain. Stronger check nodes imply a higher threshold and the iterative nature of the BP decoder propagates this influence from the boundaries into the graph. In fact, the BP threshold of the coupled ensemble converges to the MAP threshold of the uncoupled ensemble. Additionally, when the degree  $J$  increases while the ratio  $J/K$  is kept constant,  $\epsilon^{MAP}$  converges to the Shannon limit  $\epsilon^{Sh}$ . For uncoupled LDPC ensembles a BP decoding performance close to capacity requires the introduction of irregular graphs. In the case of coupled LDPC codes, the threshold saturation behavior stems purely from the structural properties of the code

which induces the use of simple regular LDPC codes that are easy to implement.

### III. REGULAR LDPC CODES

#### A. Rate-Flexibility of Regular LDPC Codes

A regular  $(J, K)$  LDPC code ensemble is defined via its variable degree  $J$  and its check degree  $K$ . The design rate of such an ensemble is defined as

$$R = 1 - \frac{J}{K}. \quad (3)$$

Given the above definition, regular LDPC code ensembles can achieve every rational rate  $R \in \mathbb{Q}$  with  $R \in [0, 1]$ . The rate is only determined by the variable and check degree. To yield a tuple  $(J, K)$  for a given rate  $R$ , the solution to the equation  $K(1 - R) = J$  needs to be determined. The solution to this equation is not unique, so a given rate  $R$  can be achieved by an infinite number of tuples  $(J, K)$ . Due to the sparse nature of their parity-check matrix, regular LDPC codes are predestined for decoding with the BP decoder, whose complexity is determined by the number of edges in the decoding graph. This stems from the fact, that a BP update operation has to be done per edge. The complexity in terms of average operations per bit is then given as follows.

**Definition III.1** (Complexity of an LDPC code ensemble). The complexity  $\mathcal{C}$  of an LDPC code ensemble with average variable node degree  $\bar{J}$  and rate  $R$  is defined as

$$\mathcal{C} = \frac{\bar{J}}{R}. \quad (4)$$

Achieving low complexity for a wide range of code rates is a central goal throughout the remainder of this paper. Therefore in the following we discuss a simple construction of regular rate-flexible LDPC code ensembles with bounded complexity.

Assume a given rate  $R$  whose decimal representation has a bounded number of digits  $i$  after the radix point. Then the accuracy of the rate is  $A = 10^{-i}$  with  $i = 1, 2, \dots$ . An achievable check degree  $K$  can be found by  $K = 1/A$ . Accordingly  $J$  is then given by  $J = (1 - R)K$ . Not all rates  $R \in [0, 1]$  with  $R \in \mathbb{Q}$  can be achieved with this procedure as the simple counterexample  $R = 1/3$  shows. Omitting the rates with unbounded complexity, we can state the following

**Proposition III.2.** *Given a code rate  $R$  with  $i$  decimal digits,  $i < \infty$ , and accuracy  $A = 10^{-i}$ , the degrees of a regular  $(J, K)$  code ensemble can be upper bounded by*

$$\begin{aligned} K_{Max} &= \frac{1}{A} \\ J_{Max} &= (1 - R)K_{Max}. \end{aligned}$$

*A degree tuple  $(J, K)$  can always be found, that suffices  $J \leq J_{Max}$  and  $K \leq K_{Max}$ .*

While Proposition III.2 states that one can always achieve a check degree  $K_{Max}$  or lower, given a rate  $R$  with accuracy  $A$ , still smaller values can be found for various  $J$  and  $K$ . This is summarized in the following

**Proposition III.3.** *If for given  $(J, K)$  with rate  $R$ ,  $\gcd(J, K) = 1$ , then  $J_{Min} = J$  and  $K_{Min} = K$  are the smallest degrees for given rate  $R$ .*

