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Evolutionary dynamics and a refinement of the neutral stability criterion

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Abstract

We introduce two refinements of the neutral stability criterion, namely the *ascending* and the *eroding* neutrally stable strategies (NSS). These criteria take into account how well the NSS preform against all pure strategies in symmetric two-player games. We also present a dynamic model which supports the refinements.

Keywords: Evolutionary dynamics; Neutrally stable strategies; Ascending NSS; Eroding NSS.

JEL classification: C73.

1 Introduction

Evolutionary game theory mainly focus on frequency-dependent selection, i.e. the fitness of a strategy depends on the frequency of other strategies in the population. Typically, individuals are repeatedly drawn at random from a large population to play a game which they are programmed to play a certain way, i.e. to use a particular strategy σ. Mutation are viewed as a rare phenomena that only happens to a small fraction of the population at a time.[1] The incumbent strategy σ is said to satisfy an evolutionary stability criteria if mutants, i.e. individuals who are programmed to use another strategy μ, perform poorly in comparison. The key concept is the criterion of evolutionary stability (Maynard Smith and Price, 1973); σ is a

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[1] Mutations are treated as isolated events.

evolutionarily stable strategy (ESS) if it is a best reply to itself and it is a better reply to any alternative best reply μ than μ is to itself. Unfortunately, many games lack ESS. To deal with mutant strategies that preform equally good as the incumbent strategy, the weaker criterion of neutral stability (Maynard Smith, 1982) is used; σ is a *neutrally stable strategy (NSS)* if it is a best reply to itself and it is an at least as good reply to any alternative best reply μ than μ is to itself.

These evolutionary stability criteria are robustness tests and never explain how a strategy became the incumbent strategy in the first place. Thus, by definition, several NSS can coexist in a population. In this paper we wish to distinguish between the NSS and therefor subject the neutral stability criterion to refinements. For symmetric two-player games, we distinguish between the *ascending* and the *eroding* NSS. These criteria take into account the performance of the NSS against all other (out-of-equilibrium) strategies in the game.

Let σ be an NSS and let μ be another strategy such that the two strategies are best replies to themselves and to each other. In this case, we refer to μ as an alternative best reply to σ. If μ also happens to be an NSS, we refer to it as a competitor to σ. Strategy σ is said to be an *ascending neutrally stable strategy (ANSS)* if it is; (i) an ESS, or (ii) an NSS which weakly dominates all of its alternative best replies. Strategy σ is said to be an *eroding neutrally stable strategy (ENSS)* if it is an NSS and is weakly dominated by at least one of its alternative best replies.

Dynamic support for evolutionary stability has mainly been sought in the research field of population dynamics. The basic idea of population dynamics is the same as in evolutionary game theory; when a population is subjected to a small change in its composition, it should not drift away by the evolutionary forces. Taylor and Jonker (1978) was first to study the connection between the static concepts of evolutionary stability and the analyses of population dynamics.[2] In their classical model of *replicator dynamics*, strategies are genetically determined and individuals using high-payoff strategies have more offsprings than competitors using low-payoff strategies. Since a population state $x = (x_1, ..., x_n)$, where x_s is the relative frequency of individuals using the pure strategy s, can be interpreted as a mixed strategy, it is easy to check for stability. We derive one obvious result with this model. From a population state where all types are present, including an ENSS, the replicator dynamics ensures that either the population share using the ENSS or the population share using the pure strategy that makes the ENSS worse than

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[2] Other related studies are Thomas (1985a; 1985b) and Bomze and Weibull (1995), among others.

at least one of its alternative best replies (or both these population shares) vanish in time. We cannot, however, use the classical replicator dynamics to motivate the refinements of the neutral stability concept. It focus entirely on the frequency-dependent selection, while the new criteria rely on the presence of all types (strategies) in the population. One possible way is to accommodate for mutations. Models that include mutation have a long history in population dynamics (e.g. see Kingman, 1978; Hines, 1982; Hofbauer and Sigmund, 1988; Bürger, 1989b; Foster and Young, 1990; Eshel, 1991; Eshel *et al.*, 1997; Boylan, 1994; Bomze and Bürger, 1994; 1995).[3] It is evident from the literature that mutation can be modeled in a variety of ways (see Boylan, 1994).[4]

In our model, types breed true (i.e. no mutation) and the rate of reproduction is frequency-dependent. The perturbations, normally accomplished by mutations, is achieved by letting a fraction γ of the population emigrate and be replaced by the same amount of immigrants. This is no isolated event, it happens continuously. We assume that the immigrants are distributed uniformly on the pure strategies in the game. Our replicator dynamics with migration is then given by

$$\dot{x}_s(t) = [\pi(s,x) - \pi(x,x)]\, x_s + \gamma(1/n - x_s) \quad \text{for } s \in S,$$

where $\pi(s,x)$ is the payoff from playing strategy s against (the population) strategy x. We are interested in the solution path, denoted $\xi(x,t)$, to this system of differential equations. It turns out that it for many games converge to a state, i.e. $\lim_{t\to\infty}\xi(x,t) = x^*$. We are especially interested in the limit state, i.e. $\lim_{t\to\infty}\xi(x,t) = x^*$ as we let γ slowly tend to zero. Unfortunately, the conjecture that successive perturbations would shift the population from using ENSS to using ANSS do not generally hold. By definition, the pure ANSS will do better than a competing pure ENSS in this dynamics. Still, the limit state can be a mixed ENSS or even a mixed strategy that is an alternative best reply to the NSS.[5]

In order to support the refinements, we present a dynamics with reassessment and life-support. There is no population growth and no reproduction in the model. Instead a fraction of the population reassess their choice of strategy continuously. Only pure strategies may be used. As long as a pure strategy, say s, is currently performing better than the others, a fraction of

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[3] Other related work are Zeeman (1980, 1981), Eshel (1983), Schuster and Sigmund (1983), Rowe (1985), Bomze (1986) and Bürger (1989a) Binmore et al. (1995).

[4] An alternative approach is to describe the mutation-selection mechanism with a Markov process, see e.g. Kandori *et al.* (1993) and Young (1993).

[5] Given that the limit state exists and that it home in on the set of competing NSS or on the set of alternative best replies to an NSS.

them using the other strategies will change to s. The proportion of the population using a particular strategy can, however, never fall below $\mu > 0$. It turns out that the limit state in this dynamics puts all weight on the strictly or weakly dominant pure strategy, and no weight on pure strategies that are strictly or weakly dominated by another pure strategy. Thus, the limit state in this dynamic cannot be an ENSS.

The rest of the paper is organized as follows. Section 2 presents notation and the evolutionary stability criteria. Section 3 consists of three parts. In the first part, we describe the replicator dynamics, define concepts and present some fundamental results with this dynamics. In the second part, we derive a replicator dynamics with migration and study its effect on the limit states. In the third part, we present a dynamics with reassessment and life-support, which means that all pure strategies are played by a minimum proportion of the population. We close the paper with some final comments in Section 4.

