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Introduction

Professor Don Ihde introduced the term 'visualism' in his book *Expanding Hermeneutics*; *Visualism in Science*, from 1998. The introduction of the term aimed to the development of a hermeneutics of science focusing in 'visualization' of mental contents as the typical bearer of scientific truth. Ihde proposes the expansion of hermeneutical studies to the field of *technoscience* making a phenomenological interpretation of visual contents in the technoscientific discourse. In this project Ihde concluded that from the earlier times of modernity, hermeneutics grew apart from science, making rationalism, empiricism and later positivism the standard interpretations of science:

> The overarching aim here is to argue that we have often misconstrued what science is and how it operates because, in part, we have for so long ceded the interpretation of science to forms of positivism. In what I call the 'H-P Binary'—the contestation between hermeneutics and positivism—hermeneutics first finds itself divorced from the sciences, and then by its own historical proponents made semiautonomous with respect to its interpretive activities in such a way that positivism simply became the standard for framing the understanding of the sciences. What I call the 'P-H tradition'—the phenomenological version of hermeneutics—often itself simply accepted this binary, and until recently tended to ignore attempts to enter the domains of science praxis and the understanding of same.[1]

Consequently, the 'H-P Binary' is the point of departure of Ihde's project, and the actualisation of the 'P—H tradition' to the post-modern era, is its actual goal. Ihde structures his project in a 'weak' respectively a 'strong' research program. In the frame of a 'weak' program for the implicit hermeneutics within science, Ihde distinguished between pure Gestalt features—as the appearance of a figure against a ground—and "a related, but different, set of visualizations, which bear much stronger relations to what be taken as 'textlike' features. […]." This second group of depictions with textlike qualities (we will label this group of depictions as "text-depictions") is not the group of "journals, electronic publications, books" generated within the scientific activity which "always remain secondary or tertiary with respect to science" but a kind of hybrid between pure visualizations and texts:

> So this is not the textlike phenomenon I have in mind; instead, I am pointing to those analogues of texts which permeate science:

[1] Ihde, Don. *Expanding Hermeneutics. Visualism in Science*. Northwestern University Press, 1998; p. 3.

> charts, graphs, models, and the whole range of "readable" inscriptions which remain visual, but which are no longer isomorphic with the referent objects or "things themselves."[2]

It is our intention here to follow professor Ihde's research project but moving inquires from the field of natural sciences to the field of mathematics. Our intention is to discern how some textlike depictions have been taken from the everyday life into a mathematical reality, without much consideration about the implications of such transcriptions. We think that mathematics is the field of science in which a renewed 'P-H tradition' can be most useful. The differences between pure Gestalt features and textlike visualizations are very important for the purpose of our study and we will reinforce their importance introducing the idea of 'dimensionality of thought' as their main intrinsic difference. These visualizations are special aspects of the visual scientific imaginary because they have the power of being 'proof-producer'. From the time of Galileo these proof-producers depictions have played an essential role in the historical divorce between hermeneutics and science, *hiding* the nearer connection between proof and everyday life.

The development of modern mathematics and logic shows from its beginnings a marked inclination to the handling of visual representations whose 'dimensional' character (as visual imaginary) has not yet been considered from the point of view of the 'P—H tradition'. It could be said that from the origin of Western thought (e.g. Porphyry's tree) but specially from the flourishing of modernity after Galileo and Descartes' analytical geometry, all modern knowledge has been impregnated of visual constructions whose *dimensional character* continue being 'unconscious' to us. A common denominator of all these visual constructions is to represent a certain type of 'logic visual reality', which could be illustrated by John Venn's (1834 –1923) configurations of circles. The geometric constructions in logic, works generally as analogies but they are more than that, they are *parallel phenomenological worlds.* In any case, the conclusion must be that text-depictions and their relations can express logical realities because the logical process can be followed visually without any other help. It is as we could speak of a 'visual logic' that can be used *instead* of symbolic languages.

The 'diagonal' proof of Cantor

The truthfulness of a visual transcription of a mental contents can be studied for instance in the cases of numerical representations in modern mathematics. The study of series of numbers, 'which are matched 1–1'; or the case of the idea of 'cut' in the series of the Real numbers; or the case of the 'diagonal' proof of Cantor.

[2] Ihde, Don. Op.cit. p. 158-167.