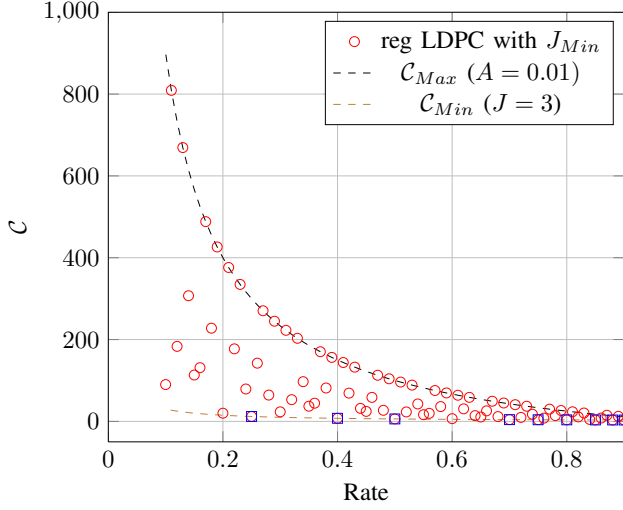


Fig. 2. Complexity  $\mathcal{C}$  of regular LDPC code ensemble tuples  $(J_{Min}, K_{Min})$  from set  $\mathbb{S}$  for  $A = 0.01$ . Blue squares show LDPC code ensembles with  $J = 3$ .

Given a set  $\mathcal{R}$  of rates  $R$  with accuracy  $A$ , the code ensemble set  $\mathbb{S}$  is built by all tuples  $(J, K)$  for all rates  $R \in \mathcal{R}$ . Proposition III.2 and III.3 define upper and lower bounds on the achievable variable and check degrees within the set  $\mathbb{S}$ . Note that without further restriction Proposition III.2 and III.3 permit variable degrees  $J \leq 2$ . As the decoding process does not benefit from such variable nodes we require  $J \geq 3$  for the remainder of the paper.

Recapitulating the definition of decoder complexity, the design goal of a low complexity code requires the use of the smallest variable degree possible. Therefore, ensembles with lowest complexity are the ones with  $(J_{Min}, K_{Min})$ . Using the above described simple construction for the exemplary rate interval  $R \in [0.1, 0.9]$  with rate accuracy  $A = 0.01$  we get the set  $\mathcal{R} = \{0.1, 0.11, 0.12, \dots, 0.89, 0.9\}$  of possible rates and the accompanying set  $\mathbb{S}$  of tuples  $(J, K)$ . The complexity of all tuples  $(J_{Min}, K_{Min})$  from  $\mathbb{S}$  is shown in Fig. 2. Certain ensembles do exhibit very high complexity. Especially those where  $J_{Min} = J_{Max}$  and  $K_{Min} = K_{Max}$ . In fact, the lower  $\gcd(J_{Max}, K_{Max})$  is, the higher becomes the complexity of the regular code ensemble. Additionally note, that the jumps of complexity do not scale proportionally with the rate, although lower rates in general have higher complexity. The lowest possible complexity  $\mathcal{C}_{Min}$  would be accomplished with a variable node degree of  $J = 3$  which forms a lower bound on the complexity for LDPC code ensembles. The construction also gives an upper bound  $\mathcal{C}_{Max}$  on the complexity of the code ensembles which is additionally shown in Fig. 2. It is given by

$$\mathcal{C}_{Max} = \frac{1}{A} \frac{1 - R}{R}. \quad (5)$$

### B. Thresholds of Coupled Regular Codes

Using spatial coupling of regular codes, the Shannon limit can be achieved with increasing degrees. In contradiction to this stands the need for low complexity that is connected to low node degrees. We investigate the performance of the LDPC code ensembles with  $(J_{Min}, K_{Min})$  from the set  $\mathbb{S}$  when spatial coupling is applied. The smoothing parameter  $w$  of the