2 Evolutionary stability criteria

Throughout the paper we make our arguments for symmetric two-player games $G = (I, \mathcal{S}, \pi)$, where $I = \{1, 2\}$ is the set of *players*, $\mathcal{S} = S \times S$ is the *pure strategy space* and $\pi : \mathcal{S} \to \mathbb{R}^2$ is the *combined payoff function*. For the finite set of pure strategies $S = \{1, ..., n\}$, the *set of mixed strategies* is $\Delta = \left\{\sigma \in \mathbb{R}^n_+ \mid \sum_{s \in S} \sigma_s = 1\right\}$. Let e_s denote the unit vectors, e.g. $e_2 = (0, 1, 0, ..., 0)$. Hence, $\{e_s\}_{s=1}^n$ is the set of *vertices* of Δ which assigns probability one to the pure strategy $s \in S$. Let $C(\sigma)$ denote the *support* of the (mixed) strategy σ, i.e. $C(\sigma) \subset S$ is the set of pure strategies that is assigned positive probabilities by σ. The subset of completely mixed strategies is called the *interior* of Δ, $\text{int}(\Delta) = \{\sigma \in \Delta \mid \sigma_s > 0 \quad \forall s \in S\}$. Obviously, $C(\sigma) = S$ if $\sigma \in \text{int}(\Delta)$. The subset of noninterior strategies is called the *boundary* of Δ, $\text{bd}(\Delta) = \{\sigma \in \Delta \mid \sigma \notin \text{int}(\Delta)\}$. The mixed-strategy space is $\Delta^2 = \Delta \times \Delta$.

A (strategy) *profile* is a vector $(\alpha, \sigma) \in \Delta^2$ of individual strategies. Let $\pi(\alpha, \sigma)$ denote the *payoff* from playing strategy α when the opponent is playing strategy σ. A strategy α is said to be a best reply to strategy σ if and only if $\pi(\alpha, \sigma) \geq \pi(\tau, \sigma)$ for all $\tau \in \Delta$. The *set of best replies* to any strategy $\sigma \in \Delta$ is denoted $B(\sigma)$. A profile $(\alpha, \sigma) \in \Delta^2$ is a *Nash equilibrium (NE)* if and only if $\alpha \in B(\sigma)$ and $\sigma \in B(\alpha)$. Let Δ^2_{NE} denote the *set of Nash equilibria* in G.

2.1 Evolutionary and neutral stability

Two of the key concepts in evolutionary game theory are the criteria of *evolutionary stability* and *neutral stability.*

Definition 1 (Maynard Smith and Price, 1973) $\sigma \in \Delta$ *is an* evolutionarily stable strategy (ESS) *if* $\forall \mu \in \Delta$ *(*$\mu \neq \sigma$*) it holds that (i)* $\pi(\sigma,\sigma) \geq \pi(\mu,\sigma)$*, and (ii)* $\pi(\sigma,\sigma) = \pi(\mu,\sigma)$ *implies* $\pi(\sigma,\mu) > \pi(\mu,\mu)$*.*

Definition 2 (Maynard Smith, 1982) $\sigma \in \Delta$ *is an* neutrally stable strategy (NSS) *if* $\forall \mu \in \Delta$ *(*$\mu \neq \sigma$*) it holds that (i)* $\pi(\sigma,\sigma) \geq \pi(\mu,\sigma)$*, and (ii)* $\pi(\sigma,\sigma) = \pi(\mu,\sigma)$ *implies* $\pi(\sigma,\mu) \geq \pi(\mu,\mu)$*.*

Let Δ^{ESS} denote the (possibly empty) subset of evolutionary stable strategies in Δ, and let Δ^{NSS} denote the (possibly empty) subset of neutrally stable strategies in Δ. The relationship between the criteria is $\Delta^{ESS} \subset \Delta^{NSS}$.

Remark 1 *Obviously, there exist other criteria besides the two presented above. Bomze and Pötscher (1989) calls an ESS "uninvadable" if it has a uniform invasion barrier against all other strategies. Other examples are "weakly evolutionarily stable strategies" by Thomas (1985b), "robustness against equilibrium entrants" by Swinkel (1992), "continuously stable strategy" by Eshel and Motro (1981), "neighborhood invader strategy" by Apaloo (1997) and "evolutionary robustness" by Oechssler and Riedel (2002), to name a few. Measures to strengthen the criterion of neutral stability has also been taken before this paper. Bomze and Weibull (1995) calls an NSS "unbeatable" if it has a uniform invasion barrier against all other strategies.*[6] *Another example is the criterion of "modified evolutionary stability", introduced by Binmore and Samuelson (1992), in which complexity considerations are taken if the strategies are alternative best replies to each other.*

2.2 Ascending and eroding neutrally stable strategies

Let $\mu \in \Delta$ be such that $\pi(\sigma,\sigma) = \pi(\mu,\sigma)$ and $\pi(\sigma,\mu) = \pi(\mu,\mu)$ when $\sigma \in \Delta^{NSS}$ in some game G. We refer to μ as an *alternative best reply* to σ. If, and only if, $\mu \in \Delta^{NSS}$ we also refer to μ as a competitor to σ. The refinements of the neutral stability criterion we suggest are defined as follows.[7]

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[6] They also defines "strongly unbeatable strategies".

[7] The idea behind these refinements resembles the idea behind a concept by Hofbauer and Sigmund (1988, p.288) called *weak evolutionarily stable strategy*, which incorporates the effect of trembles for asymmetric normal form games.

Definition 3 *$\sigma \in \Delta^{NSS}$ is an* ascending NSS *if it weakly dominates all of its alternative best replies (if such strategies exist). That is, if there exists $\mu \in \Delta$ $(\mu \neq \sigma)$ such that $\pi(\sigma,\sigma) = \pi(\mu,\sigma)$ and $\pi(\sigma,\mu) = \pi(\mu,\mu)$, then $\pi(\sigma,\gamma) \geq \pi(\mu,\gamma)$ for all $\gamma \in \Delta$ and $\exists\alpha \in \Delta$ $(\alpha \neq \sigma,\mu)$ such that $\pi(\sigma,\alpha) > \pi(\mu,\alpha)$.*

Definition 4 *$\sigma \in \Delta^{NSS}$ is an* eroding NSS *if it is weakly dominated by at least one of its alternative best replies , i.e. there exists $\mu \in \Delta$ $(\mu \neq \sigma)$, such that $\pi(\sigma,\sigma) = \pi(\mu,\sigma)$ and $\pi(\sigma,\mu) = \pi(\mu,\mu)$, where $\pi(\sigma,\gamma) \leq \pi(\mu,\gamma)$ for all $\gamma \in \Delta$ and $\exists\alpha \in \Delta$ $(\alpha \neq \sigma,\mu)$ such that $\pi(\sigma,\alpha) < \pi(\mu,\alpha)$.*

Let Δ^{ANSS} and Δ^{ENSS} denote the (possibly empty) subsets of ascending and eroding NSS in Δ^{NSS}. The relationship between the criteria are $\Delta^{ESS} \subset \Delta^{ANSS} \subset \Delta^{NSS}$ and $\Delta^{ENSS} \subset \Delta^{NSS}$. Let $\sigma \in \Delta^{NSS}$ such that $\sigma \notin \Delta^{ANSS}$ and $\sigma \notin \Delta^{ENSS}$ be called a *firm NSS*.