Many logicians and mathematicians have noticed the importance of images in the validating process of truth and understood this as an epistemological problem for positivism. Influenced by the traditional view that the visual representation is *less accurate* than the symbolic, they have tried to substitute images with symbols. We think that it is possible to relate this understanding to the rationalist epistemology of the scientific revolution (Galileo and Descartes) rather than to the empiricist aspect of the same epistemological process. However, as a consequence of rationalism, the efforts of formalism in logic and mathematics to replace the visual representations by pure symbols, had ignored the fact that all symbols, all symbolic series, every term or sentence of an artificial language is as well a visual representation, because of the dimensional character of thought. Don Ihde refers to this aspect in science:

> Of course, there are always holdouts and these usually are found among physicists. Today those who want to hold to imperceptibility belong to the quantum mechanicians who often claim that the spooky parts of quantum phenomena cannot be visualized, but are understood only through mathematics—echoing Galilean metaphysics, not Galilean practice. This is not something new: on the contrary, the trajectory toward more "textlike'" hermeneutics remains within science itself. Some scientists do not like "pictures" and prefer formulas. Others recognize the value of the "aha" quality of getting a depiction. Here is a precise counterpart to the tension between the "textualists" among post-modern critical theory and phenomenological perceptualist hermeneuts as found in the humanities.[3]

Normally the term 'dimension' is used meaning two very different realities: first the *size* of something and secondly the *dignity* of a representation (it is to say the character 0–dimensional, 1–dimensional, 2–dimensional, 3–dimensional, etc. of one representation). When for example mathematicians work with the idea of infinity, they do it referring to the notion of the size of a set. However, considering the *dignity* of the visualization of an infinite set, we immediately understand the *relative* character of its *size*. The *size* and *dignity* of the representation of a set depends of the observer's own dimensionality. We could say that for God there is no 'infinity'.

I will illustrate the complexity of the problem with an example. The diagonal–proof of Georg Cantor (1845–1918) is one of the fundamental keys of modern mathematics. It consists of a triangular representation of an imagined succession of numbers. Cantor introduced the method in question to make *rigorous* the study of infinite sets of numbers and its relations.

[3] Ihde, Don. Op.cit. p. 5.

The method allowed the ordering of infinite sets as transfinite, that is, as infinite sets of *different size*. Cantor tried to demonstrate among other things, that the set of real numbers is not countable; that is, that it cannot be put in a 1–1 relation (in pairs) together with the set of natural numbers. The property of being 'countable', supposes the congruence between any set with the set of natural numbers. The conclusion of the proof would be to demonstrate that the set of real numbers is of a *higher infinity* than that of the natural numbers. The analysis of the dignity of the representation of Cantor's proof reveals that it handles two different scales of dimensionality *simultaneously*. The construction of Cantor aligns the real numbers and the natural numbers in pairs in the following way:

```
(1) 0.x d d d d d....
       \
(2) 0.d x d d d d.....
          \
(3) 0.d d x d d d.....
             \
(4) 0.d d d x d d x...
.....................
(k).................\
```

Figure 1: Cantor's diagonal proof

A 'diagonal number' that does not belong to the original list may be constructed replacing the 'x' in the diagonal list. Observe the geometric character of the construction. Notice that the construction of the 'missing' real number of the list arises as the hypotenuse of a triangle. Nevertheless, the hypotenuse of a triangle is always larger than the triangle's legs; therefore, the 'new number' may only be a new representation of a number of the original list but now in a 'new size–dimension'.

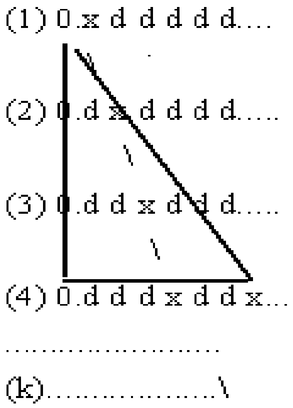

Figure 2: Cantor's diagonal proof 'visualized'