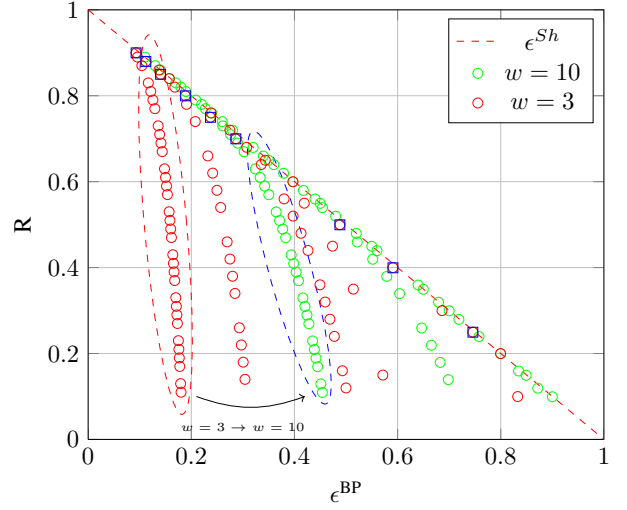


Fig. 3. Density evolution thresholds  $\epsilon^{BP}$  for regular LDPC convolutional code ensemble set  $\mathbb{S}_C$  with  $J_{Min}$  and  $R \in [0.1, 0.9]$ . The smoothing parameter is varied:  $w \in \{3, 10\}$ ; Shannon limit  $\epsilon^{Sh}$ . Blue squares show LDPC code ensembles with  $J = 3$ .

spatially coupled LDPC code ensemble is of additional importance as the complexity of a possible windowed BP decoder is directly proportional to  $w$ . Using a windowed BP decoder is the natural choice for spatially coupled LDPC codes [14]. Therefore, we seek for codes with low variable degree, small smoothing parameter  $w$  and BP thresholds close to Shannon limit  $\epsilon^{Sh}$ . The DE thresholds  $\epsilon^{BP}$  of the investigated code ensembles with  $A = 0.01$ ,  $w \in \{3, 10\}$  and rates  $R \in [0.1, 0.9]$  using always the tuples  $(J_{Min}, K_{Min})$  are shown in Fig. 3. It can be seen, that many tuples  $(J_{Min}, K_{Min})$  are close to the Shannon limit  $\epsilon^{Sh}$  and these are the ones with low degrees, e.g.,  $J = 3$ . These are the ones with lowest complexity too as the comparison of Fig. 2 and Fig. 3 shows. Blue squares denote the same LDPC code ensembles. On the other hand, there is a "branch" of code ensembles with very high degrees that exhibit very bad thresholds far away from capacity (emphasized with the dashed circles). The performance can still be improved by increasing  $w$ . By changing  $w = 3$  to  $w = 10$  (the blue dashed branch), the thresholds are pushed further to the Shannon limit but at the price of increased complexity due to larger window sizes for the BP decoder. Spatially coupled regular LDPC codes can get very close to capacity but with low complexity, this is only possible for a subset of rates within  $R \in [0, 1]$ . We therefore introduce slight irregularity to the ensemble definition to overcome this issue.

## IV. NEARLY-REGULAR COUPLED LDPC CODE ENSEMBLES

The construction of nearly-regular LDPC code ensembles is based on the idea of mixing different but few regular ensembles based on given rules to overcome the shortcomings of regular code constructions but still remain fairly regular degree distributions. Therefore, the degree distributions structure is restricted. We define a two step approach to the design of rate-adaptive nearly-regular LDPC code ensembles. The basis for the design is the use of the aforementioned set  $\mathbb{S}$  of regular LDPC codes of given rate accuracy  $A$  as described in Section III. Next, we define a set  $\mathbb{S}_C \subseteq \mathbb{S}$  according to a specific criterion  $\mathcal{C}$ , that includes all the regular codes that suffice the

criterion  $C$ . The criterion can e.g. restrict the members of  $\mathbb{S}_C$  to have variable node degree  $J = 3$  or  $J < 5$ . Other criteria could be based on complexity, belief propagation threshold or similar metrics. Once the subset  $\mathbb{S}_C$  of  $\mathbb{S}$  is chosen, a rule needs to be fixed that defines how the codes within  $\mathbb{S}_C$  have to be mixed to yield a specific nearly-regular LDPC code ensemble  $NR(\lambda, \rho)$  with rate  $R$ . Now we can define a nearly-regular LDPC code ensemble as follows.