Notice that the mixed strategy α in Definitions 3 and 4 can be replaced by a pure strategy. As a consequence, σ is better (worse) than μ against at least all $\alpha \in \text{int}(\Delta)$ in Definition 3 (Definition 4). Clearly, weak dominance ensures that $\pi(\sigma,s) \geq \pi(\mu,s)$ for all $s \in S$ in Definition 3. Since σ is better than μ against α, there exists at least one $s \in S$ for which $\pi(\sigma,s) > \pi(\mu,s)$. Let $D \subset S$ be the set of these pure strategies. We have that $\pi(\sigma,\alpha) > \pi(\mu,\alpha)$ for all $\alpha \in \{\alpha \in \Delta \mid C(\alpha) \subset D\}$ in Definition 3. Obviously, the opposite is true in Definition 4.

Example 1 *Consider the symmetric two-player game G_1 with payoff matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (1)$$

where $a_{xy} \in A$ is interpreted as the payoff from playing strategy $x \in S$ when the opponent is playing strategy $y \in S$, $S = \{1,2,3\}$. It is easy to verify that $\Delta^{NSS} \neq \emptyset$ and $\Delta^{ESS} = \emptyset$. The set of neutrally stable strategies is also large, $\Delta^{NSS} = \{\sigma \in \Delta \mid \sigma_3 = 0\}$.[8]

Obviously, for any $\alpha \in B(e_1) = \{\alpha \in \Delta \mid \alpha_3 = 0\}$ we have $\pi(\alpha,\alpha) = \pi(e_1,\alpha) = \pi(\alpha,e_1) = \pi(e_1,e_1) = 2$. Let $\alpha \in \Delta^{NSS}$ $(\alpha \neq e_1)$. For any $\tau = \{\tau \in \Delta \mid \tau_3 > 0\}$, we have $\pi(\alpha,\tau) \neq \pi(e_1,\tau)$. The exact payoffs are

$$\pi(\alpha,\tau) = \alpha_1(2\tau_1 + 2\tau_2 + \tau_3) + \alpha_2(2\tau_1 + 2\tau_2), \qquad (2)$$

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[8] Strategy 1 (which can also be expressed by e_1) can invade every mixed strategy whose support contains strategy 3.

$$\pi\left(e_1,\tau\right)=2\tau_1+2\tau_2+\tau_3. \tag{3}$$

So $\pi\left(\alpha,\tau\right)<\pi\left(e_1,\tau\right)$ *if* $\alpha_1<1$*, which is the case for all* $\alpha\in B\left(e_1\right)$ *besides* e_1 *itself. Thus,* e_1 *will earn more against all* $\tau\in\Delta\setminus\Delta^{NSS}$ *compared to* $\alpha\in\Delta^{NSS}$ $(\alpha\neq e_1)$*. Hence,* $\Delta^{ANSS}=\{e_1\}$ *and* $\Delta^{ENSS}=\Delta^{NSS}\setminus\{e_1\}$*.*

3 Evolutionary dynamics

This section consists of three parts. In the first part, we present the classic "replicator dynamics" developed by Taylor and Jonker (1978). In the second part, we derive a model in which a proportion of the population emigrates and are replaced by an equal number of immigrants. In the third and final part, we presents a dynamic model which supports our refinement of the neutral stability criterion.

3.1 The replicator dynamics

The replicator dynamics considers a population whose members are playing a symmetric two-player game $G=(I,\mathcal{S},\pi)$ with random match. In contrast to evolutionary game theory, the (classic) replicator dynamics presumes that individuals only can play pure strategies.[9] This does not exclude mixed strategies from the model. A mixed strategy $\sigma\in\Delta$ corresponds to the (polymorphic) population strategy, whose weights σ_s are represented by the population shares programmed to use the pure strategy $s\in S$. The fitness of a strategy is given by its (relative) payoff and measured by the number of replicators it produces.

Formally, let a nonempty subset of the set of pure strategies be called a *repertoire*, denoted $R\subset S$. The strategies $s\in R$ are also referred to as *types* currently present in the population. At any point of time, the *relative frequency* of type $s\in R$ in the population is denoted by $x_s\left(t\right)\geq0$. The *population state* is represented by the vector $x\left(t\right)=\left(x_s\left(t\right),...,x_v\left(t\right)\right)$, where $s,v\in R$. Since $x\left(t\right)$ is a point in the simplex Δ, it corresponds to the mean strategy of the population. For a population state $x\left(t\right)$, the *expected payoff* to player type $s\in R$ with random match is denoted $\pi\left(e_s,x\left(t\right)\right)$. The average payoff in the population is $\pi\left(x\left(t\right),x\left(t\right)\right)=\sum_{s\in R}x_s\left(t\right)\pi\left(e_s,x\left(t\right)\right)$.

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[9] Replicator dynamics with mixed strategies have been investigated by Weissing (1990) and Cressman (1990), but the repertoire of types (i.e. mixed strategies) in the population is still limited to a finite subset of the simplex Δ (see Hammarstein and Selten, 1994, p. 950). An exception is the model by Bomze and Bürger (1994; 1995).

The *replicator dynamics* is given by[10]

$$\dot{x}_s(t) = [\pi(e_s, x(t)) - \pi(x(t), x(t))]\, x_s(t) \quad \text{for } s \in R. \tag{4}$$

Hence, all types present in the population that are associated with a better-than-average payoff grows at the expense of those types associated with a lower-than-average payoff. There is no population growth and no mutations (i.e. types breed true). The repertoire only contains the types that were present in the initial population, so by default only their relative frequency $x_s(t)$ may change.