How can we know that Cantor's proof in fact proves what it is meant to prove? It is evident that the visual diagonal in the demonstration of Cantor tries only to be a selection–method and does not work as a truthful 'geometric' diagonal (the diagonal is in some sense a 'text-depiction'). However, the success of the selection–technique rests precisely in the fact of being a *geometric construction*. Without the visual representation of a diagonal, there is no proof. Without the change of status in the representation of real numbers, a 'new number' cannot be produced. The great dilemma is then to know if the number constructed by Cantor is nothing else but the original representation presented through a new dimensionality. Wittgenstein wrote about this:

> The following sentence sounds to sober: 'If something is called to series of real numbers, then the expansion given by the diagonal procedure is also called too 'real number' and is to moreover said to be different from all members of the series'. Our suspicion ought always to be aroused when to proof proves means dwells than it allow it: Something of this sort might be called 'to puffed–up proof'.[4]

The underlying problem is that of the notion of dignity of a representation, a problem that still lacks philosophical precision. The demonstration in diagonal supposes the handling of depictions that represent numbers. The effectiveness of the proof rests in a dimensional *incongruence,* in the fact that pictures are understood as symbolic representations. The value of the proof is then comparable to the value of the following proof of my own:

> If all the men of the world align themselves in a row properly arranged, it is possible to proof that it shall always be possible to construct a man diagonally with the parts of the aligned men of the original row. A diagonal–man with the hair of the first man, the eyes of the second, [...] etc.

The proof ignores the forceful fact that the constructed man is nothing else but the representation of an individual man on a completely different size–scale.

[4] Wittgenstein L. *Remarks on the Foundation of Mathematics*. Basil Blackwell. Oxford, 1956. II-3, 56.

Numerals as text-depictions of numbers

When working with logic and mathematics, we handled semantic and syntactic aspects simultaneously and it is important to pay attention to their differences. Working with 'numerals' for example, would seem to impose the abandonment of the complex visual semantics of mathematical representations of different size and different dignity in benefit of pure syntactical considerations of the pure d=1–level. Numerals in fact, as text-depictions of numbers, occult the intuitive connections underlying mathematical thought and its visual representations, and convert mathematical language to an artificial one. In fact, the natural connections between numbers as symbols, and between numbers as spatial representations, disappear behind the numeral. The numeral is in this sense *opaque*.

The use of numerals became regular with the work of the Italian mathematician Giuseppe Peano (1858–1932) and his work with the axiomatization of arithmetic. A numeral is a logic predicate as 'next to…' In this way if the '0' is defined, it is possible to derivate the number '1' as 'next to 0'. If we identify the expression 'next to…' = S, then can we express '1' as S (0). The addition '2+2 = 4' can then be expressed as 'S(S (0)) + S(S0)) = S(S(S(S(0)))). The introduction of numerals can be justified because it reduces the number of symbols needed to express mathematical contents. However, as a negative consequence, the mathematical terms became much longer, took up more space and time to read and much more time to understand because they became *unintuitive*. The numeral introduces the problem of the intuitive perception of mathematical content. The human mind does not work as well with logical 'reductions' as machines do. The human mind needs in any case, to translate numerals to numbers to think them mathematically. About this Wittgenstein wrote:

> […] you can easily come to believe that the expression of an equation is a tautology. That e.g. 28 + 16 = 44 might be expressed in the following way:
>
> (E28x) bx . (E16x) mx ind.: > (E44x) bx v mx
>
> This expression is a tautology. But in order to find the number on the right–hand side that turns this expression into a tautology, you have to use calculus, and this calculus is entirely independent of tautology. Tautology is an application of the calculus, not its expression.[5]

Although Wittgenstein never handled the notion of the size and dignity of a logical or a mathematical representation, and phenomenology was not his project, it would be possible to say that he anticipated it in a remarkable way.

[5] Waismann F. *Ludwig Wittgenstein and the Vienna Circle*. Blackwell. Oxford, 1979; p. 106.

The process towards the simplification of representation in mathematics and logic was justified as the natural defence of science against the vagueness of intuition. Connected to this problem was the idea of 'rigorous thinking'.