**Definition IV.1.** Given a set  $\mathbb{S}_C$ , a mixing rule and a desired rate  $R$ , a nearly-regular LDPC code ensemble  $NR(\lambda, \rho)$  is defined by the degree distributions

$$\lambda(x) = x^{J-1} \quad (6)$$

$$\rho(x) = \rho_{K_i} x^{K_i-1} + \rho_{K_j} x^{K_j-1} \quad (7)$$

that suffice

$$R(\lambda, \rho) = 1 - \frac{\int_0^1 \rho(x) dx}{\int_0^1 \lambda(x) dx} \quad (8)$$

and where  $(J, K_i)$  and  $(J, K_j)$  are taken from  $\mathbb{S}_C$  according to the specified mixing rule.

An extension to the above definition in terms of a mixture of different variable node degrees is also possible but not subject of this paper. For the mixing rule different options are possible. Two options that are investigated within this paper are the *direct neighbor mix* as well as the *boundary mix*. The boundary mix takes the two regular codes with highest and lowest rate from the set  $\mathbb{S}_C$  and mixes them. Assuming the criterion for the set  $\mathbb{S}_C$  to be  $J = 3$ , the degree distribution for the boundary mix are

$$\lambda(x) = x^2 \quad (9)$$

$$\rho(x) = \rho_{K_i} x^{K_i-1} + \rho_{K_j} x^{K_j-1}. \quad (10)$$

where  $K_i$  and  $K_j$  denote the check node degree for the regular code ensemble in  $\mathbb{S}_C$  with highest rate  $R_i$  and lowest rate  $R_j$ , respectively. The coefficients  $\rho_{K_i}$  and  $\rho_{K_j}$  have then to be chosen to yield the target rate  $R$ , which has to suffice  $R_i \geq R \geq R_j$ . A descriptive analogy for the method is the interpolation of rates in between the highest and lowest rate with the nearly-regular mixture. The boundary mix does not necessarily make use of all the members of  $\mathbb{S}_C$  as it only uses the highest and lowest rate members.

**Example IV.2** (Boundary Mix for  $R = 0.67$  with  $J = 3$ ). For obtaining the rate  $R = 0.67$  with  $J = 3$ , we first choose the set  $\mathbb{S}_C$  according to the constraint  $J = 3$ . We obtain a highest rate check degree of  $K_{30} = 30$  and lowest rate check degree  $K_4 = 4$ . Solving the rate constraint (8) and the degree distribution constraint (10) for the coefficients, we obtain

$$\lambda(x) = x^2 \quad (11)$$

$$\rho(x) = 0.3536x^3 + 0.6464x^{29}. \quad (12)$$

This ensemble has an average variable node degree  $\bar{J} = 3$  and average check node degree  $\bar{K} = 9.096$ .

The direct neighbor mix approach incorporates all members of  $\mathbb{S}_C$  as it implies a pairwise "interpolation" between members of the set. To get a direct neighbor mix, take two regular ensembles  $\mathcal{E}_i$  and  $\mathcal{E}_j$  with consecutive rates from the set  $\mathbb{S}_C$ , i.e. with  $R_i \geq R \geq R_j$  and  $R \notin \mathbb{S}_C$ . Mixing  $\mathcal{E}_i$  and  $\mathcal{E}_j$  yields an ensemble  $\mathcal{E}_k$  with rate  $R \in [R_j, R_i]$ . The resulting degree

distribution is given by (10). The union of all rate intervals formed by the mixture of consecutive ensembles from  $\mathbb{S}_C$  covers the rate range employed by the highest and lowest rate ensemble from  $\mathbb{S}_C$ . Therefore, all code rates within  $\mathbb{S}_C$  are achievable with the neighboring mixture.

**Example IV.3** (Direct neighbor mix for  $R = 0.67$  with  $J = 4$ ). To obtain the rate  $R = 0.67$  with given constraint, e.g.  $J = 4$ , first choose  $\mathbb{S}_C$  according to  $J = 4$ . The  $(4, 10)$  and  $(4, 16)$  regular LDPC codes from  $\mathbb{S}_C$  cover the rate interval  $R \in [0.6, 0.75]$ . The two regular LDPC code ensembles are mixed according to (10). The coefficients are solved to obtain the desired rate  $R = 0.67$  which yields

$$\lambda(x) = x^3 \quad (13)$$

$$\rho(x) = 0.5372x^9 + 0.4628x^{15}. \quad (14)$$

The average check degree of this ensemble is  $\bar{K} = 12.1$ , while  $\bar{J} = J = 4$ .