It can be proved that the system of differential equations (4) defines a continuous solution mapping $\xi : \Delta \times \mathbb{R} \rightarrow \Delta$, which to each initial population state $x^o \in \Delta$ assigns as the population state $\xi(x^o, t)$ at time t.[11] The system (4) can be expressed in vector form, i.e.

$$\dot{x} = \varphi(x), \tag{5}$$

where $\varphi : \Delta \rightarrow \mathbb{R}^n$ $(\Delta \subset \mathbb{R}^n)$. The function φ is a *vector field* and it defines, at each state $x \in \Delta$, the direction and velocity of the change of the state. For each component x_s of the state x, $\varphi_s(x)$ is its time derivative $(s \in S)$. A *solution* to (5) is a function $\xi : \Delta \times \mathbb{R} \rightarrow \Delta$ such that

$$\frac{d}{dt}\xi(x,t) = \varphi[\xi(x,t)] \qquad \text{for all } x \in \Delta \text{ and } t \in \mathbb{R}. \tag{6}$$

If the vector field is sufficiently smooth, i.e. satisfy Lipschitz continuity, the existence and uniqueness of a solution is guaranteed. The vector field is *Lipschitz continuous* on a domain Δ^* if there exists a nonnegative real number L (the Lipschitz constant) such that $\|\varphi(x) - \varphi(y)\| \leq L\|x - y\|$ for all $x, y \in \Delta^*$. The right hand side of (5) is Lipschitz-continuous since it is a polynomial in the population shares.[12] The existence and uniqueness of a solution is then ensured by the following theorem.[13]

Theorem 1 (The Picard-Lindelöf Theorem) *If φ is Lipschitz continuous on an open domain Δ^* containing Δ, then the system (5) has a unique solution where $\xi(x^o, 0) = x^o$ and $\xi(x^o, \cdot) : \Delta \times \mathbb{R} \rightarrow \Delta$ through every state $x^o \in \Delta$. Moreover $\xi(x,t)$ is continuous in x^o and t.*

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[10] See Weibull (1995, pp. 72-73, 124-125) or Samuelson (1997, pp. 64-66).

[11] x^o denotes the state of origin in the analysis.

[12] See Samuelson (1997, p. 67) or Weibull (1995, p. 74 and p. 232).

[13] For proof, see Hirsch and Small (1974, Chapter 8) or Hale (1969). For more on Lipschitz continuity, see Weibull (1995, Chapter 6).

Next we present some useful concepts. The (solution) *trajectory* (or path) passing through x^o is the graph of how the state evolves over time,

$$\tau\left(x^o\right)=\left\{(x,t)\in\Delta\times\mathbb{R}\mid x=\xi\left(x^o,t\right)\right\}. \tag{7}$$

The (solution) *orbit* passing through x^o is the graph of which states are reached (but it does not say when), $\omega\left(x^o\right)=\left\{x\in\Delta\mid x=\xi\left(x^o,t\right)\ \text{ for } t\in\mathbb{R}\right\}$. The *forward orbit* is $\omega^+\left(x^o\right)=\left\{x\in\Delta\mid x=\xi\left(x^o,t\right)\ \text{ for } t\geq 0\right\}$. A set Δ is *invariant* if $\omega\left(x^o\right)\subset\Delta$ for all $x^o\in\Delta$ and it is *forward invariant* if $\omega^+\left(x^o\right)\subset\Delta$ for all $x^o\in\Delta$.

It is easy to see that the simplex Δ is invariant in the replicator dynamics, and so is bd(Δ) and int(Δ). By definition, $\sum_{s\in R}x_s\left(t\right)=1$ and $x_s\left(t\right)>0$. This implies two things. First, the growth rate of all types are finite, i.e. $\sum_{s\in R}\dot{x}_s=0$. Secondly, the relative frequency of a type in the population can get very small but never reach zero. Hence, an interior orbit can only converge to the boundary as time goes to infinity, i.e. $\omega\left(x^o\right)\in$ int(Δ) if $x^o\in$ int(Δ). Since strategies excluded from the repertoire cannot be introduced later on, a orbit starting from the boundary of the simplex can never leave it, i.e. $\omega\left(x^o\right)\in$ bd(Δ) if $x^o\in$ bd(Δ).

The standard concepts of stable population states are defined as follows.

Definition 5 *The state $x\in\Delta$ is* ***stationary*** *if $\xi\left(x,t\right)=x$ for all $t\in\mathbb{R}$. The state x is* ***Lyapunov stable*** *if it is a stationary state with the property that for any neighborhood U of x, there exists another neighborhood V contained in U such that if $x^o\in V$ then $\xi\left(x^o,t\right)\in U$ for all $t>0$, i.e. any state path which starts in V remains in U. The state x is* ***asymptotically stable*** *if it is Lyapunov stable and there exists a neighborhood W of x such that $\lim_{t\to\infty}\xi\left(x^o,t\right)=x$ if $x^o\in W$, i.e. any state path which starts in W converges to x.*

Figure 1 informally describes orbits, stable states and unstable states in the simplex of game G_1 (described in Example 1). A stationary state x coincides with the null vectors ($\varphi\left(x\right)=0$) and there are several stationary states in G_1. Three of them (x,y and z) are indicated in Figure 1, but all the NSS located along the base of the simplex are also stationary states.[14] The other two stationary states, $y=(0,0,1)$ and $z=(0,1/3,2/3)$, are unstable. State y is vulnerable to any perturbation which includes strategy 1, and z is vulnerable to any perturbation.

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[14] Strategies 1 and 2 are best replies to themselves and to each other, so the vector field are null for any mixed population (e.g. x) of the two types.

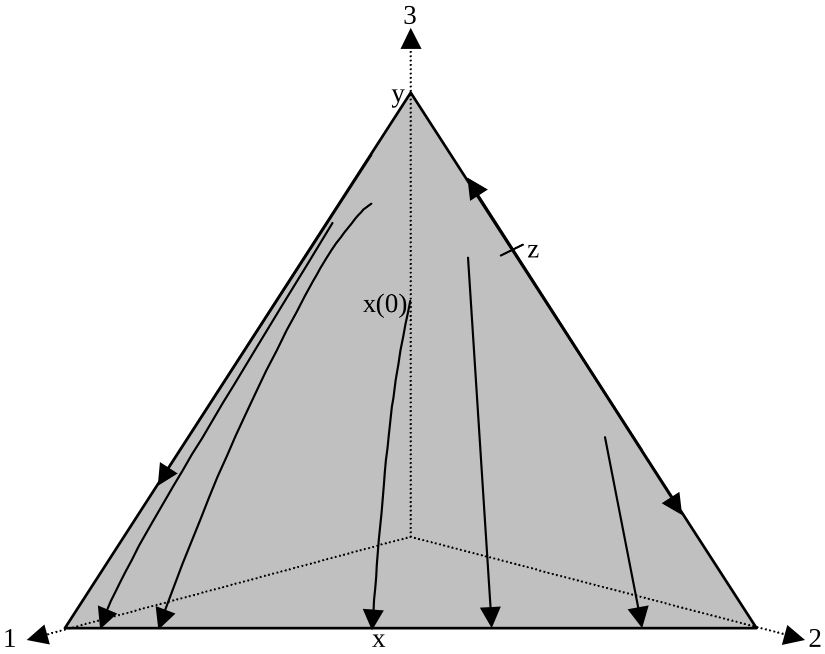

Figure 1: Orbits and stationary states in G_1.