The properties of depiction and its relation to hermeneutics

The connection between texts and depictions in mathematics is characterized by the following rules:

a) The interpretation is *relatively independent* of the circumstances. As in the rules of chess, it is not too difficult to see the same patterns and describe them as rules, which are different from the factual game–situation.

b) The connection reveals phenomenological properties and is a part of the everyday praxis associated to the depiction. A mathematical 'cut' is exactly the same 'cut' as it is in everyday life.

c) The connection does not depend on the semantic level. It is not e.g. the contents of an equation—its factual meaning—that make the interpretation possible, but the text-depicting figure features of the equation—e.g. that which we call the *kind* of equation—that makes it work. It is not some hidden mathematical meaning, which makes Cantor's diagonal proof work, it works because the proof release the power of intuition associated to everyday praxis.

d) The connection is *operable*, it arises from the work with rules that are known and can be described. There is a mechanical component that reinforces the artificial character of the proof assuring its epistemological value.

Our study begins then, putting aside the obvious meaning of a mathematical content trying to find phenomenological features that connects texts-depictions with everyday realities. Strictness in logic and mathematics, reaches through putting representations in 'opaque' terms, trying to elude everydayness intuition. A great part of modern work in mathematics and logic has been done through the blackout of everydayness intuitions. However, this blocking of everydayness in mathematics can be uncovered by hermeneutic studies.

Julius Dedekind's 'cut'

Julius Dedekind (1831-1916) produced a historical definition based on the idea of a 'cut' in the series of Real numbers. The ideal 'cut' divides the rational numbers into two sets, in which all the members of one upper-set are greater than the members of the lower-set.

An irrational number is then defined as the number that fills up the gap between the upper and lower class. For instance, taken the example of the square root of 2, we put all the negative numbers and the numbers whose squares are less than 2 into the lower class, and the positive numbers whose squares are greater than 2 into the upper class. Once again we quoted Wittgenstein's criticism:

> The misleading thing about Dedekind's conception is the idea that the real number are there spread out in the number line. They may be known or not; that does not matter. And in this way all that one need to do is to cut or divide into classes, and one has dealt with them all. It is by combining calculation and construction that one gets the idea that there must be a point left out on the straight line, […]. What is the application of the concept of a straight line in which a point is missing?[6]

There is a very important and unconscious manipulation of text-depicting gestalts in Dedekind's construction, the praxis of cutting and separating, the praxis of finding things spread around in suitable successions make this proof a master piece of art rather than a scientific result. That talks a lot about the nature of mathematical knowledge, which is in fact deeply rooted in the everyday world.

Visualism inside and outside science

In an article from 1980[7], W. J. T. Mitchell proposed the systematic study of visual representation in literature in all its forms, the study of text-depictions without limits. Mitchell proposed a study of the structures and forms imbedded in the literary work as intended and not intended depictions, as obvious and not obvious visual contents, as explicit and as metaphorical depicting relations. Mitchell's project reminds Ihde's; both Ihde's and Mitchell's projects works against a very old institution within Western thought, the canon of the *idea*. There is 'the power of the idea over the image', the authority of 'pure thought' as the bearer of truth. This tradition begins with Plato and Greek philosophy and gets its definitive consolidation with Descartes' antagonism between the body and the mind. Responding to such critical standpoint with the article *Diagrammatology* from 1981[8], W. J. T. Mitchell criticizes the Platonist point of view 'that asserts that form is an abstract, non sensible concept' and defends the opposite understanding of form as 'concrete, spatial, visible, and diagrammatic'. Mitchell asserts that:

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6 Wittgenstein L. *Remarks on the Foundation of Mathematics*..IV-37, p.151e.

7 Mitchell, W. J. T. *Spatial Form in Literature: Toward a General Theory*. Critical Inquiry, Vol. 6, No. 3. (Spring, 1980), pp. 539-567.

8 Surette, Leon. *Rational Form in Literature.* Critical Inquiry, Vol. 7, No. 3. (Spring, 1981), pp. 612-621.

'[…] we seem unable to articulate our intuitions or interpretations of formal characteristics in literature and the other arts except by recourse to 'sensible' or 'spatial' constructs (not just diagrams and not just visual forms)[…]'.[9]

Consequently, if our thinking needs the recourse of 'sensible' or 'spatial' constructs:

'[…] why not do it explicitly, consciously, and, most important, systematically? If we cannot get at form except through the mediation of things like diagrams, do we not then need something like a diagrammatology, a systematic study of the way that relationships among elements are represented and interpreted by graphic constructions?'