Using the above mentioned two step construction, various mixing ensembles can be defined. Other mixing options incorporating more than two check degrees are also possible but restricting the degree distributions to be only slightly irregular is beneficial in terms of implementation. Note, that as we allow only ensembles with variable node degrees  $J \geq 3$ , a linear distance growth is guaranteed for the nearly-regular LDPC code ensembles [15].

## V. RESULTS AND DISCUSSION

In the following we make use of the threshold saturation effect of spatial coupling for the nearly-regular LDPC code ensembles. We therefore use the irregular extension of density evolution for coupled irregular LDPC code ensembles  $(\lambda, \rho)$ . The difference to the above mentioned regular spatially coupled case is that at each time instant  $t$  we now assume variable nodes with irregular degree distributions. The randomized spreading of edges over  $w$  time instants is similar to the regular case. The ensemble definition is an extension to [4]. This defines a  $(\lambda, \rho, L, w)$  coupled irregular LDPC code ensemble. The design rate is given in the following

**Lemma V.1** (Design Rate). The design rate of a  $(\lambda, \rho, L, w)$  coupled irregular LDPC code ensemble with  $w \leq L$  is given by

$$R = 1 - (1 - R(\lambda, \rho)) \frac{L - w - 1 + 2 \sum_{i=0}^{w-1} (1 - \gamma(\frac{w-i-1}{w}))}{L} \quad (15)$$

with  $\gamma(x)$  as the check degree distribution of the underlying irregular ensemble from a node perspective.

*Proof:* The proof follows directly from [4, Lemma 3]. Starting from the definition of a coupled regular LDPC code ensemble, the variable and check degrees  $J$  and  $K$  appear to be random variables in the irregular LDPC code ensemble case. The irregular ensemble is built by taking the expected value over the possible regular edge connections of the regular ensemble. ■

As the fraction in (15) tends to one when  $L \rightarrow \infty$  and  $w < L$ , the rate of the irregular coupled ensemble converges to the design rate of the underlying irregular ensemble. To assess the threshold improvement for the coupled irregular LDPC code

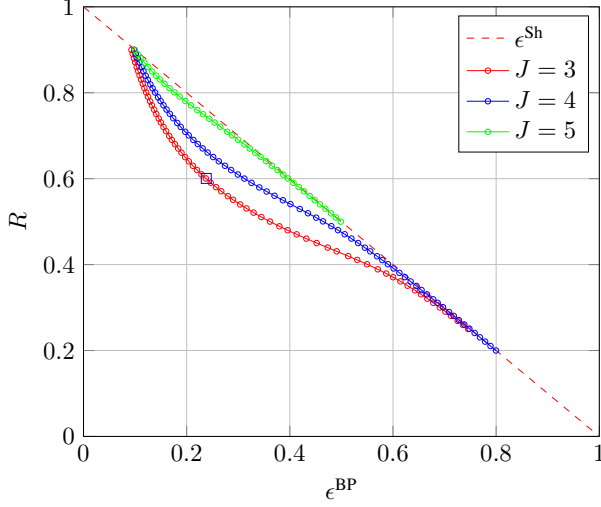


Fig. 4. BP thresholds  $\epsilon^{BP}$  of boundary mix ensembles for  $w = 3$  and  $L \rightarrow \infty$

ensembles, the BP thresholds for the binary erasure channel have to be calculated via density evolution. Here we also give the extension of the above mentioned regular DE equations to the irregular case in the following

**Definition V.2** (DE for coupled irregular LDPC code ensembles). Given a coupled irregular LDPC code ensemble  $(\lambda, \rho, L, w)$ , assume belief propagation decoding over the binary erasure channel with erasure probability  $\epsilon$ . Define  $x_i^{(l)}$  to be the erasure probability of variable nodes at position  $t$  with  $t \in [0, L-1]$  after  $l$  iterations. Then the density evolution update equation is given as

$$x_t^{(l)} = \epsilon \lambda \left( \frac{1}{w} \sum_{j=0}^{w-1} 1 - \rho \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{t+j-k}^{(l-1)} \right) \right). \quad (16)$$

Using density evolution as given in Definition V.2, a BP threshold  $\epsilon^{BP}$  for an  $NR(\lambda, \rho, L, w)$  ensemble over the BEC can be computed.