All forward orbits starting from nonstationary states, except for

$$v = \{v \in \ \mathrm{bd}\,(\Delta) \mid v_3 \in (2/3, 1) \ \text{ and } v_1 = 0\}\,, \tag{8}$$

approach the (base) boundary $\{\sigma \in \Delta \mid \sigma_3 = 0\}$ in time. Consider the forward orbit from state $x\,(0)$, where an equal share of the population use strategies 1 and 2 while the residual use strategy 3. It illustrates how strategy 1 (the ANSS) preforms much better than strategy 2 (the pure ENSS) in the presence of strategy 3. As strategy 3 is asymptotically eliminated this advantage diminish and the end result is a decrease of the relative frequency of strategy 2 (not elimination) and an increase of the relative frequency of strategy 1. Notice that even though the orbit converges to state x, this point is never reached and consequently the relative frequency of strategy 3 never reach zero. Hence, if the system starts from a nonstationary state it will never reach such a state. However, an interesting result is that if the orbit converges to a state, then this state must be stationary (Weibull, 1995, Proposition 6.3).

Notice that the system does not have to converge to a state in order to be Lyapunov stable. Definition 5 does not require that the forward orbit $\omega^{+}(z)$ return the population to state x, but more importantly, that it does not stray too far from x.

A straightforward result is:

Proposition 1 *Every* $x \in \Delta^{ANSS}$ *and* $x \in \Delta^{ENSS}$ *for a symmetric two-player game* G *are Lyapunov stable in replicator dynamics (4). Every* $x \in \Delta^{ANSS}$*, such that* $x \in \Delta^{ESS}$ *, for a symmetric two-player game* G *is asymptotically stable in replicator dynamics (4).*

Proof. Taylor and Jonker (1978) established that every $x \in \Delta^{ESS}$ for a symmetric two-player game G is asymptotically stable in replicator dynamics (4). Thomas (1985a) and Bomze and Weibull (1995) established that every $x \in \Delta^{NSS}$ for a symmetric two-player game G is Lyapunov stable in replicator dynamics (4). By definition, $\Delta^{ANSS}, \Delta^{ENSS} \subset \Delta^{NSS}$ and an ESS also is an ANSS so our claim is true. ■

In Figure 1, x is a state in which strategies 1 and 2, but not strategy 3, are represented in the repertoire. A small perturbation to a nearby state is accomplished by letting a small proportion of both types switch to strategy 3. This perturbation disturbs the dynamic equilibrium. The replicator dynamics will asymptotically eliminate type 3 (the mutant), decrease the relative frequency of type 2 (the pure ENSS) and increase the relative frequency of type 1 (the ANSS). In fact, since all ENSS are weakly dominated can we use a result by Weibull (1995, p. 83, Proposition 3.2) and claim:

Proposition 2 *Let* $\mu \in \Delta^{ENSS}$*, then for some* $v \in C(\mu)$ *there exists* $\sigma \in \Delta$ $(\sigma \neq v)$ *such that* $\pi(\sigma, s) \geq \pi(v, s)$ *for all* $s \in S$ *and strict inequality for some* $s' \in S$*, and consequently* $\xi_v(x^o, t)_{t\to\infty} \to 0$ *or* $\xi_{s'}(x^o, t)_{t\to\infty} \to 0$ *(or both) for any* $x^o \in$ *int*(Δ)*.*

Proof. From the definition of an ENSS we know that μ is weakly dominated by at least one of its alternative best replies σ. The effect, $\xi_v(x^o, t)_{t\to\infty} \to 0$ or $\xi_{s'}(x^o, t)_{t\to\infty} \to 0$ (or both) for any $x^o \in \text{int}(\Delta)$, is given by Proposition 3.2 in Weibull (1995, p. 83). ■

This result coincide with another result by Samuelson (1997, Proposition 4.5.1, p. 122); the dynamic process cannot converge from an interior initial state to an outcome that places all of the probability on a pure strategy weakly dominated by another pure strategy. Both these results are promising, since they suggests that successive perturbations might push the population from an ENSS to an ANSS.[15] Next, we accommodate for successive perturbation in the replicator dynamics to investigate this conjecture.

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[15] Given that such strategies exists in a game.

3.2 A replicator dynamics with migration

In order to accommodate for successive perturbations in the replicator dynamics we add migration as a factor in the model. Let $N_s(t)$ be the number of people playing strategy $s \in S$ and let $N(t)$ be the total number of people in the population at time t. Then, as before, $x_s(t) = N_s(t)/N(t)$ is the proportion of type s in the population and $x(t)$ the vector of proportions describing the population state. Let $x(t) \in \Delta$ also be interpreted as the population strategy. In each period of time of length τ, a fraction of the population reproduce asexually. Types breed true and a reproducing individual of type s produce $\pi(s, x(t))$ replicators of type s (nothing more, less or different). During the same length of time, a fraction γ of the population emigrates and are replaced by the same amount of immigrants. The types of the immigrants is assumed to be independent across individuals and time. We assume that each strategy is equally likely to be adopted by an immigrant. Thus the probability for a particular strategy to be adopted by an immigrant is $1/n$. In order to get continuous population shares we implicitly assumes that the population size $N \to \infty$, so even for a very small γ the law of large numbers suggests that $1/n$ of the immigrants adopt strategy $s \in S$.

The number of people playing strategy s at time $t + \tau$ is then given by

$$N_s(t+\tau) = N_s \cdot [1 + \tau\pi(s,x)](1-\gamma\tau) + \frac{\gamma\tau}{n}\sum_{i\in S} N_i \cdot [1+\tau\pi(i,x)] \tag{9}$$

and the total population by

$$N(t+\tau) = \sum_{i\in S} N_i \cdot [1+\tau\pi(i,x)]\,. \tag{10}$$

Notice that the time indexes on the right-hand side has been dropped, but they are always t (i.e. $N_s(t)$ and $x(t)$ instead of N_s and x). From now on we write π_s instead of $\pi(s,x)$. By dividing (9) with (10) we obtain

$$x_s(t+\tau) = \frac{N_s \cdot [1+\tau\pi_s](1-\gamma\tau) + \frac{1}{n}\gamma\tau\sum_{i\in S} N_i \cdot [1+\tau\pi_i]}{\sum_{i\in S} N_i \cdot [1+\tau\pi_i]}. \tag{11}$$

Next, we divide both the numerator and the denominator on the right-hand side of (11) by $N(t)$ and obtain

$$x_s(t+\tau) = \frac{x_s [1+\tau\pi_s](1-\gamma\tau) + \frac{1}{n}\gamma\tau\sum_{i\in S} x_i [1+\tau\pi_i]}{\sum_{i\in S} x_i [1+\tau\pi_i]}. \tag{12}$$

Since the average payoff in the population is given by $\pi_x = \pi(x,x) = \sum_{i\in S} x_i \pi(i,x)$ and $\sum_{i\in S} x_i = 1$ by definition, (12) can be expressed

$$x_s(t+\tau) = \frac{x_s [1+\tau\pi_s](1-\gamma\tau) + \frac{1}{n}\gamma\tau [1+\tau\pi_x]}{1+\tau\pi_x}. \tag{13}$$