Mitchell's perspective includes two very important issues that are relevant in any post-modern theory of knowledge, but also in any post-modern ontology. The first is to decide how important depictions are in the process of thinking and communicating. The second is to decide how important depictions are as constructs of the ontologies of the world. There are also ontological aspects and ontical aspects associated to the real nature of 'diagrams'. According to Mitchell's argumentation, those aspects can be the content of a new discipline that he names 'diagrammatology'. Mitchell agrees about the complexity of this task when he writes:

Our ability to anticipate, to follow, or to experience surprise in any temporally unfolding process, artistic or otherwise, is, I would suggest, based in senses of form which shape our responses. Whether these formal senses are internal representations of spatial structures that let us know what 'place' we are occupying in a discourse, a symphony, or a sentence, or whether these structures only exist as posterior and analytic representations of behavioral competence is a question that, so far as I know, has not yet been decided. Until it has, I think it best to assume that our analytic structural representations are nothing more (or less) than representations in diagrammatic space of something that occurs in a virtual or mental space.[10]

According to Mitchell, 'diagrams' have a space of their own, a place that allows the transcription of 'virtual realities' or of 'mental activities' into depictions. Nevertheless, even if we assume that 'our analytic structural representations are nothing more (or less) than representations in diagrammatic space of something that occurs in a virtual or mental space, they constitute a hermeneutical problem which has to be treated as such in any discipline which works systematically.

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[9] W. J. T. Mitchell. *Diagrammatology.* Critical Inquiry, Vol. 7, No. 3. (Spring, 1981), pp. 622-633. The University of Chicago Press, p. 622.

[10] Wittgenstein L. *Remarks on the Foundation of Mathematics*. Basil Blackwell. Oxford, 1956. p. 623-624.

The phenomenological aspects of such a discipline have to include a discussion about conventions, because any depicted ontology can only be a ‘construction’, a formal structure that support our understanding of the world but which always accepts other equivalent solutions. The idealization of the immediate world as a computer would show it, through a new kind of geometry[11], is one of the fundamental issues of science today. During the last twenty years, the traditional belief in the possibility of the representation of a thing, through the measurement of variables that are congruent which each other, has almost disappeared. With the term ‘congruent’, we mean the rules that govern how things dock with each other. Congruency expresses that the spatial properties of a thing can be compared with each other to allow a representation of that thing as an individual in a group of individuals. In this respect, Mitchell wants to criticize that the ‘concept of form is independent of both space and time and is better represented in the pure ratios of numerical or algebraic expressions.’ Mitchell writes:

> We might reply first that algebra and numbers are no ‘purer’ than geometrical figures; they are simply a kind of arbitrary notation, like writing, which can be read in a temporal sequence and which can often be represented with stunning accuracy and power by geometrical figures.[12]

As an example, Mitchell discusses Euclid’s quadratic equation: a2 + 2ab + b2 = (a + b) (a+ b) and its spatial representation:

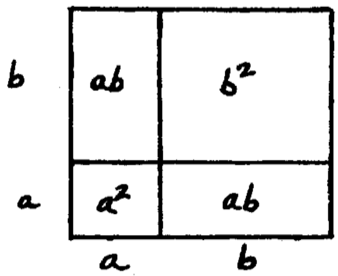

Figure 3 –Euclid’s quadratic equation

With Mitchell's words:

> Now it is true that the quantities *a* and *b* can be whatever you like, but the square can be drawn any size you like as well. In both cases, we understand that what is being represented is a ratio among elements, but I daresay we understand this ratio considerably better when we have both the algebraic expression and its graphic equivalent.

However, why is the spatial representation necessary? What is it that the spatial representation adds to the pure idea? Mitchell again:

[11] Mandelbrot, B. *The fractal Geometry of Nature*. W. H. Freeman and CO., New York.
[12] Op. cit. p.625; 1982.

> Does the existence of these alternative modes of expression and their mutual translatability prove the existence of an abstract concept that lies behind both? Or is this abstraction simply a figure of speech which arises when we are able to make this sort of translation, like our notion that there is an Ur–narrative which underlies all versions of a story?

History shows that the visual representations of abstract ideas were very common even before Descartes. Nevertheless, as an important act of foundations of modern science, the father of Analytical Geometry radically separated pure thought from the world as extension. A consequence of that is that mathematical visual representations presented a conceptualist ontology that reduced their transcendental properties to text-depictions hidden their direct connection to the intuition of everyday life.

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