While the possibilities of constructing mixed ensembles based on the above mentioned two step approach are endless, we focus on specific examples derived from the shortcomings of unfavorable regular codes in the following. The set  $\mathbb{S}_C$  is restricted to be chosen based on a variable node degree constraint from  $\mathbb{S}$  with  $A = 0.01$ . We fix  $J \in \{3, 4, 5\}$ . Using these sets  $\mathbb{S}_C$ , we then apply the direct neighbor mix and boundary mix ensembles to obtain nearly-regular LDPC code ensembles in the rate range  $R \in [0.1, 0.9]$ . We further apply spatial coupling to the nearly-regular mixing ensembles from Section IV to get the benefit of threshold improvement. Therefore in the following,  $NR(\lambda, \rho, L, w)$  ensembles are evaluated. Belief propagation decoding is assumed with transmission over the BEC and we calculate density evolution thresholds  $\epsilon^{BP}$ . Table I shows exemplary parameters and thresholds of neighboring and boundary mix ensembles for specific rates for  $J = 3$  as well as the mixing sets used in evaluation.

Figure 4 shows the boundary mixing ensembles for  $J \in \{3, 4, 5\}$ . While all three ensemble sets are covering a highest rate of  $R = 0.9$  the lowest rate of each ensemble set differs.

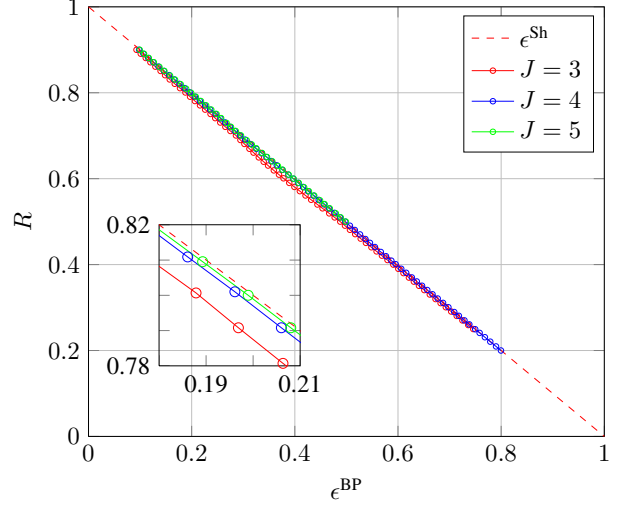


Fig. 5. BP thresholds  $\epsilon^{BP}$  of direct neighbor mix ensembles for  $w = 3$  and  $L \rightarrow \infty$

The ensemble set with  $J = 4$  achieves a rate as low as  $R = 0.2$ . The ensemble set with  $J = 5$  on the other hand restricts the interval of supported rates only down to  $R = 0.5$ . The performance of the three sets has to be compared to the Shannon limit  $\epsilon^{Sh}$ . One can observe that for high as well as for low rates, the performance for all ensemble sets is very close to the Shannon limit.

Only in the intermediate rate regime  $R \approx 0.6$  the performance drifts away from the Shannon limit for the ensemble sets with lower variable node degree  $J \in \{3, 4\}$ . This behavior can be circumvented by increasing the degree, as the ensemble set with  $J = 5$  shows. This can be explained with the threshold saturation phenomenon. The BP threshold of the coupled nearly-regular LDPC code ensemble is close to the MAP threshold of the uncoupled ensemble. When the variable degree is increased while maintaining the rate, the MAP threshold converges to the Shannon limit. The ordering of performance for the three different variable node degrees that were considered, underlines this explanation. For comparison, the MAP threshold for the nearly-regular boundary mix ensemble with  $J = 3$  and rate  $R = 0.6$  is shown in the graph as well (blue square). The spatially coupled nearly-regular ensemble achieves the MAP threshold, but the MAP threshold is simply far away from the Shannon limit. The boundary mix construction shows some limitations in terms of performance for the ensemble sets with lower variable node degree.