Subtracting $x_s(t)$ from both side of (13) and simplification yields

$$x_s(t+\tau) - x_s(t) = \frac{x_s\left[\tau\pi_s - \gamma\tau - \gamma\tau^2\pi_s\right] + \frac{1}{n}\gamma\tau\left[1+\tau\pi_x\right] - x_s\tau\pi_x}{1+\tau\pi_x}. \tag{14}$$

Finally we divide both sides of (14) by τ and take the limit as $\tau \to 0$, i.e.

$$\dot{x}_s(t) = \lim_{\tau\to 0} \frac{x_s\left[\pi_s - \gamma - \gamma\tau\pi_s\right] + \frac{1}{n}\gamma\left[1+\tau\pi_x\right] - x_s\pi_x}{1+\tau\pi_x}. \tag{15}$$

Thus implying that the replicator dynamics with migration is

$$\dot{x}_s\left(t\right) = \left[\pi_s - \pi_x\right]x_s + \gamma\left(1/n - x_s\right) \quad \text{for } s \in S. \tag{16}$$

The dynamic model describes a situation in which the population evolve partly according to the the usual selection process and partly by the perturbations which favors strategies used by small parts of the population at the expense of strategies used by large parts of the population.

Remark 2 *We could introduce mutations, instead of migration, by assuming the replication process to be error-prone. In this case, the number of people playing strategy s at time $t+\tau$ is given by*

$$N_s(t+\tau) = N_s(t)\left[1 + \left(1-\gamma\right)\tau\pi\left(s,x\right)\right] + \frac{1}{n}\sum_{i\in S}\gamma N_i\left(t\right)\tau\pi\left(i,x\right), \tag{17}$$

where γ is the mutation rate. Then the replicator dynamics becomes

$$\dot{x}_s\left(t\right) = \left[\left(1-\gamma\right)\pi\left(s,x\right) - \pi\left(x,x\right)\right]x_s + \frac{\gamma}{n}\pi\left(x,x\right) \quad \textit{for } s \in S. \tag{18}$$

Another common dynamics is

$$\dot{x}_s\left(t\right) = \left(1-\gamma\right)\left[\pi\left(s,x\right) - \pi\left(x,x\right)\right]x_s + \gamma\left(\frac{1}{n} - x_s\right), \tag{19}$$

e.g. see Samuelson (1997, p. 127). A similar dynamics is used by Hofbauer and Sigmund (1988), Foster and Young (1990), Boylan (1994) and Bomze and Bürger (1994; 1995).

The introduction of migration affect the dynamics drastically. The continuous vector field $\varphi\left(x\right)$, given by the system (16), is always directed inwards on the boundary of the simplex Δ, i.e. $\varphi_s\left(x\right) > 0$ if $x_s > 0$. From (16) it is clear that the vector field is the sum of two vectors. The first vector is the same as in the replicator dynamics without migration, in which case the boundary of the simplex Δ was invariant. Hence, it always points within the boundary. The second vector, caused by migration, is always inward-pointing so the sum of the two vectors must be inward-pointing. As a consequent, the null vectors are replaced with vectors which all have the same velocity and all are directed towards the uniform mixed state $x' = (1/n, 1/n, ..., 1/n)$.

In Figure 2, this vector field is illustrated at three states by $\varphi(x)$, $\varphi(y)$ and $\varphi(z)$ for game G_1.

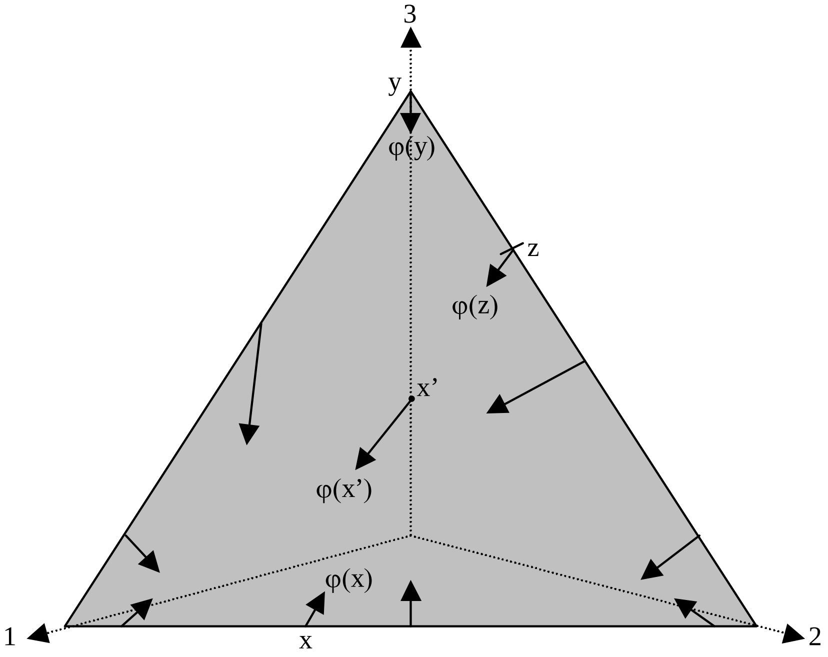

Figure 2: The vector field with migration in G_1.

This is the migration effect γ/n given by the last term in (16). The vector field at other states, except for x', is the sum of both terms in (16). The vector field at the uniform mixed state $\varphi(x')$ is only given by the first term in (16) since the emigration and immigration cancel each other out in this state ($x' = 1/n$ for all $s \in S$). The existence of a stationary point x^* in the interior of the simplex is now guaranteed by the *inward-pointing vector field theorem* (see Magill and Quinzee, 1996, Theorem 7.5, p. 59). This theorem does not ensure uniqueness or that the system converges to the stationary state(s). The forward orbit can get stuck in a cycle or in a complex pattern with many stationary states. Both existence and uniqueness of a solution is guaranteed by the Picard-Lindelöf theorem if the right-hand side of (16) is Lipschitz continuous. It is easy too see that this condition is satisfied. Let $\varphi^o(x)$ be the vector field given by (4), then the right-hand side of (16) can be expressed

$$\|\varphi(x)-\varphi(y)\| = \|\varphi^o(x)-\varphi^o(y)+\gamma(y-x)\| \quad \text{for all } x,y\in\Delta^*. \tag{20}$$

Clearly, we can still find a $L \in \mathbb{R}$ satisfying Lipschitz continuity.

Example 2 *Consider the forward orbits in game G_1, illustrated in Figure 3.*

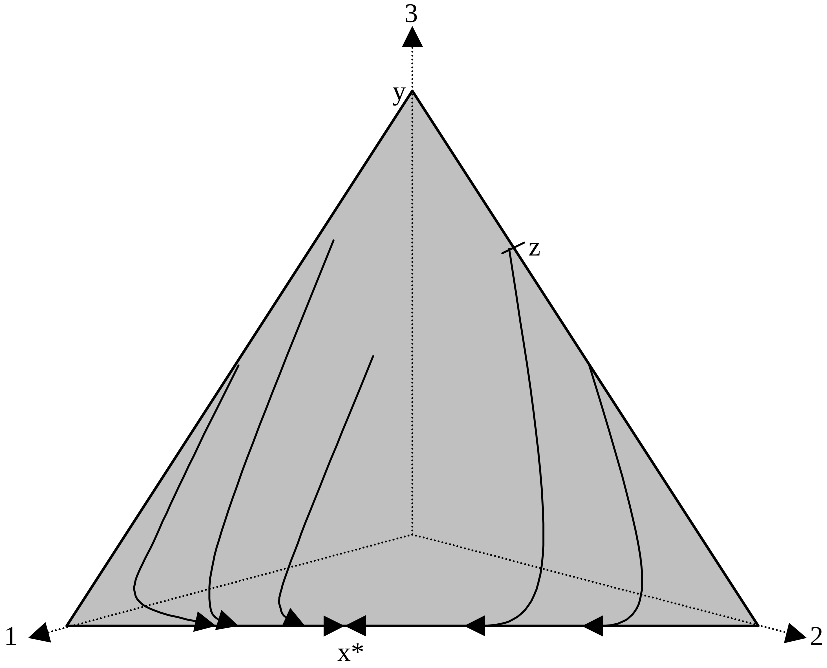

Figure 3: Forward orbits to the stationary state x^* in G_1.

For a small migration factor $\gamma = 0.000001$, corresponding to a migration of 1000 people in a population of a billion, the system converges to the state $x^* = (0.5633, 0.4367, 0)$. Somewhat surprisingly, strategy 2 (the pure ENSS) is not wiped out. The reason for this is dual effect migration has on a population. Immigration ensures that some tiny fraction always plays strategy 3, thus enforcing selection pressure from strategy 2 towards strategy 1. At the same time, migration is responsible for replacing more of the individuals playing strategy 1 with individuals playing strategies 2 and 3, than it is replacing individuals playing strategy 2 for the same purpose. The migration effect is direct, while the selection effect is diminishing the closer the population gets to the ANSS. Eventually the higher net loss of individuals from strategy 1 will halt the erosion of strategy 2 and the system converges to the mixed ENSS x^*.

Finally, we study the limit states, i.e. $\lim_{t\to\infty}\xi(x,t) \to x^*$, as we let γ slowly tend to zero.

Proposition 3 *Let x^* be the limit state of (16) for a symmetric two-player game G. Then,*

(i) $x_s^* > x_v^*$ if strategy s weakly dominates strategy v,

(ii) $x_s^* = 0$ if strategy s is strictly dominated by another pure strategy,

(iii) $x_s^* = 0$ if strategy s is iteratively strictly dominated by another pure strategy,

(iv) $x_s^* = x_v^*$ if s and v has identical payoffs against all other strategies.

Proof. Result (ii) has been shown by Akin (1980) and result (iii) by Samuelson and Zhang (1992) for $x \in \text{int}(\Delta)$. Migration will introduce all pure strategies into the population, so it will be true also when $x \in \Delta$. Result (i) follows from the fact that $\pi(s, x) > \pi(v, x)$ for all $x \in \text{int}(\Delta)$ and the migration effect will only work against strategy s when there are more people using it compared to there are people using strategy v. Result (iv) follows from the fact that migration will eventually even out the population shares, i.e. $\dot{x}_s - \dot{x}_v = \gamma(x_v - x_s) = 0$ when $x_v = x_s$. ■

In the dynamics above, the negative effect from the lesser payoff can be compensated by the positive effect from the successive perturbations. The ENSS survives in the population because eventually more people of this type immigrates than emigrates and this compensates the loss inflicted by the relatively low reproduction rate. The ANSS does not take over in the population because eventually more people of this type emigrates than immigrates and this offsets the gain caused by the relatively high reproduction rate. The same is true when migration is substituted by mutations; for the ENSS (ANSS) more (less) people mutates into its type than is lost (gained) in the same phenomenon.

3.3 A dynamics with reassessment and life-support

In order to support the refinements of the neutral stability criterion, we introduce a dynamics in which people continuously reassess their choice of strategy in the game but where strategies are protected from extinction by a life-support mechanism. There is no population growth and no reproduction in the model. In each period of time of length τ, a fraction of the population reassess their choice of strategy.

The number of people using strategy $s \in S$ in period $t + \tau$ is given by

$$N_s(t+\tau) = \max\left\{\hat{N}, \left[1 - \tau\frac{\pi(s, x(t))}{\pi(i, x(t))}\right] N_s(t)\right\} \tag{21}$$

if $\pi(s, x(t)) < \pi(i, x(t))$ for some $i \in S$. That is, a fraction of the people using strategy s will switch to strategy $i \in S$ if i is a best pure reply to the population strategy $x(t)$ while s is not. A strategy $i \in S$ is a best pure reply to population strategy $x(t)$ if $\pi(i, x(t)) \geq \pi(s, x(t))$ for all $s \in S$. The worse of a reply s is to $x(t)$ compared to i, the more people switch to i. There is, however, a limit to how low the number of people playing strategy $s \in S$ can become, which is denoted $\hat{N}$. If $s \in S$ is the best pure reply to the population strategy $x(t)$, then

$$N_s(t+\tau) = N_s(t) + \frac{\tau}{n+1} \sum_{p \in S^*} \frac{\pi(p, x(t))}{\pi(s, x(t))} N_p(t) \tag{22}$$

where $S^* \subset S$ is the set of strategies such that $\pi(s, x(t)) > \pi(p, x(t))$ for $p \in S^*$ and n is the number of pure strategies for which $\pi(s, x(t)) = \pi(i, x(t))$. Notice that $N_s(t+\tau)$ can be smaller than the expression in (22) if some p reach its lowest number of users. Dividing both sides with N yields[16]

$$x_s(t+\tau) = \max\left\{\mu, \left[1 - \tau \frac{\pi(s, x(t))}{\pi(i, x(t))}\right] x_s(t)\right\}, \tag{23}$$

and

$$x_s(t+\tau) = x_s(t) + \frac{\tau}{n+1} \sum_{p \in S^*} \frac{\pi(p, x(t))}{\pi(s, x(t))} x_p(t). \tag{24}$$

There is a subset of mixed strategies which becomes the *evolutionary playground*, i.e.

$$\Delta_{EP} = \{\sigma \in \Delta \mid x_s(t) \geq \mu \quad \forall s \in S\}. \tag{25}$$

Let $\text{int}(\Delta_{EP})$ be the *interior* of Δ_{EP}, i.e.

$$\text{int}(\Delta_{EP}) = \{\sigma \in \Delta_{EP} \mid \sigma_s > \mu \quad \forall s \in S\}. \tag{26}$$