The performance for the direct neighbor mix for the nearly-regular LDPC code ensembles is shown in Fig. 5. The three considered ensemble sets cover the same rate intervals as their boundary mix counterparts, due to the same sets  $\mathbb{S}_C$  in both cases. One can observe that all three ensemble sets perform very close to the Shannon limit over the complete rate interval that is supported. Only minor deviations from the Shannon limit can be seen in case of the lower variable node degrees. The inset in Fig. 5 again shows the convergence behavior of the threshold saturation for increasing variable node degrees but here the differences between the considered variable node degrees are negligible. A variable node degree of  $J = 3$  is already sufficient. Although the performance is very close to the Shannon limit a drawback can be that several different

TABLE I. NEARLY-REGULAR LDPC CONVOLUTIONAL CODE ENSEMBLES: THRESHOLDS AND MIXING SET EXAMPLES

Rate	Neighboring mix with $J = 3$					Boundary mix with $J = 3$ and $(K_i, K_j) = (30, 4)$		
	$\rho_{K_i}$	$K_i$	$K_j$	$\epsilon^{BP}$	Gap	$\rho_{K_i}$	$\epsilon^{BP}$	Gap
0.80	0.0132	20	15	0.1878	0.0114	0.8471	0.1215	0.0777
0.70	0.0119	12	10	0.2858	0.0135	0.6913	0.1662	0.1344
0.60	0.5042	10	6	0.3780	0.0211	0.5392	0.2372	0.1622
0.50	0	10	6	0.4880	0.0119	0.3848	0.3614	0.1384
0.40	0	6	5	0.5909	0.0090	0.2319	0.5497	0.0495
0.30	0.3380	5	4	0.6927	0.0066	0.0769	0.6908	0.0091

(a) Exemplary values for nearly regular degree distributions with gap to capacity ( $\rho_{K_j} = 1 - \rho_{K_i}$ )

	$J = 3$	$J = 4$	$J = 5$
$K$	30, 4	40, 5	50, 10

(b)  $(J, K)$  in  $\mathbb{S}_C$  for boundary mix

	$K$
$J = 3$	30, 25, 20, 15, 12, 10, 6, 5, 4
$J = 4$	40, 25, 20, 16, 10, 8, 5
$J = 5$	50, 25, 20, 10

(c)  $(J, K)$  in  $\mathbb{S}_C$  for neighboring mix

check degrees were mixed to cover the whole rate range. A trade off between higher degrees for the boundary mix or different check degree mixes depending on the rate interval has to be evaluated regarding implementation.

The complexity as mentioned above is influenced from the variable degree as well as the window size of the BP decoder, which scales with  $w$ . The boundary mix shows good performance only for the cases where additional complexity ( $J = 5$ ) has to be spent. The direct neighbor mix shows a negligible gap to the Shannon limit already for variable node degree  $J = 3$  and its complexity lies on the lower bound. This might be the best choice when an implementation can afford many different mixtures of check degrees but keeping the operations per variable node as low as possible. Note that all the coupled nearly-regular LDPC code ensembles shown in Fig. 4 and Fig. 5 have  $w = 3$  which is sufficient for good performance close to capacity while keeping the complexity of the windowed BP decoder low.

## VI. CONCLUSION

In this paper we recapitulated the performance of spatially coupled regular LDPC code ensembles. The rate-flexibility of regular LDPC codes was investigated, which unveiled that while regular LDPC codes at certain favorable rates have very low complexity and good performance, other rates turn out to perform very poorly and having high decoding complexity. To overcome this issue we introduced a new class of nearly-regular LDPC code ensembles that are built upon the mixture of two favorable regular codes of same variable node degree. These codes exhibit performance on the binary erasure channel close to the Shannon limit for all rates in the considered rate interval, while having a decoder complexity as low as for the best regular codes. The exclusion of variable nodes of degree two in the construction ensures that the minimum distance of the proposed ensembles increases linearly with the block length.

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