The subset of noninterior strategies is called the *boundary* of Δ_{EP},

$$\text{bd}(\Delta_{EP}) = \{\sigma \in \Delta_{EP} \mid \sigma \notin \text{int}(\Delta_{EP})\}. \tag{27}$$

The dynamics is;

$$\begin{array}{rll} \dot{x}_s(t) = & -\frac{\pi(s,x(t))}{\pi(i,x(t))} \cdot x_s(t) & \text{if } x_s(t) > \mu \\ \dot{x}_s(t) = & 0 & \text{if } x_s(t) = \mu \end{array} \tag{28}$$

if $\pi(s, x(t)) < \pi(i, x(t))$ for $i \in S$, and

$$\dot{x}_s(t) = \frac{1}{n+1} \sum_{p \in S^*} \frac{\pi(p, x(t))}{\pi(s, x(t))} \cdot x_p(t) \tag{29}$$

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[16] In order to get continuous population shares we implicitly assumes that the population size $N \to \infty$. Also, let $\hat{N}/N = \mu$.

if $\pi(s, x(t)) > \pi(p, x(t))$ for $p \in S$ such that $x_p(t) > \mu$. (Again n is the number of pure strategies for which $\pi(s, x(t)) = \pi(i, x(t))$.)

We study the limit states, i.e. $\lim_{t\to\infty} \xi(x^o, t) \to x^*$, as we let μ slowly tend to zero.

Proposition 4 *Let x^* be the limit state of (28)-(29) for a symmetric two-player game G. Then, for any $x^o \in \Delta_{EP}$:*

(i) $x^*_s = 1$ if strategy s is a strictly or weakly dominant strategy,

(ii) $x^*_s = 0$ if strategy s is strictly or weakly dominated by another pure strategy.

Proof. We show that the result in (i) is true when s is a weakly dominant strategy, it must then also be true when s is a strictly dominant strategy. v is weakly dominated by s if $\pi(s, p) \geq \pi(v, p)$ for all $p \in S$ with strict inequality for at least one $p \in S$. It follows that $\pi(s, x) > \pi(v, x)$ for all $v \in S$, since $\omega(x^o) \subset \Delta_{EP}$. Thus, $\lim_{t\to\infty} x_v(t) = \mu$ for all $v \in S \setminus \{s\}$ and $[\lim_{t\to\infty} x_v(t)]_{\mu\to 0} = 0$.

We show that the result in (ii) is true when s is weakly dominated, it must then also be true when s is strictly dominated by another strategy. Let s be weakly dominated by p. Then $\pi(s, x) < \pi(p, x)$ for all $x \in \Delta_{EP}$. If $x^*_p > 0$, then $x^*_s = 0$ since people would change from s to p (and to all $v \in S$ such that $x^*_v > 0$). If $x^*_p = 0$, then $x^*_s = 0$ because if $\lim_{t\to\infty} x_p(t) = \mu$ then $\lim_{t\to\infty} x_s(t) = \mu$. ■

Proposition 5 *The limit state x^* of (28)-(29) for a symmetric two-player game G cannot be an ENSS, i.e. $x^* \notin \Delta^{ENSS}$.*

Proof. First, let $x^* = s \in S$ be a pure ENSS. By definition, s is weakly dominated and thus $x^*_s = 0$ according to Proposition 4, which is a contradiction. Second, let $x^* = \sigma \in \Delta$ be a mixed ENSS. According to Proposition 4, $x^*_v = 0$ if there exists $s \in S$ such that $\pi(s, x) > \pi(v, x)$ for $x \in \Delta_{EP}$, i.e. s weakly dominates v. By definition, σ is an ENSS if and only if there exists $v \in C(\sigma)$ and $s \in S$ such that s weakly dominates v. Again we have a contradiction. ■

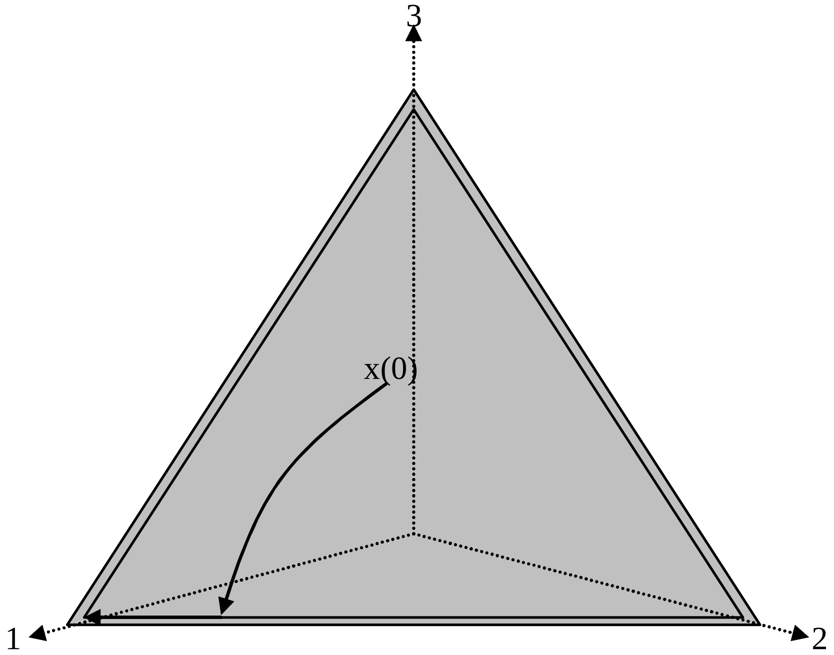

Figure 4: The evolutionary playground in G_1.

In Figure 4 we see how people using strategy 2 and 3 switch to strategy 1. Eventually, the people using strategy 3 reach the critical number and the life-support kicks in, keeping the proportion of the population using strategy 3 at μ. The people using strategy 2 is still getting lesser payoff than the people using strategy 1, thus people are changing to strategy 1 from strategy 2. We have $\lim_{t\to\infty}(x_1(t), x_2(t), x_3(t)) = (1-2\mu, \mu, \mu)$.

4 Final comments

The conjecture that successive perturbations would shift the population to using an ANSS does not generally hold in the replicator dynamics with migration. The refinement is better supported in the dynamics with reassessment and life-support. This dynamics is, in comparison, less plausible and needs to be motivated.

The reassessment part is closely related to learning and imitation, thus it is the life-support mechanism that is hard to motivate. One can think of a social game in which everybody wants to be accepted but also to be special. A real-life example is youth subcultures. The desire to be accepted causes uniformity up to a point. When almost all players are using the same strategy (e.g. likes the same music or have the same look), each individual is to some extent prepared to switch to a less profitable strategy (e.g. listen to alternative music or switch to a less common look) given that it is special enough. The limitation of the dynamics and the corresponding interpretation imply that the criteria of ANSS and ENSS cannot be used as general as the ESS and NSS criteria.

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